

## ADVANCED ANALYSIS: AMORTIZATION AND RECURRENCE RELATIONS

- amortized time complexity
- accounting method
- Java vectors
- Recurrence Relations

## Amortized Running Time

- Amortized running time considers interactions between operations by studying the total running time of a series of operations.

- **Example:** a **Clearable Stack**: supports the usual stack methods plus operation

**clearStack()**: Empty the stack by removing all its elements

Input: None; Output: None

**clearStack** takes  $O(n)$  time in the worst case

- **Proposition:** A series of  $n$  operations on an initially empty clearable stack implemented with an array takes overall  $O(n)$  time
- **Justification:**
  - Let  $M_0, \dots, M_{n-1}$  be the series of operations and  $M_{i_0}, \dots, M_{i_{k-1}}$  be the  $k$ -th **clearStack** operations in the series
  - We define  $i_{-1} = -1$
  - The run time of operation  $M_{i_j}$  is  $O(i_j - i_{j-1})$  since at most  $i_j - i_{j-1}$  elements can be on the stack

## Amortized Running Time (cont)

- Thus the running time of all the **clearStack** operations is

$$O\left(\sum_{j=0}^{k-1} (i_j - i_{j-1})\right)$$

which is a telescoping sum.

- So the run time is  $O(n)$

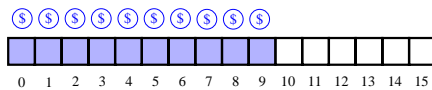
- **Definition:** the **amortized running time** of an operation within a series of operations is the worst-case running time of the entire series of operations divided by the number of operations.

## Accounting Method

- The **accounting method** performs an amortization analysis with a system of credits and debits
- Let's view the computer as vending machine that requires one cyber-dollar for a constant amount of computing time.
- An operation consists of a series of constant-time **primitive operations** that cost one cyber-dollar each.
- We will overcharge an operation that executes few primitives and use the profit to pay for operations that execute many primitives.
- We will need to set up a scheme for charging operations. This is known as the amortization scheme.

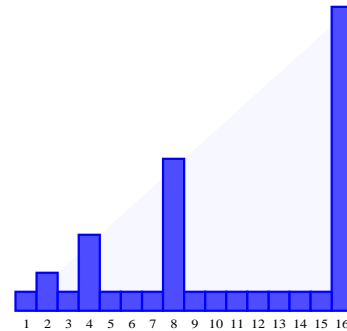
## Amortization Scheme Example for a ClearableStack

- Assume one cyber-dollar is enough to pay for the push, pop, top, size, or isEmpty and for the time spent by the clearStack to dereference one element.
- We will charge 2 cyber-dollars though.
- So we undercharge clearStack but overcharge the other operations. When a clearStack operation is executed, the cyber-dollars stored in the stack are used to pay for dereferencing the items.



## Java Vectors

- The `java.util.vector` class provides a convenient expandable data type in Java.
- A vector is a wrapper around an array that holds a variable called `capacityIncrement`. When the user inserts the  $n+1^{\text{st}}$  element into a vector of size  $n$ , the size of the array is increased by `capacityIncrement` if it is positive, or doubled if `capacityIncrement` is 0.
- Consider the case of `capacityIncrement = 0`:
  - Copying an array into a larger array takes  $O(n)$  time, but this only happens for  $\log(n)$  insertions.
  - Each insertion has  $O(1)$  amortized running time



## Java Vectors (contd.)

### Justification:

The array doubles in size with the insertion of every  $2^i$ th element ( $1^{\text{st}}$ ,  $2^{\text{nd}}$ ,  $4^{\text{th}}$ , etc.)

**Worst case:** we insert exactly  $n = 2^i$  elements, so the last operation involves copying the entire array over again.

We have  $n$  insertions, and  $n$  elements copied in the last insertion. We also have  $i-1$  previous expansions of the array, which perform the following number of element-copy operations:

$$\sum_{k=1}^{i-1} 2^k = 2^i - 1 = n - 1$$

- The overall time complexity is proportional to  $3n-1$ , which is  $O(n)$
- But what if the `capacityIncrement` is, say, 3? Do we still have the same amortization?
  - No!** Copying an array into a larger array is  $O(n)$ , but this happens once every  $n/\text{capacityIncrement}$  insertions.
  - Each insertion is amortized to  $O(n)$

## Java Vectors (contd.)

### Justification: ( $c = \text{capacityIncrement}$ )

Let us assume that the original vector size is 0.

The vector increases in size by the insertion of every  $(ic)^{\text{th}}$  element ( $1^{\text{st}}$ ,  $c^{\text{th}}$ ,  $2*c^{\text{th}}$ , etc.)

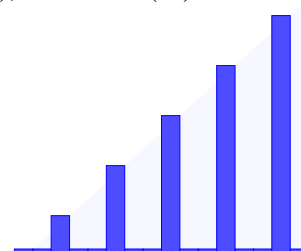
**Worst case:** we insert exactly  $n = ic$  elements, so the last operation involves copying the entire array. We have  $n$  insertions, and  $n$  elements copied on the last insertion.

We also have  $i-1$  other array copies, for a total of:

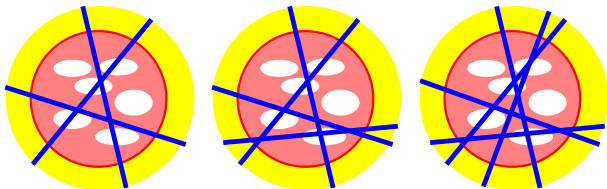
$$\sum_{k=0}^{i-1} ck = c \sum_{k=0}^{i-1} k = c \frac{i(i-1)}{2}$$

previous element copies.

- The overall time complexity is proportional to  $n(n-1/(2c))$ , which is  $O(n^2)$

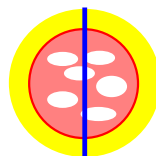


## Recurrence Relations

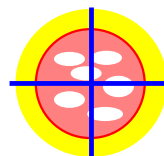


## The Pizza Slicing Problem

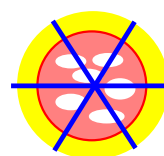
How many pieces of pizza can you get with  $N$  straight cuts ?



1 cut  
2 slices



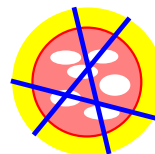
2 cuts  
4 slices



3 cuts  
6 slices

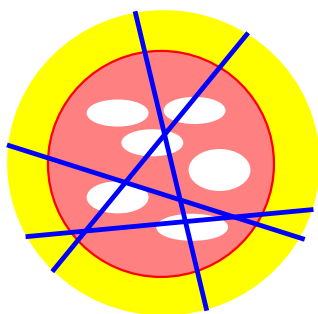
...  $N$  cuts  
 $2N$  slices

But ... who said you should cut through the center every time?

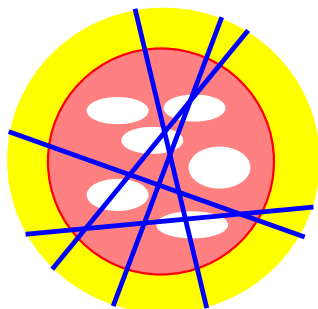


## A Better Slicing Method ...

When cutting, intersect all previous cuts and avoid previous intersection points!

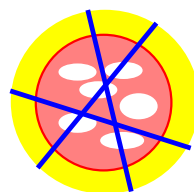


4 cuts  
11 slices!!

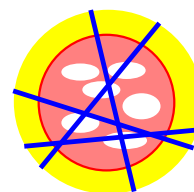


5 cuts  
16 slices!!

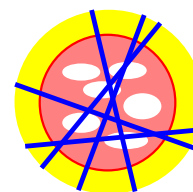
## So ... How Many Pieces?



3 cuts  
7 slices



4 cuts  
11 slices



5 cuts  
16 slices

The  $N$ -th cut creates  $N$  new pieces.  
Hence, the total number of pieces given by  $N$  cuts, denoted  $P(N)$ , is given by the following two rules:

- $P(1) = 2$
- $P(N) = P(N-1) + N$

Recursive definition of  $P(N)$ !

## Recurrence Relations

- The pizza-cutting problem is an example of **recurrence relation**, where a function  $f(N)$  is recursively defined.

(Base Case)  $f(1) = 2$

(Recursive Case)  $f(N) = f(N-1) + N$  for  $N \geq 2$

- The standard method for solving recurrence relations, called “**unfolding**”, makes repeated **substitutions** applying the recursive rule until the **base case** is reached.

$$f(N) = f(N-1) + N$$

$$f(N) = f(N-2) + (N-1) + N$$

$$f(N) = f(N-3) + (N-2) + (N-1) + N$$

...

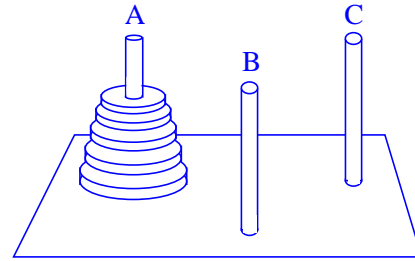
$$f(N) = f(N-i) + (N-i+1) + \dots + (N-1) + N$$

The base case is reached when  $i = N-1$

$$f(N) = 2 + 2 + 3 + \dots + (N-2) + (N-1) + N$$

$$f(N) = N \frac{(N+1)}{2} + 1 = O(N^2)$$

## Towers of Hanoi



**Goal:** transfer all  $N$  disks from peg A to peg C

**Rules:**

- move one disk at a time
- never place larger disk above smaller one

**Recursive solution:**

- transfer  $N-1$  disks from A to B
- move largest disk from A to C
- transfer  $N-1$  disks from B to C

**Total number of moves:**

- $T(N) = 2 T(N-1) + 1$

## Solution of the Recurrence for Towers of Hanoi

**Recurrence relation:**

- $T(N) = 2 T(N-1) + 1$
- $T(1) = 1$

**Solution by unfolding:**

$$\begin{aligned} T(N) &= 2 (2 T(N-2) + 1) + 1 = \\ &= 4 T(N-2) + 2 + 1 = \\ &= 4 (2 T(N-3) + 1) + 2 + 1 = \\ &= 8 T(N-3) + 4 + 2 + 1 = \\ &\dots \\ &= 2^i T(N-i) + 2^{i-1} + 2^{i-2} + \dots + 2^1 + 2^0 \end{aligned}$$

the expansion stops when  $i = N-1$

$$T(N) = 2^{N-1} + 2^{N-2} + 2^{N-3} + \dots + 2^1 + 2^0$$

This is a **geometric sum**, so that we have:

$$T(N) = 2^N - 1 = O(2^N)$$

## Another Recurrence

$$T(N) = 2T\left(\frac{N}{2}\right) + N \quad \text{for } N \geq 2$$

$$T(1) = 1$$

$$T(N) = 2\left(2T\left(\frac{N}{4}\right) + \frac{N}{2}\right) + N$$

$$= 4T\left(\frac{N}{4}\right) + 2N$$

$$= 4\left(2T\left(\frac{N}{8}\right) + \frac{N}{4}\right) + 2N$$

$$= 8T\left(\frac{N}{8}\right) + 3N$$

$$= \dots$$

$$= 2^i T\left(\frac{N}{2^i}\right) + iN$$

The expansion stops for  $i = \log N$ , so that

$$T(N) = N + N \log N$$

## Solving Recurrences by “Guess and Prove”

$$T(N) = 2T\left(\frac{N}{2}\right) + N \quad \text{for } N \geq 2$$

$$T(1) = 1$$

**Step 1:** Take a **wild guess** that

$$T(N) = N + N \log N$$

**Step 2:** Prove it by **induction**:

**Basis**

$$T(1) = 1 + \log 1 = 1$$

**Inductive Step**

$$T(N) = 2T\left(\frac{N}{2}\right) + N = 2\left(\frac{N}{2} + \frac{N}{2} \log \frac{N}{2}\right) + N$$

$$T(N) = N + N(\log N - 1) + N = N + N \log N$$

## A More Difficult Example

$$T(N) = 2T(\sqrt{N}) + 1 \quad T(2) = 0$$

$$\begin{aligned} & 2T(N^{1/2}) + 1 \\ & 2(2T(N^{1/4}) + 1) + 1 \\ & 4T(N^{1/4}) + 1 + 2 \\ & 8T(N^{1/8}) + 1 + 2 + 4 \\ & \dots \\ & 2^i T\left(N^{\frac{1}{2^i}}\right) + 2^0 + 2^1 + \dots + 2^{i-1} \end{aligned}$$

The expansion stops for  $N^{\frac{1}{2^i}} = 2$   
i.e.,  $i = \log \log N$

$$T(N) = 2^0 + 2^1 + \dots + 2^{\log \log N - 1} = \log N.$$

## Proofs by Induction

We want to show that property  $P$  is true for all integers  $n \geq n_0$

**Basis:**

prove that  $P$  is true for  $n_0$ .

**Inductive Step:**

prove that

if  $P$  is true for all  $k$  such that  $n_0 \leq k \leq n-1$

then  $P$  is also true for  $n$

## An Example of Proof by Induction

$$S(n) = \sum_{i=1}^n i = n \frac{(n+1)}{2} \quad \text{for } n \geq 1$$

**Basis:**

$$S(1) = 1 \frac{(1+1)}{2} = 1 \quad \text{Easy, Right?}$$

**Inductive Step:**

$$\text{Assume } S(k) = k \frac{(k+1)}{2} \quad \text{for } 1 \leq k \leq n-1$$

$$S(n) = \sum_{i=1}^n i = \sum_{i=1}^{n-1} i + n = S(n-1) + n$$

$$= (n-1) \frac{(n-1+1)}{2} + n = \frac{(n^2 - n + 2n)}{2}$$

$$= n \frac{(n+1)}{2}$$