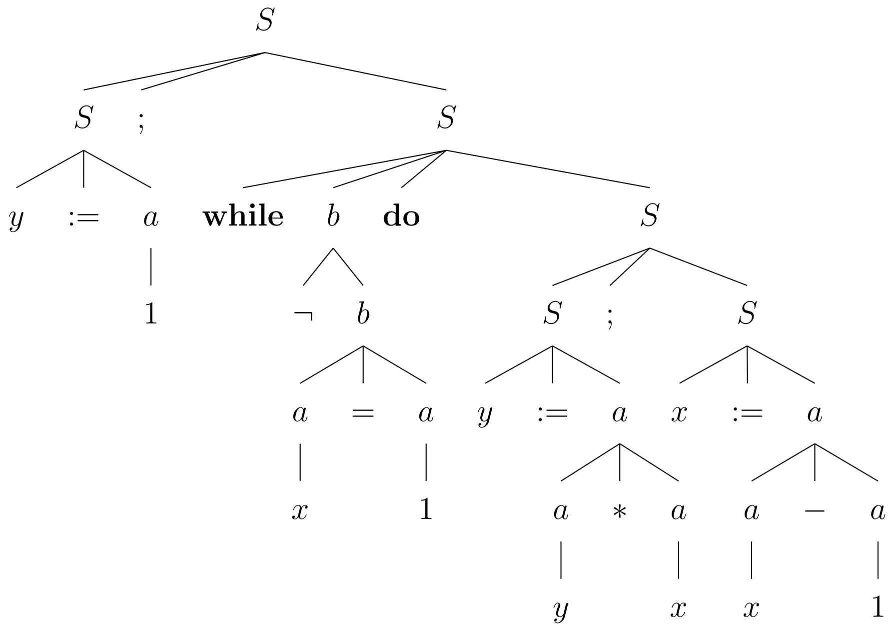
1. Introduction

Exercise 1.1

The following is a statement in **While**:

$$y := 1;\ \textbf{while}\ \neg(x = 1)\ \textbf{do}\ (y := y * x;\ x := x - 1)$$

It computes the factorial of the initial value bound to x (provided that it is positive), and the result will be the final value of y. Draw a graphical representation of the abstract syntax tree.

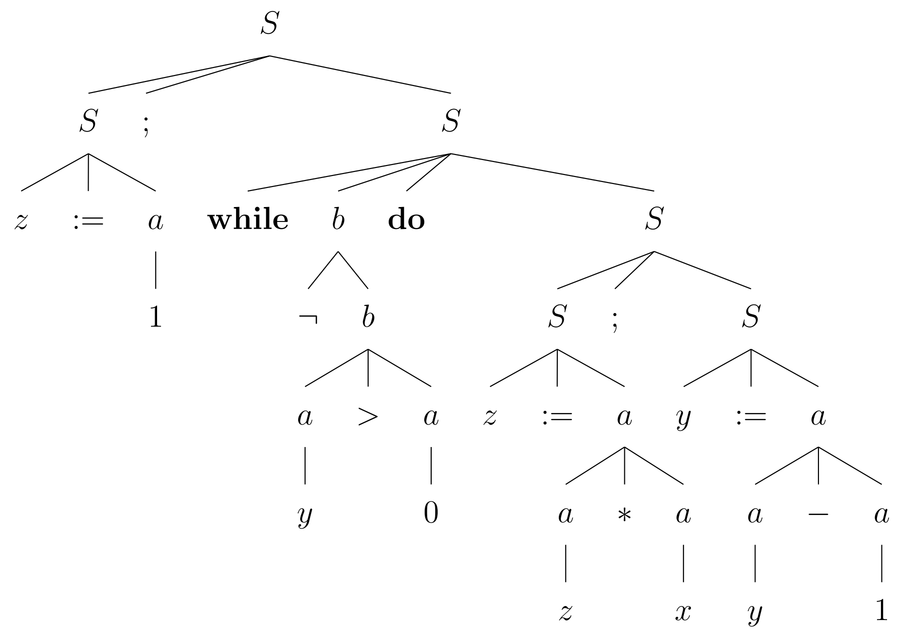


Exercise 1.2

Assume that the initial value of the variable x is n and that the initial value of y is m. Write a statement in **While** that assigns z the value of n to the power of m.

Give a linear as well as a graphical representation of the abstract syntax.

$$z := 1;\ \textbf{while}\ y > 0\ \textbf{do}\ (z := z * x;\ y := y - 1)$$



Exercise 1.4

Suppose that the grammar for n had been

$$n ::= 0 \,|\, 1 \,|\, 0\,n \,|\, 1\,n$$

Can you define $\mathcal{N}$ correctly in this case?

Of course, we can. We define $\mathcal{N}$ as follows:

$$\begin{aligned}
\mathcal{N}\,[\![0]\!] &= 0 \\
\mathcal{N}\,[\![1]\!] &= 1 \\
\mathcal{N}\,[\![0\,n]\!] &= \mathcal{N}\,[\![n]\!] \\
\mathcal{N}\,[\![1\,n]\!] &= 2^{length(n)} + \mathcal{N}\,[\![n]\!]
\end{aligned}$$

where $length(n)$ is the length of the numeral n in digits.

Exercise 1.8

Prove that the equations of Table 1.1 define a total function $\mathcal{A}$ in **Aexp** $\rightarrow$ (**State** $\rightarrow \mathcal{Z}$): First argue that it is sufficient to prove that for each $a \in$ **Aexp** and each $s \in$ **State** there is exactly one value $v \in \mathcal{Z}$ such that $\mathcal{A}\,[\![a]\!]\,s = v$. Next use structural induction on the arithmetic expressions to prove that this is indeed the case.

A total function is a function where each element in the domain is mapped to an element in the codomain. In this case, we need to prove that for each arithmetic expression we have a function that maps a state into a numeral, in other word, if we have $a \in$ **Aexp** and $s \in$ **State** we need to prove that there is only one value that is the result of the evaluation of a in s. We name this value v.

In order to prove this result, we can use structural induction on the shape of the arithmetic expression. First, we check if the property holds for the basis elements:

- If $a := n$ where $n \in \mathbf{Num}$, we have $\mathcal{A}\,[\![a]\!]\,s = \mathcal{N}\,[\![n]\!]$ which depends only on n and the function $\mathcal{N}$ which is a total function, so, there is no problem.
- If $a := x$ where $x \in \mathbf{Var}$, we have $\mathcal{A}\,[\![a]\!]\,s = s\,x$. s is itself a total function that provides only one result for each variable, so, again, we have only one value.

Now, we try to prove it for the composite elements:

- If $a := a_1 + a_2$ where $a_1, a_2 \in \mathbf{Aexp}$, we have $\mathcal{A}\,[\![a]\!]\,s = \mathcal{A}\,[\![a_1]\!]\,s + \mathcal{A}\,[\![a_2]\!]\,s$ and, assuming that the property holds for $a1$ and $a2$, because they are the immediate constituents of a, and knowing that the addition is a total function itself, the property holds in this case.
- The proof for $a := a_1 * a_2$ and $a := a_1 - a_2$ is analogous.

Exercise 1.9

Assume that $s\,x = 3$, and determine $\mathcal{B}\,[\![\neg(x = 1)]\!]\,s$.

Ok, we have the following situation:

- $\mathcal{B}\,[\![(x = 1)]\!]\,s = \mathbf{ff}$ because, using the third rule, we have that $\mathcal{A}\,[\![x]\!] = 1$ and $\mathcal{A}\,[\![3]\!] = 3$ and they are not equal.
- $\mathcal{B}\,[\![\neg(x = 1)]\!]\,s$ depends on the evaluation of $\mathcal{B}\,[\![(x = 1)]\!]\,s$ which is $\mathbf{ff}$, so, using the fifth rule, we obtain $\mathcal{B}\,[\![\neg(x = 1)]\!]\,s = \mathbf{tt}$.

Exercise 1.10

Prove that Table 1.2 defines a total function $\mathcal{B}$ in $\mathbf{Bexp} \to (\mathbf{State} \to \mathbf{T})$.

We must follow the same steps that we used in exercise 8. For each $b \in \mathbf{Bexp}$ and $s \in \mathbf{State}$ we should obtain only one result when we evaluate b in s. First, we check the property for the basis elements and then for the composite elements:

- If $b := true$, we have $\mathcal{B}\,[\![b]\!]\,s = tt$. Obviously, we obtain only one result. The same holds when $b := false$.
- If $b := (a_1 = a_2)$ where $a_1, a_2 \in \mathbf{Aexp}$, we must check whether or not $\mathcal{A}\,[\![a1]\!]\,s$ is equal to $\mathcal{A}\,[\![a2]\!]\,s$. Assuming that we have already prove that $\mathcal{A}$ is a total function, we have that the evaluation of b in s has only one result depending on s. We can use the same argument with $\leq$.

- If $b = \neg b_1$ where $b_1 \in \mathbf{Bexp}$, $\mathcal{B}\llbracket b_1 \rrbracket\, s$ could be $\mathbf{tt}$ or $\mathbf{ff}$, but only one of them, because we suppose that the property holds for b_1. In this case, using the fifth rule, we have that $\mathcal{B}\llbracket b \rrbracket\, s$ maps to only one value (the opposite of $\mathcal{B}\llbracket b_1 \rrbracket\, s$).

- If $b = b_1 \wedge b_2$ where $b_1, b_2 \in \mathbf{Bexp}$, we have $\mathcal{B}\llbracket b_1 \rrbracket\, s = v_1$ and $\mathcal{B}\llbracket b_1 \rrbracket\, s = v_2$ where v_1 and v_2 belong to $\mathbf{T}$ and are the only values returned by the evaluation of b_1 and b_2 in s. $\mathcal{B}\llbracket b \rrbracket\, s = \mathbf{tt}$, if v_1 and v_2 are both $\mathbf{tt}$, and $\mathcal{B}\llbracket b \rrbracket\, s = \mathbf{ff}$ otherwise. In both cases, there is only one possible value.