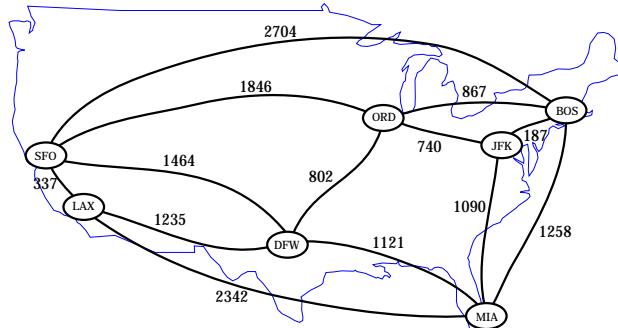


# SHORTEST PATHS

- Weighted Digraphs
- Shortest paths

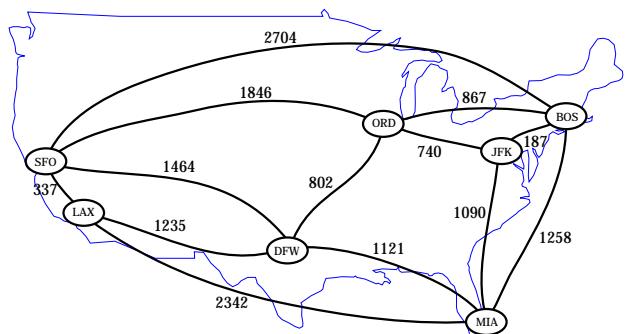


Shortest Paths

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# Weighted Graphs

- **weights** on the edges of a graph represent distances, costs, etc.
- An example of an undirected weighted graph:

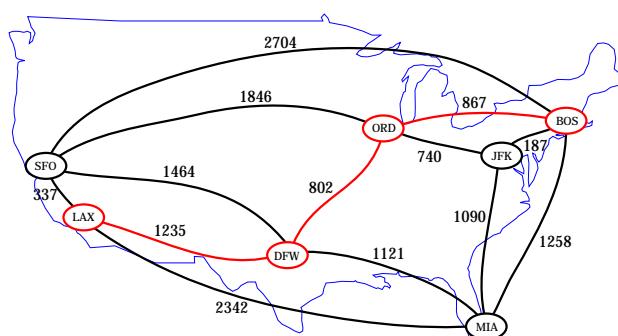


Shortest Paths

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## Shortest Path

- BFS finds paths with the minimum number of edges from the start vertex
- Hence, BFS finds shortest paths assuming that each edge has the same weight
- In many applications, e.g., transportation networks, the edges of a graph have different weights.
- How can we find paths of minimum total weight?
- Example - Boston to Los Angeles:



Shortest Paths

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## Dijkstra's Algorithm

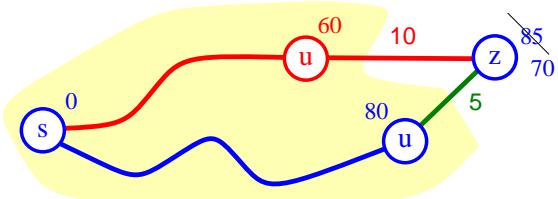
- Dijkstra's algorithm finds shortest paths from a start vertex  $s$  to all the other vertices in a graph with
  - **undirected edges**
  - **nonnegative edge weights**
- Dijkstra's algorithm uses a **greedy method** (sometimes greed works and is good ...)
- the algorithm computes for each vertex  $v$  the **distance** of  $v$  from the start vertex  $s$ , that is, the weight of a shortest path between  $s$  and  $v$ .
- the algorithm keeps track of the set of vertices for which the distance has been computed, called the **cloud**  $C$
- the algorithm uses a **label**  $D[v]$  to store an approximation of the distance between  $s$  and  $v$
- when a vertex  $v$  is added to the cloud, its label  $D[v]$  is equal to the actual distance between  $s$  and  $v$
- initially, the cloud  $C$  contains  $s$ , and we set
  - $D[s] = 0$
  - $D[v] = \infty$  for  $v \neq s$

Shortest Paths

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## Expanding the Cloud

- meaning of  $D[z]$ : length of shortest path from  $s$  to  $z$  that uses only intermediate vertices in the cloud
- after a new vertex  $u$  is added to the cloud, we need to check whether  $u$  is a better routing vertex to reach  $z$
- let  $u$  be a vertex not in the cloud that has smallest label  $D[u]$ 
  - we add  $u$  to the cloud  $C$
  - we update the labels of the adjacent vertices of  $u$  as follows
    - for each vertex  $z$  adjacent to  $u$  do
    - if  $z$  is not in the cloud  $C$  then
    - if  $D[u] + \text{weight}(u, z) < D[z]$  then
    - $D[z] = D[u] + \text{weight}(u, z)$
- the above step is called a **relaxation** of edge  $(u, z)$



Shortest Paths

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## Pseudocode

- we use a priority queue  $Q$  to store the vertices not in the cloud, where  $D[v]$  the key of a vertex  $v$  in  $Q$

Algorithm **ShortestPath**( $G, v$ ):

**Input:** A weighted graph  $G$  and a distinguished vertex  $v$  of  $G$ .

**Output:** A label  $D[u]$ , for each vertex that  $u$  of  $G$ , such that  $D[u]$  is the length of a shortest path from  $v$  to  $u$  in  $G$ .

initialize  $D[v] \leftarrow 0$  and  $D[u] \leftarrow +\infty$  for each vertex  $v \neq u$

let  $Q$  be a priority queue that contains all of the vertices of  $G$  using the  $D$  labels as keys.

while  $Q \neq \emptyset$  do

{pull  $u$  into the cloud  $C$ }

$u \leftarrow Q.\text{removeMinElement}()$

for each vertex  $z$  adjacent to  $u$  such that  $z$  is in  $Q$  do

{perform the relaxation operation on edge  $(u, z)$ }

if  $D[u] + w(u, z) < D[z]$  then

$D[z] \leftarrow D[u] + w(u, z)$

change the key value of  $z$  in  $Q$  to  $D[z]$

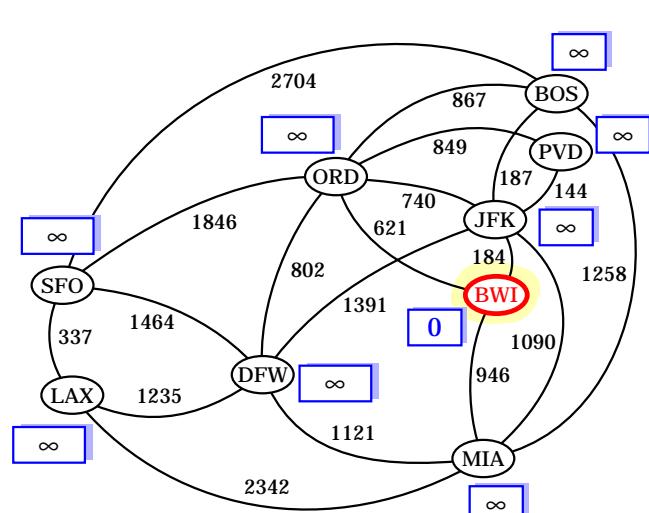
return the label  $D[u]$  of each vertex  $u$ .

Shortest Paths

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## Example

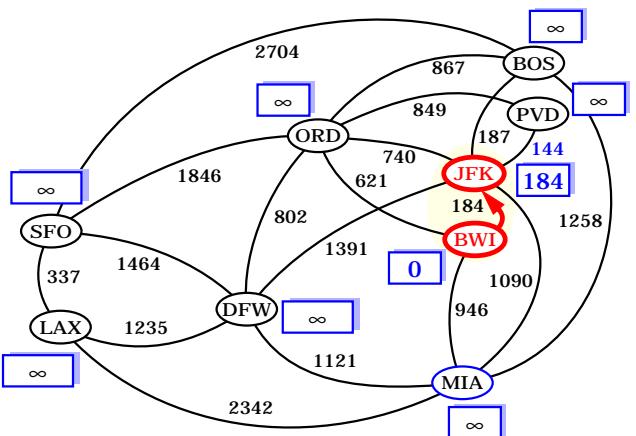
- shortest paths starting from BWI



Shortest Paths

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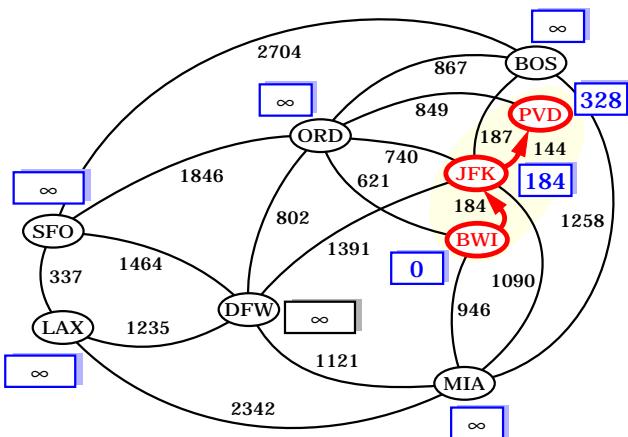
- JFK is the nearest...



Shortest Paths

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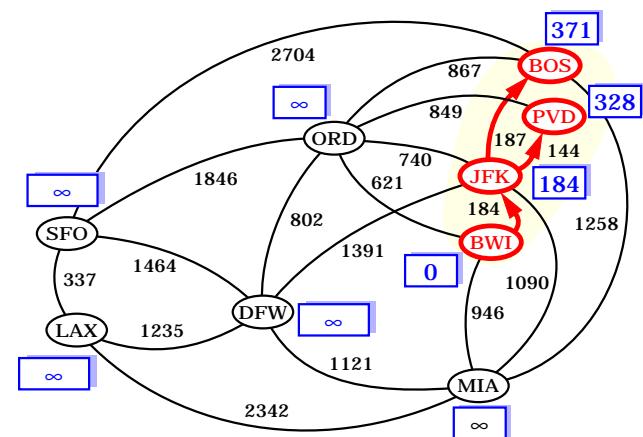
- followed by sunny PVD.



Shortest Paths

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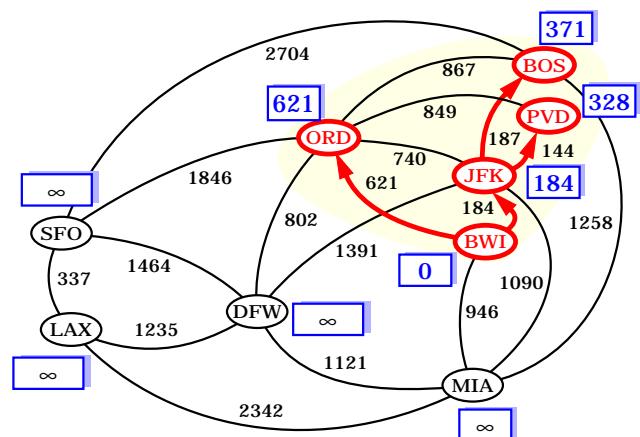
- BOS is just a little further.



Shortest Paths

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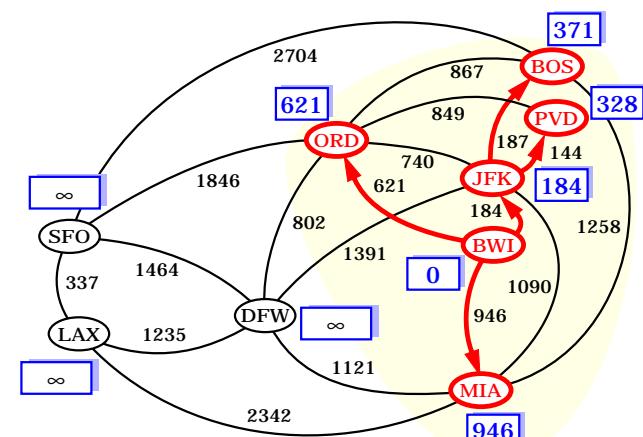
- ORD: Chicago is my kind of town.



Shortest Paths

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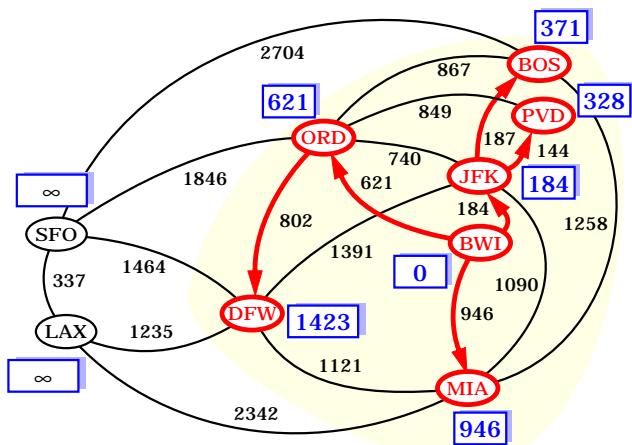
- MIA, just after Spring Break.



Shortest Paths

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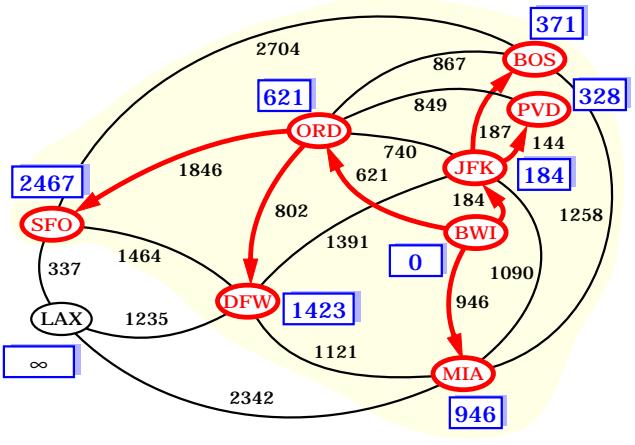
- DFW is huge like Texas.



Shortest Paths

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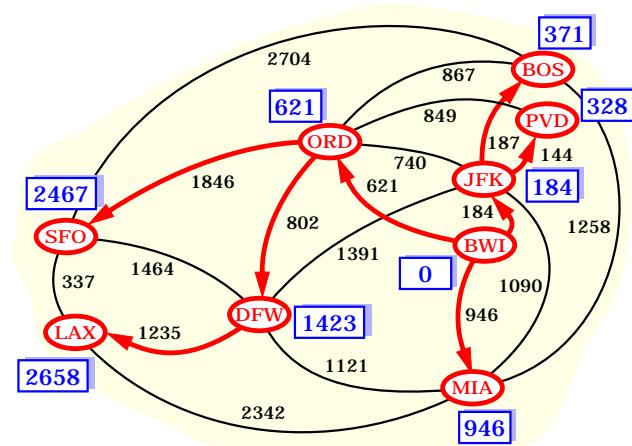
- SFO: the 49'ers will take the prize next year.



Shortest Paths

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- LAX is the last stop on the journey.



Shortest Paths

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## Running Time

- Let's assume that we represent  $G$  with an adjacency list. We can then step through all the vertices adjacent to  $u$  in time proportional to their number (i.e.  $O(j)$  where  $j$  is the number of vertices adjacent to  $u$ )
- The priority queue  $Q$ :
  - A Heap: Implementing  $Q$  with a heap allows for efficient extraction of vertices with the smallest  $D$  label ( $O(\log N)$ ). If  $Q$  is implemented with locators, key updates can be performed in  $O(\log N)$  time. The total run time is  $O((n+m)\log n)$  where  $n$  is the number of vertices in  $G$  and  $m$  is the number of edges. In terms of  $n$ , worst case time is  $(On^2\log n)$
  - Unsorted Sequence:  $O(n)$  when we extract minimum elements, but fast key updates ( $O(1)$ ). There are only  $n-1$  extractions and  $m$  relaxations. The running time is  $O(n^2+m)$
- In terms of worst case time, heap is good for small data sets and sequence for larger.
- For each vertex, its neighbors are pulled into the cloud in random order. There are only  $O(\log n)$  updates to the key of a vertex. Under this

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## Running Time (cont)

assumption, the run time of the head is  $O(n\log n + m)$ , which is always  $O(n^2)$  the heap implementation is thus preferable for all but degenerate cases.

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## Java Implementation

- we use a priority queue `Q` supporting locator-based methods in the implementation of Dijkstra's shortest path algorithm
- when we insert a vertex `u` into `Q`, we associate with `u` the locator returned by `insert` (e.g., via a dictionary)  
`Locator u_loc = Q.insert(new Integer(u_dist), u);  
setLocator(u, u_loc);`
- in the relaxation of an edge `(u,z)`, the update of the distance of `z` is performed with operation `replaceKey`  
`for (Enumeration u_edges = graph.incidentEdges(u);  
u_edges.hasMoreElements(); ) {  
 Edge e = (Edge) u_edges.nextElement();  
 Vertex z = graph.opposite(u,e);  
 Locator z_loc = getLocator(z);  
 if (z_loc.isContained()) { // test whether z is in Q  
 int e_weight = weight(e);  
 int z_dist = value(z_loc);  
 if (u_dist + e_weight < z_dist)  
 Q.replaceKey(z_loc, new Integer(u_dist + e_weight));  
 }  
}`

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## Java Implementation (contd.)

```
public abstract class Dijkstra {  
    private static final int INFINITE = Integer.MAX_VALUE;  
    protected InspectableGraph graph;  
    // priority queue used by the algorithm  
    protected PriorityQueue Q;  
    public Object execute(InspectableGraph g, Vertex start) {  
        graph = g;  
        dijkstraVisit(start);  
        return distances();  
    }  
    // initialization  
    abstract void init();  
    // create an empty priority queue  
    abstract PriorityQueue initPQ(Comparator comp);  
    // return the weight of edge e  
    abstract int weight(Edge e);  
    // attach to u its locator loc in Q  
    abstract void setLocator(Vertex u, Locator loc);  
    // return the locator attached to u  
    abstract Locator getLocator(Vertex u);
```

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## Java Implementation(cont)

```
// attach to u its distance dist  
abstract void setDistance(Vertex u, int dist);  
// return the vertex distances in a data structure  
abstract Object distances();  
// return as an int the key of a vertex in Q  
private int value(Locator u_loc) {  
    return ((Integer) u_loc.key()).intValue();  
}
```

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## Java Implementation (cont.)

```
protected void dijkstraVisit (Vertex v) {  
    // initialize the priority queue Q and store all the  
    // vertices in it  
    init();  
    Q = initPQ(new IntegerComparator());  
    for (Enumeration vertices = graph.vertices();  
        vertices.hasMoreElements(); ) {  
        Vertex u = (Vertex) vertices.nextElement();  
        int u_dist;  
        if (u==v)  
            u_dist = 0;  
        else  
            u_dist = INFINITE;  
        Locator u_loc = Q.insert(new Integer(u_dist), u);  
        setLocator(u, u_loc);  
    }  
    // grow the cloud, one vertex at a time  
    while (! Q.isEmpty()) {  
        // remove from Q and insert into cloud a vertex with  
        // minimum distance  
        Locator u_loc = Q.min();
```

Shortest Paths

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## Java Implementation (cont)

```
Q.remove(u_loc);  
setDistance(u, u_dist); // the distance of u is final  
// examine all the neighbors of u and update their  
distances  
for (Enumeration u_edges = graph.incidentEdges(u);  
    u_edges.hasMoreElements(); ) {  
    Edge e = (Edge) u_edges.nextElement();  
    Vertex z = graph.opposite(u,e);  
    Locator z_loc = getLocator(z);  
    // check if z is not in the cloud, i.e., z is in Q  
    if (z_loc.isContained()) {  
        // relaxation of edge e = (u,z)  
        int e_weight = weight(e);  
        int z_dist = value(z_loc);  
        if ( u_dist + e_weight < z_dist )  
            Q.replaceKey(z_loc, new Integer(u_dist +  
e_weight));  
    }  
}
```

Shortest Paths

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## Java Implementation (cont)

```
public class MyDijkstra extends Dijkstra {  
    protected Hashtable locators = new Hashtable();  
    protected Hashtable distances = new Hashtable();  
    protected Hashtable weights = new Hashtable();  
    public void init() {}  
    public PriorityQueue initPQ(Comparator comp) {  
        return (PriorityQueue) new  
        SequenceLocPriorityQueue(comp);  
    }  
    public int weight(Edge e) {  
        return ((Integer) weights.get(e)).intValue();  
    }  
    public void setWeight(Edge e, int w) {  
        weights.put(e, new Integer(w));  
    }  
    public void setLocator(Vertex u, Locator loc) {  
        locators.put(u, loc);  
    }  
    public Locator getLocator(Vertex u) {  
        return (Locator) locators.get(u);
```

Shortest Paths

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## Java Implementation (cont.)

```
}  
public void setDistance(Vertex u, int dist) {  
    distances.put(u, new Integer(dist));  
}  
public int distance(Vertex u) {  
    return ((Integer) distances.get(u)).intValue();  
}  
public Object distances() {  
    return distances;  
}
```

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