To begin this case study, I loaded the provided population data into MATLAB (file was modified to have year and population data in two columns)

- y is a variable with all years and P is a variable storing population data (billions)

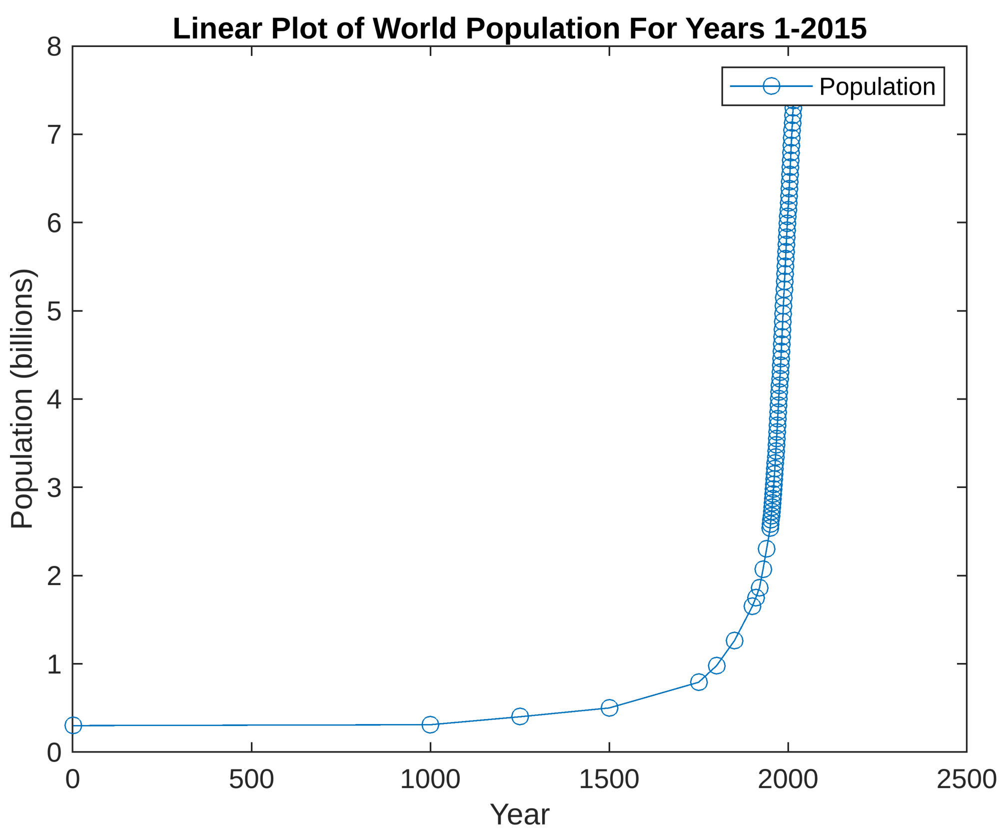
```
[y, P] = readvars("Pop_Data.txt","NumHeaderLines",1);
```

a) Initial Data Plotting

I plotted the population record on linear, semi-log, and log-log graphs

- **linear:**

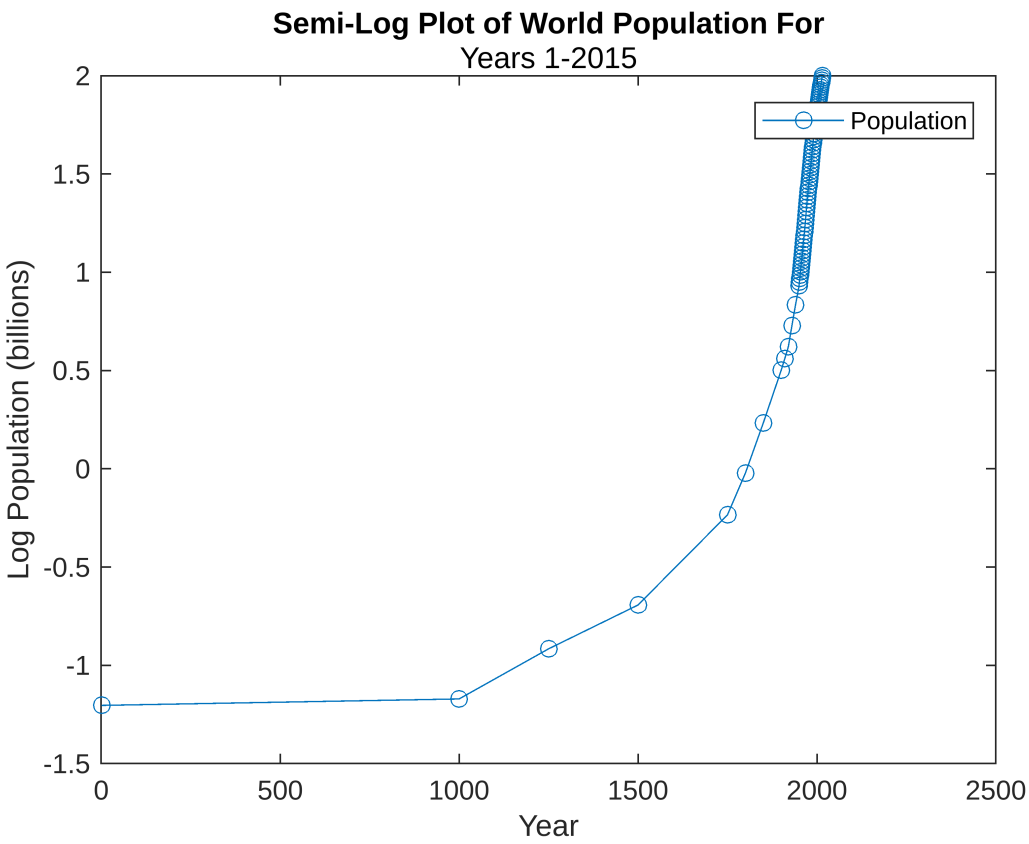
```
figure
plot(y,P,'o-')
title("Linear Plot of World Population For Years 1-2015")
ylabel("Population (billions)")
xlabel("Year")
legend("Population")
```



This plot shows world population (y axis) from the years 1 to 2015 (x axis). It has the classic exponential growth shape, which shows that population began to increase rapidly beginning around the 17th to 19th centuries and continuing to today.

- **semi-log:**

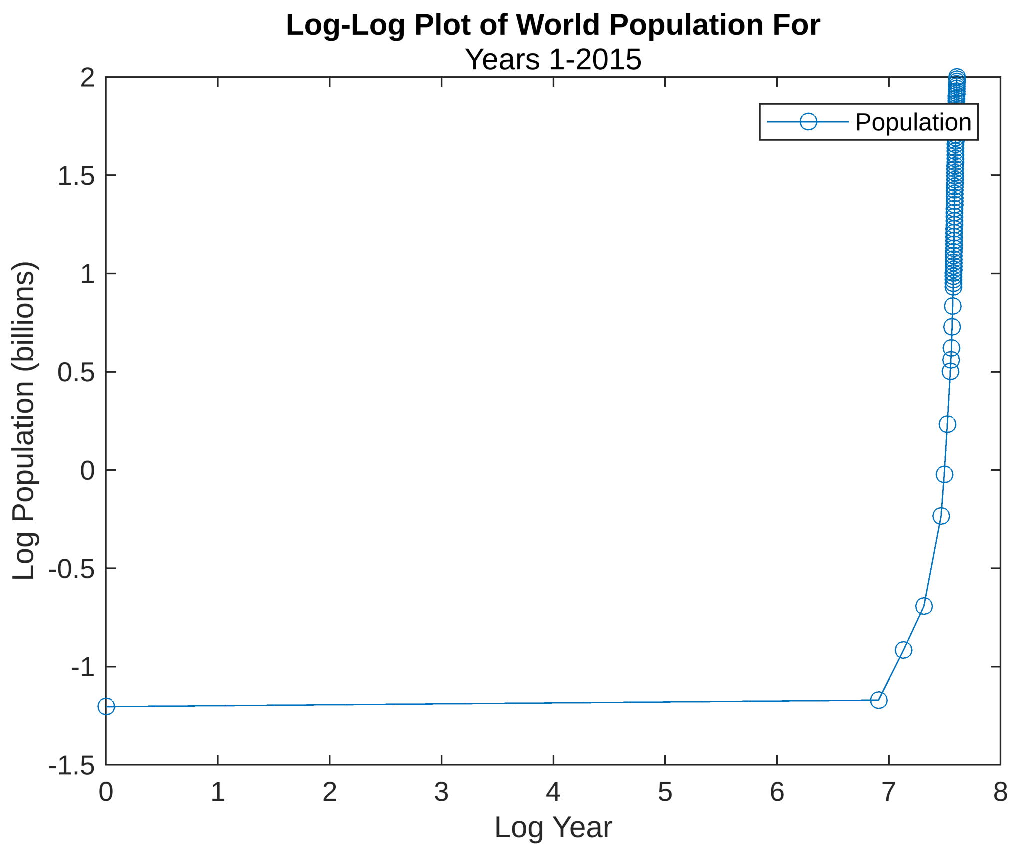
```
figure
plot(y,log(P),'o-')
title("Semi-Log Plot of World Population For", "Years 1-2015")
ylabel("Log Population (billions)")
xlabel("Year")
legend("Population")
```



The shape of this semi-log plot of the log of population (y axis) versus year (x axis) is the closest to a straight line of the three plots, however, the curvature of the line indicates that the world population is experiencing exponential growth and may be best represented using an exponential model.

- **log-log:**

```
figure
plot(log(y),log(P),'o-')
title("Log-Log Plot of World Population For", "Years 1-2015")
ylabel("Log Population (billions)")
xlabel("Log Year")
legend("Population")
```



The shape of this log-log plot of the logarithms of population (y axis) and year (x-axis) is similar to the linear plot and is skewed to the right likely due to the lack of data in ancient times and the aforementioned population growth in more recent times. This likely would not work well with a power law-based model.

b) Exponential Function Plotting

To fit the data using an exponential function, a MATLAB function file was created called "expofunc.m". This function file contained the following two lines to compute the model function P(y), which is global population as a function of time and assumes a pre-existing population prior to exponential growth:

function out=expofunc(x, a)

out = a(1)*exp(a(2)*y);

The provided "nleasqr.m" and "dfdp.m" files were used for nonlinear least squares fitting.

- All data before the year 1500 was removed due to poor model fitting.

```
[y, P] = readvars("Pop_Data_mod.txt","NumHeaderLines",1);
```

Initial Parameters:

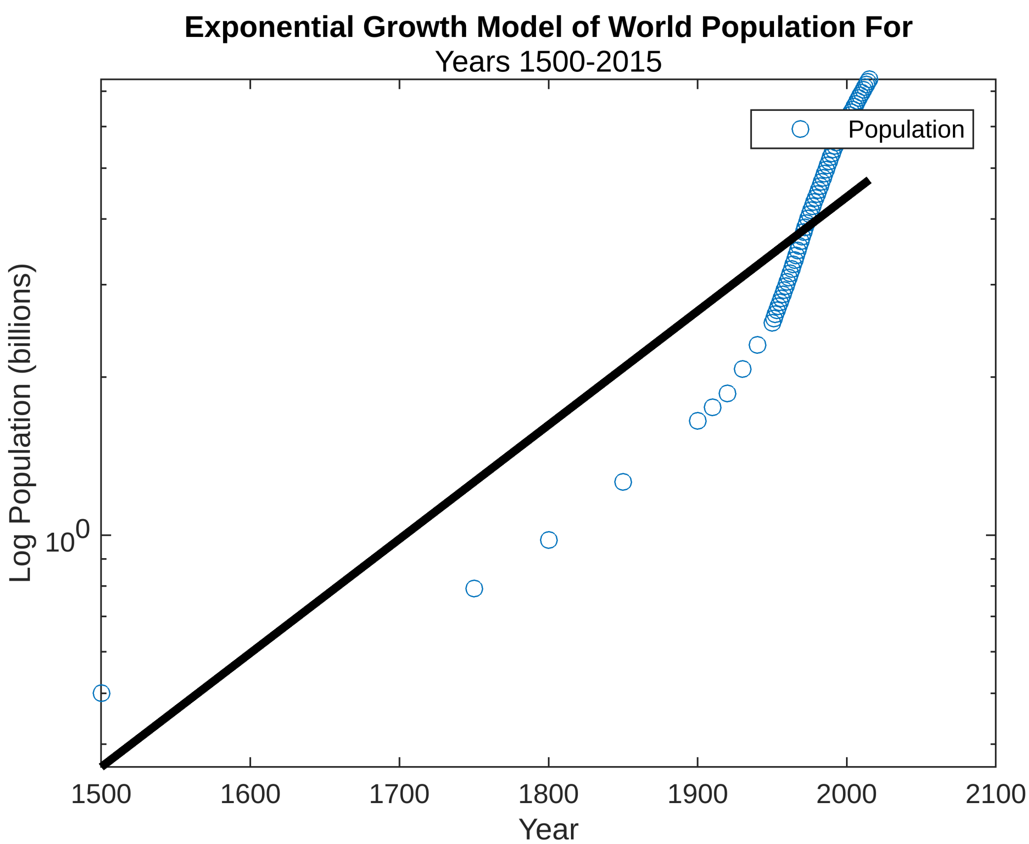
- Based on the semi-log plot from part a, the beginning of exponential growth appeared to occur around the year 1800 at a log population of 0.0002 (a(1)), and the specific growth rate was estimated as 0.005 (a(2)).
- The equation is P = a1 + a2e^ut

```
ain = [0.0002 0.005];
[f,a,kvg,iter,corp,covp,covr,stdresid,Z,r2] = nlleasqr(y,P,ain,'expofunc');
```

```
CONVERGENCE NOT ACHIEVED!
```

Plotted intital guess models with the global population data:

```
figure
semilogy(y,P,'o')
hold on
semilogy(y,expofunc(y,ain),'k','LineWidth',3)
title("Exponential Growth Model of World Population For", "Years 1500-2015")
ylabel("Log Population (billions)")
xlabel("Year")
legend("Population")
```



Estimated coefficient of determination

```
coefficients = a'
```

```
coefficients = 1×2
    0.0000    0.0094
```

```
sigmas = sqrt(diag(covp))'
```

```
sigmas = 1×2
10^-3 ×
    0.0001    0.7877
```

```
R2 = r2
```

```
R2 = 0.9240
```

The line appears to fit decently well after removing outliers (data for years before 1500), however, the data is not completely linear. The coefficient of determination was calculated as 0.9240.

c) Power Law Fitting

For this model, the whole dataset was used including years prior to 1500:

```
[y, P] = readvars("Pop_Data.txt","NumHeaderLines",1);
```

A function file called "powerfunc.m" was created to define the power law function:

$$\frac{\partial}{\partial t}P = a(t^b)$$

function out = powerfunc(y,a)

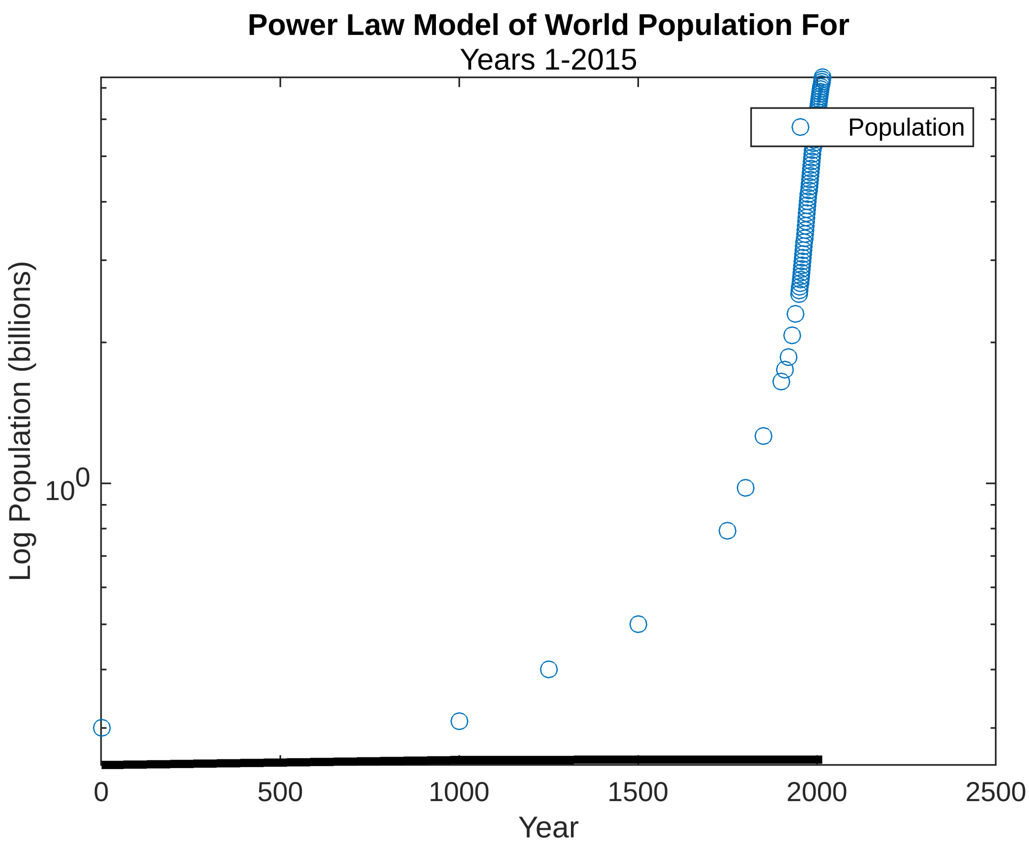
out = a(1)*y.^a(2);

In this case, P represents population (billions), t represents time (years), a is a constant, and b is the power law exponent. Initial parameters were estimated as a1 = 0.25 (beginning of dataset) and a2 = 0.0036 (slope of nonlinear regression line).

```
ain = [0.25 0.0036];
```

Plotted semi-log plot with power law fitting based on initial parameter estimates:

```
figure
semilogy(y,P,'o')
hold on
semilogy(y,powerfunc(y,ain),'k','LineWidth',3)
title("Power Law Model of World Population For", "Years 1-2015")
ylabel("Log Population (billions)")
xlabel("Year")
legend("Population")
```



The power law does not appear to fit the dataset as well as the previous exponential model. There appears to be some sort of a rise around where the critical time may be expected to occur.

d) Critical Time Estimation

In this case, the critical time (tc) was estimated as the year 2030. Thus, the x-axis of the model was changed from time (t) to $(t_c - t)$

- Convert variable y (years) to $(t_c - t)$ = y1

```
y1 = 2030 - y;
```

- A new function file (crittime.m) was created with the following equation to represent the critical time model (from Johansen and Sornette (2001):

$$P(y) = (t_c - t)^{\frac{-1}{\delta}}$$

Function file:

function out = crittime(y,a)

out = y1.^(-1/a(1));

- Define input of the critical time function (δ). Used population of 7.4 billion in year 2015 to solve for δ.

$$\log(7.4) = (2030 - 2015)^{\frac{-1}{\delta}}$$

$$0.87 = 15^{\frac{-1}{\delta}}$$

$$\log\left(0.87\right) = \frac{-1}{\delta} * \log(15)$$

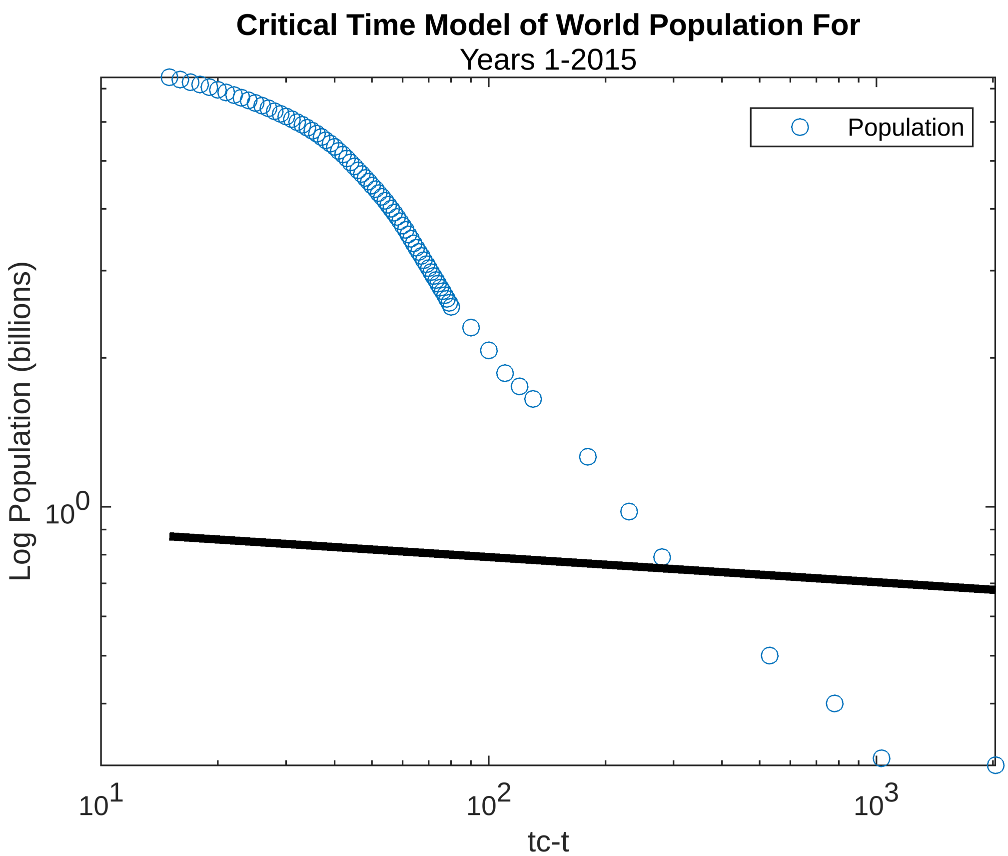
$$\delta * \log(0.87) = -1 * \log(15)$$

$$\delta = \frac{-\log(15)}{\log(0.87)} = 19.7$$

```
ain = 19.7;
```

Repeat previous plot with $(t_c - t)$ in place of t on the x-axis and using the critical time function:

```
figure
loglog(y1,P,'o')
hold on
loglog(y1,crittime(y1,ain),'k','LineWidth',3)
title("Critical Time Model of World Population For", "Years 1-2015")
ylabel("Log Population (billions)")
xlabel("tc-t")
legend("Population")
```



In this case, employing the same power law equation and converting time to $(t_c - t)$ yielded a more linear downward-sloping trend in the data with one obvious outlying point for the earliest data point. This may have caused the trendline to not properly fit the slope of the data although it does appear to be centered on the data, implying that solving for δ as the input variable worked.

d) Part II: Critical Time Manipulation

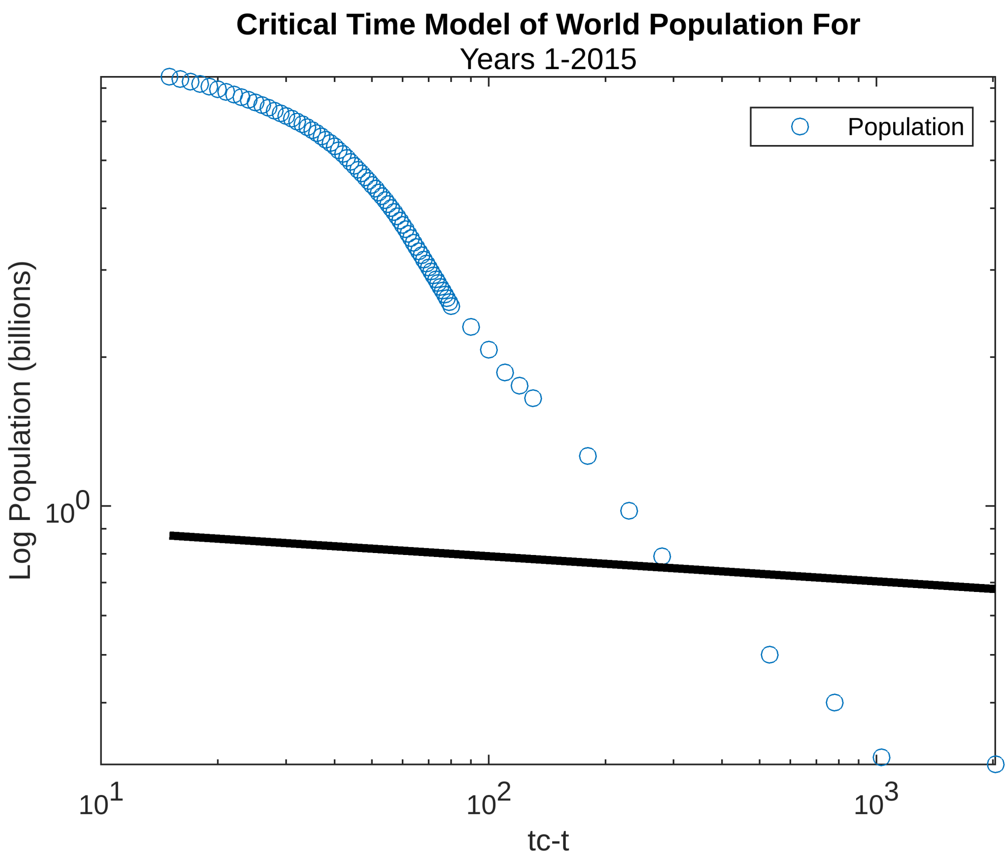
First, the critical time was adjusted to visualize effects on the model fit:

- tc = 2050:

```
y1 = 2030 - y;
ain = 19.7;
```

```
figure
loglog(y1,P,'o')
hold on
loglog(y1,crittime(y1,ain),'k','LineWidth',3)
title("Critical Time Model of World Population For", "Years 1-2015")
ylabel("Log Population (billions)")
xlabel("tc-t")
```

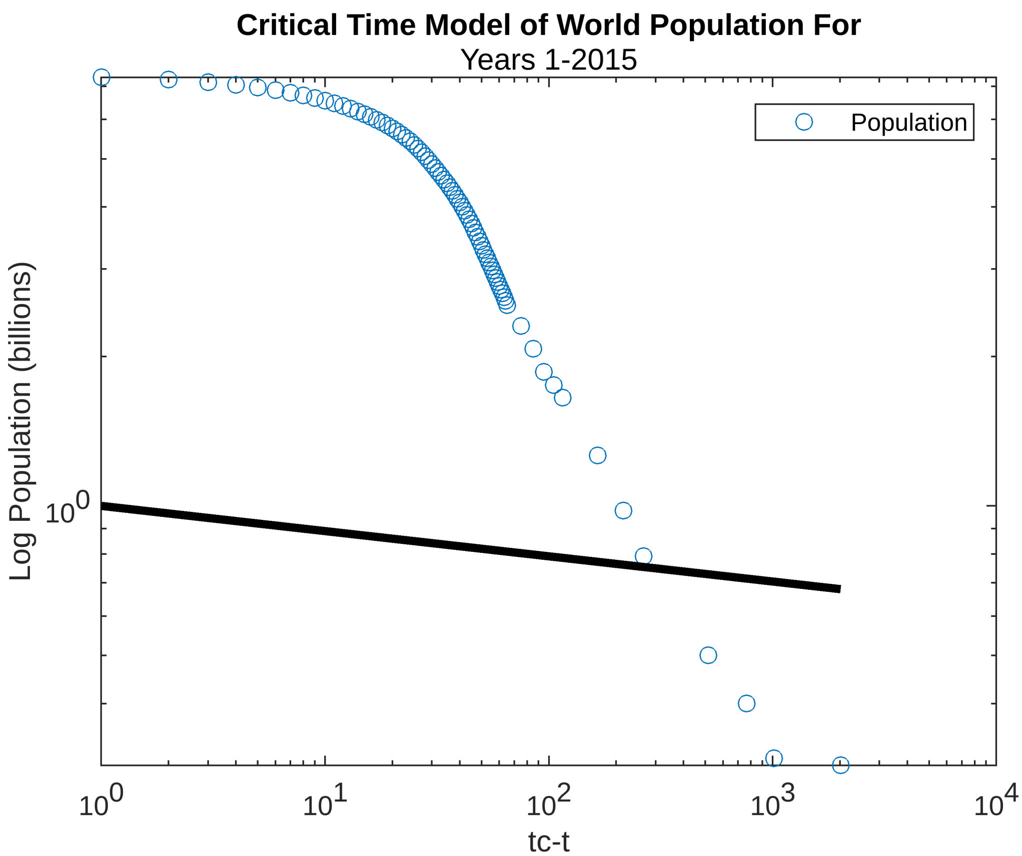
```
legend("Population")
```



- tc = 2015

```
y1 = 2015 - y;
ain = 19.7;
```

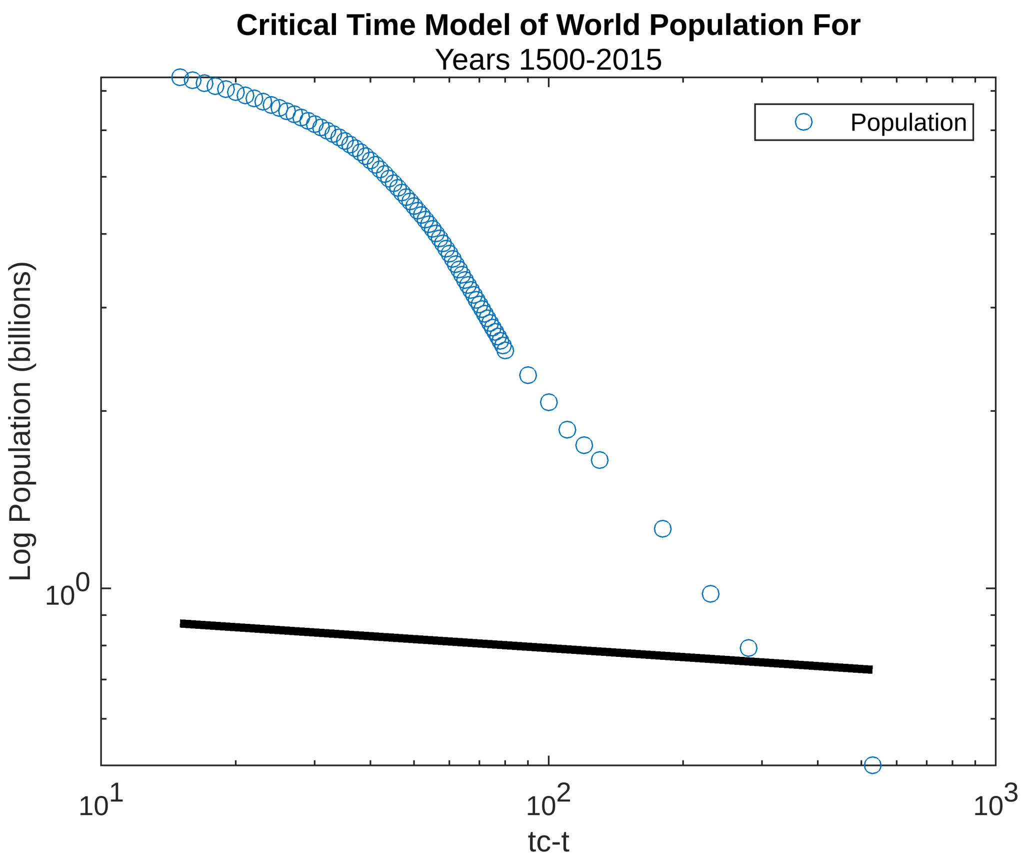
```
figure
loglog(y1,P,'o')
hold on
loglog(y1,crittime(y1,ain),'k','LineWidth',3)
title("Critical Time Model of World Population For", "Years 1-2015")
ylabel("Log Population (billions)")
xlabel("tc-t")
legend("Population")
```



Changing the critical time dit not appear to appreciably improve the model fit. Removing some outlying data points (population data for years before 1500) will be investigated to see if there is any positive effect.

```
[y, P] = readvars("Pop_Data_mod.txt","NumHeaderLines",1);
y1 = 2030 - y; %tc = 2030
ain = 19.7;
```

```
figure
loglog(y1,P,'o')
hold on
loglog(y1,crittime(y1,ain),'k','LineWidth',3)
title("Critical Time Model of World Population For", "Years 1500-2015")
ylabel("Log Population (billions)")
xlabel("tc-t")
legend("Population")
```

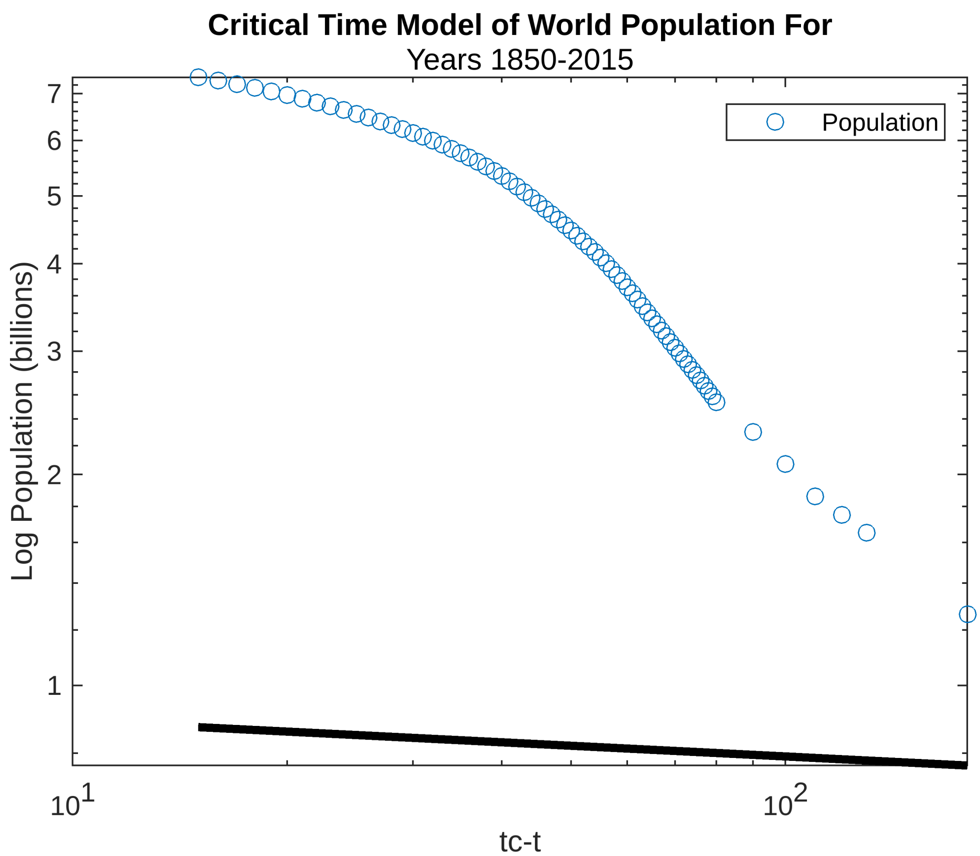


This trendline appears to be a better fit for the data, more years will be removed to see if there is a positive effect:

- Modified original data file to remove all years before 1850:

```
[y, P] = readvars("Pop_Data_mod2.txt","NumHeaderLines",1);
y1 = 2030 - y; %tc = 2030
ain = 19.7;
```

```
figure
loglog(y1,P,'o')
hold on
loglog(y1,crittime(y1,ain),'k','LineWidth',3)
title("Critical Time Model of World Population For", "Years 1850-2015")
ylabel("Log Population (billions)")
xlabel("tc-t")
legend("Population")
```



Removing earlier years for which there were large data gaps and outliers appeared to smooth out the data into a relatively linear shape, however, the trendline does not fit the data well although the slope appears to be accurate. More manipulation of the critical time function concept will be needed to improve the model fit.