

# Geometry

## Problem booklet

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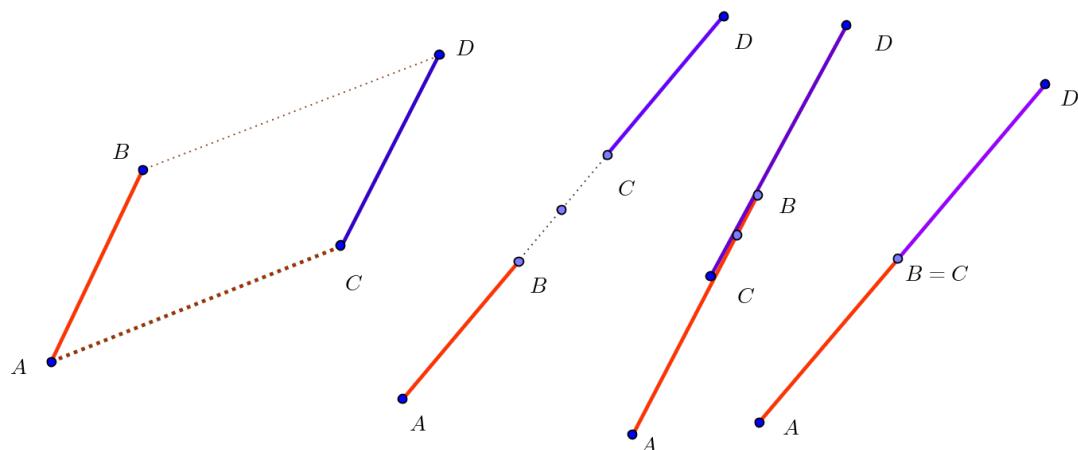
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# 1 Week 1: Vector algebra

## 1.1 Free vectors

**Vectors** Let  $\mathcal{P}$  be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If  $(A, B) \in \mathcal{P} \times \mathcal{P}$  is an ordered pair, then  $A$  is called the *original point* or the *origin* and  $B$  is called the *terminal point* or the *extremity* of  $(A, B)$ .

**Definition 1.1.** The ordered pairs  $(A, B), (C, D)$  are said to be equipollent, written  $(A, B) \sim (C, D)$ , if the segments  $[AD]$  and  $[BC]$  have the same midpoint.



Pairs of equipollent points  $(A, B) \sim (C, D)$

**Remark 1.1.** If the points  $A, B, C, D \in \mathcal{P}$  are not collinear, then  $(A, B) \sim (C, D)$  if and only if  $ABDC$  is a parallelogram. In fact the length of the segments  $[AB]$  and  $[CD]$  is the same whenever  $(A, B) \sim (C, D)$ .

**Proposition 1.1.** If  $(A, B)$  is an ordered pair and  $O \in \mathcal{P}$  is a given point, then there exists a unique point  $X$  such that  $(A, B) \sim (O, X)$ .

**Proposition 1.2.** The equipollence relation is an equivalence relation on  $\mathcal{P} \times \mathcal{P}$ .

**Definition 1.2.** The equivalence classes with respect to the equipollence relation are called *(free) vectors*.

Denote by  $\overrightarrow{AB}$  the equivalence class of the ordered pair  $(A, B)$ , that is  $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$  and let  $\mathcal{V} = \mathcal{P} \times \mathcal{P} / \sim = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$  be the set of (free) vectors. The *length* or the *magnitude* of the vector  $\overrightarrow{AB}$ , denoted by  $\|\overrightarrow{AB}\|$  or by  $|\overrightarrow{AB}|$ , is the length of the segment  $[AB]$ .

**Remark 1.2.** If two ordered pairs  $(A, B)$  and  $(C, D)$  are equipollent, i.e. the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

**Proposition 1.3.** 1.  $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$ .

2.  $\forall A, B, O \in \mathcal{P}, \exists ! X \in \mathcal{P}$  such that  $\overrightarrow{AB} = \overrightarrow{OX}$ .

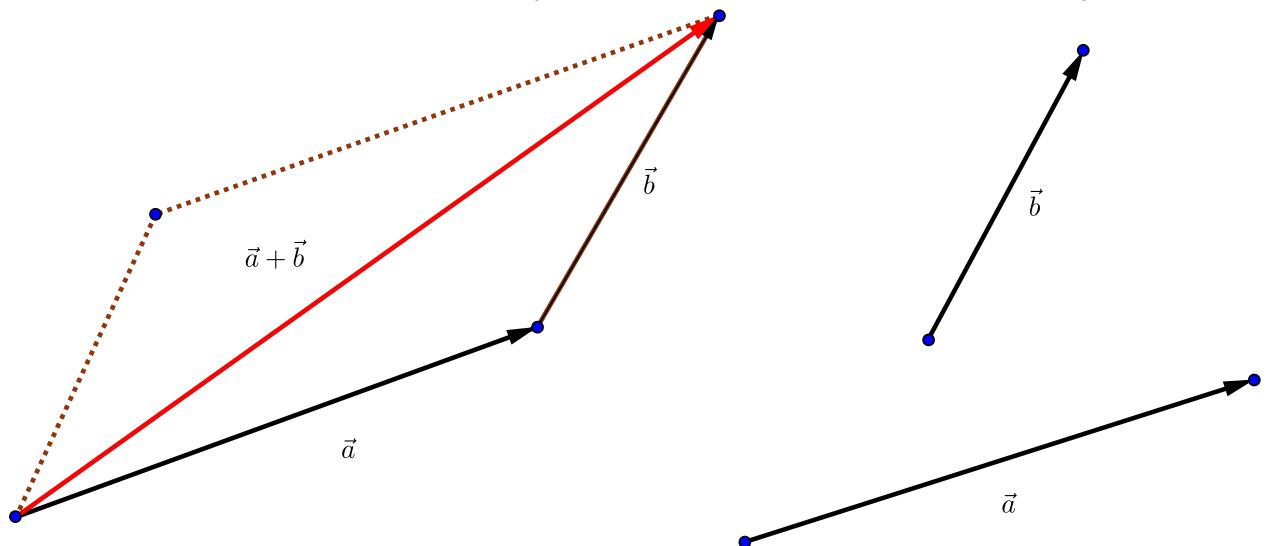
3.  $\overrightarrow{AB} = \overrightarrow{A'B'}, \overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$ .

**Definition 1.3.** If  $O, M \in \mathcal{P}$ , the vector  $\overrightarrow{OM}$  is denoted by  $\vec{r}_M$  and is called the *position vector* of  $M$  with respect to  $O$ .

**Corollary 1.4.** The map  $\varphi_O : \mathcal{P} \rightarrow \mathcal{V}$ ,  $\varphi_O(M) = \vec{r}_M$  is one-to-one and onto, i.e. bijective.

### 1.1.1 Operations with vectors

• **The addition of vectors** Let  $\vec{a}, \vec{b} \in \mathcal{V}$  and  $O \in \mathcal{P}$  be such that  $\overrightarrow{a} = \overrightarrow{OA}$ ,  $\overrightarrow{b} = \overrightarrow{AB}$ . The vector  $\overrightarrow{OB}$  is called the *sum* of the vectors  $\vec{a}$  and  $\vec{b}$  and is written  $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$ .



Let  $O'$  be another point and  $A', B' \in \mathcal{P}$  be such that  $\overrightarrow{O'A'} = \vec{a}$ ,  $\overrightarrow{A'B'} = \vec{b}$ . Since  $\overrightarrow{OA} = \overrightarrow{O'A'}$  and  $\overrightarrow{AB} = \overrightarrow{A'B'}$  it follows, according to Proposition 1.3(3), that  $\overrightarrow{OB} = \overrightarrow{O'B'}$ . Therefore the vector  $\vec{a} + \vec{b}$  is independent on the choice of the point  $O$ .

**Proposition 1.5.** The set  $\mathcal{V}$  endowed to the binary operation  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ ,  $(\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$ , is an abelian group whose zero element is the vector  $\overrightarrow{AA} = \overrightarrow{BB} = \vec{0}$  and the opposite of  $\overrightarrow{AB}$ , denoted by  $-\overrightarrow{AB}$ , is the vector  $\overrightarrow{BA}$ .

In particular the addition operation is associative and the vector

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

is usually denoted by  $\vec{a} + \vec{b} + \vec{c}$ . Moreover the expression

$$((\cdots (\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n) \cdots), \quad (1.1)$$

is independent of the distribution of parenthesis and it is usually denoted by

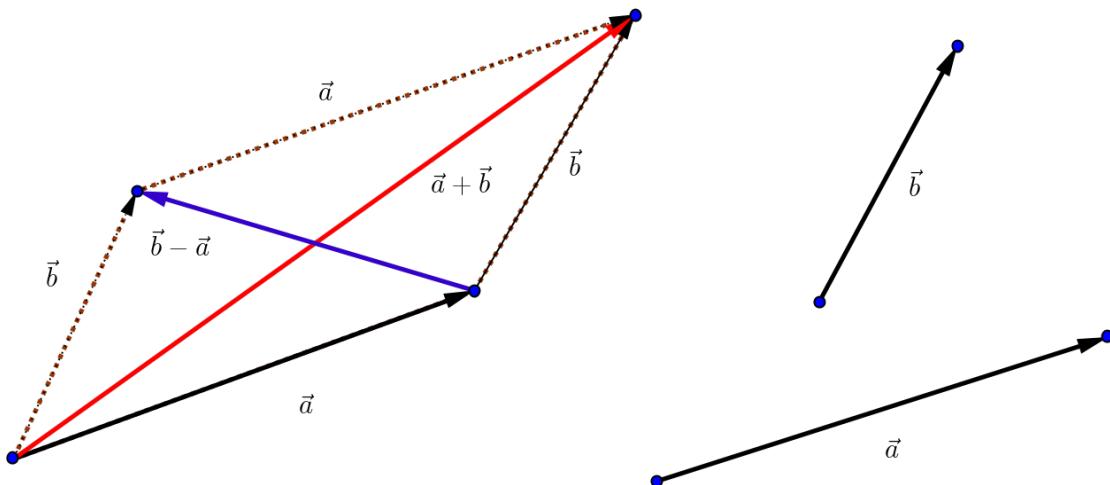
$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n.$$

**Example 1.1.** If  $A_1, A_2, A_3, \dots, A_n \in \mathcal{P}$  are some given points, then

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n}.$$

This shows that  $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \overrightarrow{0}$ , namely the sum of vectors constructed on the edges of a closed broken line is zero.

**Corollary 1.6.** If  $\vec{a} = \overrightarrow{OA}$ ,  $\vec{b} = \overrightarrow{OB}$  are given vectors, there exists a unique vector  $\vec{x} \in \mathcal{V}$  such that  $\vec{a} + \vec{x} = \vec{b}$ . In fact  $\vec{x} = \vec{b} + (-\vec{a}) = \overrightarrow{AB}$  and is denoted by  $\vec{b} - \vec{a}$ .



### • The multiplication of vectors with scalars

Let  $\alpha \in \mathbb{R}$  be a scalar and  $\vec{a} = \overrightarrow{OA} \in \mathcal{V}$  be a vector. We define the vector  $\alpha \cdot \vec{a}$  as follows:  $\alpha \cdot \vec{a} = \vec{0}$  if  $\alpha = 0$  or  $\vec{a} = \vec{0}$ ; if  $\vec{a} \neq \vec{0}$  and  $\alpha > 0$ , there exists a unique point on the half line  $]OA$  such that  $\|OB\| = \alpha \cdot \|OA\|$  and define  $\alpha \cdot \vec{a} = \overrightarrow{OB}$ ; if  $\alpha < 0$  we define  $\alpha \cdot \vec{a} = -(|\alpha| \cdot \vec{a})$ . The external binary operation

$$\mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}, (\alpha, \vec{a}) \mapsto \alpha \cdot \vec{a}$$

is called the *multiplication of vectors with scalars*.

**Proposition 1.7.** *The following properties hold:*

- (v1)  $(\alpha + \beta) \cdot \vec{a} = \alpha \cdot \vec{a} + \beta \cdot \vec{a}$ ,  $\forall \alpha, \beta \in \mathbb{R}, \vec{a} \in \mathcal{V}$ .
- (v2)  $\alpha \cdot (\vec{a} + \vec{b}) = \alpha \cdot \vec{a} + \alpha \cdot \vec{b}$ ,  $\forall \alpha \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$ .
- (v3)  $\alpha \cdot (\beta \cdot \vec{a}) = (\alpha\beta) \cdot \vec{a}$ ,  $\forall \alpha, \beta \in \mathbb{R}$ .
- (v4)  $1 \cdot \vec{a} = \vec{a}$ ,  $\forall \vec{a} \in \mathcal{V}$ .

**Application 1.1.** Consider two parallelograms,  $A_1A_2A_3A_4, B_1B_2B_3B_4$  in  $\mathcal{P}$ , and  $M_1, M_2, M_3, M_4$  the midpoints of the segments  $[A_1B_1], [A_2B_2], [A_3B_3], [A_4B_4]$  respectively. Then:

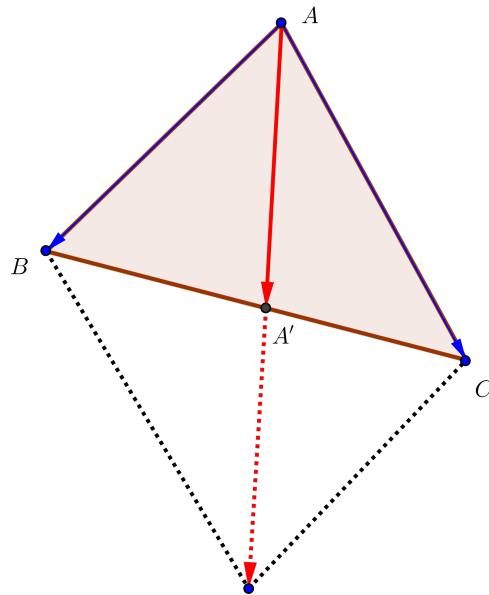
- $2 \vec{M_1M_2} = \vec{A_1A_2} + \vec{B_1B_2}$  and  $2 \vec{M_3M_4} = \vec{A_3A_4} + \vec{B_3B_4}$ .
- $M_1, M_2, M_3, M_4$  are the vertices of a parallelogram.

### 1.1.2 The vector structure on the set of vectors

**Theorem 1.8.** *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

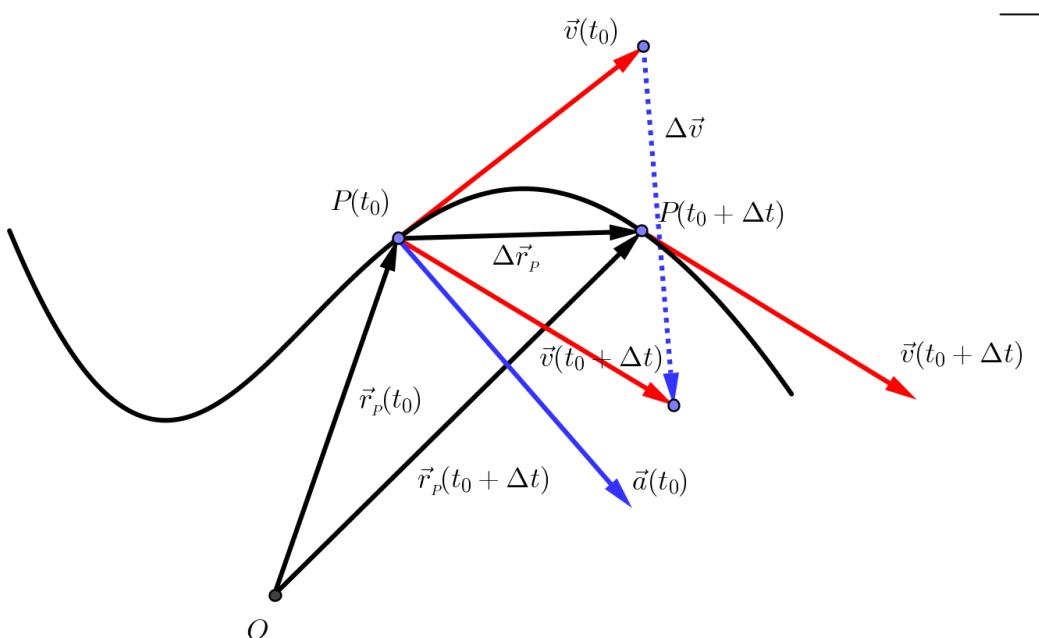
**Example 1.2.** If  $A'$  is the midpoint of the edge  $[BC]$  of the triangle  $ABC$ , then

$$\vec{AA'} = \frac{1}{2}(\vec{AB} + \vec{AC}).$$



A few vector quantities:

1. The force, usually denoted by  $\vec{F}$ .
2. The velocity  $\frac{d\vec{r}_p}{dt}$  of a moving particle  $P$ , is usually denoted by  $\vec{v}_p$  or simply by  $\vec{v}$ .
3. The acceleration  $\frac{d\vec{v}_p}{dt}$  of a moving particle  $P$ , is usually denoted by  $\vec{a}_p$  or simply by  $\vec{a}$ .



- **Newton's law of gravitation**, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force  $F$  is equal to  $G$  (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses ( $m_1$  and  $m_2$ ) and divided by the square of the distance  $R$ :  $F = G(m_1 m_2)/R^2$ . (Encyclopdia

Britannica)

• **Newton's second law** is a quantitative description of the changes that a force can produce on the motion of a body. It states that the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it. The momentum of a body is equal to the product of its mass and its velocity. Momentum, like velocity, is a vector quantity, having both magnitude and direction. A force applied to a body can change the magnitude of the momentum, or its direction, or both. Newton's second law is one of the most important in all of physics. For a body whose mass  $m$  is constant, it can be written in the form  $F = ma$ , where  $F$  (force) and  $a$  (acceleration) are both vector quantities. If a body has a net force acting on it, it is accelerated in accordance with the equation. Conversely, if a body is not accelerated, there is no net force acting on it. (Encyclopdia Britannica)

## 1.2 Problems

1. Consider a tetrahedron  $ABCD$ . Find the the following sums of vectors:

- (a)  $\vec{AB} + \vec{BC} + \vec{CD}$ .
- (b)  $\vec{AD} + \vec{CB} + \vec{DC}$ .
- (c)  $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD}$ .

*Solution.*

2. ([4, Problem 3, p. 1]) Let  $OABCDE$  be a regular hexagon in which  $\overrightarrow{OA} = \vec{a}$  and  $\overrightarrow{OE} = \vec{b}$ . Express the vectors  $\overrightarrow{OB}$ ,  $\overrightarrow{OC}$ ,  $\overrightarrow{OD}$  in terms of the vectors  $\vec{a}$  and  $\vec{b}$ . Show that  $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3 \overrightarrow{OC}$ .

*Solution.*

3. Consider a pyramid with the vertex at  $S$  and the basis a parallelogram  $ABCD$  whose diagonals are concurrent at  $O$ . Show the equality  $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4 \overrightarrow{SO}$ .

*Solution.*

4. Let  $E$  and  $F$  be the midpoints of the diagonals of a quadrilateral  $ABCD$ . Show that

$$\overrightarrow{EF} = \frac{1}{2} \left( \overrightarrow{AB} + \overrightarrow{CD} \right) = \frac{1}{2} \left( \overrightarrow{AD} + \overrightarrow{CB} \right).$$

*Solution.*

5. In a triangle  $ABC$  we consider the height  $AD$  from the vertex  $A$  ( $D \in BC$ ). Find the decomposition of the vector  $AD$  in terms of the vectors  $\vec{c} = \overrightarrow{AB}$  and  $\vec{b} = \overrightarrow{AC}$ .

*Solution.*

6. ([4, Problem 12, p. 3]) Let  $M, N$  be the midpoints of two opposite edges of a given quadrilateral  $ABCD$  and  $P$  be the midpoint of  $[MN]$ . Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

*Solution.*

7. ([4, Problem 12, p. 7]) Consider two perpendicular chords  $AB$  and  $CD$  of a given circle and  $\{M\} = AB \cap CD$ . Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2 \overrightarrow{OM}.$$

*Solution.*

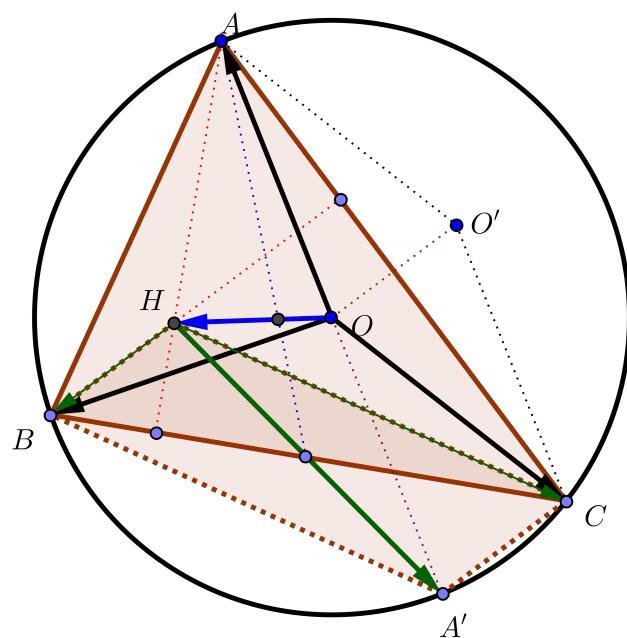
8. ([4, Problem 13, p. 3]) If  $G$  is the centroid of a triangle  $ABC$  and  $O$  is a given point, show that

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

*Solution.*

9. ([4, Problem 14, p. 4]) Consider the triangle  $ABC$  alongside its orthocenter  $H$ , its circumcenter  $O$  and the diametrically opposed point  $A'$  of  $A$  on the latter circle. Show that:

- (a)  $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$ .
- (b)  $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$ .
- (c)  $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2 \overrightarrow{HO}$ .



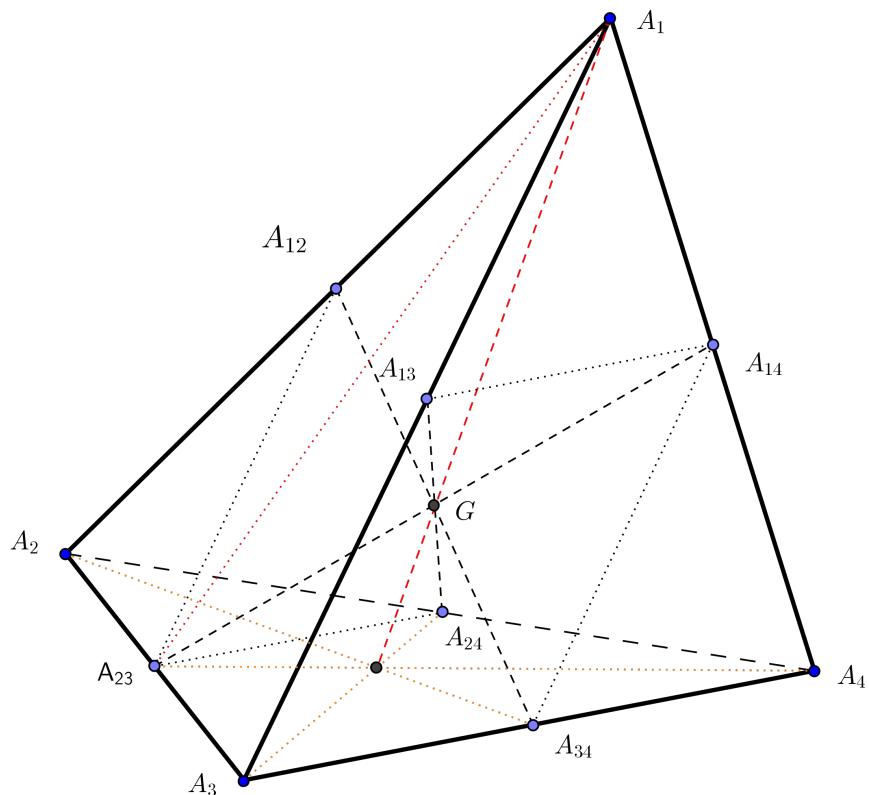
*Solution.*

10. ([4, Problem 15, p. 4]) Consider the triangle  $ABC$  alongside its centroid  $G$ , its orthocenter  $H$  and its circumcenter  $O$ . Show that  $O, G, H$  are collinear and  $3 \overrightarrow{HG} = 2 \overrightarrow{HO}$ .

*Solution.*

11. ([4, Problem 27, p. 13]) Consider a tetrahedron  $A_1A_2A_3A_4$  and the midpoints  $A_{ij}$  of the edges  $A_iA_j$ ,  $i \neq j$ . Show that:

- (a) The lines  $A_{12}A_{34}$ ,  $A_{13}A_{24}$  and  $A_{14}A_{23}$  are concurrent in a point  $G$ .
- (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at  $G$ .
- (c) Determine the ratio in which the point  $G$  divides each median.
- (d) Show that  $\vec{GA}_1 + \vec{GA}_2 + \vec{GA}_3 + \vec{GA}_4 = \vec{0}$ .
- (e) If  $M$  is an arbitrary point, show that  $\vec{MA}_1 + \vec{MA}_2 + \vec{MA}_3 + \vec{MA}_4 = 4 \vec{MG}$ .



*Solution.*



12. In a triangle  $ABC$  consider the points  $M, L$  on the side  $AB$  and  $N, T$  on the side  $AC$  such that  $3 \overrightarrow{AL} = 2 \overrightarrow{AM} = \overrightarrow{AB}$  and  $3 \overrightarrow{AT} = 2 \overrightarrow{AN} = \overrightarrow{AC}$ . Show that  $\overrightarrow{AB} + \overrightarrow{AC} = 5 \overrightarrow{AS}$ , where  $\{S\} = MT \cap LN$ .

*Solution.*

13. Consider two triangles  $A_1B_1C_1$  and  $A_2B_2C_2$ , not necessarily in the same plane, alongside their centroids  $G_1, G_2$ . Show that  $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{G_1G_2}$ .

*Solution.*

## 2 Week 2: Straight lines and planes

### 2.1 Linear dependence and linear independence of vectors

**Definition 2.1.** 1. The vectors  $\overrightarrow{OA}, \overrightarrow{OB}$  are said to be *collinear* if the points  $O, A, B$  are collinear. Otherwise the vectors  $\overrightarrow{OA}, \overrightarrow{OB}$  are said to be *noncollinear*.

2. The vectors  $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$  are said to be *coplanar* if the points  $O, A, B, C$  are coplanar. Otherwise the vectors  $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$  are *noncoplanar*.

**Remark 2.1.** 1. The vectors  $\overrightarrow{OA}, \overrightarrow{OB}$  are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors  $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$  are linearly (in)dependent if and only if they are (non)coplanar.

**Proposition 2.1.** The vectors  $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$  form a basis of  $\mathcal{V}$  if and only if they are noncoplanar.

**Corollary 2.2.** The dimension of the vector space of free vectors  $\mathcal{V}$  is three.

**Proposition 2.3.** Let  $\Delta$  be a straight line and let  $A \in \Delta$  be a given point. The set

$$\vec{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of  $\mathcal{V}$ . It is independent on the choice of  $A \in \Delta$  and is called the director subspace of  $\Delta$  or the direction of  $\Delta$ .

**Remark 2.2.** The straight lines  $\Delta, \Delta'$  are parallel if and only if  $\vec{\Delta} = \vec{\Delta}'$

**Definition 2.2.** We call director vector of the straigh line  $\Delta$  every nonzero vector  $\vec{d} \in \vec{\Delta}$ .

If  $\vec{d} \in \mathcal{V}$  is a nonzero vector and  $A \in \mathcal{P}$  is a given point, then there exists a unique straight line which passes through  $A$  and has the direction  $\langle \vec{d} \rangle$ . This straight line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \vec{d} \rangle\}.$$

$\Delta$  is called the straight line which passes through  $O$  and is parallel to the vector  $\vec{d}$ .

**Proposition 2.4.** Let  $\pi$  be a plane and let  $A \in \pi$  be a given point. The set  $\vec{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$  is a two dimensional subspace of  $\mathcal{V}$ . It is independent on the position of  $A$  inside  $\pi$  and is called the director subspace, the director plane or the direction of the plane  $\pi$ .

**Remark 2.3.** • The planes  $\pi, \pi'$  are parallel if and only if  $\vec{\pi} = \vec{\pi}'$ .

• If  $\vec{d}_1, \vec{d}_2$  are two linearly independent vectors and  $A \in \mathcal{P}$  is a fixed point, then there exists a unique plane through  $A$  whose direction is  $\langle \vec{d}_1, \vec{d}_2 \rangle$ . This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \vec{d}_1, \vec{d}_2 \rangle\}.$$

We say that  $\pi$  is the plane which passes through the point  $A$  and is parallel to the vectors  $\vec{d}_1$  and  $\vec{d}_2$ .

**Remark 2.4.** Let  $\Delta \subset \mathcal{P}$  be a straight line and  $\pi \subset \mathcal{P}$  be given plane.

1. If  $A \in \Delta$  is a given point, then  $\varphi_O(\Delta) = \vec{r}_A + \vec{\Delta}$ .

2. If  $B \in \Delta$  is a given point, then  $\varphi_O(\pi) = \vec{r}_B + \vec{\pi}$ .

Generally speaking, a subset  $X$  of a vector space is called *linear variety* if either  $X = \emptyset$  or there exists  $a \in V$  and a vector subspace  $U$  of  $V$ , such that  $X = a + U$ .

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

**Proposition 2.5.** *The bijection  $\varphi_O$  transforms the straight lines and the planes of the affine space  $\mathcal{P}$  into the one and two dimensional linear varieties of the vector space  $\mathcal{V}$  respectively.*

## 2.2 The vector equations of the straight lines and planes

**Proposition 2.6.** *Let  $\Delta$  be a straight line, let  $\pi$  be a plane,  $\{\vec{d}\}$  be a basis of  $\vec{\Delta}$  and let  $[\vec{d}_1, \vec{d}_2]$  be an ordered basis of  $\vec{\pi}$ .*

1. *The points  $M \in \Delta$  are characterized by the vector equation of  $\Delta$*

$$\vec{r}_M = \vec{r}_A + \lambda \vec{d}, \quad \lambda \in \mathbb{R} \quad (2.1)$$

*where  $A \in \Delta$  is a given point.*

2. *The points  $M \in \pi$  are characterized by the vector equation of  $\pi$*

$$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad (2.2)$$

*where  $A \in \pi$  is a given point.*

PROOF.

□

**Corollary 2.7.** If  $A, B \in \mathcal{P}$  are different points, then the vector equation of the line  $AB$  is

$$\vec{r}_M = (1 - \lambda) \vec{r}_A + \lambda \vec{r}_B, \quad \lambda \in \mathbb{R}. \quad (2.3)$$

PROOF.

□

**Corollary 2.8.** If  $A, B, C \in \mathcal{P}$  are three noncollinear points, then the vector equation of the plane  $(ABC)$  is

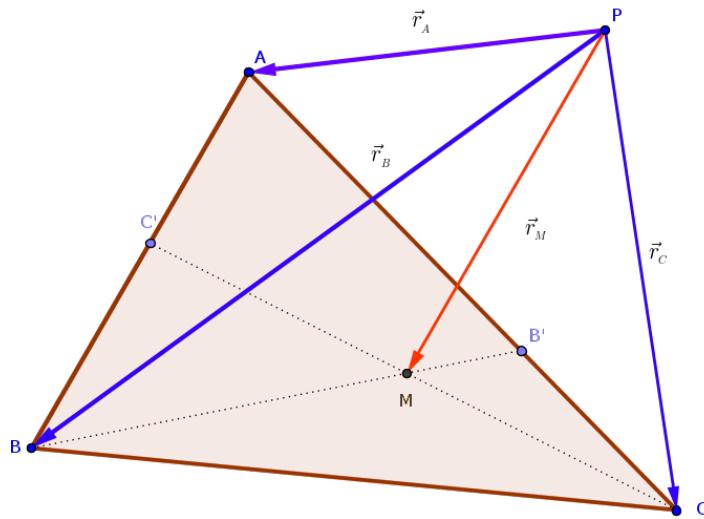
$$\vec{r}_M = (1 - \lambda_1 - \lambda_2) \vec{r}_A + \lambda_1 \vec{r}_B + \lambda_2 \vec{r}_C, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \quad (2.4)$$

PROOF.

□

**Example 2.1.** Consider the points  $C'$  and  $B'$  on the sides  $AB$  and  $AC$  of the triangle  $ABC$  such that  $\vec{AC}' = \lambda \vec{BC}', \vec{AB}' = \mu \vec{CB}'$ . The lines  $BB'$  and  $CC'$  meet at  $M$ . If  $P \in \mathcal{P}$  is a given point and  $\vec{r}_A = \vec{PA}, \vec{r}_B = \vec{PB}, \vec{r}_C = \vec{PC}$  are the position vectors, with respect to  $P$ , of the vertices  $A, B, C$  respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda \vec{r}_B - \mu \vec{r}_C}{1 - \lambda - \mu}. \quad (2.5)$$



SOLUTION.

□

## 2.3 Problems

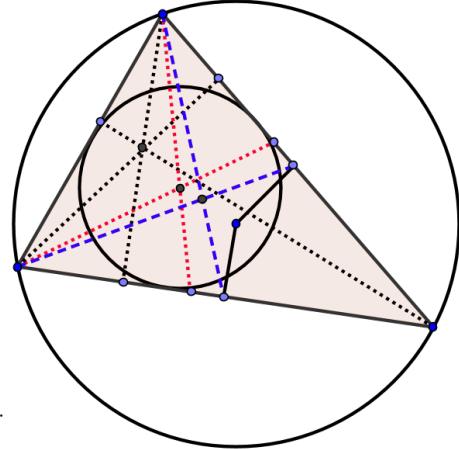
1. ([4, Problem 17, p. 5]) Consider the triangle  $ABC$ , its centroid  $G$ , its orthocenter  $H$ , its incenter  $I$  and its circumcenter  $O$ . If  $P \in \mathcal{P}$  is a given point and  $\vec{r}_A = \overrightarrow{PA}$ ,  $\vec{r}_B = \overrightarrow{PB}$ ,  $\vec{r}_C = \overrightarrow{PC}$  are the position vectors with respect to  $P$  of the vertices  $A, B, C$  respectively, show that:

$$\vec{r}_G := \overrightarrow{PG} = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}.$$

$$\vec{r}_I := \overrightarrow{PI} = \frac{a \vec{r}_A + b \vec{r}_B + c \vec{r}_C}{a + b + c}.$$

$$\vec{r}_H := \overrightarrow{PH} = \frac{(\tan A) \vec{r}_A + (\tan B) \vec{r}_B + (\tan C) \vec{r}_C}{\tan A + \tan B + \tan C}.$$

$$\vec{r}_O := \overrightarrow{PO} = \frac{(\sin 2A) \vec{r}_A + (\sin 2B) \vec{r}_B + (\sin 2C) \vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}.$$



*Solution.*

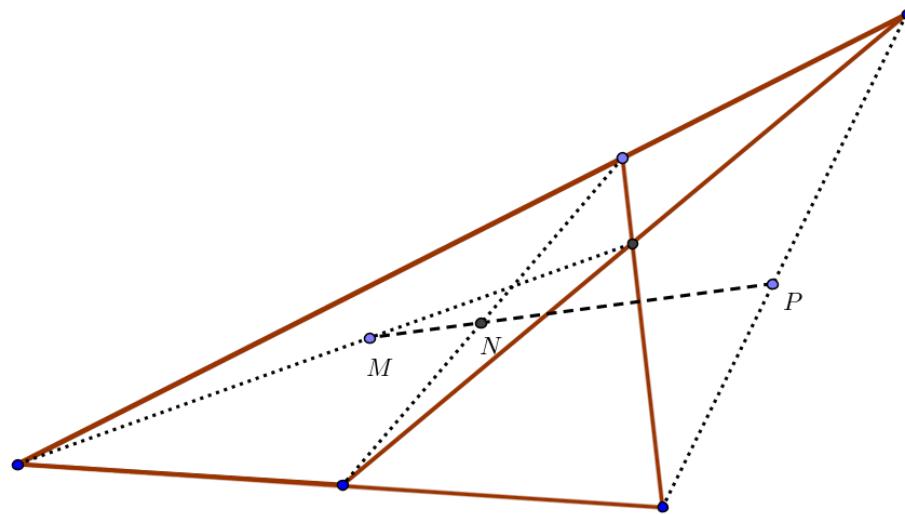
2. Consider the angle  $BOB'$  and the points  $A \in [OB]$ ,  $A' \in [OB']$ . Show that

$$\begin{aligned}\overrightarrow{OM} &= m \frac{1-n}{1-mn} \overrightarrow{OA} + n \frac{1-m}{1-mn} \overrightarrow{OA}' \\ \overrightarrow{ON} &= m \frac{n-1}{n-m} \overrightarrow{OA} + n \frac{m-1}{m-n} \overrightarrow{OA}' .\end{aligned}$$

where  $\{M\} = AB' \cap A'B$ ,  $\{N\} = AA' \cap BB'$ ,  $\vec{u} = \overrightarrow{OA}$ ,  $\vec{v} = \overrightarrow{OA}'$ ,  $\overrightarrow{OB} = m \overrightarrow{OA}$  and  $\overrightarrow{OB'} = n \overrightarrow{OA}'$ .

*Solution.*

3. Show that the midpoints of the diagonals of a complete quadrilateral are collinear (Newton's theorem).



*Solution.*

4. Let  $d, d'$  be concurrent straight lines and  $A, B, C \in d, A', B', C' \in d'$ . If the following relations  $AB' \parallel A'B, AC' \parallel A'C, BC' \parallel B'C$  hold, show that the points  $\{M\} := AB' \cap A'B, \{N\} := AC' \cap A'C, \{P\} := BC' \cap B'C$  are collinear (Pappus' theorem).

SOLUTION.

5. Let  $d, d'$  be two straight lines and  $A, B, C \in d, A', B', C' \in d'$  three points on each line such that  $AB' \parallel BA', AC' \parallel CA'$ . Show that  $BC' \parallel CB'$  (the affine Pappus' theorem).

SOLUTION.

6. Let us consider two triangles  $ABC$  and  $A'B'C'$  such that the lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent at a point  $O$  and  $AB \nparallel A'B'$ ,  $BC \nparallel B'C'$  and  $CA \nparallel C'A'$ . Show that the points  $\{M\} = AB \cap A'B'$ ,  $\{N\} = BC \cap B'C'$  and  $\{P\} = CA \cap C'A'$  are collinear (Desargues).

SOLUTION.

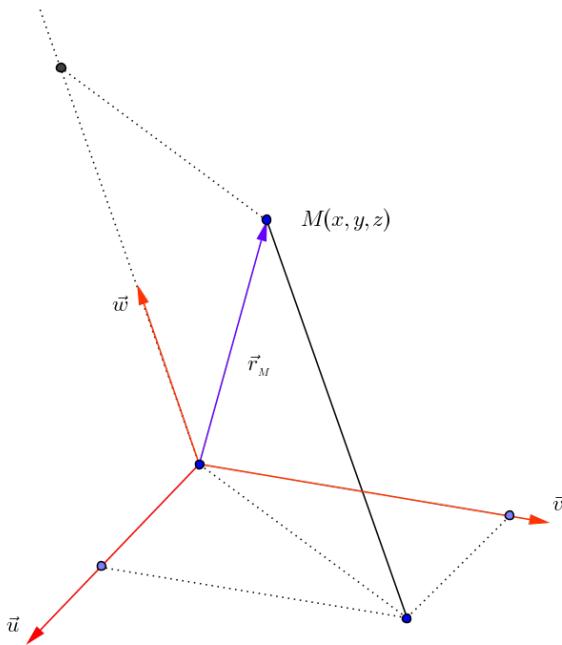
### 3 Week 3: Cartesian equations of lines and planes

#### 3.1 Cartesian and affine reference systems

If  $b = [\vec{u}, \vec{v}, \vec{w}]$  is an ordered basis of  $\mathcal{V}$  and  $\vec{x} \in \mathcal{V}$ , recall that the column vector of the coordinates of  $\vec{x}$  with respect to  $b$  is denoted by  $[\vec{x}]_b$ . In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever  $\vec{x} = x_1 \vec{u} + x_2 \vec{v} + x_3 \vec{w}$ . To emphasize the coordinates of  $\vec{x}$  with respect to  $b$ , we shall use the notation  $\vec{x} (x_1, x_2, x_3)$ .



**Definition 3.1.** A *cartesian reference system*  $R = (O, \vec{u}, \vec{v}, \vec{w})$  of the space  $\mathcal{P}$ , consists in a point  $O \in \mathcal{P}$  called the *origin* of the reference system and an ordered basis  $b = [\vec{u}, \vec{v}, \vec{w}]$  of the vector space  $\mathcal{V}$ .

Denote by  $E_1, E_2, E_3$  the points for which  $\vec{u} = \overrightarrow{OE_1}$ ,  $\vec{v} = \overrightarrow{OE_2}$ ,  $\vec{w} = \overrightarrow{OE_3}$ .

**Definition 3.2.** The system of points  $(O, E_1, E_2, E_3)$  is called *the affine reference system associated to the cartesian reference system  $R = (O, \vec{u}, \vec{v}, \vec{w})$* .

The straight lines  $OE_i$ ,  $i \in \{1, 2, 3\}$ , oriented from  $O$  to  $E_i$  are called *the coordinate axes*. The coordinates  $x, y, z$  of the position vector  $\vec{r}_M = \overrightarrow{OM}$  with respect to the basis  $[\vec{u}, \vec{v}, \vec{w}]$  are called the coordinates of the point  $M$  with respect to the cartesian system  $R$  written  $M(x, y, z)$ . Also, for the column matrix of coordinates of the vector  $\vec{r}_M$  we are going to use the notation  $[M]_R$ . In other words, if  $\vec{r}_M = x \vec{u} + y \vec{v} + z \vec{w}$ , then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

**Remark 3.1.** If  $A(x_A, y_A, z_A)$ ,  $B(x_B, y_B, z_B)$  are two points, then

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B \vec{u} + y_B \vec{v} + z_B \vec{w} - (x_A \vec{u} + y_A \vec{v} + z_A \vec{w}) \\ &= (x_B - x_A) \vec{u} + (y_B - y_A) \vec{v} + (z_B - z_A) \vec{w},\end{aligned}$$

i.e. the coordinates of the vector  $\overrightarrow{AB}$  are being obtained by performing the differences of the coordinates of the points  $A$  and  $B$ .

**Remark 3.2.** If  $R = (O, b)$  is a cartesian reference system, where  $b = [\vec{u}, \vec{v}, \vec{w}]$  is an ordered basis of  $\mathcal{V}$ , recall that  $\varphi_O : \mathcal{P} \longrightarrow \mathcal{V}$ ,  $\varphi_O(M) = \overrightarrow{OM}$  is bijective and  $\psi_b : \mathbb{R}^3 \longrightarrow \mathcal{V}$ ,  $\psi_b(x, y, z) = x \vec{u} + y \vec{v} + z \vec{w}$  is a linear isomorphism. The bijection  $\varphi_O$  defines a unique vector structure over  $\mathcal{P}$  such that  $\varphi_O$  becomes an isomorphism. This vector structure depends on the choice of  $O \in \mathcal{P}$ . Therefore a point  $M \in \mathcal{P}$  could be identified either with its position vector  $\vec{r}_M = \varphi_O(M)$ , or, with the triplet  $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$  of its coordinates with respect to the reference system  $R$ . If  $f : X \longrightarrow \mathbb{R}^3$  is a given application, then  $\varphi_O^{-1} \circ \psi_b \circ f : X \longrightarrow \mathcal{P}$  will be denoted by  $M_f$ . A similar discussion can be done for a cartesian reference system  $R' = (O', b')$  of a plane  $\pi$ , where  $b' = [\vec{u}', \vec{v}']$  is an ordered basis of  $\pi$ .

**Example 3.1 (Homework).** Consider the tetrahedron  $ABCD$ , where  $A(1, -1, 1)$ ,  $B(-1, 1, -1)$ ,  $C(2, 1, -1)$  and  $D(1, 1, 2)$ . Find the coordinates of:

1. the centroids  $G_A$ ,  $G_B$ ,  $G_C$ ,  $G_D$  of the triangles  $BCD$ ,  $ACD$ ,  $ABD$  and  $ABC^1$  respectively.
2. the midpoints  $M$ ,  $N$ ,  $P$ ,  $Q$ ,  $R$  and  $S$  of its edges  $[AB]$ ,  $[AC]$ ,  $[AD]$ ,  $[BC]$ ,  $[CD]$  and  $[DB]$  respectively.

SOLUTION.

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<sup>1</sup>The centroids of its faces

### 3.2 The cylindrical coordinate System

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated with a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the Cartesian system of coordinates.

Let  $P(x, y, z)$  be a point in a Cartesian system of coordinates  $Oxyz$  and  $P'$  be the orthogonal projection of  $P$  on the plane  $xOy$ . One can associate to the point  $P$  the triple  $(r, \theta, z)$ , where  $(r, \theta)$  are the polar coordinates of  $P'$  (see Figure 1). The polar coordinates of  $P'$  can be obtained by specifying the distance  $\rho$  from  $O$  to  $P'$  and the angle  $\theta$  (measured in radians), whose "initial" side is the polar axis, i.e. the  $x$ -axis, and whose "terminal" side is the ray  $OP$ . The *polar coordinates* of the point  $P$  are  $(\rho, \theta)$  (See also section (3.6.2) of the Appendix). The triple  $(r, \theta, z)$  gives the *cylindrical coordinates* of the point  $P$ . There is the bijection

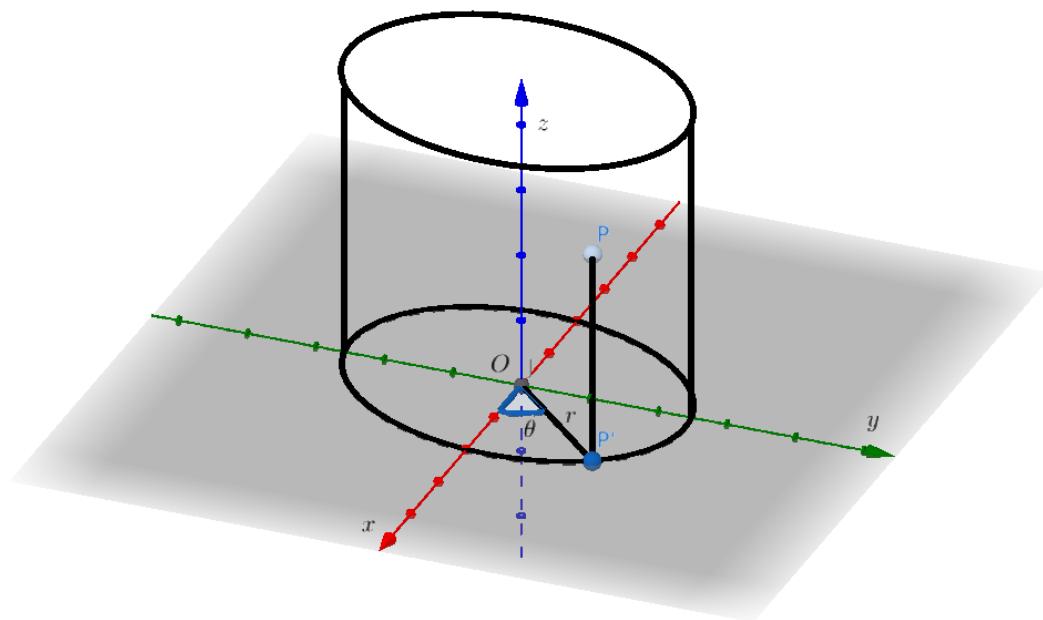


Figure 1: cylindrical coordinates

$$h_1 : \mathcal{P} \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}, P \rightarrow (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* in  $\mathcal{P}$ . For the conversion formulas between the cylindrical coordinates and the Cartesian coordinates we refere the reader to [1, p. 19]. Note however that once we have the cylindrical coordinates  $(r, \theta, z)$  of a point  $P$ , then its Cartesian coordinates are

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} .$$

### 3.3 The spherical coordinate system

Another way to associate to each point  $P$  in  $\mathcal{P}$  a triple of real numbers is illustrated in Figure 2. If  $P(x, y, z)$  is a point in a rectangular system of coordinates  $Oxyz$  and  $P'$  its or-

thogonal projection on  $Oxy$ , let  $\rho$  be the length of the segment  $[OP]$ ,  $\theta$  be the oriented angle determined by  $[Ox]$  and  $[OP']$  and  $\varphi$  be the oriented angle between  $[Oz]$  and  $[OP]$ . The triple

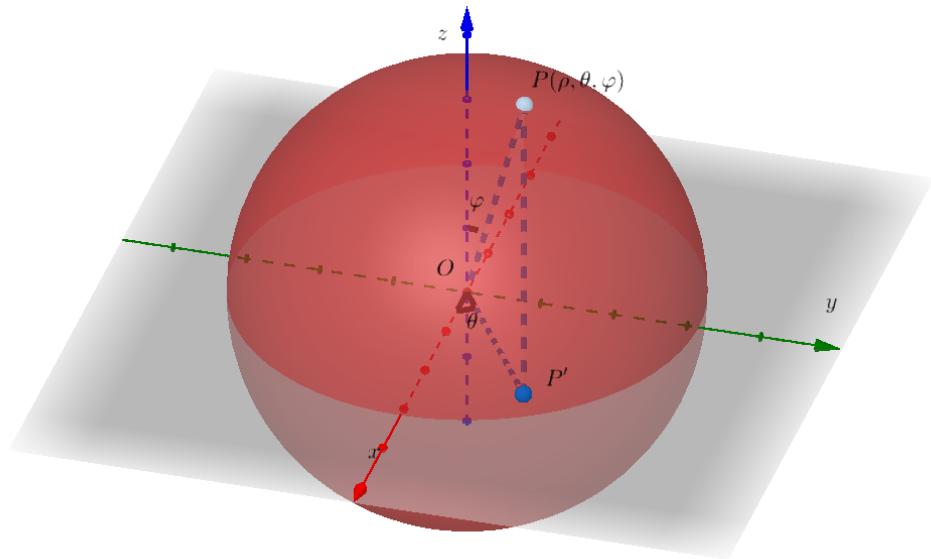


Figure 2: spherical coordinates

$(\rho, \theta, \varphi)$  gives the *spherical coordinates* of the point  $P$ . This way, one obtains the bijection

$$h_2 : \mathcal{P} \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \rightarrow (\rho, \theta, \varphi),$$

which defines a new coordinate system in  $\mathcal{P}$ , called the *spherical coordinate system*. For the conversion formulas between the spherical coordinate system and the Cartesian coordinate system we refere the reader to [1, p. 20]. Note however that once we have the cylindrical coordinates  $(\rho, \theta, \varphi)$  of a point  $P$ , then its Cartesian coordinates are

$$\begin{cases} x = r \cos \theta \sin \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \varphi \end{cases}.$$

### 3.4 The Cartesian equations of the straight lines

Let  $\Delta$  be the straight line passing through the point  $A_0(x_0, y_0, z_0)$  which is parallel to the vector  $\vec{d} = (p, q, r)$ . Its vector equation is

$$\vec{r}_M = \vec{r}_{A_0} + \lambda \vec{d}, \lambda \in \mathbb{R}. \quad (3.1)$$

Denoting by  $x, y, z$  the coordinates of the generic point  $M$  of the straight line  $\Delta$ , its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}, \lambda \in \mathbb{R} \quad (3.2)$$

Indeed, the vector equation of  $\Delta$  can be written, in terms of the coordinates of the vectors  $\vec{r}_M$ ,  $\vec{r}_{A_0}$  and  $\vec{d}$ , as follows:

$$\begin{aligned} x \vec{u} + y \vec{v} + z \vec{w} &= x_0 \vec{u} + y_0 \vec{v} + z_0 \vec{w} + \lambda(p \vec{u} + q \vec{v} + r \vec{w}) \\ \iff x \vec{u} + y \vec{v} + z \vec{w} &= (x_0 + p\lambda) \vec{u} + (y_0 + q\lambda) \vec{v} + (z_0 + r\lambda) \vec{w}, \quad \lambda \in \mathbb{R} \end{aligned}$$

which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line  $\Delta$  and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \quad (3.3)$$

If  $r = 0$ , for instance, the canonical equations of the straight line  $\Delta$  are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \wedge z = z_0.$$

If  $A(x_A, y_A, z_A)$ ,  $B(x_B, y_B, z_B)$  are different points of the line  $\Delta$ , then

$$\overrightarrow{AB} (x_B - x_A, y_B - y_A, z_B - z_A)$$

is a director vector of  $\Delta$ , its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \quad (3.4)$$

**Example 3.2.** Consider the tetrahedron  $ABCD$ , where  $A(1, -1, 1)$ ,  $B(-1, 1, -1)$ ,  $C(2, 1, -1)$  and  $D(1, 1, 2)$ , as well as the centroids  $G_A$ ,  $G_B$ ,  $G_C$ ,  $G_D$  of the triangles  $BCD$ ,  $ACD$ ,  $ABD$  and  $ABC^2$  respectively. Show that the medians  $AG_A$ ,  $BG_B$ ,  $CG_C$  and  $DG_D$  are concurrent and find the coordinates of their intersection point.

SOLUTION. One can easily see that the coordinates of the centroids  $G_A$ ,  $G_B$ ,  $G_C$ ,  $G_D$  are  $(2/3, 1, 0)$ ,  $(4/3, 1/3, 2/3)$ ,  $(1/3, 1/3, 2/3)$  and  $(2/3, 1/3, -1/3)$  respectively. The equations of the medians  $AG_A$  and  $BG_B$  are

$$\begin{aligned} (AG_A) \quad \frac{x - 1}{2/3 - 1} &= \frac{y + 1}{1 - (-1)} = \frac{z - 1}{0 - 1} \iff \frac{x - 1}{-1/3} = \frac{y + 1}{2} = \frac{z - 1}{-1} \\ (BG_B) \quad \frac{x + 1}{4/3 + 1} &= \frac{y - 1}{1/3 - 1} = \frac{z + 1}{2/3 + 1} \iff \frac{x + 1}{7/3} = \frac{y - 1}{-2/3} = \frac{z + 1}{5/3}. \end{aligned}$$

Thus, the director space of the median  $AG_A$  is  $\left\langle \left( -\frac{1}{3}, 2, -1 \right) \right\rangle = \langle (-1, 6, -3) \rangle$  and the director space of the median  $BG_B$  is  $\left\langle \left( \frac{7}{3}, -\frac{2}{3}, \frac{5}{3} \right) \right\rangle = \langle (7, -2, 5) \rangle$ . Consequently, the parametric equations of the medians  $AG_A$  and  $BG_B$  are

$$(AG_A) \quad \begin{cases} x = 1 - t \\ y = -1 + 6t \\ z = 1 - 3t \end{cases}, \quad t \in \mathbb{R} \text{ and } (BG_B) \quad \begin{cases} x = -1 + 7s \\ y = 1 - 2s \\ z = -1 + 5s \end{cases}, \quad s \in \mathbb{R}.$$

Thus, the two medians  $AG_A$  and  $BG_B$  are concurrent if and only if there exist  $s, t \in \mathbb{R}$  such that

$$\begin{cases} 1 - t = -1 + 7s \\ -1 + 6t = 1 - 2s \\ 1 - 3t = -1 + 5s \end{cases} \iff \begin{cases} 7s + t = 2 \\ 2s + 6t = 2 \\ 5s + 3t = 2 \end{cases} \iff \begin{cases} 7s + t = 2 \\ s + 3t = 1 \\ 5s + 3t = 2. \end{cases}$$

<sup>2</sup>The centroids of its faces

This system is compatible and has the unique solution  $s = t = \frac{1}{4}$ , which shows that the two medians  $AG_A$  and  $BG_B$  are concurrent and

$$AG_A \cap BG_B = \left\{ G \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right) \right\}.$$

One can similarly show that  $BG_B \cap CG_C = CG_C \cap AG_A = \left\{ G \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right) \right\}$ .

**Example 3.3 (Homework).** Consider the tetrahedron  $ABCD$ , where  $A(1, -1, 1)$ ,  $B(-1, 1, -1)$ ,  $C(2, 1, -1)$  and  $D(1, 1, 2)$ , as well as the midpoints  $M$ ,  $N$ ,  $P$ ,  $Q$ ,  $R$  and  $S$  of its edges  $[AB]$ ,  $[AC]$ ,  $[AD]$ ,  $[BC]$ ,  $[CD]$  and  $[DB]$  respectively. Show that the lines  $MR$ ,  $PQ$  and  $NS$  are concurrent and find the coordinates of their intersection point.

SOLUTION.

### 3.5 The Cartesian equations of the planes

Let  $A_0(x_0, y_0, z_0) \in \mathcal{P}$  and  $\vec{d}_1(p_1, q_1, r_1)$ ,  $\vec{d}_2(p_2, q_2, r_2) \in \mathcal{V}$  be linearly independent vectors, that is

$$\text{rank} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane  $\pi$  passing through  $A_0$  which is parallel to the vectors  $\vec{d}_1(p_1, q_1, r_1)$ ,  $\vec{d}_2(p_2, q_2, r_2)$  is

$$\vec{r}_M = \vec{r}_{A_0} + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \quad (3.5)$$

If we denote by  $x, y, z$  the coordinates of the generic point  $M$  of the plane  $\pi$ , then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases}, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \quad (3.6)$$

Indeed, the vector equation of  $\pi$  can be written, in terms of the coordinates of the vectors  $\vec{r}_M$ ,  $\vec{r}_{A_0}$ ,  $\vec{d}_1$  and  $\vec{d}_2$ , as follows:

$$\begin{aligned} x \vec{u} + y \vec{v} + z \vec{w} &= x_0 \vec{u} + y_0 \vec{v} + z_0 \vec{w} + \lambda_1(p_1 \vec{u} + q_1 \vec{v} + r_1 \vec{w}) + \lambda_2(p_2 \vec{u} + q_2 \vec{v} + r_2 \vec{w}) \\ \iff x \vec{u} + y \vec{v} + z \vec{w} &= (x_0 + \lambda_1 p_1 + \lambda_2 p_2) \vec{u} + (y_0 + \lambda_1 q_1 + \lambda_2 q_2) \vec{v} + (z_0 + \lambda_1 r_1 + \lambda_2 r_2) \vec{w}, \\ \lambda_1, \lambda_2 &\in \mathbb{R}, \end{aligned}$$

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane  $\pi$  and are called the *parametric equations* of the plane  $\pi$ . More precisely, the compatibility of the linear system (3.6) with the unknowns  $\lambda_1, \lambda_2$  is a necessary and sufficient condition for the point  $M(x, y, z)$  to be contained within the plane  $\pi$ . On the other hand the compatibility of the linear system (3.6) is equivalent to

$$\left| \begin{array}{ccc} x - x_0 & y - y_0 & z - z_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{array} \right| = 0, \quad (3.7)$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane  $\pi$  in terms of the Cartesian coordinates of the generic point  $M$  and is called the *cartesian equation* of the plane  $\pi$ . One can put the equation (3.7) in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{ or} \quad (3.8)$$

$$Ax + By + Cz + D = 0, \quad (3.9)$$

where the coefficients  $A, B, C$  satisfy the relation  $A^2 + B^2 + C^2 > 0$ . It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if  $A \neq 0$ , then the equation (3.9) is equivalent to

$$\left| \begin{array}{ccc} x + \frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{array} \right| = 0.$$

We observe that one can put the equation (3.8) in the form

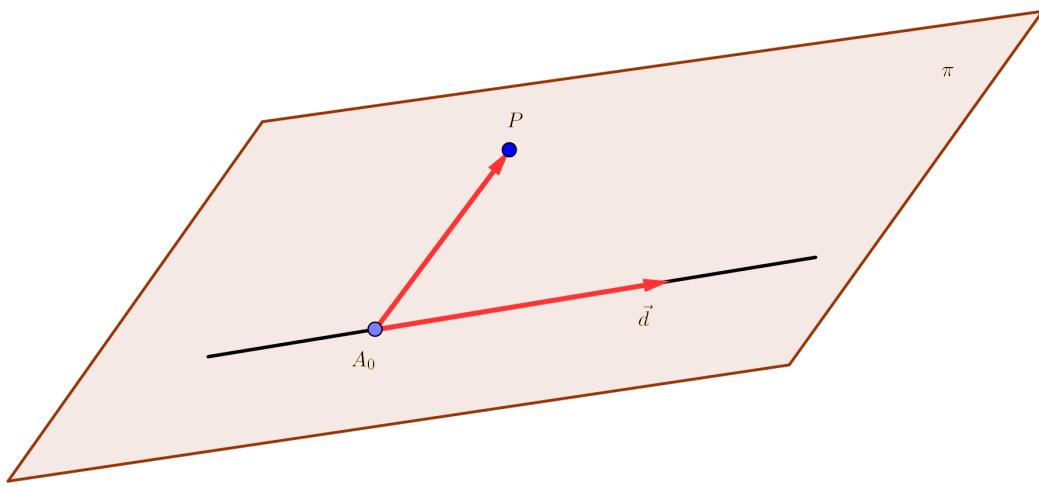
$$AX + BY + CZ = 0 \quad (3.10)$$

where  $X = x - x_0$ ,  $Y = y - y_0$ ,  $Z = z - z_0$  are the coordinates of the vector  $\vec{A_0M}$ .

**Example 3.4.** Write the equation of the plane determined by the point  $P(-1, 1, 2)$  and the line  $(\Delta)$   $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{-1}$ .

**SOLUTION.** Note that  $P \notin \Delta$ , as  $\frac{-1-1}{3} \neq \frac{1}{2} \neq -3 = \frac{2+1}{-1}$ , i.e. the point  $P$  and the line  $\Delta$  determine, indeed, a plane, say  $\pi$ . One can regard  $\pi$  as the plane through the point  $A_0(1, 0, -1)$  which is parallel to the vectors  $\vec{A_0P} (-1 - 1, 1 - 0, 2 - (-1)) = \vec{A_0P} (-2, 1, 3)$  and  $\vec{d} (3, 2, -1)$ . Thus, the equation of  $\pi$  is

$$\left| \begin{array}{ccc} x - 1 & y & z + 1 \\ -2 & 1 & 3 \\ 3 & 2 & -1 \end{array} \right| = 0 \iff x - y + z = 0.$$



**Example 3.5 (Homework).** Generalize Example 3.4: Write the equation of the plane determined by the line  $(\Delta)$   $\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$  and the point  $M(x_M, y_M, z_M) \notin \Delta$ .

SOLUTION.

**Remark 3.3.** If  $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$  are noncollinear points, then the plane  $(ABC)$  determined by the three points can be viewed as the plane passing through the point  $A$  which is parallel to the vectors  $\vec{d}_1 = \overrightarrow{AB}, \vec{d}_2 = \overrightarrow{AC}$ . The coordinates of the vectors  $\vec{d}_1$  și  $\vec{d}_2$  are

$(x_B - x_A, y_B - y_A, z_B - z_A)$  and  $(x_C - x_A, y_C - y_A, z_C - z_A)$  respectively.

Thus, the equation of the plane  $(ABC)$  is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \quad (3.11)$$

or, equivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \quad (3.12)$$

Thus, four points  $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$  and  $D(x_D, y_D, z_D)$  are coplanar if and only if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \quad (3.13)$$

**Example 3.6 (Homework).** Write the equation of the plane determined by the points  $M_1(3, -2, 1)$ ,  $M_2(5, 4, 1)$  and  $M_3(-1, -2, 3)$ .

SOLUTION.

**Remark 3.4.** If  $A(a, 0, 0)$ ,  $B(0, b, 0)$ ,  $C(0, 0, c)$  are three points ( $abc \neq 0$ ), then for the equation of the plane  $(ABC)$  we have successively:

$$\begin{aligned} \left| \begin{array}{cccc} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{array} \right| = 0 &\iff \left| \begin{array}{cccc} x & y & z - c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{array} \right| = 0 \iff \left| \begin{array}{ccc} x & y & z - c \\ a & 0 & -c \\ 0 & b & -c \end{array} \right| = 0 \\ &\iff ab(z - c) + bcx + acy = 0 \iff bcx + acy + abz = abc \\ &\iff \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \end{aligned} \tag{3.14}$$

The equation (3.14) of the plane  $(ABC)$  is said to be in *intercept form* and the  $x, y, z$ -intercepts of the plane  $(ABC)$  are  $a, b, c$  respectively.

**Example 3.7 (Homework).** Write the equation of the plane  $(\pi)$   $3x - 4y + 6z - 24 = 0$  in intercept form.

SOLUTION.

## 3.6 Appendix: The Cartesian equations of lines in the two dimensional setting

### 3.6.1 Cartesian and affine reference systems

If  $b = [\vec{e}, \vec{f}]$  is an ordered basis of the director subspace  $\vec{\pi}$  of the plane  $\pi$  and  $\vec{x} \in \vec{\pi}$ , recall that the column vector of  $\vec{x}$  with respect to  $b$  is being denoted by  $[\vec{x}]_b$ . In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever  $\vec{x} = x_1 \vec{e} + x_2 \vec{f}$ .

**Definition 3.3.** A *cartesian reference system* of the plane  $\pi$ , is a system  $R = (O, \vec{e}, \vec{f})$ , where  $O$  is a point from  $\pi$  called the *origin* of the reference system and  $b = [\vec{e}, \vec{f}]$  is a basis of the vector space  $\pi$ .

Denote by  $E, F$  the points for which  $\vec{e} = \overrightarrow{OE}$ ,  $\vec{f} = \overrightarrow{OF}$ .

**Definition 3.4.** The system of points  $(O, E, F)$  is called *the affine reference system associated to the cartesian reference system  $R = (O, \vec{e}, \vec{f})$* .

The straight lines  $OE$ ,  $OF$ , oriented from  $O$  to  $E$  and from  $O$  to  $F$  respectively, are called *the coordinate axes*. The coordinates  $x, y$  of the position vector  $\vec{r}_M = \overrightarrow{OM}$  with respect to the basis  $[\vec{e}, \vec{f}]$  are called the coordinates of the point  $M$  with respect to the cartesian system  $R$  written  $M(x, y)$ . Also, for the column matrix of coordinates of the vector  $\vec{r}_M$  we are going to use the notation  $[M]_R$ . In other words, if  $\vec{r}_M = x \vec{e} + y \vec{f}$ , then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Remark 3.5.** If  $A(x_A, y_A)$ ,  $B(x_B, y_B)$  are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = x_B \vec{e} + y_B \vec{f} - (x_A \vec{e} + y_A \vec{f}) \\ &= (x_B - x_A) \vec{e} + (y_B - y_A) \vec{f}, \end{aligned}$$

i.e. the coordinates of the vector  $\overrightarrow{AB}$  are being obtained by performing the differences of the coordinates of the points  $A$  and  $B$ .

### 3.6.2 The Polar Coordinate System [1, p. 17]

As an alternative to a Cartesian coordinate system (RS) one considers in the plane  $\pi$  a fixed point  $O$ , called *pole* and a half-line directed to the right of  $O$ , called *polar axis* (see Figure 3). By specifying the distance  $\rho$  from  $O$  to a point  $P$  and an angle  $\theta$  (measured in radians), whose

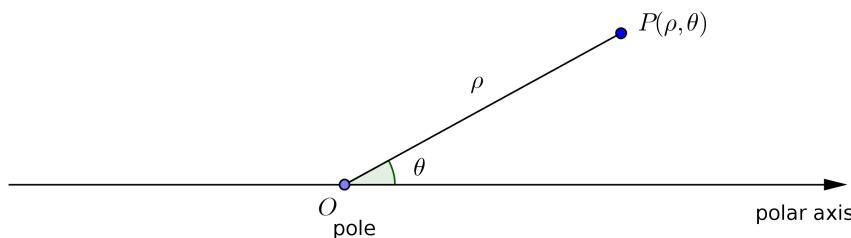


Figure 3: The pole and the polar axis related to a polar coordinate system

"initial" side is the polar axis and whose "terminal" side is the ray  $OP$ , the *polar coordinates* of the point  $P$  are  $(\rho, \theta)$ . One obtains a bijection

$$\pi \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi), \quad P \rightarrow (\rho, \theta)$$

which associates to any point  $P$  in  $\pi \setminus \{O\}$  the pair  $(\rho, \theta)$  (suppose that  $O(0, 0)$ ). The positive real number  $\rho$  is called the *polar ray* of  $P$  and  $\theta$  is called the *polar angle* of  $P$ .

Consider the Cartesian coordinate system in  $\pi$ , whose origin  $O$  is the pole and whose positive half-axis  $Ox$  is the polar axis (see Figure 4). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

Let  $P$  be a given point of polar coordinates  $(\rho, \theta)$ . Its Cartesian coordinates are

$$\begin{cases} x_P = \rho \cos \theta \\ y_P = \rho \sin \theta \end{cases}. \quad (3.15)$$

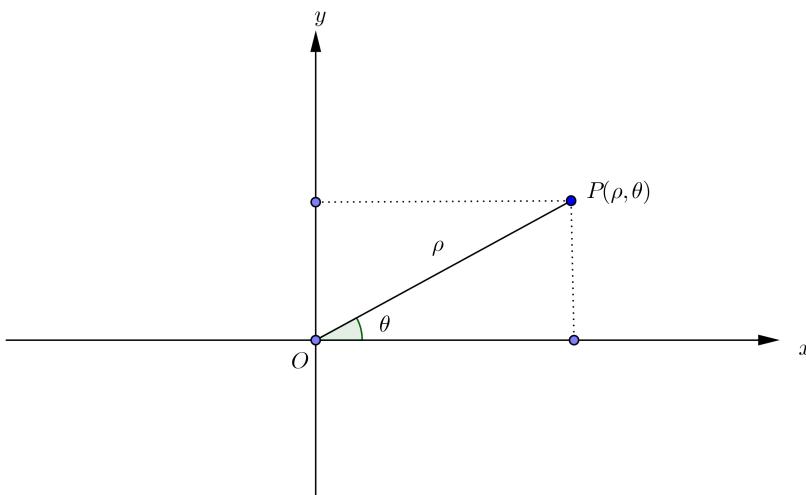


Figure 4: Polar coordinates

Let  $P$  be a point of Cartesian coordinates  $(x, y)$ . It is clear that the polar ray of  $P$  is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \quad (3.16)$$

In order to obtain the polar angle of  $P$ , it must be considered the quadrant where  $P$  is situated. One obtains the following formulas:

*Case 1.* If  $x \neq 0$ , then using  $\tan \theta = \frac{y}{x}$ , one has

$$\theta = \arctan \frac{y}{x} + k\pi, \quad \text{where } k = \begin{cases} 0 & \text{if } P \in I \cup ]Ox} \\ 1 & \text{if } P \in II \cup III \cup ]Ox' \\ 2 & \text{if } P \in IV; \end{cases};$$

*Case 2.* If  $x = 0$  and  $y \neq 0$ , then

$$\theta = \begin{cases} \frac{\pi}{2} & \text{when } P \in ]Oy} \\ \frac{3\pi}{2} & \text{when } P \in ]Oy'; \end{cases}$$

*Case 3.* If  $x = 0$  and  $y = 0$ , then  $\theta = 0$ .

### 3.6.3 Parametric and Cartesian equations of Lines

Let  $\Delta$  be a line passing through the point  $A_0(x_0, y_0) \in \pi$  which is parallel to the vector  $\vec{d} = (p, q)$ . Its vector equation is

$$\vec{r}_M = \vec{r}_{A_0} + t \vec{d}, \quad t \in \mathbb{R}. \quad (3.17)$$

If  $(x, y)$  are the coordinates of a generic point  $M \in \Delta$ , then its vector equation (3.17) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases}, \quad t \in \mathbb{R}. \quad (3.18)$$

The relations are called the *parametric equations* of the line  $\Delta$  and they are equivalent to the following equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \quad (3.19)$$

called the *canonical equation* of  $\Delta$ . If  $q = 0$ , then the equation (3.19) becomes  $y = y_0$ .

If  $A(x_A, y_A)$  are two different points of the plane  $\pi$ , then  $\vec{AB} = (x_B - x_A, y_B - y_A)$  is a director vector of the line  $AB$  and the canonical equation of the line  $AB$  is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \quad (3.20)$$

The equation (3.20) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \iff \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three points  $P_1(x_1, y_1)$ ,  $P_2(x_2, y_2)$  and  $P_3(x_3, y_3)$  are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \quad (3.21)$$

### 3.6.4 General Equations of Lines

We can put the equation (3.19) in the form

$$ax + by + c = 0, \quad \text{with } a^2 + b^2 > 0, \quad (3.22)$$

which means that any line from  $\pi$  is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.22) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through  $P_0\left(-\frac{c}{a}, 0\right)$  and parallel to  $\bar{v}\left(-\frac{b}{a}, 1\right)$ . The equation (3.22) is called *general equation* of the line.

**Remark 3.6.** The lines

$$(d) ax + by + c = 0 \text{ and } (\Delta) \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

are parallel if and only if  $ap + bq = 0$ . Indeed, we have:

$$\begin{aligned} d \parallel \Delta &\iff \vec{d} = \vec{\Delta} \iff \langle \vec{u}(p, q) \rangle = \langle \vec{v}\left(-\frac{b}{a}, 1\right) \rangle \iff \exists t \in \mathbb{R} \text{ s.t. } \vec{u}(p, q) = t \vec{v}\left(-\frac{b}{a}, 1\right) \\ &\iff \exists t \in \mathbb{R} \text{ s.t. } p = -t\frac{b}{a} \text{ and } q = t \iff ap + bq = 0. \end{aligned}$$

### 3.6.5 Reduced Equations of Lines

Consider a line given by its general equation  $Ax + By + C = 0$ , where at least one of the coefficients  $A$  and  $B$  is nonzero. One may suppose that  $B \neq 0$ , so that the equation can be divided by  $B$ . One obtains

$$y = mx + n \quad (3.23)$$

which is said to be the *reduced equation* of the line.

*Remark:* If  $B = 0$ , (3.22) becomes  $Ax + C = 0$ , or  $x = -\frac{C}{A}$ , a line parallel to  $Oy$ . (In the same way, if  $A = 0$ , one obtains the equation of a line parallel to  $Ox$ ).

Let  $d$  be a line of equation  $y = mx + n$  in a Cartesian system of coordinates and suppose that the line is not parallel to  $Oy$ . Let  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  be two different points on  $d$  and  $\varphi$  be the angle determined by  $d$  and  $Ox$  (see Figure 5);  $\varphi \in [0, \pi] \setminus \{\pi/2\}$ . The points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  belong to  $d$ , hence

$$\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n, \end{cases}$$

and  $x_2 \neq x_1$ , since  $d$  is not parallel to  $Oy$ . Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \varphi. \quad (3.24)$$

The number  $m = \tan \varphi$  is called the *angular coefficient* of the line  $d$ . It is immediate that the equation of the line passing through the point  $P_0(x_0, y_0)$  and of the given angular coefficient  $m$  is

$$y - y_0 = m(x - x_0). \quad (3.25)$$

### 3.6.6 Intersection of Two Lines

Let  $d_1 : a_1x + b_1y + c_1 = 0$  and  $d_2 : a_2x + b_2y + c_2 = 0$  be two lines in  $\mathcal{E}_2$ . The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of  $d_1$  and  $d_2$ .

- 1) If  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ , the system has a unique solution  $(x_0, y_0)$  and the lines have a unique intersection point  $P_0(x_0, y_0)$ . They are *secant*.

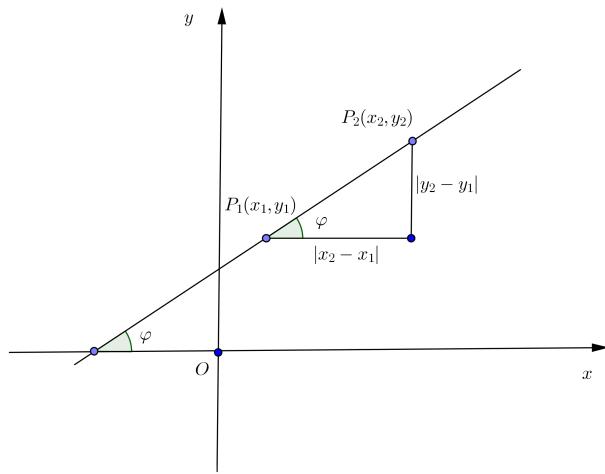


Figure 5:

- 2) If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ , the system is not compatible, and the lines have no points in common. They are *parallel*.
- 3) If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ , the system has an infinity of solutions, and the lines coincide. They are *identical*.

If  $d_i : a_i x + b_i y + c_i = 0, i = \overline{1,3}$  are three lines in  $\mathcal{E}_2$ , then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \quad (3.26)$$

### 3.6.7 Bundles of Lines ([1])

The set of all the lines passing through a given point  $P_0$  is said to be a *bundle* of lines. The point  $P_0$  is called the *vertex* of the bundle.

If the point  $P_0$  is of coordinates  $P_0(x_0, y_0)$ , then the equation of the bundle of vertex  $P_0$  is

$$r(x - x_0) + s(y - y_0) = 0, \quad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \quad (3.27)$$

*Remark:* The *reduced bundle* of line through  $P_0$  is,

$$y - y_0 = m(x - x_0), \quad m \in \mathbb{R}, \quad (3.28)$$

and covers the bundle of lines through  $P_0$ , except the line  $x = x_0$ . Similarly, the family of lines

$$x - x_0 = k(y - y_0), \quad k \in \mathbb{R}, \quad (3.29)$$

covers the bundle of lines through  $P_0$ , except the line  $y = y_0$ .

If the point  $P_0$  is given as the intersection of two lines, then its coordinates are the solution of the system

$$\left\{ \begin{array}{l} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{array} \right. ,$$

assumed to be compatible. The equation of the bundle of lines through  $P_0$  is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \quad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \quad (3.30)$$

*Remark:* As before, if  $r \neq 0$  (or  $s \neq 0$ ), one obtains the reduced equation of the bundle, containing all the lines through  $P_0$ , except  $d_1$  (respectively  $d_2$ ).

### 3.6.8 The Angle of Two Lines ([1])

Let  $d_1$  and  $d_2$  be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of  $d_1$  and  $d_2$  are  $m_1 = \tan \varphi_1$  and  $m_2 = \tan \varphi_2$  (see Figure 6). One may suppose that  $\varphi_1 \neq \frac{\pi}{2}$ ,  $\varphi_2 \neq \frac{\pi}{2}$ ,  $\varphi_2 \geq \varphi_1$ , such that  $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$ .

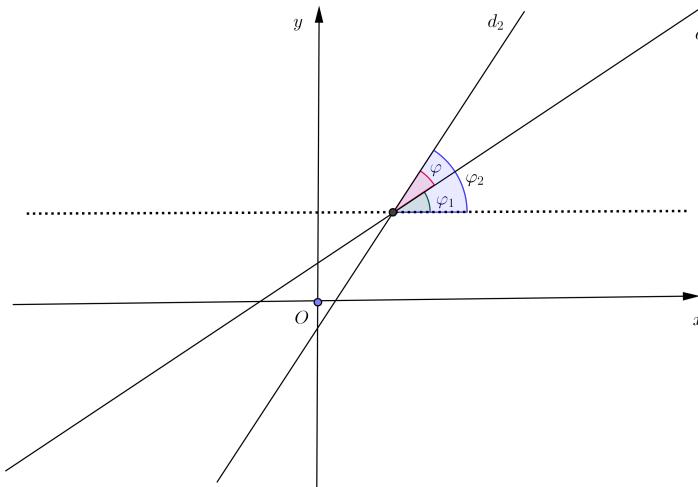


Figure 6:

The angle determined by  $d_1$  and  $d_2$  is given by

$$\tan \varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan \varphi_2 - \tan \varphi_1}{1 + \tan \varphi_1 \tan \varphi_2},$$

hence

$$\tan \varphi = \frac{m_2 - m_1}{1 + m_1 m_2}. \quad (3.31)$$

- 1) The lines  $d_1$  and  $d_2$  are parallel if and only if  $\tan \varphi = 0$ , therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \quad (3.32)$$

- 2) The lines  $d_1$  and  $d_2$  are orthogonal if and only if they determine an angle of  $\frac{\pi}{2}$ , hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \quad (3.33)$$

### 3.7 Problems

1. Write the equation of the plane which passes through  $M_0(1, -2, 3)$  and is parallel to the vectors  $\vec{v}_1(1, -1, 0)$  and  $\vec{v}_2(-3, 2, 4)$ .

HINT.

$$\begin{vmatrix} x - 0 & y + 2 & z - 3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

vspace\*2cm

2. Write the equation of the line which passes through  $A(1, -2, 6)$  and is parallel to
- (a) The  $x$ -axis;
  - (b) The line  $(d_1)$   $\frac{x-1}{2} = \frac{y+5}{-3} = \frac{z-1}{4}$ .
  - (c) The vector  $\vec{v}(1, 0, 2)$ .

SOLUTION.

3. Write the equation of the plane which contains the line

$$(d_1) \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$

and is parallel to the line

$$(d_2) \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}.$$

HINT.

$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0.$$

4. Consider the points  $A(\alpha, 0, 0)$ ,  $B(0, \beta, 0)$  and  $C(0, 0, \gamma)$  such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{a} \text{ where } a \text{ is a constant.}$$

Show that the plane  $(A, B, C)$  passes through a fixed point.

SOLUTION. The equation of the plane  $(ABC)$  can be written in intercept form, namely

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

The given relation shows that the point  $P(a, a, a) \in (ABC)$  whenever  $\alpha, \beta, \gamma$  verifies the given relation.

5. Write the equation of the line which passes through the point  $M(1, 0, 7)$ , is parallel to the plane  $(\pi)$   $3x - y + 2z - 15 = 0$  and intersects the line

$$(d) \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

*Solution.*

6. Write the equation of the plane which passes through  $M_0(1, -2, 3)$  and cuts the positive coordinate axes through equal intercepts.

SOLUTION. The general equation of such a plane is  $x + y + z = a$ . In this particular case  $a = 1 + (-2) + 3 = 2$  and the equation of the required plane is  $x + y + z = 2$ .

7. Write the equation of the plane which passes through  $A(1, 2, 1)$  and is parallel to the straight lines

$$(d_1) \begin{cases} x + 2y - z + 1 = 0 \\ x - y + z - 1 = 0 \end{cases} \quad (d_2) \begin{cases} 2x - y + z = 1 \\ x - y + z = 0. \end{cases}$$

SOLUTION. We need to find some director parameters of the lines  $(d_1)$  and  $(d_2)$ . In this respect we may solve the two systems. The general solution of the first system is

$$\begin{cases} x = -\frac{1}{3}t + \frac{1}{3} \\ y = \frac{2}{3}t - \frac{2}{3} \\ z = t \end{cases}, t \in \mathbb{R}$$

and the general solution of the second system is

$$\begin{cases} x = 1 \\ y = t + 1 \\ z = t \end{cases}, t \in \mathbb{R}$$

and these are the parametric equations of the lines  $(d_1)$  and  $(d_2)$ . Thus, the direction of the line  $(d_1)$  is the 1-dimensional subspace

$$\left\langle \left( -\frac{1}{3}, \frac{2}{3}, 1 \right) \right\rangle = \langle (-1, 2, 3) \rangle,$$

and the direction of the line  $(d_2)$  is the 1-dimensional subspace  $\langle (0, 1, 1) \rangle$ .

Consequently, some director parameters of the line  $(d_1)$  are  $p_1 = -1, q_1 = 2, r_1 = 3$  and some director parameters of the line  $(d_2)$  are  $p_2 = 0, q_2 = r_2 = 1$ . Finally, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$

The computation of the determinant is left to the reader.

#### A few questions in the two dimensional setting ([1])

8. The sides  $[BC]$ ,  $[CA]$ ,  $[AB]$  of the triangle  $\Delta ABC$  are divided by the points  $M, N$  respectively  $P$  into the same ratio  $k$ . Prove that the triangles  $\Delta ABC$  and  $\Delta MNP$  have the same center of gravity.

SOLUTION.



9. Sketch the graph of  $x^2 - 4xy + 3y^2 = 0$ .

SOLUTION.

10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point  $M$  which divides the segment  $[AB]$ ,  $A(4, -3)$ ,  $B(-1, 2)$ , into the ratio  $k = \frac{2}{3}$ .

SOLUTION.

11. Let  $A$  be a mobile point on the  $Ox$  axis and  $B$  a mobile point on  $Oy$ , so that  $\frac{1}{OA} + \frac{1}{OB} = k$  (constant). Prove that the lines  $AB$  passes through a fixed point.

SOLUTION.

12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of  $Oy$  at the point  $A$  with  $OA = 3$ .

SOLUTION.

13. Find the parametric equations of the line through  $P_1$  and  $P_2$ , when

- (a)  $P_1(3, -2)$ ,  $P_2(5, 1)$ ;
- (b)  $P_1(4, 1)$ ,  $P_2(4, 3)$ .

SOLUTION.

14. Find the parametric equations of the line through  $P(-5, 2)$  and parallel to  $\bar{v}(2, 3)$ .

SOLUTION.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \quad \text{and} \quad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

SOLUTION.

16. Find the vector equation of the line  $P_1P_2$ , where

- (a)  $P_1(2, -1)$ ,  $P_2(-5, 3)$ ;
- (b)  $P_1(0, 3)$ ,  $P_2(4, 3)$ .

SOLUTION.

17. Given the line  $d : 2x + 3y + 4 = 0$ , find the equation of a line  $d_1$  through the point  $M_0(2, 1)$ , in the following situations:

- (a)  $d_1$  is parallel with  $d$ ;
- (b)  $d_1$  is orthogonal on  $d$ ;
- (c) the angle determined by  $d$  and  $d_1$  is  $\varphi = \frac{\pi}{4}$ .

SOLUTION.

18. The vertices of the triangle  $\Delta ABC$  are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

- (a) Find the coordinates of  $A, B, C$ .
- (b) Find the equations of the median lines of the triangle.
- (c) Find the equations of the heights of the triangle.

SOLUTION.

## 4 Week 4

### 4.1 Analytic conditions of parallelism and nonparallelism

#### 4.1.1 The parallelism between a line and a plane

**Proposition 4.1.** *The equation of the director subspace  $\vec{\pi}$ , of the plane  $\pi : Ax + By + Cz + D = 0$  is  $AX + BY + CZ = 0$ .*

*Proof.* We first recall that

$$\vec{\pi} = \{A_0\vec{M} \mid M \in \pi\}, \quad (4.1)$$

where  $A_0 \in \pi$  is an arbitrary point, and the representation (4.1) of  $\vec{\pi}$  is independent on the choice of  $A_0 \in \pi$ . According to equation (3.8), the equation of a plane  $\pi$  can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where  $A_0(x_0, y_0, z_0)$  is a point in  $\pi$ . In other words,

$$M(x, y, z) \in \pi \iff A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which shows that

$$\begin{aligned} \vec{\pi} &= \{A_0\vec{M} (x - x_0, y - y_0, z - z_0) \mid M(x, y, z) \in \pi\} \\ &= \{A_0\vec{M} (x - x_0, y - y_0, z - z_0) \mid A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\} \\ &= \{\vec{v} (X, Y, Z) \in \mathcal{V} \mid AX + BY + CZ = 0\}. \end{aligned}$$

Thus, the equation  $AX + BY + CZ = 0$  is a necessary and sufficient condition for the vector  $\vec{v} (X, Y, Z)$  to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

In other words, the *equation of the director subspace*  $\vec{\pi}$  is  $AX + BY + CZ = 0$ . □

**Corollary 4.2.** *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

*is parallel to the plane  $\pi : Ax + By + Cz + D = 0$  if and only if*

$$Ap + Bq + Cr = 0 \quad (4.2)$$

*Proof.* Indeed,

$$\begin{aligned} \Delta \parallel \pi &\iff \vec{\Delta} \subseteq \vec{\pi} \iff \langle(p, q, r) \rangle \subseteq \vec{\pi} \\ &\iff \vec{d} (p, q, r) \in \vec{\pi} \iff Ap + Bq + Cr = 0. \end{aligned}$$

□

### 4.1.2 The intersection point of a straight line and a plane

**Proposition 4.3.** Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane  $\pi : Ax + By + Cz + D = 0$  which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

The coordinates of the intersection point  $d \cap \pi$  are

$$\begin{cases} x_0 - p \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}, \end{cases} \quad (4.3)$$

where  $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $F(x, y, z) = Ax + By + Cz + D$ .

*Proof.* The parametric equations of  $(d)$  are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}. \quad (4.4)$$

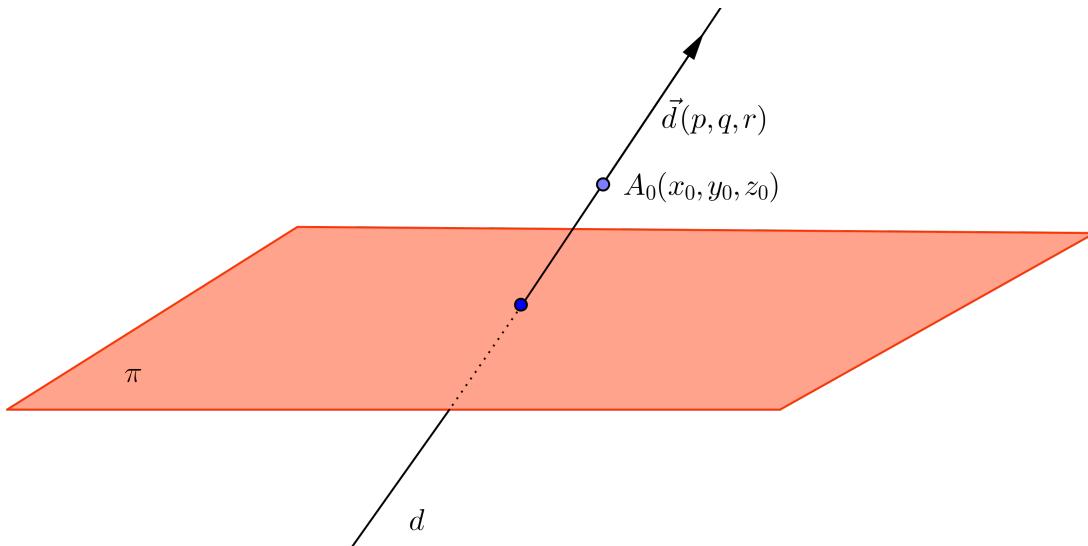
The unique value of  $t \in \mathbb{R}$ , which corresponds to the intersection point  $d \cap \pi$ , can be found by solving the equation

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + rt) + D = 0.$$

Its unique solution is

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}$$

and can be used to obtain the required coordinates (4.3) by replacing this value in (4.4).  $\square$



**Example 4.1 (Homework).** Decide whether the line  $d$  and the plane  $\pi$  are parallel or concurrent and find the coordinates of the intersection point of  $\Delta$  and  $\pi$  whenever  $\Delta \nparallel \pi$ :

1.  $d : \frac{x+2}{1} = \frac{y-1}{3} = \frac{z-3}{1}$  and  $\pi : x - y + 2z = 1$ .
2.  $d : \frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-2}{-1}$  and  $\pi : 2x - y + 3z - 1 = 0$ .

SOLUTION.

### 4.1.3 Parallelism of two planes

**Proposition 4.4.** Consider the planes

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0.$$

Then  $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) \in \{1, 2\}$  and the following statements are equivalent

1.  $\pi_1 \parallel \pi_2$ .
2.  $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 2$ , i.e.  $\vec{\pi}_1 = \vec{\pi}_2$ .
3.  $\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$ .
4. The vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$  are linearly dependent.

**Remark 4.1.** Note that

$$\begin{aligned} \text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1 &\Leftrightarrow \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0 \\ &\Leftrightarrow A_1B_2 - A_2B_1 = A_1C_2 - A_2C_1 = B_1C_2 - C_2B_1 = 0. \end{aligned} \quad (4.5)$$

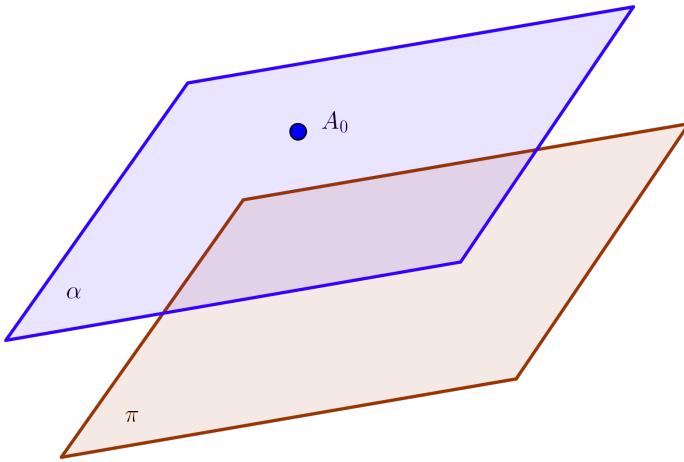
The relations (4.5) are often written in the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \quad (4.6)$$

although at most two of the coefficients  $A_2, B_2$  or  $C_2$  might be zero. In fact relations (4.6) should be understood in terms of linear dependence of the vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$ , i.e.  $(A_1, B_1, C_1) = k(A_2, B_2, C_2)$ , where  $k \in \mathbb{R}$  is the common value of those ratios (4.6) which do not involve any zero coefficients. Let us finally mention that the equivalences (4.5) prove the equivalence (3)  $\Leftrightarrow$  (4) of Proposition 4.4.

**Example 4.2.** The equation of the plane  $\alpha$  passing through the point  $A_0(x_0, y_0, z_0)$ , which is parallel to the plane  $\pi$ :  $Ax + By + Cz + D = 0$  is

$$\alpha : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



#### 4.1.4 Straight lines as intersections of planes

**Corollary 4.5.** Consider the planes

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1.  $\pi_1 \nparallel \pi_2$ .
2.  $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 1$ .
3.  $\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$ .
4. The vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$  are linearly independent.

By using the characterization of parallelism between a line and a plane, given by Proposition 4.2, we shall find the direction of a straight line which is given as the intersection of two planes. Consider the planes  $(\pi_1) A_1x + B_1y + C_1z + D_1 = 0$ ,  $(\pi_2) A_2x + B_2y + C_2z + D_2 = 0$  such that

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersection straight line  $\Delta = \pi_1 \cap \pi_2$  of equations

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

Thus,  $\vec{\Delta} = \vec{\pi}_1 \cap \vec{\pi}_2$  and therefore, by means of some previous Proposition, it follows that the equations of  $\vec{\Delta}$  are

$$(\vec{\Delta}) \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \quad (4.7)$$

By solving the system (4.7) one can therefore deduce that  $\vec{d} = (p, q, r) \in \vec{\Delta} \Leftrightarrow \exists \lambda \in \mathbb{R}$  such that

$$(p, q, r) = \lambda \left( \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \quad (4.8)$$

The relation is usually (4.8) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \quad (4.9)$$

Let us finally mention that we usually choose the values

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ și } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \quad (4.10)$$

for the director parameters  $(p, q, r)$  of  $\Delta$ .

**Example 4.3.** Write the equations of the plane through  $P(4, -3, 1)$  which is parallel to the lines

$$(\Delta_1) \begin{cases} 2x - z + 1 = 0 \\ 3y + 2z - 2 = 0 \end{cases} \text{ and } (\Delta_2) \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0 \end{cases}$$

SOLUTION. One can see the required plane as the one through  $P(4, -3, 1)$  which is parallel to the director vectors  $\vec{d}_1(p_1, q_1, r_1)$  and  $\vec{d}_2(p_2, q_2, r_2)$  of  $\Delta_1$  and  $\Delta_2$  respectively. One can choose

$$\begin{aligned} p_1 &= \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 & p_2 &= \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4 \\ q_1 &= \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4 & \text{and} & q_2 = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \\ r_1 &= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 & r_2 &= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3. \end{aligned}$$

Thus, the equation of the required plane is

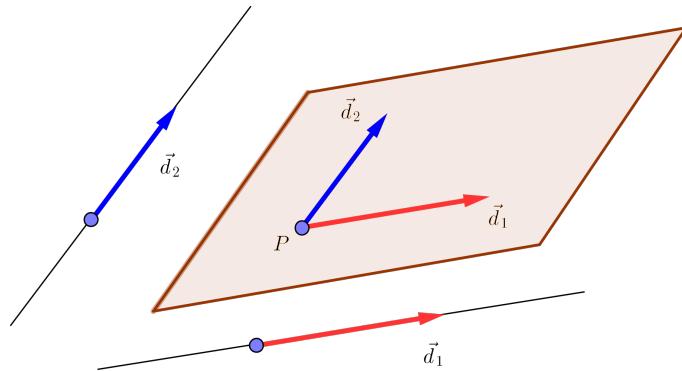


Figure 7:

$$\begin{vmatrix} x-4 & y+3 & z-1 \\ 3 & -4 & 6 \\ 4 & -1 & -3 \end{vmatrix} = 0 \iff 12(x-4) - 3(z-1) + 24(y+3) + 16(z-1) + 6(x-4) + 9(y+3) = 0 \iff 18(x-4) + 33(y+3) + 13(z-1) = 0 \iff 18x + 33y + 13z - 72 + 99 - 13 = 0 \iff 18x + 33y + 13z + 14 = 0.$$

## 4.2 Pencils of planes

**Definition 4.1.** The collection of all planes containing a given straight line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

is called the *pencil* or the *bundle* of planes through  $\Delta$ .

**Proposition 4.6.** *The plane  $\pi$  belongs to the pencil of planes through the straight line  $\Delta$  if and only if the equation of the plane  $\pi$  is*

$$\lambda(A_1x + B_1y + C_1z + D_1) + \mu(A_2x + B_2y + C_2z + D_2) = 0. \quad (4.11)$$

for some  $\lambda, \mu \in \mathbb{R}$  such that  $\lambda^2 + \mu^2 > 0$ .

*Proof.* Every plane in the family (4.11) obviously contains the line  $\Delta$ .

Conversely, assume that  $\pi$  is a plane through the line  $\Delta$ . Consider a point  $M \in \pi \setminus \Delta$  and recall that  $\pi$  is completely determined by  $\Delta$  and  $M$ . On the other hand  $M$  and  $\Delta$  are obviously contained in the plane  $F_1(x_M, y_M, z_M)F_2(x, y, z) - F_2(x_M, y_M, z_M)F_1(x, y, z) = 0$  of the family (4.11), where  $F_1, F_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $F_i(x, y, z) = A_i x + B_i y + C_i z + D_i$ , for  $i = 1, 2$ . Thus the plane  $\pi$  belongs to the family (4.11) and its equation is

$$F_1(x_M, y_M, z_M)F_2(x, y, z) - F_2(x_M, y_M, z_M)F_1(x, y, z) = 0.$$

□

**Remark 4.2.** The family of planes  $A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0$ , where  $\lambda$  covers the whole real line  $\mathbb{R}$ , is the so called *reduced pencil of planes* through  $\Delta$  and it consists in all planes through  $\Delta$  except the plane of equation  $A_2x + B_2y + C_2z + D_2 = 0$ .

**Example 4.4.** Write the equations of the plane parallel to the line  $d : x = 2y = 3z$  passing through the line

$$\Delta : \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0. \end{cases}$$

SOLUTION. Note that none of the planes  $x + y + z = 0$  and  $x - y + 3z = 0$ , passing through  $(\Delta)$ , is parallel to  $(d)$ , as  $1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} \neq 0$  and  $2 \cdot 1 + (-1) \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} \neq 0$ . Thus, the required plane is in a reduced pencil of planes, such as the family  $\pi_\lambda : x + y + z + \lambda(2x - y + 3z) = 0$ ,  $\lambda \in \mathbb{R}$ . The parallelism relation between  $(d)$  and  $\pi_\lambda : (2\lambda + 1)x + (1 - \lambda)y + (3\lambda + 1)z = 0$  is

$$(2\lambda + 1) \cdot 1 + (1 - \lambda) \cdot \frac{1}{2} + (3\lambda + 1) \cdot \frac{1}{3} = 0 \iff 12\lambda + 6 + 3 - 3\lambda + 6\lambda + 2 = 0 \iff \lambda = -\frac{11}{15}.$$

Thus, the required plane is

$$\pi_{-11/15} : \left(-2\frac{11}{15} + 1\right)x + \left(1 + \frac{11}{15}\right)y + \left(-3\frac{11}{15} + 1\right)z = 0 \iff -7x + 26y - 18z = 0.$$

## Appendix

### 4.3 Projections and symmetries

#### 4.3.1 The projection on a plane parallel with a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane  $\pi : Ax + By + Cz + D = 0$  which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection  $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$  of  $\mathcal{P}$  on  $\pi$  parallel to  $d$ , whose value  $p_{\pi,d}(M)$  at  $M \in \mathcal{P}$  is the intersection point between  $\pi$  and the line through  $M$  which is parallel to  $d$ . Due to relations (4.3), the coordinates of  $p_{\pi,d}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_M - p \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \end{cases} \quad (4.12)$$

where  $F(x, y, z) = Ax + By + Cz + D$ .

Consequently, the position vector of  $p_{\pi,d}(M)$  is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (4.13)$$

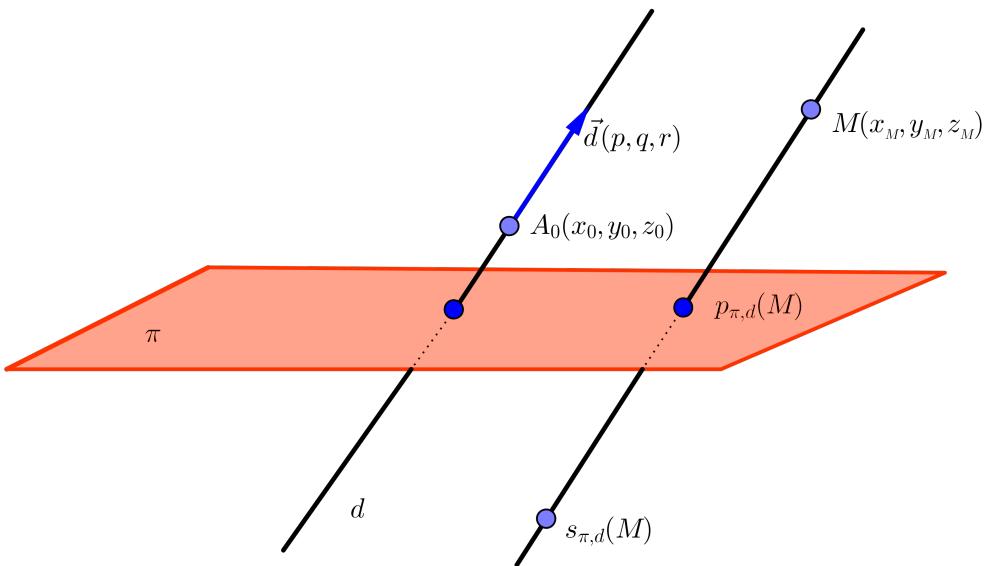
**Proposition 4.7.** If  $R = (O, b)$  is the Cartesian reference system behind the equations of the line

$$(d) \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and the plane  $(\pi)$   $Ax + By + Cz + D = 0$ , concurrent with  $(d)$ , then

$$[p_{\pi,d}(M)]_R = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix} [M]_R - \frac{D}{Ap + Bq + Cr} [\vec{d}]_b,$$

where  $\vec{d} (p, q, r)$  stands for the director vector of the line  $(d)$ .



### 4.3.2 The symmetry with respect to a plane parallel with a given line

We call the function  $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$ , whose value  $s_{\pi,d}(M)$  at  $M \in \mathcal{P}$  is the symmetric point of  $M$  with respect to  $p_{\pi,d}(M)$  the symmetry of  $\mathcal{P}$  with respect to  $\pi$  parallel to  $d$ . The direction of  $d$  is equally

called the *direction* of the symmetry and  $\pi$  is called the *axis* of the symmetry. For the position vector of  $s_{\pi,d}(M)$  we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \quad (4.14)$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2 \overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (4.15)$$

**Proposition 4.8.** If  $R = (O, b)$  is the Cartesian reference system behind the equations of the line

$$(d) \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and the plane  $(\pi)$   $Ax + By + Cz + D = 0$ , concurrent with  $(d)$ , then

$$(Ap + Bq + Cr)[s_{\pi,d}(M)]_R = \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix} [M]_R - 2D[\vec{d}]_b, \quad (4.16)$$

where  $\vec{d} (p, q, r)$  stands for the director vector of the line  $(d)$ .

### 4.3.3 The projection on a straight line parallel with a given plane

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane  $\pi : Ax + By + Cz + D = 0$  which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection  $p_{d,\pi} : \mathcal{P} \longrightarrow d$  of  $\mathcal{P}$  on  $d$ , whose value  $p_{d,\pi}(M)$  at  $M \in \mathcal{P}$  is the intersection point between  $d$  and the plane through  $M$  which is parallel to  $\pi$ . Due to relations (4.3), the coordinates of  $p_{d,\pi}(M)$ , in terms of the coordinates of  $M$ , are

$$\left\{ \begin{array}{l} x_0 - p \frac{G_M(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q \frac{G_M(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r \frac{G_M(x_0, y_0, z_0)}{Ap + Bq + Cr} \end{array} \right. \quad (4.17)$$

where  $G_M(x, y, z) = A(x - x_M) + B(y - y_M) + C(z - z_M)$ . Consequently, the position vector of  $p_{d,\pi}(M)$  is

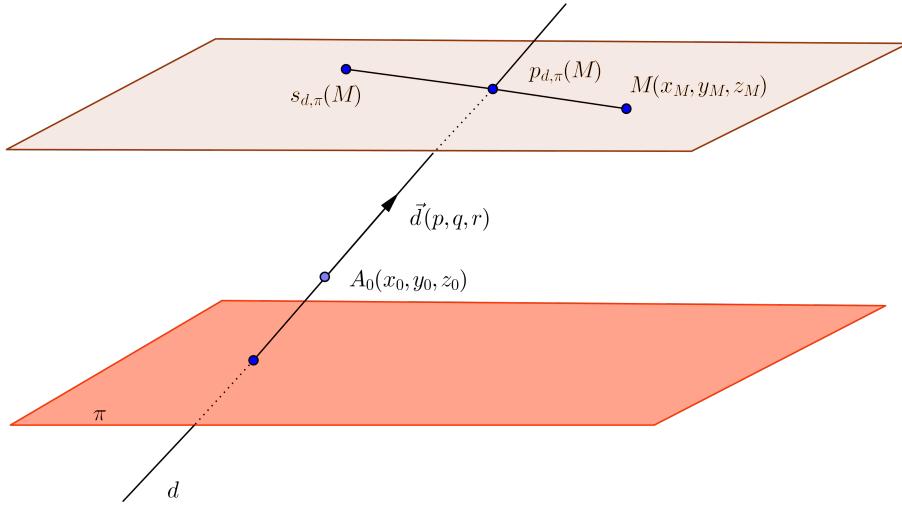
$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} - \frac{G_M(A_0)}{Ap + Bq + Cr} \vec{d}, \text{ where } A_0(x_0, y_0, z_0). \quad (4.18)$$

Note that  $G_M(A_0) = A(x_0 - x_M) + B(y_0 - y_M) + C(z_0 - z_M) = F(A_0) - F(M)$ , where  $F(x, y, z) = Ax + By + Cz + D$ . Consequently the coordinates of  $p_{d,\pi}(M)$ , in terms of the coordinates of  $M$ , are

$$\left\{ \begin{array}{l} x_0 + p \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \\ y_0 + q \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \\ z_0 + r \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \end{array} \right. \quad (4.19)$$

and the position vector of  $p_{d,\pi}(M)$  is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \vec{d}, \text{ where } A_0(x_0, y_0, z_0). \quad (4.20)$$



#### 4.3.4 The symmetry with respect to a line parallel with a plane

We call the function  $s_{d,\pi} : \mathcal{P} \rightarrow \mathcal{P}$ , whose value  $s_{d,\pi}(M)$  at  $M \in \mathcal{P}$  is the symmetric point of  $M$  with respect to  $p_{d,\pi}(M)$ , the *symmetry of  $\mathcal{P}$  with respect to  $d$  parallel to  $\pi$* . The direction of  $\pi$  is equally called the *direction* of the symmetry and  $d$  is called the *axis* of the symmetry. For the position vector of  $s_{d,\pi}(M)$  we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \quad (4.21)$$

$$\begin{aligned} \overrightarrow{Os_{d,\pi}(M)} &= 2 \overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM} \\ &= 2 \overrightarrow{OA_0} - \overrightarrow{OM} + 2 \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \vec{d}. \end{aligned} \quad (4.22)$$

## 4.4 Projections and symmetries in the two dimensional setting

### 4.4.1 The intersection point of two concurrent lines

Consider two lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

și  $\Delta : ax + by + c = 0$  which are not parallel to each other, i.e.

$$ap + bq \neq 0.$$

The parametric equations of  $d$  are:

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases}, t \in \mathbb{R} \quad (4.23)$$

The value of  $t \in \mathbb{R}$  for which this line (4.23) punctures the line  $\Delta$  can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt)$$

to verify the equation of the line  $\Delta$ , namely

$$a(x_0 + pt) + b(y_0 + qt) + c = 0.$$

Thus

$$t = -\frac{ax_0 + by_0 + c}{ap + bq} = -\frac{F(x_0, y_0)}{ap + bq},$$

where  $F(x, y) = ax + by + c$ .

The coordinates of the intersection point  $d \cap \Delta$  are:

$$\begin{aligned} x_0 - p \frac{F(x_0, y_0)}{ap + bq} \\ y_0 - q \frac{F(x_0, y_0)}{ap + bq}. \end{aligned} \tag{4.24}$$

#### 4.4.2 The projection on a line parallel with another given line

Consider two straight non-parallel lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and  $\Delta : ax + by + c = 0$  which are not parallel to each other, i.e.  $ap + bq \neq 0$ . For these given data we may define the projection  $p_{\Delta,d} : \pi \rightarrow \Delta$  of  $\pi$  on  $\Delta$  parallel cu  $d$ , whose value  $p_{\Delta,d}(M)$  at  $M \in \pi$  is the intersection point between  $\Delta$  and the line through  $M$  which is parallel to  $d$ . Due to relations (4.24), the coordinates of  $p_{\Delta,d}(M)$ , in terms of the coordinates of  $M$  are:

$$\begin{aligned} x_M - p \frac{F(x_M, y_M)}{ap + bq} \\ y_M - q \frac{F(x_M, y_M)}{ap + bq}, \end{aligned}$$

where  $F(x, y) = ax + by + c$ .

Consequently, the position vector of  $p_{\Delta,d}(M)$  is

$$\overrightarrow{Op_{\Delta,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{ap + bq} \overrightarrow{d},$$

where  $\overrightarrow{d} = p \overrightarrow{e} + q \overrightarrow{f}$ .

**Proposition 4.9.** If  $R$  is the Cartesian reference system of the plane  $\pi$  behind the equations of the concurrent lines

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[p_{\Delta,d}(M)]_R = \frac{1}{ap + bq} \begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix} [M]_R - \frac{c}{ap + bq} [\overrightarrow{d}]_b. \tag{4.25}$$

#### 4.4.3 The symmetry with respect to a line parallel with another line

We call the function  $s_{\Delta,d} : \pi \rightarrow \pi$ , whose value  $s_{\Delta,d}(M)$  at  $M \in \pi$  is the symmetric point of  $M$  with respect to  $p_{\Delta,d}(M)$ , the *symmetry of  $\pi$  with respect to  $\Delta$  parallel to  $d$* . The direction of  $d$  is equally called the direction of the symmetry and  $\pi$  is called the *axis of the symmetry*. For the position vector of  $s_{\Delta,d}(M)$  we have

$$\overrightarrow{Op_{\Delta,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\Delta,d}(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Os_{\Delta,d}(M)} = 2\overrightarrow{Op_{\Delta,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{ap+bq}\overrightarrow{d},$$

where  $F(x, y) = ax + by + c$ . Thus, the coordinates of  $s_{\Delta,d}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_M - 2p\frac{F(x_M, y_M)}{ap+bq} \\ y_M - 2q\frac{F(x_M, y_M)}{ap+bq}. \end{cases}$$

**Proposition 4.10.** If  $R$  is the Cartesian reference system of the plane  $\pi$  behind the equations of the concurrent lines

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[s_{\Delta,d}(M)]_R = \frac{1}{ap+bq} \begin{pmatrix} -ap+bq & -2bp \\ -2aq & ap-bq \end{pmatrix} [M]_R - \frac{2c}{ap+bq} [\vec{d}]_b. \quad (4.26)$$

## 4.5 Problems

1. Write the equation of the plane determined by the line

$$(d) \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

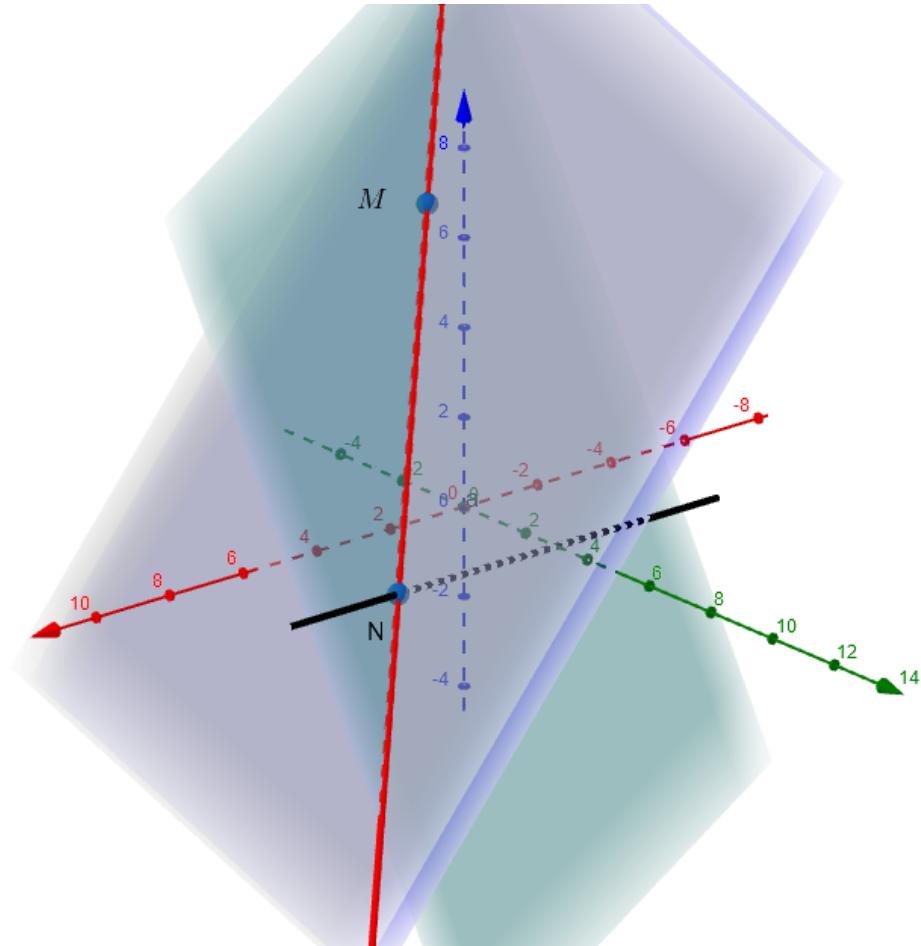
and the point  $A(-1, 2, 6)$ .

SOLUTION.

2. Write the equation of the line which passes through the point  $M(1, 0, 7)$ , is parallel to the plane  $(\pi)$   $3x - y + 2z - 15 = 0$  and intersects the line

$$(d) \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

SOLUTION 1. The equation of the plane  $\alpha$  passing through the point  $M(1, 0, 7)$ , which is parallel to the plane  $(\pi)$   $3x - y + 2z - 15 = 0$ , is  $(\alpha)$   $3(x-1) - (y-0) + 2(z-7) = 0$ , i.e.  $(\alpha)$   $3x - y + 2z - 17 = 0$ .



The parametric equations of the line  $d$  are

$$\begin{cases} x = 1 + 4t \\ y = 3 + 2t \\ z = t \end{cases}, t \in \mathbb{R}.$$

The coordinates of the intersection point  $N$  between the line  $(d)$  and the plane  $\alpha$  can be obtained by solving the equation  $3((1+4t) - (3+2t)) + 2t - 17 = 0$ . The required line is  $MN$ .

SOLUTION 2. The required line can be equally regarded as the intersection line between the plane  $\alpha$  (passing through the point  $M(1, 0, 7)$ , which is parallel to the plane  $(\pi)$ ) and the plane determined by the given line  $(d)$  and the point  $M$ . While the equation  $3x - y + 2z - 17 = 0$  of  $\alpha$  was already used above, the equation of the plane determined by the line  $(d)$  and the point  $M$  can be determined via the pencil of planes through

$$(d) \begin{cases} \frac{x-1}{4} = \frac{y-3}{2} \\ \frac{y-3}{2} = \frac{z}{1} \end{cases} \Leftrightarrow (d) \begin{cases} x - 2y + 5 = 0 \\ y - 2z - 3 = 0. \end{cases}$$

Note that none of the planes  $x - 2y + 5 = 0$  or  $y - 2z - 3 = 0$  passes through  $M$ , which means that the plane determined by  $d$  and  $M$  is in the reduced pencil of planes

$$(\pi_\lambda) \quad x - 2y + 5 = 0 + \lambda(y - 2z - 3) = 0.$$

The plane determined by  $d$  and  $M$  can be found by imposing on the coordinates of  $M$  to verify the equation of  $\pi_\lambda$ .

3. Write the equations of the projection of the line

$$(d) \quad \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane  $\pi : x + 2y - z = 0$  parallel to the direction  $\overrightarrow{u} (1, 1, -2)$ . Write the equations of the symmetry of the line  $d$  with respect to the plane  $\pi$  parallel to the direction  $\overrightarrow{u} (1, 1, -2)$ .

SOLUTION.

**4. Prove Proposition 4.7****SOLUTION.**

5. Prove Proposition 5.6

SOLUTION.

6. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection  $p_{\pi,d}$ , where

$$\pi : Ax + By + Cz + D = 0, \quad d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and  $\pi \nparallel d$ .

SOLUTION.

7. Show that two different parallel lines are mapped onto parallel lines by a symmetry  $s_{\pi,d}$ , where

$$\pi : Ax + By + Cz + D = 0, \quad d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and  $\pi \nparallel d$ .

SOLUTION.

8. Assume that  $R = (O, b)$  ( $b = [\vec{u}, \vec{v}, \vec{w}]$ ) is the Cartesian reference system behind the equations of a plane  $\pi : Ax + By + Cz + D = 0$  and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If  $\pi \nparallel d$ , show that

- (a)  $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = p(\overrightarrow{MN})$ , for all  $M, N \in \mathcal{V}$ , where  $p : \mathcal{V} \longrightarrow \mathcal{V}$  is the linear transformation whose matrix representation is

$$[p]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}.$$

SOLUTION.

- (b)  $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = \overrightarrow{s(MN)}$ , for all  $M, N \in \mathcal{V}$ , where  $s : \mathcal{V} \longrightarrow \mathcal{V}$  is the linear transformation whose matrix representation is

$$[s]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix}.$$

SOLUTION.

9. Consider a plane  $\pi : Ax + By + Cz + D = 0$  and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If  $\pi \nparallel d$ , show that

- (a)  $p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d}$ .
- (b)  $s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}$ .

SOLUTION.

10. Prove Proposition 4.9.

SOLUTION.

11. Prove Proposition 4.10.

SOLUTION.

## 5 Week 5: Products of vectors

### 5.1 The dot product

**Definition 5.1.** The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 & \text{if } \vec{a} = 0 \text{ or } \vec{b} = 0 \\ \|\vec{a}\| \cdot \|\vec{b}\| \cos(\widehat{\vec{a}, \vec{b}}) & \text{if } \vec{a} \neq 0 \text{ and } \vec{b} \neq 0 \end{cases} \quad (5.1)$$

is called the *dot product* of the vectors  $\vec{a}, \vec{b}$ .

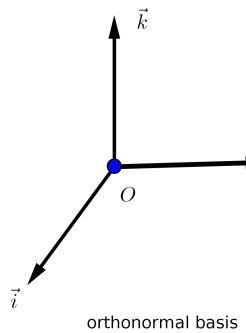
**Remark 5.1.** 1.  $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$ .

$$2. \vec{a} \cdot \vec{a} = \|\vec{a}\| \cdot \|\vec{a}\| \cos 0 = \|\vec{a}\|^2.$$

**Proposition 5.1.** The dot product has the following properties:

1.  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \forall \vec{a}, \vec{b} \in \mathcal{V}$ .
2.  $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$ .
3.  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ .
4.  $\vec{a} \cdot \vec{a} \geq 0, \forall \vec{a} \in \mathcal{V}$ .
5.  $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}$ .

**Definition 5.2.** A basis of the vector space  $\mathcal{V}$  is said to be *orthonormal*, if  $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1, \vec{i} \perp \vec{j}, \vec{j} \perp \vec{k}, \vec{k} \perp \vec{i} (\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1, \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0)$ . A Cartesian reference system  $R = (O, \vec{i}, \vec{j}, \vec{k})$  is said to be *orthonormal* if the basis  $[\vec{i}, \vec{j}, \vec{k}]$  is orthonormal.



**Proposition 5.2.** Let  $[\vec{i}, \vec{j}, \vec{k}]$  be an orthonormal basis and  $\vec{a}, \vec{b} \in \mathcal{V}$ . If  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ ,  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ , then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (5.2)$$

*Proof.* Indeed,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\ &= a_1 b_1 \vec{i} \cdot \vec{i} + a_1 b_2 \vec{i} \cdot \vec{j} + a_1 b_3 \vec{i} \cdot \vec{k} \\ &\quad + a_2 b_1 \vec{j} \cdot \vec{i} + a_2 b_2 \vec{j} \cdot \vec{j} + a_2 b_3 \vec{j} \cdot \vec{k} \\ &\quad + a_3 b_1 \vec{k} \cdot \vec{i} + a_3 b_2 \vec{k} \cdot \vec{j} + a_3 b_3 \vec{k} \cdot \vec{k} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3. \end{aligned}$$

□

**Remark 5.2.** Let  $[\vec{i}, \vec{j}, \vec{k}]$  be an orthonormal basis and  $\vec{a}, \vec{b} \in \mathcal{V}$ . If  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$  and  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ , then

$$1. \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 \text{ and we conclude that } \|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

2.

$$\begin{aligned} \cos(\widehat{\vec{a}, \vec{b}}) &= \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} \\ &= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}. \end{aligned} \quad (5.3)$$

In particular

$$\begin{aligned} \cos(\widehat{\vec{a}, \vec{i}}) &= \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}; \\ \cos(\widehat{\vec{a}, \vec{j}}) &= \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}; \\ \cos(\widehat{\vec{a}, \vec{k}}) &= \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}. \end{aligned}$$

$$3. \vec{a} \perp \vec{b} \Leftrightarrow a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$$

### 5.1.1 Applications of the dot product

#### ◊ The two dimensional setting

- **The distance between two points** Consider two points  $A(x_A, y_A), B(x_B, y_B) \in \pi$ . The norm of the vector  $\vec{AB}$  ( $x_B - x_A, y_B - y_A$ ) is

$$\|\vec{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

- **The equation of the circle**

Recall that the circle  $\mathcal{C}(O, r)$  is the locus of points  $M$  in the plane such that  $\text{dist}(O, M) = r \iff \|\vec{OM}\| = r$ . If  $(a, b)$  are the coordinates of  $O$  and  $(x, y)$  are the coordinates of  $M$ , then

$$\begin{aligned} \|\vec{OM}\| = r &\iff \sqrt{(x - a)^2 + (y - b)^2} = r \iff (x - a)^2 + (y - b)^2 = r^2 \\ &\iff x^2 + y^2 - 2ax - 2by + c = 0, \end{aligned} \quad (5.4)$$

where  $c = a^2 + b^2 - r^2$ . Conversely, every equation of the form  $x^2 + y^2 + 2ex + 2fy + g = 0$  is the equation of the circle centered at  $(-e, -f)$  and having the radius  $r = \sqrt{e^2 + f^2 - g}$ , whenever  $e^2 + f^2 \geq g$ . One can find the equation of the circle circumscribed to the triangle  $ABC$  by imposing the requirement on the coordinates  $(x_A, y_A), (x_B, y_B)$  and  $(x_C, y_C)$  of its vertices  $A, B, C$  to verify the equation  $x^2 + y^2 + 2ex + 2fy + g = 0$ . A point  $M(x, y)$  belongs to this circumcircle if and only if

$$\left\{ \begin{array}{l} x^2 + y^2 + 2ex + 2fy + g = 0 \\ x_A^2 + y_A^2 + 2ex_A + 2fy_A + g = 0 \\ x_B^2 + y_B^2 + 2ex_B + 2fy_B + g = 0 \\ x_C^2 + y_C^2 + 2ex_C + 2fy_C + g = 0 \end{array} \right. \quad (5.5)$$

On can regard the system (5.5) as linear with the unknowns  $e, g, f$ , whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_A^2 + y_A^2 & x_A & y_A & 1 \\ x_B^2 + y_B^2 & x_B & y_B & 1 \\ x_C^2 + y_C^2 & x_C & y_C & 1 \end{vmatrix} = 0,$$

which is the equation of the circumcircle of the triangle  $ABC$ .

- **The normal vector of a line** If  $R = (O, b)$  is the orthonormal Cartesian reference system behind the equation of a line  $(d)$   $ax + by + c = 0$ , then  $\vec{n} (a, b)$  is a normal vector to the direction  $\vec{d}$  of  $d$ . Indeed, every vector of the direction  $\vec{d}$  of  $d$  has the form  $\vec{PM}$ , where  $P(x_p, y_p)$  and  $M(x, y)$  are two points on the line  $d$ . Thus,  $ax_p + by_p + c = 0 = ax_M + by_M + c$ , which shows that

$$a(x_M - x_p) + b(y_M - y_p) = 0,$$

namely

$$\vec{n} \cdot \vec{PM} = 0 \iff \vec{n} \perp \vec{PM}.$$

- **The distance from a point to a line** If  $(d)$   $ax + by + c = 0$  is a line and  $M(x_M, y_M) \in \pi$  a given point, then the distance from  $M$  to  $d$  is

$$\delta(M, d) = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \quad (5.6)$$

Indeed,  $\delta(M, d) = |\delta|$ , where  $\delta$  is the real scalar with the property  $\vec{PM} = \delta \frac{\vec{n}}{\|\vec{n}\|}$  and  $P(x_p, y_p)$  is the orthogonal projection of  $M(x_M, y_M)$  on  $d$ . Thus  $\vec{PM} (x_M - x_p, y_M - y_p)$  andul

$$\begin{aligned} \delta(M, d) &= |\delta| = \left| \vec{PM} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right| = \frac{|\vec{PM} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|a(x_M - x_p) + b(y_M - y_p)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_M + by_M - ax_p - by_p|}{\sqrt{a^2 + b^2}} = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \end{aligned}$$

### ◊ The three dimensional setting

- **The distance between two points** Consider two points  $A(x_A, y_A, z_A), B(x_B, y_B, z_B) \in \mathcal{P}$ . The norm of the vector  $\vec{AB} (x_B - x_A, y_B - y_A, z_B - z_A)$  is

$$\|\vec{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

- **The equation of the sphere**

Recall that the sphere  $\mathcal{S}(O, r)$  is the locus of points  $M$  in space such that  $\text{dist}(O, M) = r \iff \|\vec{OM}\| = r$ . If  $(a, b, c)$  are the coordinates of  $O$  and  $(x, y, z)$  are the coordinates of  $M$ , then

$$\begin{aligned} \|\vec{OM}\| = r &\iff \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} = r \iff (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \\ &\iff x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0, \end{aligned}$$

where  $d = a^2 + b^2 + c^2 - r^2$ . Conversely, every equation of the form

$$x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0$$

is the equation of the sphere centered at  $(-e, -g, -f)$  and having the radius  $r = \sqrt{e^2 + f^2 + g^2 - h}$ , whenever  $e^2 + f^2 + g^2 \geq h$ . One can find the equation of the sphere circumscribed to the tetrahedron  $ABCD$  by imposing the requirement on the coordinates  $(x_A, y_A, z_A)$ ,  $(x_B, y_B, z_B)$  and  $(x_C, y_C, z_C)$  and  $(x_D, y_D, z_D)$  of its vertices  $A, B, C, D$  to verify the equation  $x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0$ . A point  $M(x, y, z)$  belongs to this circumcircle if and only if

$$\begin{cases} x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0 \\ x_A^2 + y_A^2 + z_A^2 + 2ex_A + 2fy_A + 2gz_A + h = 0 \\ x_B^2 + y_B^2 + z_B^2 + 2ex_B + 2fy_B + 2gz_B + h = 0 \\ x_C^2 + y_C^2 + z_C^2 + 2ex_C + 2fy_C + 2gz_C + h = 0 \\ x_D^2 + y_D^2 + z_D^2 + 2ex_D + 2fy_D + 2gz_D + h = 0 \end{cases} \quad (5.7)$$

One can regard the system (5.7) as linear with the unknowns  $e, g, f, h$ , whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\left| \begin{array}{ccccc} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_A^2 + y_A^2 + z_A^2 & x_A & y_A & z_A & 1 \\ x_B^2 + y_B^2 + z_B^2 & x_B & y_B & z_B & 1 \\ x_C^2 + y_C^2 + z_C^2 & x_C & y_C & z_C & 1 \\ x_D^2 + y_D^2 + z_D^2 & x_D & y_D & z_D & 1 \end{array} \right| = 0,$$

which is the equation of the circumsphere of the tetrahedron  $ABCD$ .

- **The normal vector of a plane.** Consider the plane  $\pi : Ax + By + Cz + D = 0$  and the point  $P(x_0, y_0, z_0) \in \pi$ . The equation of  $\pi$  becomes

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \quad (5.8)$$

If  $M(x, y, z) \in \pi$ , the coordinates of  $\vec{PM}$  are  $(x - x_0, y - y_0, z - z_0)$  and the equation (5.8) tells us that  $\vec{n} \cdot \vec{PM} = 0$ , for every  $M \in \pi$ , that is  $\vec{n} \perp \vec{PM} = 0$ , for every  $M \in \pi$ , which is equivalent to  $\vec{n} \perp \vec{\pi}$ , where  $\vec{n} (A, B, C)$ . This is the reason to call  $\vec{n} (A, B, C)$  the *normal vector* of the plane  $\pi$ .

- **The distance from a point to a plane.** Consider the plane  $\pi : Ax + By + Cz + D = 0$ , a point  $P(x_P, y_P, z_P) \in \mathcal{P}$  and  $M$  the orthogonal projection of  $P$  on  $\pi$ . The real number  $\delta$  given by  $\vec{MP} = \delta \cdot \vec{n}_0$  is called the *oriented distance* from  $P$  to the plane  $\pi$ , where  $\vec{n}_0 = \frac{1}{\|\vec{n}\|} \vec{n}$  is the versor of the normal vector  $\vec{n} (A, B, C)$ . Since  $\vec{MP} = \delta \cdot \vec{n}_0$ , it follows that  $\delta(P, M) = \|\vec{MP}\| = |\delta|$ , where  $\delta(P, M)$  stands for the distance from  $P$  to  $\pi$ . We shall show that

$$\delta = \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.$$

Indeed, since  $\vec{MP} = \delta \cdot \vec{n}_0$ , we get successively:

$$\begin{aligned} \delta &= \vec{n}_0 \cdot \vec{MP} = \left( \frac{1}{\|\vec{n}\|} \vec{n} \right) \cdot \vec{MP} = \frac{\vec{n} \cdot \vec{MP}}{\|\vec{n}\|} \\ &= \frac{A(x_P - x_M) + B(y_P - y_M) + C(z_P - z_M)}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{Ax_P + By_P + Cz_P - (Ax_M + By_M + Cz_M)}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}. \end{aligned}$$

Consequently, the distance from  $P$  to the plane  $\pi$  is

$$\delta(P, \pi) = \|\vec{MP}\| = |\delta| = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

**Example 5.1.** Compute the distance from the point  $A(3, 1, -1)$  to the plane

$$\pi : 22x + 4y - 20z - 45 = 0.$$

**SOLUTION.**

$$\delta(A, \pi) = \frac{|22 \cdot 3 + 4 \cdot 1 - 20 \cdot (-1) - 45|}{\sqrt{22^2 + 4^2 + (-20)^2}} = \frac{45}{\sqrt{900}} = \frac{45}{30} = \frac{3}{2}.$$

## 5.2 Appendix: Orthogonal projections and reflections

### 5.2.1 The two dimensional setting

Asssume that  $R = (O, \vec{i}, \vec{j})$  is the orthonormal Cartesian system of a plane  $\pi$  behind the equation of the line  $\Delta : ax + by + c = 0$ .

• **The orthogonal projection of a point on a line.** We define the projection of the ambient plane  $p_\Delta : \pi \rightarrow \Delta$  on  $\Delta$ , whose value  $p_\Delta$  at  $M \in \pi$  is the intersection point between  $\Delta$  and the line through  $M$  perpendicular to  $\Delta$ . Due to relations (4.24), the coordinates of  $p_\Delta(M)$ , in terms of the coordinates of  $M$  are:

$$\begin{aligned} x_M - p \frac{F(x_M, y_M)}{a^2 + b^2} \\ y_M - q \frac{F(x_M, y_M)}{a^2 + b^2}, \end{aligned}$$

where  $F(x, y) = ax + by + c$ . Consequently, the position vector of  $p_\Delta(M)$  is

$$\overrightarrow{Op_\Delta(M)} = \overrightarrow{OM} - \frac{F(M)}{a^2 + b^2} \vec{n}_\Delta,$$

where  $\vec{n}_\Delta = a \vec{i} + b \vec{j}$ .

**Proposition 5.3.** If  $R = (O, \vec{i}, \vec{j})$  is the orthonormal Cartesian reference system of the plane  $\pi$  behind the equations of the line

$$\Delta : ax + by + c = 0,$$

then

$$[p_\Delta(M)]_R = \frac{1}{a^2 + b^2} \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} [M]_R - \frac{c}{a^2 + b^2} [\vec{n}_\Delta]_b, \quad (5.9)$$

where  $b$  stands for the orthonormal basis  $[\vec{i}, \vec{j}]$  of  $\pi$ .

• **The reflection of the plane about a line.** We call the function  $r_\Delta : \pi \rightarrow \pi$ , whose value  $r_\Delta$  at  $M \in \pi$  is the symmetric point of  $M$  with respect to  $p_\Delta(M)$ , the *reflection of  $\pi$  about  $\Delta$* . For the position vector of  $r_\Delta(M)$  we have

$$\begin{aligned} \overrightarrow{Op_\Delta(M)} &= \frac{\overrightarrow{OM} + \overrightarrow{Or_\Delta(M)}}{2}, \text{ i.e.} \\ \overrightarrow{Or_\Delta(M)} &= 2\overrightarrow{Op_\Delta(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{a^2 + b^2} \vec{n}_\Delta, \end{aligned}$$

where  $F(x, y) = ax + by + c$  and  $\vec{n}_\Delta = a \vec{i} + b \vec{j}$ . Thus, the coordinates of  $s_{\Delta,d}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_M - 2p \frac{F(x_M, y_M)}{a^2 + b^2} \\ y_M - 2q \frac{F(x_M, y_M)}{a^2 + b^2}. \end{cases}$$

**Proposition 5.4.** If  $R = (O, \vec{i}, \vec{j})$  is the orthonormal Cartesian reference system of the plane  $\pi$  behind the equations of the line

$$\Delta : ax + by + c = 0,$$

then

$$[r_\Delta(M)]_R = \frac{1}{a^2 + b^2} \begin{pmatrix} -a^2 + b^2 & -2ab \\ -2ab & a^2 - b^2 \end{pmatrix} [M]_R - \frac{2c}{a^2 + b^2} [\vec{n}_\Delta]_b, \quad (5.10)$$

where  $b$  stands for the orthonormal basis  $[\vec{i}, \vec{j}]$  of  $\pi$ .

**Example 5.2.** Find the coordinates of the reflected point of  $P(-5, 13)$  with respect to the line

$$d : 2x - 3y - 3 = 0,$$

knowing that the Cartesian reference system  $R$  behind the coordinates of  $A$  and the equation of  $(d)$  is orthonormal.

HINT. According to 5.11 it follows that

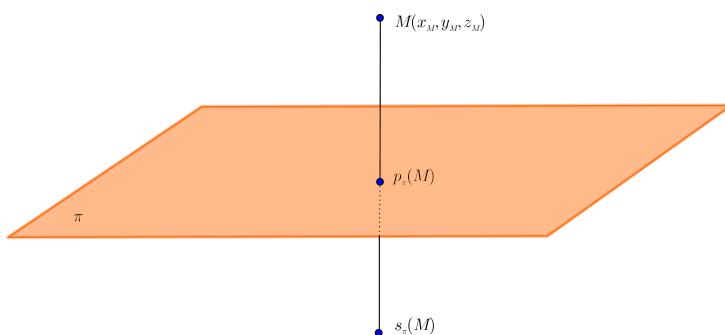
$$[r_d(P)]_R = \frac{1}{2^2 + (-3)^2} \begin{pmatrix} -2^2 + (-3)^2 & -2 \cdot 2 \cdot (-3) \\ -2 \cdot 2 \cdot (-3) & 2^2 - (-3)^2 \end{pmatrix} \begin{bmatrix} -5 \\ 13 \end{bmatrix} - \frac{2 \cdot (-3)}{2^2 + (-3)^2} \begin{bmatrix} 2 \\ -3 \end{bmatrix}. \quad (5.11)$$

## 5.2.2 The three dimensional setting

- The orthogonal projection of a point on a plane. For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point  $M(x_M, y_M, z_M)$ , we shall determine the coordinates of its orthogonal projection on the plane  $\pi$ , as well as the coordinates of its (orthogonal) symmetric with respect to  $\pi$ . The equation of the plane and the coordinates of  $M$  are considered with respect to some cartesian coordinate system  $R = (O, \vec{i}, \vec{j}, \vec{k})$ . In this respect we consider the orthogonal line on  $\pi$  which passes through  $M$ .



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, t \in \mathbb{R}. \quad (5.12)$$

The orthogonal projection  $p_\pi(M)$  of  $M$  on the plane  $\pi$  is at its intersection point with the orthogonal line (5.12) and the value of  $t \in \mathbb{R}$  for which this orthogonal line (5.12) puncture the plane  $\pi$  can

be determined by imposing the condition on the point of coordinates  $(x_M + At, y_M + Bt, z_M + Ct)$  to verify the equation of the plane, namely  $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$ . Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\|\vec{n}_\pi\|^2},$$

where  $F(x, y, z) = Ax + By + Cz + D$  și  $\vec{n}_\pi = A\vec{i} + B\vec{j} + C\vec{k}$  is the normal vector of the plane  $\pi$ .

- **The orthogonal projection of the space on a plane.**

The coordinates of the orthogonal projection  $p_\pi(M)$  of  $M$  on the plane  $\pi$  are

$$\begin{cases} x_M - A \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection  $p_\pi(M)$  is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\|\vec{n}_\pi\|^2} \vec{n}_\pi. \quad (5.13)$$

**Proposition 5.5.** If  $R = (O, b)$  is the orthonormal Cartesian reference system behind the equation of the plane  $(\pi) Ax + By + Cz + D = 0$ , then

$$(A^2 + B^2 + C^2)[p_\pi(M)]_R = \begin{pmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{pmatrix} [M]_R - D[\vec{n}_\pi]_b. \quad (5.14)$$

**Remark 5.3.** The distance from the point  $M(x_M, y_M, z_M)$  to the plane  $\pi : Ax + By + Cz + D = 0$  can be equally computed by means of (5.13). Indeed,

$$\begin{aligned} \delta(M, \pi) &= \| \overrightarrow{Mp_\pi(M)} \| = \| \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} \| \\ &= \left| -\frac{F(M)}{\|\vec{n}_\pi\|^2} \right| \cdot \|\vec{n}_\pi\| = \frac{|F(M)|}{\|\vec{n}_\pi\|}. \end{aligned}$$

• **The reflection of the space about a plane.** In order to find the position vector of the orthogonally symmetric point  $r_\pi(M)$  of  $M$  w.r.t.  $\pi$ , we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2} \left( \overrightarrow{OM} + \overrightarrow{Or_\pi(M)} \right),$$

namely

$$\overrightarrow{Or_\pi(M)} = 2 \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{\|\vec{n}_\pi\|^2} \vec{n}_\pi.$$

The correspondence which associate to some point  $M$  its orthogonally symmetric point w.r.t.  $\pi$ , is called the *reflection* in the plane  $\pi$  and is denoted by  $r_\pi$ .

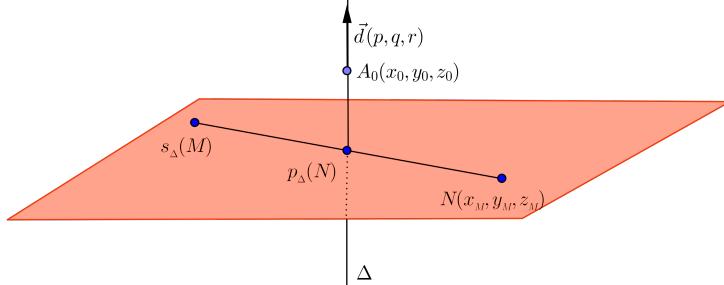
**Proposition 5.6.** If  $R = (O, b)$  is the orthonormal Cartesian reference system behind the equation of the plane  $(\pi) Ax + By + Cz + D = 0$ , then

$$(A^2 + B^2 + C^2)[r_\pi(M)]_R = \begin{pmatrix} -A^2 + B^2 + C^2 & -2AB & -2AC \\ -2AB & A^2 - B^2 + C^2 & -2BC \\ -2AC & -2BC & A^2 + B^2 - C^2 \end{pmatrix} [M]_R - 2D[\vec{n}_\pi]_b. \quad (5.15)$$

- **The orthogonal projection of the space on a line.** For a given line

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a point  $N(x_N, y_N, z_N)$ , we shall find the coordinates of its orthogonal projection on the line  $\Delta$ , as well as the coordinates of the orthogonally symmetric point  $M$  with respect to  $\Delta$ . The equations of the line and the coordinates of the point  $N$  are considered with respect to an orthonormal coordinate system  $R = (O, \vec{i}, \vec{j}, \vec{k})$ . In this respect we consider the plane  $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$  orthogonal on the line  $\Delta$  which passes through the point  $N$ .



The parametric equations of the line  $\Delta$  are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}. \quad (5.16)$$

The orthogonal projection of  $N$  on the line  $\Delta$  is at its intersection point with the plane

$$p(x - x_N) + q(y - y_N) + r(z - z_N) = 0,$$

and the value of  $t \in \mathbb{R}$  for which the line  $\Delta$  puncture the orthogonal plane  $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$  can be found by imposing the condition on the point of coordinate  $(x_0 + pt, y_0 + qt, z_0 + rt)$  to verify the equation of the plane, namely  $p(x_0 + pt - x_N) + q(y_0 + qt - y_N) + r(z_0 + rt - z_N) = 0$ . Thus

$$t = -\frac{p(x_0 - x_N) + q(y_0 - y_N) + r(z_0 - z_N)}{p^2 + q^2 + r^2} = -\frac{G(x_0, y_0, z_0)}{\|\vec{d}_\Delta\|^2},$$

where  $G(x, y, z) = p(x - x_N) + q(y - y_N) + r(z - z_N)$  and  $\vec{d}_\Delta = p\vec{i} + q\vec{j} + r\vec{k}$  is the director vector of the line  $\Delta$ . The coordinates of the orthogonal projection  $p_\Delta(N)$  of  $N$  on the line  $\Delta$  are therefore

$$\begin{cases} x_0 - p\frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ y_0 - q\frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ z_0 - r\frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \end{cases}$$

Thus, the position vector of the orthogonal projection  $p_\Delta(N)$  is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{G(A_0)}{\|\vec{d}_\Delta\|^2} \vec{d}_\Delta, \quad (5.17)$$

where  $A_0(x_0, y_0, z_0) \in \Delta$ .

- **The reflection of the space about a line.** In order to find the position vector of the orthogonally symmetric point  $r_\Delta(N)$  of  $N$  with respect to the line  $\Delta$  we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2} \left( \overrightarrow{ON} + \overrightarrow{Or_\Delta(N)} \right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)} = 2 \overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON} = 2 \overrightarrow{OA_0} - 2 \frac{\overrightarrow{G(A_0)}}{\|\overrightarrow{d_{\Delta}}\|^2} \overrightarrow{d_{\Delta}} - \overrightarrow{ON}.$$

The correspondence which associate to some point  $M$  its orthogonally symmetric point w.r.t.  $\delta$ , is called the *reflection* in the line  $\delta$  and is denoted by  $r_{\delta}$ .

### 5.3 Problems

1. (2p) Consider the triangle  $ABC$  and the midpoint  $A'$  of the side  $[BC]$ . Show that

$$4 \overrightarrow{AA'}^2 - \overrightarrow{BC}^2 = 4 \overrightarrow{AB} \cdot \overrightarrow{AC}.$$

*Solution.*

2. (2p) Consider the rectangle  $ABCD$  and the arbitrary point  $M$  within the space. Show that

(a)  $\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}$ .

(b)  $\overrightarrow{MA}^2 + \overrightarrow{MC}^2 = \overrightarrow{MB}^2 + \overrightarrow{MD}^2$ .

*Solution.*

3. (3p) Find the angle between:

(a) the straight lines

$$(d_1) \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \quad (d_2) \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases}$$

(b) the planes

$$\pi_1 : x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$

(c) the plane  $xOy$  and the straight line  $M_1M_2$ , where  $M_1(1, 2, 3)$  and  $M_2(-2, 1, 4)$ .

*Solution.*

4. (3p) Consider the noncoplanar vectors  $\overrightarrow{OA} (1, -1, -2)$ ,  $\overrightarrow{OB} (1, 0, -1)$ ,  $\overrightarrow{OC} (2, 2, -1)$  related to an orthonormal basis  $\vec{i}, \vec{j}, \vec{k}$ . Let  $H$  be the foot of the perpendicular through  $O$  on the plane  $ABC$ . Determine the components of the vectors  $\overrightarrow{OH}$ .

*Solution.*

5. (2p) Find the points on the  $z$ -axis which are equidistant with respect to the planes

$$\pi_1 : 12x + 9y - 20z - 19 = 0 \text{ and } \pi_2 : 16x + 12y + 15z - 9 = 0.$$

*Solution.*

6. (2p) Consider two planes

$$\begin{aligned}(\pi_1) \quad & A_1x + B_1y + C_1z + D_1 = 0 \\(\pi_2) \quad & A_2x + B_2y + C_2z + D_2 = 0\end{aligned}$$

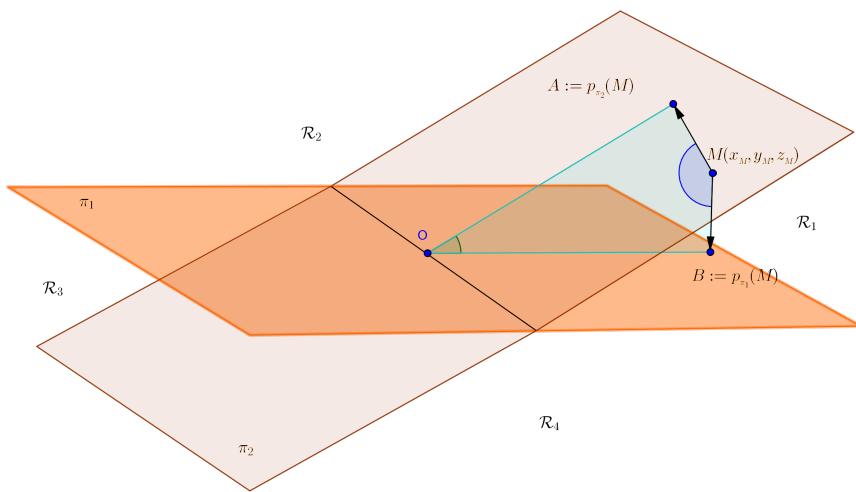
which are not parallel and not perpendicular as well. The two planes  $\pi_1, \pi_2$  devide the space into four regions  $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$  and  $\mathcal{R}_4$ , two of which, say  $\mathcal{R}_1$  and  $\mathcal{R}_3$ , correspond to the acute dihedral angle of the two planes. Show that  $M(x, y, z) \in \mathcal{R}_1 \cup \mathcal{R}_3$ , if and only if

$$F_1(x, y, z) \cdot F_2(x, y, z)(A_1A_2 + B_1B_2 + C_1C_2) < 0,$$

where  $F_1(x, y, z) = A_1x + B_1y + C_1z + D_1$  and  $F_2(x, y, z) = A_2x + B_2y + C_2z + D_2$ .

*Hint.* The non-parallelism relation between the two planes is equivalent with the condition

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$



The point  $M$  belongs to the union  $\mathcal{R}_1 \cup \mathcal{R}_3$  if and only if the angle of the vectors  $\overrightarrow{Mp_{\pi_1}(M)}$  and  $\overrightarrow{Mp_{\pi_2}(M)}$  is at least  $90^\circ$ , as the quadrilateral  $OAMB$  is inscriptible. More formally

$$\begin{aligned}M(x, y, z) \in \mathcal{R}_1 \cup \mathcal{R}_3 &\Leftrightarrow m(\overrightarrow{Mp_{\pi_1}(M)}, \overrightarrow{Mp_{\pi_2}(M)}) > 90^\circ \\&\Leftrightarrow \overrightarrow{Mp_{\pi_1}(M)} \cdot \overrightarrow{Mp_{\pi_2}(M)} < 0,\end{aligned}$$

where  $p_{\pi_1}(M), p_{\pi_2}(M)$  are the orthogonal projections of  $M$  on the planes  $\pi_1$  and  $\pi_2$  respectively.

*Solution.*

7. (3p) Consider the planes  $(\pi_1)$   $2x + y - 3z - 5 = 0$ ,  $(\pi_2)$   $x + 3y + 2z + 1 = 0$ . Find the equations of the bisector planes of the dihedral angles formed by the planes  $\pi_1$  and  $\pi_2$  and select the one contained into the acute regions of the dihedral angles formed by the two planes.

*Solution.*

8. (3p) Let  $a, b$  be two real numbers such that  $a^2 \neq b^2$ . Consider the planes:

$$(\alpha_1) ax + by - (a + b)z = 0$$

$$(\alpha_2) ax - by - (a - b)z = 0$$

and the quadric  $(\mathcal{C}) : a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$ . If  $a^2 < b^2$ , show that the quadric  $\mathcal{C}$  is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary,  $a^2 > b^2$ , show that the quadric  $\mathcal{C}$  is contained in the obtuse regions of the dihedral angles formed by the two planes.

*Solution.*

9. If two pairs of opposite edges of the tetrahedron  $ABCD$  are perpendicular ( $AB \perp CD$ ,  $AD \perp BC$ ), show that

- (a) The third pair of opposite edges are perpendicular too ( $AC \perp BD$ ).
- (b)  $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$ .
- (c) The heights of the tetrahedron are concurrent.  
(Such a tetrahedron is said to be orthocentric)

*Solution.* Denote by  $\vec{AB} = \vec{b}$ ,  $\vec{AC} = \vec{c}$  and  $\vec{AD} = \vec{d}$ .

$$(a) AB \perp CD \implies \vec{b}(\vec{d} - \vec{c}) = 0 \implies \vec{b}\vec{d} = \vec{b}\vec{c} = k$$

$$AD \perp BC \implies \vec{d}(\vec{c} - \vec{b}) = 0 \implies \vec{c}\vec{d} = \vec{b}\vec{d} = k,$$

$$\text{deci } \vec{c}\vec{b} = \vec{c}\vec{d} \implies \vec{c}(\vec{b} - \vec{d}) = 0 \implies AC \perp BD.$$

$$(b) AB^2 + CD^2 = \vec{b}^2 + (\vec{d} - \vec{c})^2 = \vec{b}^2 + \vec{d}^2 + \vec{c}^2 - 2k;$$

$$AC^2 + BD^2 = \vec{c}^2 + (\vec{d} - \vec{b})^2 = \vec{b}^2 + \vec{c}^2 + \vec{d}^2 - 2k;$$

$$BC^2 + AD^2 = \vec{d}^2 + (\vec{c} - \vec{b})^2 = \vec{b}^2 + \vec{c}^2 + \vec{d}^2 - 2k.$$

- (c) We shall show that there exists a point  $H$  such that  $AH \perp (DBC)$ ,  $BH \perp (ACD)$ ,  $CH \perp (ABD)$ ,  $DH \perp (ABC)$ . Let  $\vec{h} = \vec{AH} = m\vec{a} + n\vec{b} + p\vec{c}$ . Writing the conditions  $\vec{AH} \perp \vec{BC}$ ,  $\vec{CD}$ ;  $\vec{BH} \perp \vec{AC}$ ,  $\vec{AD}$ ;  $\vec{CH} \perp \vec{AB}$ ,  $\vec{AD}$ ;  $\vec{DH} \perp \vec{AB}$ ,  $\vec{AC}$  we obtain a consistent system with one single solution:

$$\begin{cases} b^2m + kn + kp = k \\ km + c^2n + kp = k \\ km + kn + d^2p = k. \end{cases} \quad (5.18)$$

Indeed the matrix of the system is

$$A = \begin{pmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{pmatrix}$$

and for its determinant we have successively

$$\begin{aligned} \det(A) &= \begin{vmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{vmatrix} = \begin{vmatrix} b \cdot b & b \cdot c & b \cdot c \\ c \cdot b & c \cdot c & c \cdot d \\ d \cdot b & d \cdot c & d \cdot d \end{vmatrix} \\ &= \begin{vmatrix} b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 & b_1d_1 + b_2d_2 + b_3d_3 \\ c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 & c_1d_1 + c_2d_2 + c_3d_3 \\ d_1b_1 + d_2b_2 + d_3b_3 & d_1c_1 + d_2c_2 + d_3c_3 & d_1^2 + d_2^2 + d_3^2 \end{vmatrix} \\ &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_1 & c_2 & d_2 \\ b_1 & c_3 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d}) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d})^2. \end{aligned}$$

The linear independence of the vectors  $\vec{b}, \vec{c}, \vec{d}$  ensure that  $(\vec{b}, \vec{c}, \vec{d}) \neq 0$  and shows that the linear system (5.18) is consistent and has one single solution.

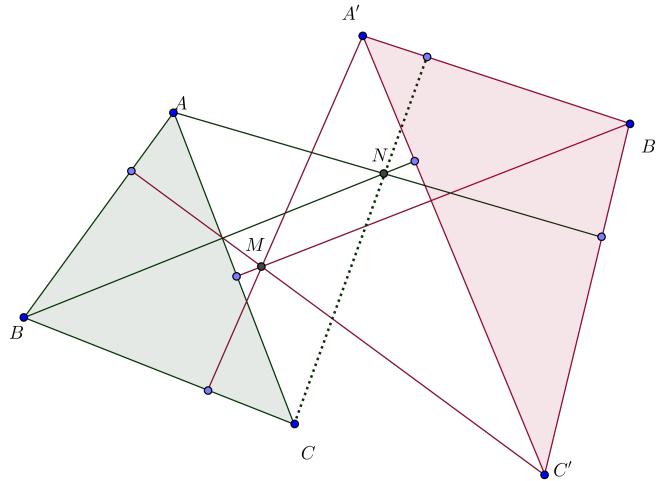
10. Two triangles  $ABC$  și  $A'B'C'$  are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices  $A', B', C'$  on the sides  $BC, CA, AB$  are concurrent. Show

that, in this case, the perpendicular lines from the vertices  $A, B, C$  on the sides  $B'C', C'A', A'B'$  are concurrent too.

*Solution* Due to the given hypothesis, we have

$$\vec{MA}' \cdot \vec{BC} = \vec{MB}' \cdot \vec{CA} = \vec{MC}' \cdot \vec{AB} = 0 \quad (5.19)$$

We now consider the perpendicular lines from the vertices  $A$  and  $B$  on the edges  $B'C'$  and  $C'A'$  and denote by  $N$  their intersection point.



Thus

$$\vec{NA} \cdot \vec{B'C'} = \vec{NB} \cdot \vec{C'A'} = 0.$$

By using the relations (5.19) we obtain

$$\begin{aligned} & \vec{MA}' \cdot \vec{BC} + \vec{MB}' \cdot \vec{CA} + \vec{MC}' \cdot \vec{AB} = 0 \\ \Leftrightarrow & \vec{MA}' \cdot (\vec{NC} - \vec{NB}) + \vec{MB}' \cdot (\vec{NA} - \vec{NC}) + \vec{MC}' \cdot (\vec{NB} - \vec{NA}) = 0 \\ \Leftrightarrow & (\vec{MB}' - \vec{MC}') \cdot \vec{NA} + (\vec{MC}' - \vec{MA}') \cdot \vec{NB} + (\vec{MA}' - \vec{MB}') \cdot \vec{NC} = 0 \\ \Leftrightarrow & \vec{C'B'} \cdot \vec{NA} + \vec{A'C'} \cdot \vec{NB} + \vec{B'A'} \cdot \vec{NC} = 0 \\ \Leftrightarrow & \vec{B'A'} \cdot \vec{NC} = 0 \Leftrightarrow NC \perp A'B'. \end{aligned}$$

11. (2p) Find the orthogonal projection

- (a) of the point  $A(1, 2, 1)$  on the plane  $\pi : x + y + 3z + 5 = 0$ .
- (b) of the point  $B(5, 0, -2)$  on the straight line  $(d) \frac{x-2}{3} = \frac{y-1}{2} = \frac{z-3}{4}$ .

*Solution.*

**A few questions in the two dimensional setting**

12. (3p) Find the coordinates of the point  $P$  on the line  $d : 2x - y - 5 = 0$  for which the sum  $AP + PB$  is minimum, when  $A(-7, 1)$  and  $B(-5, 5)$ .
13. (2p) Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines  $4x - y + 2 = 0$ ,  $x - 4y - 8 = 0$  and  $x + 4y - 8 = 0$ .

*Solution.*

14. (3p) Given the bundle of lines of equations  $(1-t)x + (2-t)y + t - 3 = 0$ ,  $t \in \mathbb{R}$  and  $x + y - 1 = 0$ , find:

- the coordinates of the vertex of the bundle;
- the equation of the line in the bundle which cuts  $Ox$  and  $Oy$  in  $M$  respectively  $N$ , such that  $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$ .

*Solution.*

15. (2p) Let  $\mathcal{B}$  be the bundle of lines of vertex  $M_0(5, 0)$ . An arbitrary line from  $\mathcal{B}$  intersects the lines  $d_1 : y - 2 = 0$  and  $d_2 : y - 3 = 0$  in  $M_1$  respectively  $M_2$ . Prove that the line passing through  $M_1$  and parallel to  $OM_2$  passes through a fixed point.

16. (3p) The vertices of the quadrilateral  $ABCD$  are  $A(4, 3)$ ,  $B(5, -4)$ ,  $C(-1, -3)$  and  $D((-3, -1))$ .

- (a) Find the coordinates of the intersection points  $\{E\} = AB \cap CD$  and  $\{F\} = BC \cap AD$ ;
- (b) Prove that the midpoints of the segments  $[AC]$ ,  $[BD]$  and  $[EF]$  are collinear.

*Solution.*

17. (3p) Let  $M$  be a point whose coordinates satisfy

$$\frac{4x + 2y + 8}{3x - y + 1} = \frac{5}{2}.$$

- (a) Prove that  $M$  belongs to a fixed line  $(d)$ ;
- (b) Find the minimum of  $x^2 + y^2$ , when  $M \in d \setminus \{M_0(-1, -2)\}$ .

*Solution.*

18. (3p) Find the locus of the points whose distances to two orthogonal lines have a constant ratio.

*Solution.*

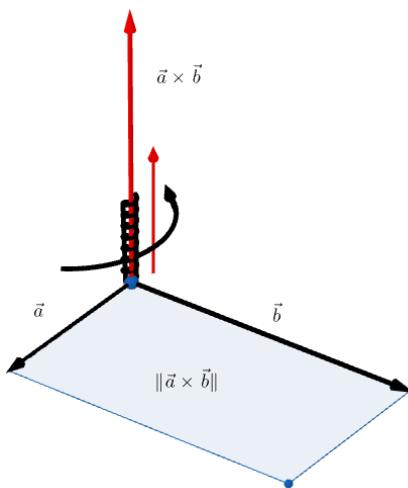
## 6 Week 6:

### 6.1 The vector product

**Definition 6.1.** The *vector product* or the *cross product* of the vectors  $\vec{a}, \vec{b} \in \mathcal{V}$  is a vector, denoted by  $\vec{a} \times \vec{b}$ , which is defined to be zero if  $\vec{a}, \vec{b}$  are linearly dependent (collinear), and if  $\vec{a}, \vec{b}$  are linearly independent (noncollinear), then it is defined by the following data:

1.  $\vec{a} \times \vec{b}$  is a vector orthogonal on the two-dimensional subspace  $\langle \vec{a}, \vec{b} \rangle$  of  $\mathcal{V}$ ;
2. if  $\vec{a} = \overrightarrow{OA}$ ,  $\vec{b} = \overrightarrow{OB}$ , then the sense of  $\vec{a} \times \vec{b}$  is the one in which a right-handed screw, placed along the line passing through  $O$  orthogonal to the vectors  $\vec{a}$  and  $\vec{b}$ , advances when it is being rotated simultaneously with the vector  $\vec{a}$  from  $\vec{a}$  towards  $\vec{b}$  within the vector subspace  $\langle \vec{a}, \vec{b} \rangle$  and the support half line of  $\vec{a}$  sweeps the interior of the angle  $\widehat{AOB}$  (Screw rule).
3. the *norm (magnitude or length)* of  $\vec{a} \times \vec{b}$  is defined by

$$\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \cdot \| \vec{b} \| \sin(\widehat{\vec{a}, \vec{b}}).$$



**Remark 6.1.** 1. The norm (magnitude or length) of the vector  $\vec{a} \times \vec{b}$  is actually the area of the parallelogram constructed on the vectors  $\vec{a}, \vec{b}$ .

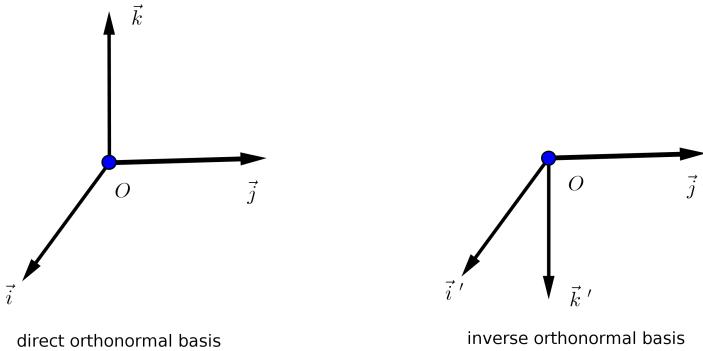
2. The vectors  $\vec{a}, \vec{b} \in \mathcal{V}$  are linearly dependent (collinear) if and only if  $\vec{a} \times \vec{b} = \vec{0}$ .

**Proposition 6.1.** The vector product has the following properties:

1.  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}, \forall \vec{a}, \vec{b} \in \mathcal{V};$
2.  $(\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) = \lambda(\vec{a} \times \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V};$
3.  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}.$

## 6.2 The vector product in terms of coordinates

If  $[\vec{i}, \vec{j}, \vec{k}]$  is an orthonormal basis, observe that  $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$ . We say that the orthonormal basis  $[\vec{i}, \vec{j}, \vec{k}]$  is *direct* if  $\vec{i} \times \vec{j} = \vec{k}$ . If, on the contrary,  $\vec{i} \times \vec{j} = -\vec{k}$ , we say that the orthonormal basis  $[\vec{i}, \vec{j}, \vec{k}]$  is *inverse*.



Therefore, if  $[\vec{i}, \vec{j}, \vec{k}]$  is a direct orthonormal basis, then  $\vec{i} \times \vec{j} = \vec{k}$ ,  $\vec{j} \times \vec{k} = \vec{i}$ ,  $\vec{k} \times \vec{i} = \vec{j}$  and obviously  $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$ .

**Proposition 6.2.** If  $[\vec{i}, \vec{j}, \vec{k}]$  is a direct orthonormal basis and  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ ,  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ , then

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}, \quad (6.1)$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \quad (6.2)$$

*Proof.* Indeed,

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \times (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\ &= a_1 b_1 \vec{i} \times \vec{i} + a_1 b_2 \vec{i} \times \vec{j} + a_1 b_3 \vec{i} \times \vec{k} \\ &\quad + a_2 b_1 \vec{j} \times \vec{i} + a_2 b_2 \vec{j} \times \vec{j} + a_2 b_3 \vec{j} \times \vec{k} \\ &\quad + a_3 b_1 \vec{k} \times \vec{i} + a_3 b_2 \vec{k} \times \vec{j} + a_3 b_3 \vec{k} \times \vec{k} \\ &= a_1 b_2 \vec{k} - a_1 b_3 \vec{j} - a_2 b_1 \vec{k} + a_2 b_3 \vec{i} + a_3 b_1 \vec{j} - a_3 b_2 \vec{i} \\ &= (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k} \end{aligned}$$

□

One can rewrite formula (6.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (6.3)$$

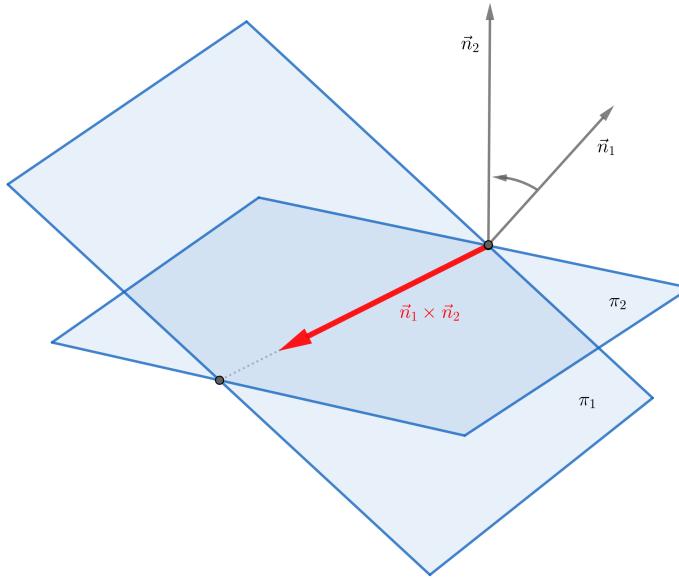
the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

**Remark 6.2.** If  $R = (O, \vec{i}, \vec{j}, \vec{k})$  is the direct Cartesian orthonormal reference system behind the equations of the line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

then we can recover the director parameters (4.10) of  $\Delta$ , in this particular case of orthonormal Cartesian reference systems, by observing that  $\vec{n}_1 \times \vec{n}_2$  is a director vector of  $\Delta$ , where

$$\begin{aligned} \vec{n}_1 &= A_1 \vec{i} + B_1 \vec{j} + C_1 \vec{k} \\ \vec{n}_2 &= A_2 \vec{i} + B_2 \vec{j} + C_2 \vec{k}. \end{aligned}$$



Recall that

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \vec{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \vec{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \vec{k}.$$

Note however that the director parameters were obtained before for arbitrary Cartesian reference systems (See (4.10)).

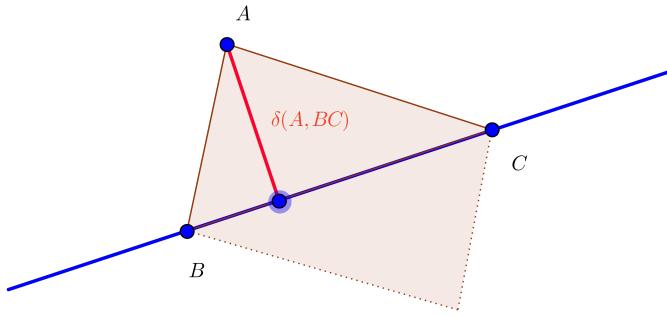
### 6.3 Applications of the vector product

- **The area of the triangle ABC.**  $S_{ABC} = \frac{1}{2} ||\vec{AB}|| \cdot ||\vec{AC}|| \sin \widehat{BAC} = \frac{1}{2} ||\vec{AB} \times \vec{AC}||$ . On the other hand

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix},$$

as the coordinates of  $\vec{AB}$  and  $\vec{AC}$  are  $(x_B - x_A, y_B - y_A, z_B - z_A)$  and  $(x_C - x_A, y_C - y_A, z_C - z_A)$  respectively. Thus,

$$4S_{ABC}^2 = \left| \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix} \right|^2 + \left| \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix} \right|^2 + \left| \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix} \right|^2.$$



- **The distance from one point to a straight line.**

- (a) The distance  $\delta(A, BC)$  from the point  $A(x_A, y_A, z_A)$  to the straight line  $BC$ , where  $B(x_B, y_B, z_B)$  and  $C(x_C, y_C, z_C)$ . Since

$$S_{ABC} = \frac{\|\overrightarrow{BC}\| \cdot \delta(A, BC)}{2}$$

it follows that

$$\delta^2(A, BC) = \frac{4S_{ABC}^2}{\|\overrightarrow{BC}\|^2}.$$

Thus, we obtain

$$\delta^2(A, BC) = \frac{\left| \begin{matrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{matrix} \right|^2 + \left| \begin{matrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{matrix} \right|^2 + \left| \begin{matrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{matrix} \right|^2}{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}.$$

- (b) The distance from  $\delta(A, d)$  from one point  $A(x_A, y_A, z_A)$  to the straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

$$\delta(A, d) = \frac{\|\overrightarrow{d} \times \overrightarrow{A_0 A}\|}{\|d\|}, \quad (6.4)$$

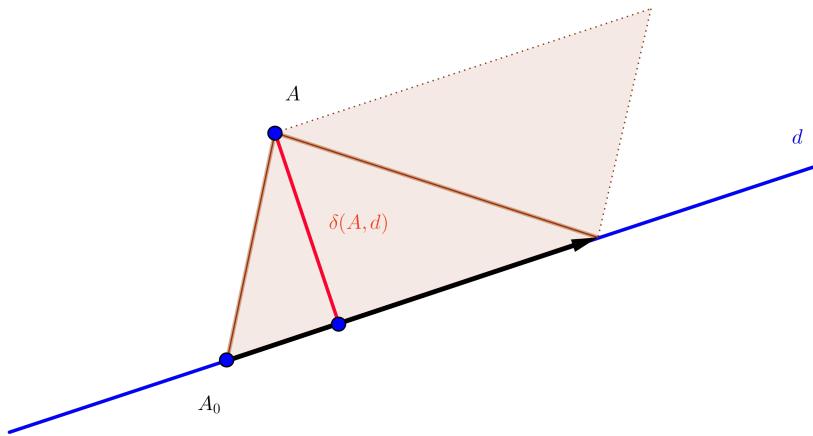
where  $A_0(x_0, y_0, z_0) \in d$ .

Since

$$\begin{aligned} \overrightarrow{d} \times \overrightarrow{A_0 A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ \frac{x_A - x_0}{p} & \frac{y_A - y_0}{q} & \frac{z_A - z_0}{r} \end{vmatrix} \\ &= \begin{vmatrix} x_A - x_0 & y_A - y_0 & z_A - z_0 \\ \frac{y_A - y_0}{q} & \frac{z_A - z_0}{r} & \frac{x_A - x_0}{p} \\ y_A - y_0 & z_A - z_0 & x_A - x_0 \end{vmatrix} \vec{i} + \begin{vmatrix} p & q & r \\ \frac{y_A - y_0}{q} & \frac{z_A - z_0}{r} & \frac{x_A - x_0}{p} \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q & r \\ \frac{y_A - y_0}{q} & \frac{z_A - z_0}{r} & \frac{x_A - x_0}{p} \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A, d) = \frac{\sqrt{\left| \begin{matrix} q & r \\ y_A - y_0 & z_A - z_0 \end{matrix} \right|^2 + \left| \begin{matrix} r & p \\ z_A - z_0 & x_A - x_0 \end{matrix} \right|^2 + \left| \begin{matrix} p & q \\ x_A - x_0 & y_A - y_0 \end{matrix} \right|^2}}{\sqrt{p^2 + q^2 + r^2}}.$$



## 6.4 The double vector (cross) product

The *double vector (cross) product* of the vectors  $\vec{a}, \vec{b}, \vec{c}$  is the vector  $\vec{a} \times (\vec{b} \times \vec{c})$

**Proposition 6.3.**

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \end{vmatrix}, \quad \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}. \quad (6.5)$$

*Proof.* (Sketch) If the vectors  $\vec{b}$  and  $\vec{c}$  are linearly dependent, then both sides are obviously zero. Otherwise one can choose an orthonormal basis  $[\vec{i}, \vec{j}, \vec{k}]$ , related to the vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$ , such that

$$\vec{b} = b_1 \vec{i}, \vec{c} = c_1 \vec{i} + c_2 \vec{j}, \vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}.$$

For example one can choose  $\vec{i}$  to be  $\vec{b} / \|\vec{b}\|$  and  $\vec{j}$  a unit vector in the subspace  $\langle \vec{b}, \vec{c} \rangle$  which is perpendicular on  $\vec{b}$ . Finally, one can choose  $\vec{k} = \vec{i} \times \vec{j}$ . By computing the two sides of the equality 6.5, in terms of coordinates and the vectors  $\vec{i}, \vec{j}, \vec{k}$ , one gets the same result.  $\square$

**Corollary 6.4.** 1.  $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{a} \end{vmatrix}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V};$

2.  $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$  (*Jacobi's identity*).

*Proof.* While the first identity follows immediately via 6.5, for the Jacobi's identity we get successively:

$$\begin{aligned} & \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} + (\vec{b} \cdot \vec{a}) \vec{c} - (\vec{b} \cdot \vec{c}) \vec{a} + (\vec{c} \cdot \vec{b}) \vec{a} - (\vec{c} \cdot \vec{a}) \vec{b} = \vec{0}. \end{aligned}$$

$\square$

## 6.5 Problems

1. (2p) Show that  $\|\vec{a} \times \vec{b}\| \leq \|\vec{a}\| \cdot \|\vec{b}\|, \forall \vec{a}, \vec{b} \in \mathcal{V}$ .

*Solution.*

2. (3p) Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be pairwise noncollinear vectors. Show that the necessary and sufficient condition for the existence of a triangle  $ABC$  with the properties  $\overrightarrow{BC} = \vec{a}$ ,  $\overrightarrow{CA} = \vec{b}$ ,  $\overrightarrow{AB} = \vec{c}$  is

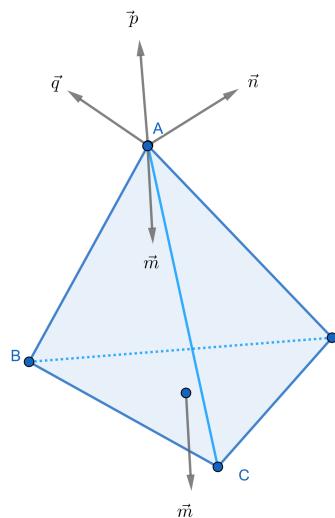
$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

From the equalities of the norms deduce the law of sines.

*Solution.*

3. (3p) Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

*Solution.*



The proportionality of  $\vec{m}$ ,  $\vec{n}$ ,  $\vec{p}$ ,  $\vec{q}$  with the areas of the corresponding faces of the tetrahedron show that

$$\begin{aligned}\vec{m} &= k\vec{BD} \times \vec{BC}, \quad \vec{n} = k\vec{AC} \times \vec{AD} \\ \vec{p} &= k\vec{AD} \times \vec{AB}, \quad \vec{q} = k\vec{AB} \times \vec{AC}\end{aligned}$$

Thus,

$$\begin{aligned}\vec{m} + \vec{n} + \vec{p} + \vec{q} &= k\vec{BD} \times \vec{BC} + k\vec{AC} \times \vec{AD} + \\ &+ k\vec{AD} \times \vec{AB} + k\vec{AB} \times \vec{AC} \\ &= k(\vec{AD} - \vec{AB}) \times (\vec{AC} - \vec{AB}) + k\vec{AC} \times \vec{AD} \\ &+ k\vec{AD} \times \vec{AB} + k\vec{AB} \times \vec{AC} = \\ &= k\vec{AD} \times \vec{AC} - k\vec{AD} \times \vec{AB} - k\vec{AB} \times \vec{AC} + k\vec{AB} \times \vec{AB} = \\ &+ k\vec{AC} \times \vec{AD} + k\vec{AD} \times \vec{AB} + k\vec{AB} \times \vec{AC} = \vec{0}.\end{aligned}$$

4. (2p) Find the distance from the point  $P(1, 2, -1)$  to the straight line  $(d)$   $x = y = z$ .

*Solution.*

5. (3p) Find the area of the triangle  $ABC$  and the lengths of its heights, where  $A(-1, 1, 2)$ ,  $B(2, -1, 1)$  and  $C(2, -3, -2)$ .

*Solution.*

6. (3p) Let  $d_1, d_2, d_3, d_4$  be pairwise skew straight lines. Assuming that  $d_{12} \perp d_{34}$  and  $d_{13} \perp d_{24}$ , show that  $d_{14} \perp d_{23}$ , where  $d_{ik}$  is the common perpendicular of the lines  $d_i$  and  $d_k$ .

*Solution.* A director vector of the common perpendicular  $d_{ij}$  is  $\vec{d}_i \times \vec{d}_j$ , where  $\vec{d}_r$  stands for a director vector of  $d_r$ . Therefore we have successively:

$$\begin{aligned} d_{12} \perp d_{34} &\Leftrightarrow \vec{d}_1 \times \vec{d}_2 \perp \vec{d}_3 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_3 \times \vec{d}_4) = 0 \\ &\Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_3 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_2 \cdot \vec{d}_3 & \vec{d}_2 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3). \end{aligned}$$

Similalry

$$\begin{aligned} d_{13} \perp d_{24} &\Leftrightarrow \vec{d}_1 \times \vec{d}_3 \perp \vec{d}_2 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_3) \cdot (\vec{d}_2 \times \vec{d}_4) = 0 \\ &\Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_3 \cdot \vec{d}_2 & \vec{d}_3 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_3 \cdot \vec{d}_2). \end{aligned}$$

Therefore we have

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3) = (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4),$$

which shows that

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) - (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = 0 \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_3 \\ \vec{d}_4 \cdot \vec{d}_2 & \vec{d}_4 \cdot \vec{d}_3 \end{vmatrix} = 0 \Leftrightarrow d_{14} \perp d_{23}.$$

## 7 Week 7: The triple scalar product

The *triple scalar product*  $(\vec{a}, \vec{b}, \vec{c})$  of the vectors  $\vec{a}, \vec{b}, \vec{c}$  is the real number  $(\vec{a} \times \vec{b}) \cdot \vec{c}$ .

**Proposition 7.1.** If  $[\vec{i}, \vec{j}, \vec{k}]$  is a direct orthonormal basis and

$$\begin{aligned}\vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \\ \vec{b} &= b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k} \\ \vec{c} &= c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}\end{aligned}$$

then

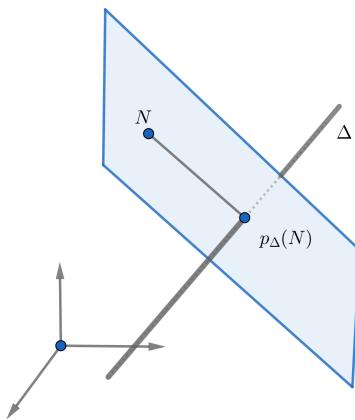
$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (7.1)$$

*Proof.* Indeed, we have successively:

$$\begin{aligned}(\vec{a}, \vec{b}, \vec{c}) &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right) \cdot (c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.\end{aligned}$$

□

**Remark 7.1.** Taking into account the formula (7.2) for the distance  $\delta(N, \Delta)$  from the point  $N(x_N, y_N, z_N)$  to the straight line  $\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$  as well as Proposition 6.3 we deduce that



$$\begin{aligned}\delta(N, \Delta) &= \| \overrightarrow{Np_\Delta(N)} \| \\ &= \| \overrightarrow{NO} + \overrightarrow{Op_\Delta(N)} \| = \left\| \overrightarrow{NA_0} - \frac{\overrightarrow{d}_\Delta \cdot \overrightarrow{NA_0}}{\| \overrightarrow{d}_\Delta \|^2} \overrightarrow{d}_\Delta \right\|\end{aligned} \quad (7.2)$$

$$\begin{aligned}
&= \frac{\| (\vec{d}_\Delta \cdot \vec{d}_\Delta) \vec{NA}_0 - (\vec{d}_\Delta \cdot \vec{NA}_0) \vec{d}_\Delta \|}{\| \vec{d}_\Delta \|^2} \\
&= \frac{\| \vec{d}_\Delta \times (\vec{NA}_0 \times \vec{d}_\Delta) \|}{\| \vec{d}_\Delta \|^2} = \frac{\| \vec{NA}_0 \times \vec{d}_\Delta \|}{\| \vec{d}_\Delta \|}.
\end{aligned}$$

Thus, we recovered the distance formula from one point to one straight line (see formula 6.4) by using different arguments.

- Corollary 7.2.**
1. The free vectors  $\vec{a}, \vec{b}, \vec{c}$  are linearly dependent (collinear) iff  $(\vec{a}, \vec{b}, \vec{c}) = 0$
  2. The free vectors  $\vec{a}, \vec{b}, \vec{c}$  are linearly independent (noncollinear) if and only if  $(\vec{a}, \vec{b}, \vec{c}) \neq 0$
  3. The free vectors  $\vec{a}, \vec{b}, \vec{c}$  form a basis of the space  $\mathcal{V}$  if and only if  $(\vec{a}, \vec{b}, \vec{c}) \neq 0$ .
  4. The correspondence  $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ ,  $F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$  is trilinear and skew-symmetric, i.e.

$$\begin{aligned}
(\alpha \vec{a} + \alpha' \vec{a}', \vec{b}, \vec{c}) &= \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}', \vec{b}, \vec{c}) \\
(\vec{a}, \beta \vec{b} + \beta' \vec{b}', \vec{c}) &= \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}', \vec{c}) \\
(\vec{a}, \vec{b}, \gamma \vec{c} + \gamma' \vec{c}') &= \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}').
\end{aligned} \tag{7.3}$$

$\forall \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}, \forall \vec{a}, \vec{b}, \vec{c}, \vec{a}', \vec{b}', \vec{c}' \in \mathcal{V}$  și

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = \text{sgn}(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}), \quad \forall \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ și } \forall \sigma \in S_3 \tag{7.4}$$

**Remark 7.2.** One can rewrite the relations (7.4) as follows:

$$\begin{aligned}
(\vec{a}_1, \vec{a}_2, \vec{a}_3) &= (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\
&= -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = -(\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1),
\end{aligned}$$

$\forall \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

- Corollary 7.3.**
1.  $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ .

2. For every  $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$  the Laplace formula holds:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \left| \begin{array}{cc} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{array} \right|.$$

*Proof.* While the first identity is obvious, for the Laplace formula we have successively:

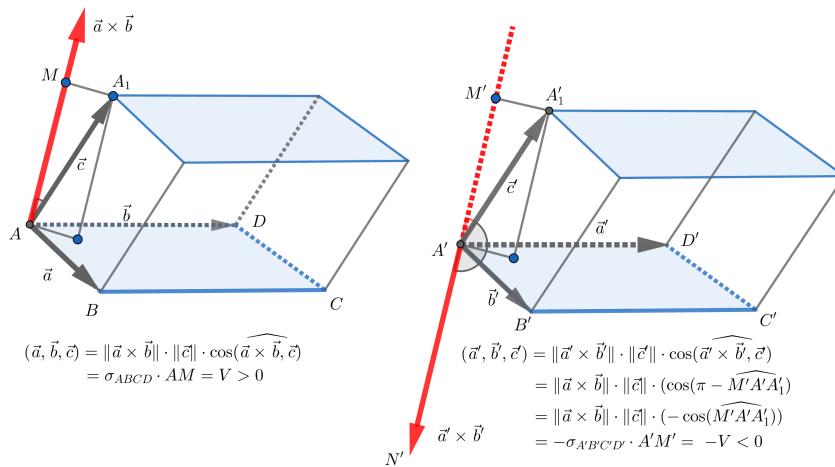
$$\begin{aligned}
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= (\vec{a}, \vec{b}, \vec{c} \times \vec{d}) = (\vec{c} \times \vec{d}, \vec{a}, \vec{b}) \\
&= [(\vec{c} \times \vec{d}) \times \vec{a}] \cdot \vec{b} = -[(\vec{a} \cdot \vec{d}) \vec{c} - (\vec{a} \cdot \vec{c}) \vec{d}] \cdot \vec{b} \\
&= -(\vec{a} \cdot \vec{d})(\vec{c} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})(\vec{d} \cdot \vec{b}) = \left| \begin{array}{cc} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{array} \right|.
\end{aligned}$$

□

**Definition 7.1.** The basis  $[\vec{a}, \vec{b}, \vec{c}]$  of the space  $\mathcal{V}$  is said to be *directe* if  $(\vec{a}, \vec{b}, \vec{c}) > 0$ . If, on the contrary,  $(\vec{a}, \vec{b}, \vec{c}) < 0$ , we say that the basis  $[\vec{a}, \vec{b}, \vec{c}]$  is *inverse*.

**Definition 7.2.** The *oriented volume* of the parallelepiped constructed on the noncoplanar vectors  $\vec{a}, \vec{b}, \vec{c}$  is  $\varepsilon \cdot V$ , where  $V$  is the volume of this parallelepiped and  $\varepsilon = +1$  or  $-1$  insomuch as the basis  $[\vec{a}, \vec{b}, \vec{c}]$  is directe or inverse respectively.

**Proposition 7.4.** The triple scalar product  $(\vec{a}, \vec{b}, \vec{c})$  of the noncoplanar vectors  $\vec{a}, \vec{b}, \vec{c}$  is equal with the oriented volume of the parallelepiped constructed on these vectors.



## 7.1 Applications of the triple scalar product

### 7.1.1 The distance between two straight lines

If  $d_1, d_2$  are two straight lines, then the distance between them, denoted by  $\delta(d_1, d_2)$ , is being defined as

$$\min\{||\overrightarrow{M_1 M_2}|| \mid M_1 \in d_1, M_2 \in d_2\}.$$

1. If  $d_1 \cap d_2 \neq \emptyset$ , then  $\delta(d_1, d_2) = 0$ .
2. If  $d_1 \parallel d_2$ , then  $\delta(d_1, d_2) = ||\overrightarrow{MN}||$  where  $\{M\} = d \cap d_1$ ,  $\{N\} = d \cap d_2$  and  $d$  is a straight line perpendicular to the lines  $d_1$  and  $d_2$ . Obviously  $||\overrightarrow{MN}||$  is independent on the choice of the line  $d$ .
3. We now assume that the straight lines  $d_1, d_2$  are noncoplanar (skew lines). In this case there exists a unique straight line  $d$  such that  $d \perp d_1, d_2$  and  $d \cap d_1 = \{M_1\}$ ,  $d \cap d_2 = \{M_2\}$ . The straight line  $d$  is called the *common perpendicular* of the lines  $d_1, d_2$  and obviously  $\delta(d_1, d_2) = ||\overrightarrow{M_1 M_2}||$ .

Assume that the straight lines  $d_1, d_2$  are given by their points  $A_1(x_1, y_1, z_1)$ ,  $A_2(x_2, y_2, z_2)$  and their vectors și au vectorii directori  $\vec{d}_1(p_1, q_1, r_1)$   $\vec{d}_2(p_2, q_2, r_2)$ , that is, thei equations are

$$d_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$d_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines  $d_1, d_2$  is the intersection line between the plane containing the line  $d_1$  which is parallel to the vector  $\vec{d}_1 \times \vec{d}_2$ , and the plane containing the line  $d_2$  which is parallel to  $\vec{d}_1 \times \vec{d}_2$ . Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \left| \begin{matrix} q_1 & r_1 \\ q_2 & r_2 \end{matrix} \right| \vec{i} + \left| \begin{matrix} r_1 & p_1 \\ r_2 & p_2 \end{matrix} \right| \vec{j} + \left| \begin{matrix} p_1 & q_1 \\ p_2 & q_2 \end{matrix} \right| \vec{k}$$

it follows that the equations of the common perpendicular are

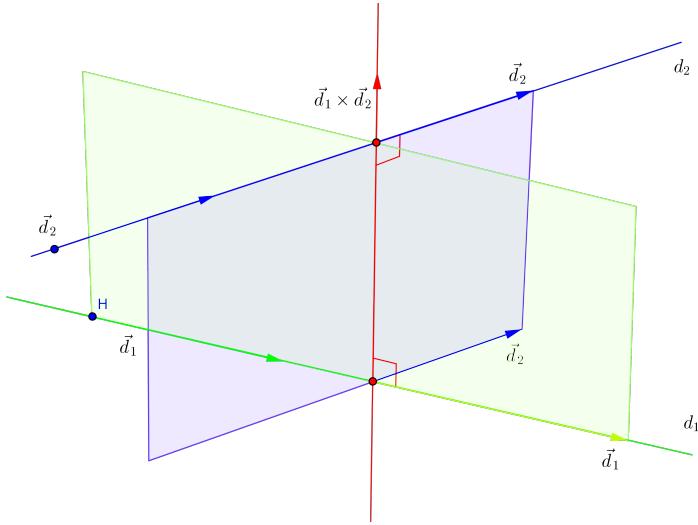


Figure 8: Perpendiculara comună a dreptelor  $d_1$  și  $d_2$

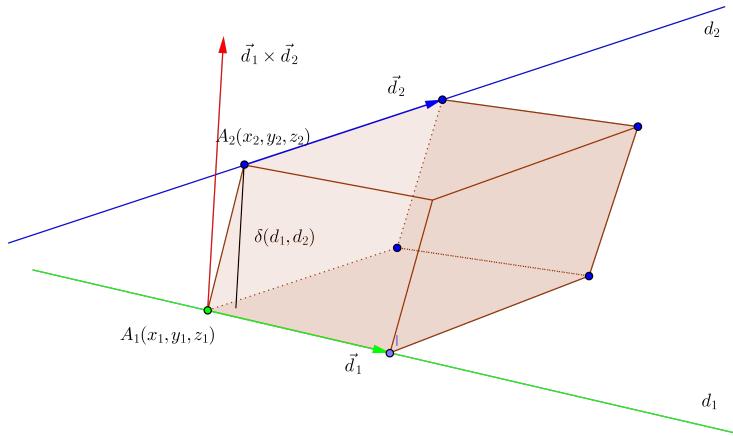
$$\left\{ \begin{array}{l} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \left| \begin{matrix} q_1 & r_1 \\ q_2 & r_2 \end{matrix} \right| & \left| \begin{matrix} r_1 & p_1 \\ r_2 & p_2 \end{matrix} \right| & \left| \begin{matrix} p_1 & q_1 \\ p_2 & q_2 \end{matrix} \right| \end{vmatrix} = 0 \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \left| \begin{matrix} q_1 & r_1 \\ q_2 & r_2 \end{matrix} \right| & \left| \begin{matrix} r_1 & p_1 \\ r_2 & p_2 \end{matrix} \right| & \left| \begin{matrix} p_1 & q_1 \\ p_2 & q_2 \end{matrix} \right| \end{vmatrix} = 0. \end{array} \right. \quad (7.5)$$

The distance between the straight lines  $d_1, d_2$  can be also regarded as the height of the parallelogram constructed on the vectors  $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$ . Thus

$$\delta(d_1, d_2) = \frac{|(A_1 \vec{A}_2, \vec{d}_1, \vec{d}_2)|}{\|\vec{d}_1 \times \vec{d}_2\|}. \quad (7.6)$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{matrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{matrix} \right|}{\sqrt{\left| \begin{matrix} q_1 & r_1 \\ q_2 & r_2 \end{matrix} \right|^2 + \left| \begin{matrix} r_1 & p_1 \\ r_2 & p_2 \end{matrix} \right|^2 + \left| \begin{matrix} p_1 & q_1 \\ p_2 & q_2 \end{matrix} \right|^2}} \quad (7.7)$$



### 7.1.2 The coplanarity condition of two straight lines

Using the notations of the previous section, observe that the straight lines  $d_1, d_2$  are coplanar if and only if the vectors  $\vec{A_1 A_2}, \vec{d}_1, \vec{d}_2$  are linearly dependent (coplanar), or equivalently  $(\vec{A_1 A_2}, \vec{d}_1, \vec{d}_2) = 0$ . Consequently the straight lines  $d_1, d_2$  are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \quad (7.8)$$

## 7.2 Problems

1. (2p) Show that

(a)  $|(\vec{a}, \vec{b}, \vec{c})| \leq \|\vec{a}\| \cdot \|\vec{b}\| \cdot \|\vec{c}\|$ ;

*Solution.*

$$(b) \text{ (2p)} (\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}) = 2(\vec{a}, \vec{b}, \vec{c}).$$

*Solution.*

2. (3p) Prove the following identity:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{c}, \vec{d}) \vec{b} - (\vec{b}, \vec{c}, \vec{d}) \vec{a} = (\vec{a}, \vec{b}, \vec{d}) \vec{c} - (\vec{a}, \vec{b}, \vec{c}) \vec{d}.$$

*Solution.* By using the identity  $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w}$  for  $\vec{u} = \vec{a} \times \vec{b}$ ,  $\vec{v} = \vec{c}$  and  $\vec{w} = \vec{d}$  we obtain

$$\begin{aligned} (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w} \\ &= [(\vec{a} \times \vec{b}) \cdot \vec{d}] \vec{c} - [(\vec{a} \times \vec{b}) \cdot \vec{c}] \vec{d} \\ &= (\vec{a}, \vec{b}, \vec{d}) \vec{c} - (\vec{a}, \vec{b}, \vec{c}) \vec{d}. \end{aligned}$$

By using the identity  $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{v} \cdot \vec{w}) \vec{u}$  for  $\vec{u} = \vec{a}$ ,  $\vec{v} = \vec{b}$  and  $\vec{w} = \vec{c} \times \vec{d}$  we obtain

$$\begin{aligned} (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= (\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{v} \cdot \vec{w}) \vec{u} \\ &= [\vec{a} \cdot (\vec{c} \times \vec{d})] \vec{b} - [\vec{b} \cdot (\vec{c} \times \vec{d})] \vec{a} \\ &= (\vec{a}, \vec{c}, \vec{d}) \vec{b} - (\vec{b}, \vec{c}, \vec{d}) \vec{a}. \end{aligned}$$

3. (3p) Prove the following identity:  $(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$ .

*Solution.* We have successively:

$$\begin{aligned} (\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) &= [(\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w})] \cdot (\vec{w} \times \vec{u}) \\ &= [(\vec{u}, \vec{v}, \vec{w}) \vec{v} - (\vec{u}, \vec{v}, \vec{v}) \vec{w}] \cdot (\vec{w} \times \vec{u}) \\ &= (\vec{u}, \vec{v}, \vec{w}) [\vec{v} \cdot (\vec{w} \times \vec{u})] = (\vec{u}, \vec{v}, \vec{w})(\vec{v}, \vec{w}, \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2. \end{aligned}$$

4. (3p) The *reciprocal vectors* of the noncoplanar vectors  $\vec{u}, \vec{v}, \vec{w}$  are defined by

$$\vec{u}' = \frac{\vec{v} \times \vec{w}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{v}' = \frac{\vec{w} \times \vec{u}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{w}' = \frac{\vec{u} \times \vec{v}}{(\vec{u}, \vec{v}, \vec{w})}.$$

Show that:

(a)

$$\begin{aligned} \vec{a} &= (\vec{a} \cdot \vec{u}') \vec{u} + (\vec{a} \cdot \vec{v}') \vec{v} + (\vec{a} \cdot \vec{w}') \vec{w} \\ &= \frac{(\vec{a}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} \vec{u} + \frac{(\vec{u}, \vec{a}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} \vec{v} + \frac{(\vec{u}, \vec{v}, \vec{a})}{(\vec{u}, \vec{v}, \vec{w})} \vec{w}. \end{aligned}$$

(b) the reciprocal vectors of  $\vec{u}', \vec{v}', \vec{w}'$  are the vectors  $\vec{u}, \vec{v}, \vec{w}$ .

*Solution.* (4a) Obviously  $\vec{a} = \alpha \vec{u} + \beta \vec{v} + \gamma \vec{w}$ , as  $\vec{u}, \vec{v}, \vec{w}$  are three linearly independent vectors of the three dimensional vector space  $\mathcal{V}$ , i.e.  $\vec{u}, \vec{v}, \vec{w}$  form a basis of  $\mathcal{V}$ . Moreover we have

$$\begin{aligned} \vec{a} \cdot \vec{u}' &= \frac{\vec{a} \cdot (\vec{v} \times \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \frac{(\vec{a}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \frac{(\alpha \vec{u} + \beta \vec{v} + \gamma \vec{w}) \cdot (\vec{v} \times \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} \\ &= \frac{\alpha(\vec{u}, \vec{v}, \vec{w}) + \beta(\vec{v}, \vec{v}, \vec{w}) + \gamma(\vec{w}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \alpha. \end{aligned}$$

One can similarly show that

$$\vec{a} \cdot \vec{v}' = \frac{(\vec{u}, \vec{a}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \beta \text{ and } \vec{a} \cdot \vec{w}' = \frac{(\vec{u}, \vec{v}, \vec{a})}{(\vec{u}, \vec{v}, \vec{w})} = \gamma.$$

(4b) Let us first observe that

$$(\vec{u}', \vec{v}', \vec{w}') = (\vec{w}, \vec{u}, \vec{v}) = \frac{(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u})}{(\vec{u}, \vec{v}, \vec{w})^3} = \frac{(\vec{u}, \vec{v}, \vec{w})^2}{(\vec{u}, \vec{v}, \vec{w})^3} = \frac{1}{(\vec{u}, \vec{v}, \vec{w})}.$$

On the other hand we have:

$$\frac{\vec{v}' \times \vec{w}'}{(\vec{u}', \vec{v}', \vec{w}')} = (\vec{u}, \vec{v}, \vec{w})(\vec{v}' \times \vec{w}') = (\vec{u}, \vec{v}, \vec{w}) \frac{(\vec{w} \times \vec{u}) \times (\vec{u} \times \vec{v})}{(\vec{u}, \vec{v}, \vec{w})^2} = \frac{(\vec{w}, \vec{u}, \vec{v}) \vec{u} - (\vec{w}, \vec{u}, \vec{u}) \vec{v}}{(\vec{u}, \vec{v}, \vec{w})} = \vec{u}.$$

One can similarly show that

$$\frac{\vec{w}' \times \vec{u}'}{(\vec{u}', \vec{v}', \vec{w}')} = \vec{v} \text{ and } \frac{\vec{u}' \times \vec{v}'}{(\vec{u}', \vec{v}', \vec{w}')} = \vec{w}.$$

5. (2p) Find the value of the parameter  $\alpha$  for which the pencil of planes through the straight line  $AB$  has a common plane with the pencil of planes through the straight line  $CD$ , where  $A(1, 2\alpha, \alpha)$ ,  $B(3, 2, 1)$ ,  $C(-\alpha, 0, \alpha)$  and  $D(-1, 3, -3)$ .

*Solution.*

6. (2p) Find the value of the parameter  $\lambda$  for which the straight lines

$$(d_1) \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1}, \quad (d_2) \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. Find the coordinates of their intersection point in that case.

*Solution.*

7. (2p) Find the distance between the straight lines

$$(d_1) \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}, \quad (d_2) \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$$

as well as the equations of the common perpendicular.

*Solution.*

8. (2p) Find the distance between the straight lines  $M_1M_2$  and  $d$ , where  $M_1(-1, 0, 1)$ ,  $M_2(-2, 1, 0)$  and

$$(d) \begin{cases} x + y + z = 1 \\ 2x - y - 5z = 0. \end{cases}$$

as well as the equations of the common perpendicular.

*Solution.*

## 8 Week 8: Curves and surfaces

### 8.1 Regular curves

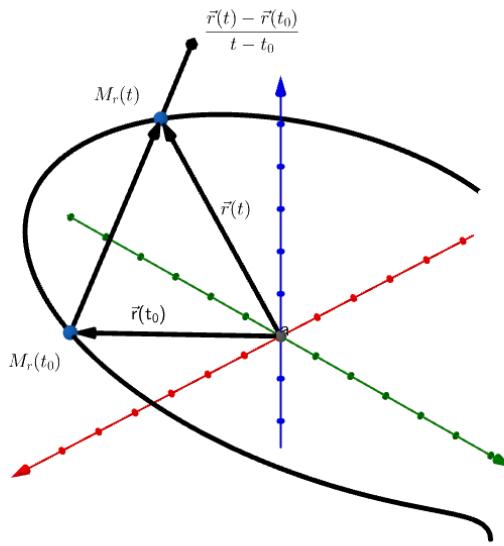
**Definition 8.1.** A subset  $C$  of  $\mathbb{R}^2$  or  $\mathbb{R}^3$  is said to be a *regular curve* if for every  $p \in C$  there exists a neighbourhood  $V$  of  $p$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  respectively and a *parametrized differentiable curve*  $r : I \rightarrow U \cap C$ , where  $I \subseteq \mathbb{R}$  is an open set, such that

1.  $r$  is smooth;
2.  $r : I \rightarrow U \cap C$  is a homeomorphism;
3.  $r$  is regular, i.e.  $\vec{r}'(t) \neq \vec{0}$ ,  $\forall t \in I$ .

The parametrized differentiable curve  $r : I \rightarrow V \cap C$  is called *local parametrization* or *local system of coordinates* at  $p$  and  $V \cap C$  is called *coordinate neighbourhood* at  $p$ . Recall that the tangent line of the local parametrization  $r : I \rightarrow U \cap C$  at  $r(t_0)$ , for some  $t_0 \in I$ , is defined as the limit position of the line  $M_r(t_0)M_r(t)$  as  $t \rightarrow t_0$ . This tangent line is denoted by  $(Tr)(t_0)$ . A director vector of the line  $M_r(t_0)M_r(t)$  is obviously

$$\frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0},$$

which shows that  $\vec{r}'(t_0)$  is a director vector of  $(Tr)(t_0)$  and the direction of  $(Tr)(t_0)$  is therefore  $(d\vec{r})_{t_0}(\mathbb{R})$ .



If  $r_1 : I_1 \rightarrow U_1 \cap C$  and  $r_2 : I_2 \rightarrow U_2 \cap C$  are two local parametrizations of  $C$  at  $p \in C$ , then  $r_1(t_1) = r_2(t_2) = p$  for some  $t_1 \in I_1$  and  $t_2 \in I_2$  and one can easily show that  $(d\vec{r}_1)_{t_1}(\mathbb{R}) = (d\vec{r}_2)_{t_2}(\mathbb{R})$ . This shows that  $r_1$  and  $r_2$  have the same tangent line at  $r_1(t_1) = r_2(t_2) = p$ .

**Proposition 8.1.** *The equation of the parametrized differentiable curve  $r : I \rightarrow \mathbb{R}^2$ ,  $r(t) = (x(t), y(t))$  at  $r(t_0)$ , for some regular point  $t_0 \in I$ , i.e.  $\vec{r}'(t_0) \neq \vec{0}$  is*

$$(Tr)(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)}. \quad (8.1)$$

*The equation of the normal line to  $r$  at  $r(t_0)$ , i.e. the line through  $M_r(t_0)$  which is perpendicular to  $(Tr)(t_0)$  is*

$$(Nr)(t_0) x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0. \quad (8.2)$$

**Proposition 8.2.** *The equation of the parametrized differentiable curve  $r : I \rightarrow \mathbb{R}^3$ ,  $r(t) = (x(t), y(t), z(t))$  at  $r(t_0)$ , for some regular point  $t_0 \in I$ , i.e.  $\vec{r}'(t_0) \neq \vec{0}$  is*

$$(Tr)(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}. \quad (8.3)$$

*The equation of the normal plane to  $r$  at  $r(t_0)$ , i.e. the plane through  $M_r(t_0)$  which is perpendicular to  $(Tr)(t_0)$  is*

$$(Nr)(t_0) x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)(z - z(t_0)) = 0. \quad (8.4)$$

**Remark 8.1.** 1. The requirement (3) of definition (8.1), is equivalent with  $(dr)_t \neq 0$ ,  $\forall t \in \mathbb{R}$ ;

2.  $V \cap C$  is the image of a regular one-to-one parametrized differentiable curve. On the other hand, there are regular one-to-one parametrized differentiable curves whose images are not parts of regular curves;
3. The role of requirement (2) in definition (8.1) is to prevent the self-intersections of the regular curves, which is not the case with the images of regular parametrized differentiable curves.
4. The requirement (3) combined with (2) ensure the existence of a unique tangent line at every point of a regular curve. The tangent line  $T_p(C)$  of  $C$  at  $p \in C$  is defined as the tangent line at  $p$  of a local parametrization  $r : I \rightarrow U \cap C$  of  $C$  at  $p$ . The tangent line  $T_p(C)$  is well-defined as the tangent at  $p$  of a local parametrization  $r : I \rightarrow U \cap C$  at  $p$  is independent of  $r$ .

**Definition 8.2.** If  $U \subseteq \mathbb{R}^2$  is an open set,  $f : U \rightarrow \mathbb{R}$  is a  $C^1$ -smooth function, then the value  $a \in \text{Im}(f)$  of  $f$  is said to be *regular* if  $(\nabla f)(x, y) \neq 0$ ,  $\forall (x, y) \in f^{-1}(a)$ , i.e.  $(df)_{(x,y)} \neq 0$ ,  $\forall (x, y) \in f^{-1}(a)$ .

**Theorem 8.3.** *(The preimage theorem) If  $U \subseteq \mathbb{R}^2$  is an open set,  $f : U \rightarrow \mathbb{R}$  is a  $C^1$ -smooth function and  $a \in \text{Im}f$  is a regular value of  $f$ , then the inverse image of  $a$  through  $f$ ,*

$$f^{-1}(a) = \{(x, y) \in U | f(x, y) = a\}$$

*is a planar regular curve called the regular curve of implicit cartesian equation  $f(x, y) = a$ .*

**Definition 8.3.** Let  $U \subset \mathbb{R}^2$  be an open set such that  $tx \in U$  for every  $t \in \mathbb{R}_+^*$  and every  $x \in U$ . The function  $f : U \rightarrow \mathbb{R}$  is said to be *homogeneous of order p* whenever  $f(tx) = t^p f(x)$ ,  $\forall t \in \mathbb{R}_+^*, x \in U$ .

For example a homogeneous polynomial function of degree  $n \in \mathbb{N}$  is a homogeneous function of order  $p$ .

**Example 8.1.** If  $f : U \rightarrow \mathbb{R}$  is a  $C^1$ -smooth homogeneous function of order  $p \in \mathbb{R}^*$  and  $c \in \text{Im } f \setminus \{0\}$ , then  $f^{-1}(c)$  is a regular curve.

Indeed, it is enough to show that  $c$  is a regular value of  $f$ . By differentiating the relation  $f(tx) = t^p f(x)$  with respect to  $t$  we obtain:

$$(df)_{tx}(x) = pt^{p-1}f(x), \quad \forall t \in \mathbb{R}_+^*, x \in U,$$

and the Euler's relation

$$(df)_x(x) = pf(x), \quad \forall x \in U. \quad (8.5)$$

follow for  $t = 1$ . But for  $x \in C(f)$  we have  $(df)_x = 0$  and thus  $(df)_x(x) = 0$ , namely  $f(x) = 0$ . We therefore showed that  $B(f) = f(C(f)) \subset \{0\}$ , or, equivalently,  $\mathbb{R}^* \subset \mathbb{R} \setminus B(f)$ , where  $C(f) \subseteq U$  stands for the closed set of critical points of  $f$ , i.e.  $C(f) := \{(x, y) \in U | (df)_{(x,y)} = 0\}$ . But since  $c \in \text{Im } f \setminus \{0\}$  we deduce that  $c$  is a regular value of  $f$  and  $f^{-1}(c)$  is a regular curve therefore.

**Proposition 8.4.** The equation of the tangent line  $T_{(x_0, y_0)}(C)$  of the planar regular curve  $C$  of implicit cartesian equation  $f(x, y) = a$  at the point  $p = (x_0, y_0) \in C$ , is

$$T_{(x_0, y_0)}(C) : f'_x(p)(x - x_0) + f'_y(p)(y - y_0) = 0,$$

and the equation of the normal line  $N_{(x_0, y_0)}(C)$  of  $C$  at  $p$  is

$$N_{(x_0, y_0)}(C) : \frac{x - x_0}{f'_x(p)} = \frac{y - y_0}{f'_y(p)}.$$

**Example 8.2.** The tangent line of the general conic

$$C : a_{00} + 2a_{10}x + 2a_{20}y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 0$$

at some of its regular point  $(x_0, y_0) \in C$  is

$$a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 0 \quad (8.6)$$

and can be obtained by polarizing the conic's equation, i.e. by replacing:

1.  $x^2$  with  $x_0x$
2.  $y^2$  with  $y_0y$
3.  $2x$  with  $x + x_0$
4.  $2y$  with  $y + y_0$
5.  $2xy$  with  $x_0y + xy_0$ .

Indeed,  $C = f^{-1}(0)$ , where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a second degree polynomial function given by  $f(x, y) = a_{00} + 2a_{10}x + 2a_{20}y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2$ . Since

$$f_x = 2a_{10} + 2a_{11}x + 2a_{12}y \text{ and } f_y = 2a_{20} + 2a_{12}x + 2a_{22}y,$$

it follows that

$$\begin{aligned} T_{(x_0, y_0)}(C) &: (2a_{10} + 2a_{11}x + 2a_{12}y)(x - x_0) + (2a_{20} + 2a_{12}x + 2a_{22}y)(y - y_0) = 0 \\ &\iff a_{10}x + a_{11}x_0x + a_{12}y_0x + a_{20}y + a_{12}x_0y + a_{22}y_0y = a_{10}x_0 + a_{11}x_0^2 + a_{12}y_0x_0 + a_{20}y_0 + a_{12}x_0y_0 + a_{22}y_0^2 \\ &\iff a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 2a_{10}x_0 + 2a_{20}y_0 + a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 \\ &\iff a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 0. \end{aligned}$$

## 8.2 Parametrized differentiable surfaces

**Definition 8.4.** Let  $U \subseteq \mathbb{R}^2$  be an open set. A smooth map  $r : U \rightarrow \mathbb{R}^3$  is said to be a *parametrized differentiable surface*. The set  $r(U)$  is called the *trace*, the *support*, or the *image* of  $r$ . If the differential  $(dr)_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is injective for  $q \in U$ , then the parametrized differentiable surface  $r$  is said to be *regular* at  $q$ . If the differential  $(dr)_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is injective for all  $q \in U$ , then the parametrized differentiable surface  $r$  is said to be *regular*.

**Remark 8.2.** Let  $U \subseteq \mathbb{R}^2$  be an open set and  $r : U \rightarrow \mathbb{R}^3$ ,  $r(u, v) = (x(u, v), y(u, v), z(u, v))$  be a parametrized differentiable surface. Then  $r$  is regular at  $q \in U$  if and only if

$$\vec{r}_u(q) \times \vec{r}_v(q) \neq \vec{0}.$$

Indeed,

$$\begin{aligned} r \text{ is regular at } q \in U &\iff (dr)_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ is one-to-one} \\ &\iff (dr)_q(e_1), (dr)_q(e_2) \text{ are linearly independent } (e_1 = (1, 0), e_2 = (0, 1)) \\ &\iff \vec{r}_u(q) = (d\vec{r})_q(e_1), \vec{r}_v(q) = (d\vec{r})_q(e_2) \text{ are linearly independent} \\ &\iff \vec{r}_u(q) \times \vec{r}_v(q) \neq \vec{0}, \end{aligned}$$

where  $\vec{r} : U \rightarrow \mathcal{V}$ ,  $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ .

The image of a parametrized differentiable surface might have self-intersections.

### 8.2.1 The tangent plane and the normal line to a parametrized surface

**Definition 8.5.** Let  $r : U \rightarrow \mathbb{R}^3$ ,  $r(u, v) = (x(u, v), y(u, v), z(u, v))$  be a regular parametrized differentiable surface and  $q = (u_0, v_0) \in U$ . The plane  $(Tr)(q)$  through  $M_r(u_0, v_0)$ , whose direction is  $(d\vec{r})_q(\mathbb{R}^2)$ , is called the *tangent plane* to  $r$  at  $M_r(q)$  corresponding to the pair  $(u_0, v_0)$  of the parameters. The perpendicular line  $(Nr)(q)$  on  $(Tr)(q)$  at  $M_r(q)$  is called the *normal line* to  $r$  at  $M_r(q)$  corresponding to the pair  $(u_0, v_0)$  of the parameters.

**Remark 8.3.** If  $r : U \rightarrow \mathbb{R}^3$ ,  $r(u, v) = (x(u, v), y(u, v), z(u, v))$  is a regular parametrized differentiable surface and  $q = (u_0, v_0) \in U$ , then the vectors  $\vec{r}_u(q) = (d\vec{r})_q(1, 0)$ ,  $\vec{r}_v(q) = (d\vec{r})_q(0, 1)$  form a basis of the two dimensional vector subspace  $(d\vec{r})(\mathbb{R}^2)$  of  $\mathcal{V}$  și and  $\vec{v}(q) = \vec{r}_u(q) \times \vec{r}_v(q)$  is therefore a director vector of the normal line to  $r$  at  $M_r(q)$  corresponding to the pair  $(u_0, v_0)$  of the parameters.

$$\begin{aligned}\vec{v}(q) &= \vec{r}_u(q) \times \vec{r}_v(q) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u(q) & y_u(q) & z_u(q) \\ x_v(q) & y_v(q) & z_v(q) \end{vmatrix} \\ &= \frac{\partial(y, z)}{\partial(u, v)}(q) \vec{i} + \frac{\partial(z, x)}{\partial(u, v)}(q) \vec{j} + \frac{\partial(x, y)}{\partial(u, v)}(q) \vec{k},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial(x, y)}{\partial(u, v)}(q) &= \begin{vmatrix} x_u(q) & y_u(q) \\ x_v(q) & y_v(q) \end{vmatrix}, \\ \frac{\partial(z, x)}{\partial(u, v)}(q) &= \begin{vmatrix} z_u(q) & x_u(q) \\ z_v(q) & x_v(q) \end{vmatrix}, \\ \frac{\partial(y, z)}{\partial(u, v)}(q) &= \begin{vmatrix} y_u(q) & z_u(q) \\ y_v(q) & z_v(q) \end{vmatrix}.\end{aligned}$$

**Proposition 8.5.** If  $r : U \rightarrow \mathbb{R}^3$   $r(u, v) = (x(u, v), y(u, v), z(u, v))$  regular parametrized differentiable surface and  $q = (u_0, v_0) \in U$ , then the equation of the tangent plane to  $r$  at  $M_r(q)$ , corresponding to the pair  $(u_0, v_0)$  of the parameters, is

$$\begin{vmatrix} x - x(q) & y - y(q) & z - z(q) \\ x_u(q) & y_u(q) & z_u(q) \\ x_v(q) & y_v(q) & z_v(q) \end{vmatrix} = 0,$$

i.e.

$$\frac{\partial(y, z)}{\partial(u, v)}(q)(x - x(q)) + \frac{\partial(z, x)}{\partial(u, v)}(q)(y - y(q)) + \frac{\partial(x, y)}{\partial(u, v)}(q)(z - z(q)) = 0 \quad (8.7)$$

Also, the equation of the normal line to  $r$  at  $M_r(q)$ , corresponding to the pair  $(u_0, v_0)$  of the parameters, is:

$$\frac{x - x(q)}{\frac{\partial(y, z)}{\partial(u, v)}(q)} = \frac{y - y(q)}{\frac{\partial(z, x)}{\partial(u, v)}(q)} = \frac{z - z(q)}{\frac{\partial(x, y)}{\partial(u, v)}(q)} \quad (8.8)$$

### 8.3 Regular surfaces

**Definition 8.6.** A subset  $S \subseteq \mathbb{R}^3$  is called *regular surface* if, for every point  $p \in S$ , there exists a neighbourhood  $V$  of  $p$ , in  $\mathbb{R}^3$ , and a mapping  $r : U \rightarrow V \cap S$ ,  $r(u, v) = (x(u, v), y(u, v), z(u, v))$ , where  $U \subseteq \mathbb{R}^2$  is an open set, with the following properties:

1.  $r$  is smooth, i.e. its coordinate functions  $x, y, z$  have arbitrary high continuous partial derivatives;
2.  $r$  is a homeomorphism;
3. For every  $q \in U$ , the differential  $(dr)_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is injective.

The function  $r : U \rightarrow V \cap S$  is called *local parametrization* at  $p$  or *local chart* at  $p$  or *local coordinate system* at  $p$ . The neighbourhood  $V \cap S$  of  $p$  in  $S$  is called *coordinate neighbourhood*. The equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in U,$$

are called the *parametric equations* of the coordinate neighbourhood  $V \cap S$ . The equation

$$\vec{r} = \vec{r}(u, v) \text{ where } \vec{r}(u, v) = x(u, v) \vec{i} + y(u, v) \vec{j} + z(u, v) \vec{k}$$

is called the *vector equation* of the coordinate neighbourhood  $V \cap S$ .

**Remark 8.4.** 1. Every open subset  $O$  of a regular surface  $S \subseteq \mathbb{R}^3$  is a regular surface. Indeed every local parametrization  $r : U \rightarrow S \cap V$  of  $S$  at some point  $p \in O$  produces a local parametrization

$$U \cap r^{-1}(O) \rightarrow S \cap C \cap V, q \mapsto r(q)$$

of  $O$  at  $p$ .

2. Every regular surface can be covered by the traces of some families of local charts. Such a family of local charts is called an *atlas* of the surface. If the regular surface is compact, then it obviously admits finite atlases. For example the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

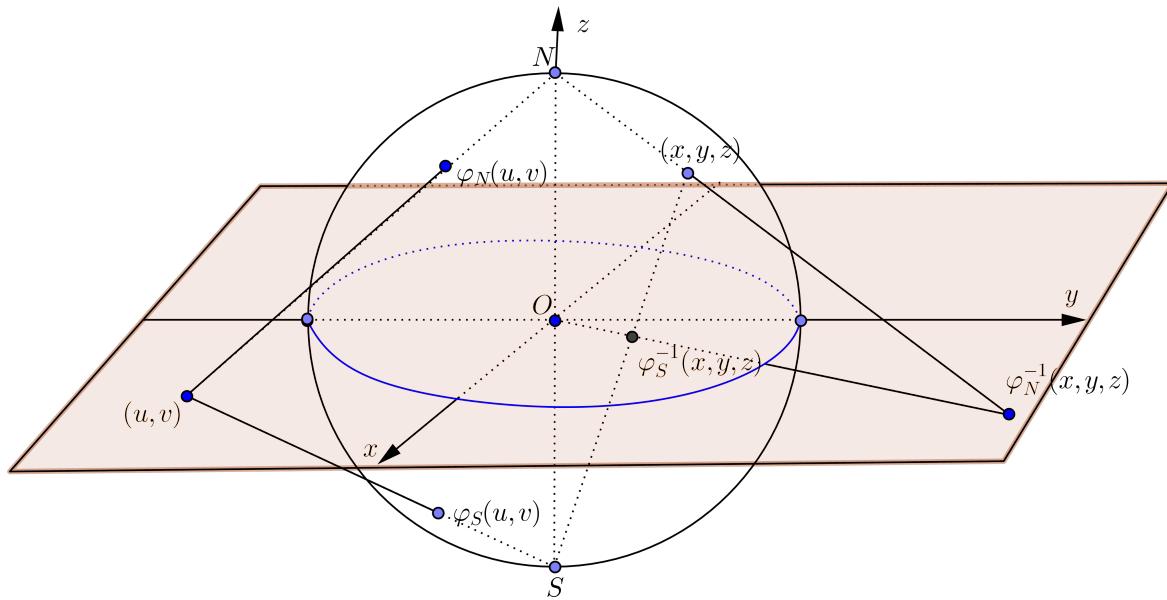
admits an atlas with two local charts  $\mathcal{A} = \{\varphi_S, \varphi_N\}$ , where

$$\begin{aligned} \varphi_S : \mathbb{R}^2 &\longrightarrow S^2 \setminus \{S\}, \varphi_S(u, v) = \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right) \\ \varphi_N : \mathbb{R}^2 &\longrightarrow S^2 \setminus \{N\}, \varphi_N(u, v) = \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2} \right) \end{aligned}$$

and  $S = (0, 0, -1)$ ,  $N = (0, 0, 1)$  are the south and north poles of  $S^2$ .

Note that the inverses of  $\varphi_S$  and  $\varphi_N$  are the stereographic projections

$$\begin{aligned} \varphi_S^{-1} : S^2 \setminus \{S\} &\longrightarrow \mathbb{R}^2, \varphi_S^{-1}(x, y, z) = \left( \frac{x}{1+z}, \frac{y}{1+z} \right) \\ \varphi_N^{-1} : S^2 \setminus \{N\} &\longrightarrow \mathbb{R}^2, \varphi_N^{-1}(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right). \end{aligned}$$

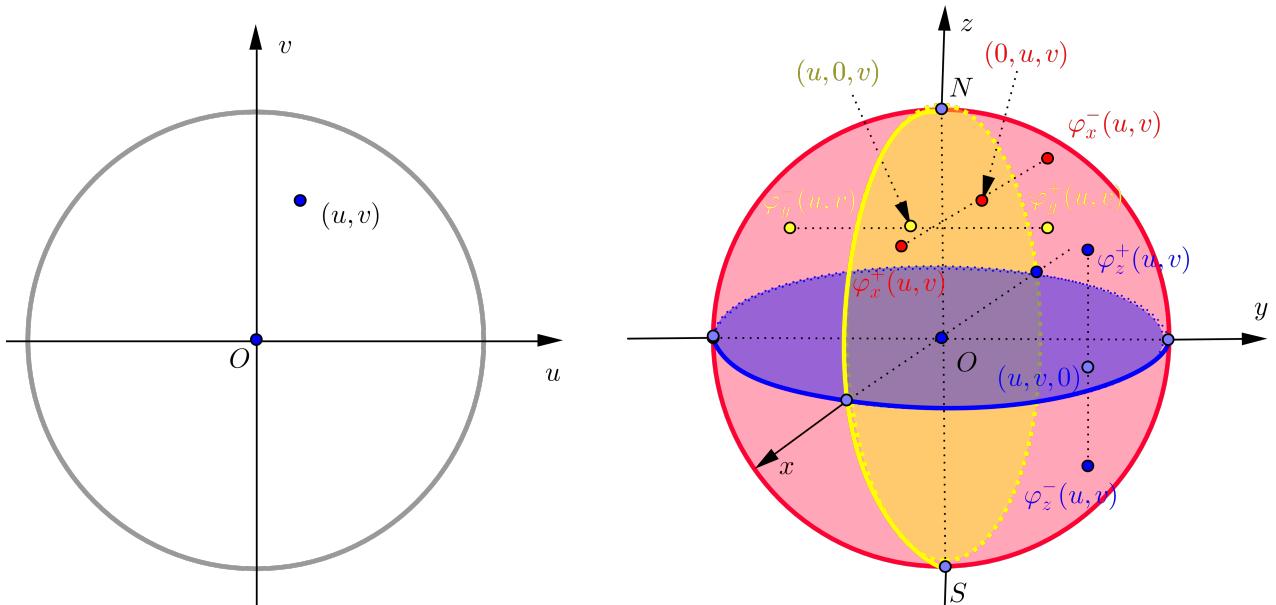


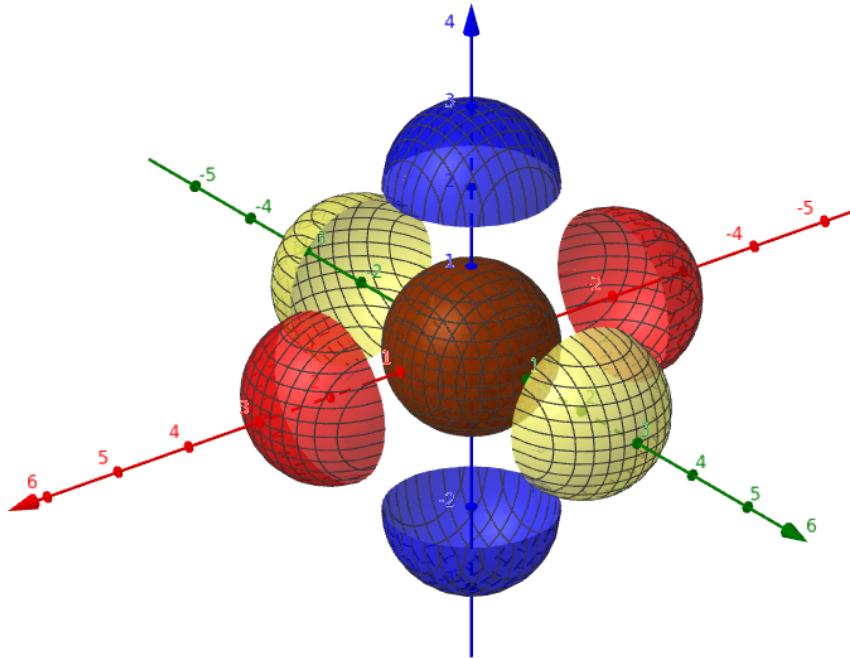
Another atlas of the sphere  $S^2$  has 6 local charts, namely

$$\mathcal{A}_1 = \{\varphi_x^\pm, \varphi_y^\pm, \varphi_z^\pm : B(0, 1) \longrightarrow S^2\},$$

where  $B(0, 1)$  is the unit ball of  $\mathbb{R}^2$  centered at the origin  $0 \in \mathbb{R}^2$  and

$$\begin{aligned}\varphi_x^\pm(u, v) &= (\pm\sqrt{1 - u^2 - v^2}, u, v), \\ \varphi_y^\pm(u, v) &= (u, \pm\sqrt{1 - u^2 - v^2}, v), \\ \varphi_z^\pm(u, v) &= (u, v, \pm\sqrt{1 - u^2 - v^2}).\end{aligned}$$





**Proposition 8.6.** If  $U \subseteq \mathbb{R}^2$  is an open set and  $f : U \rightarrow \mathbb{R}$  is a smooth function, then its graph  $G_f = \{(x, y, f(x, y)) \mid (x, y) \in U\}$  is a regular surface.

For example

1. The elliptic paraboloid  $P_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z$ ,  $p, q > 0$  is a regular surface, as  $P_e$  is the graph of the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x, y) = \frac{1}{2} \left( \frac{x^2}{p} + \frac{y^2}{q} \right)$ .
2. The hyperbolic paraboloid  $P_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z$ ,  $p, q > 0$  is a regular surface, as  $P_h$  is the graph of the function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $g(x, y) = \frac{1}{2} \left( \frac{x^2}{p} - \frac{y^2}{q} \right)$ .

**Theorem 8.7.** (The third preimage theorem). If  $U \subseteq \mathbb{R}^3$  is an open set,  $f : U \rightarrow \mathbb{R}$  is a smooth function and  $a \in \text{Im } f$  is a regular value of  $f$ , then

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}$$

is a regular surface in  $\mathbb{R}^3$  called the regular surface of implicit Cartesian equation  $f(x, y, z) = a$ .

**Proposition 8.8.** Let  $U \subset \mathbb{R}^3$  be an open set such that  $tx \in U$  for every  $t \in \mathbb{R}_+^*$  and every  $x \in U$ . A function  $f : U \rightarrow \mathbb{R}$  is said to be homogeneous of order  $p \in \mathbb{R}$  if  $f(tx) = t^p f(x)$ ,  $\forall t \in \mathbb{R}_+^*, x \in U$ . If  $f : U \rightarrow \mathbb{R}$  is a differentiable and homogeneous function of order  $p \in \mathbb{R}^*$  and  $c \in \text{Im } f \setminus \{0\}$ , then  $f^{-1}(c)$  is a regular surface.

*Proof.* Indeed, it is enough to show that  $c$  is a regular value of  $f$ . Differentiating with respect to  $t$ , the relation  $f(tx) = t^p f(x)$  we obtain

$$(df)_{tx}(x) = pt^{p-1}f(x), \forall t \in \mathbb{R}_+^*, \forall x \in U,$$

which shows, by taking  $t = 1$ , the Euler relation

$$(df)_x(x) = pf(x), \forall x \in U. \quad (8.9)$$

But for  $x \in C(f)$  we have  $(df)_x = 0$  and thus  $(df)_x(x) = 0$ , which shows that  $f(x) = 0$ . We therefore showed that  $B(f) = f(C(f)) \subset \{0\}$ , or, equivalently,  $\mathbb{R}^* \subset \mathbb{R} \setminus B(f)$ . But since  $c \in \text{Im } f \setminus \{0\}$  we deduce that  $c$  is a taken regular value of  $f$ , which shows that  $f^{-1}(c)$  is a regular surface.  $\square$

In particular,

1. the ellipsoid  $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ,
2. the hyperboloid of one sheet  $H_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ ,
3. the hyperboloid of two sheets  $H_2 : \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ .

are all regular surfaces. Let us finally observe that the cone  $C : x^2 + y^2 - z^2 = 0$  is not a regular surface.

## 8.4 The tangent vector space

Let  $S \subseteq \mathbb{R}^3$  be a regular surface and  $p \in S$ . A *tangent vector* to  $S$  at  $p$  is the tangent vector  $\vec{\alpha}'(0)$  of a parametrized differentiable curve  $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$  with  $\alpha(0) = p$

**Proposition 8.9.** *Let  $U \subseteq \mathbb{R}^2$  be an open set, let  $q \in U$  and let  $r : U \rightarrow S$  be a local parametrization of  $S$ . The 2-dimensional subspace  $(d\vec{r})_q(\mathbb{R}^2) \subseteq \mathcal{V}$  coincides with the set of all tangent vectors to  $S$  at  $r(q)$ .*

**Definition 8.7.** The plane through a point  $p$  of a regular surface  $S$ , whose direction is the tangent space to  $S$  at  $p$ ,  $\vec{T}_p(S)$ , is called the *tangent plane* to  $S$  at  $p$  and is denoted by  $T_p(S)$ . The perpendicular line on the tangent plane of the surface  $S$  at  $p$  is called the *normal line* to the surface  $S$  at  $p$ .

**Proposition 8.10.** *If  $V \subseteq \mathbb{R}^3$  is an open set,  $f : V \rightarrow \mathbb{R}$  is a smooth function,  $a \in \text{Im } f$  is a regular value of  $f$  and  $p \in f^{-1}(a)$ , then the equation of the tangent plane to the regular surface  $S = f^{-1}(a)$ , of implicit equation  $f(x, y, z) = a$ , at some point  $p \in S$  is:*

$$f_x(p)(x - x_0) + f_y(p)(y - y_0) + f_z(p)(z - z_0) = 0. \quad (8.10)$$

and the equation of the normal line to  $S$  at  $p$  is:

$$\frac{x - x_0}{f_x(p)} = \frac{y - y_0}{f_y(p)} = \frac{z - z_0}{f_z(p)} \quad (8.11)$$

For example the tangent plane of the quadric

$$(Q) a_{00} + 2a_{10}x + 2a_{20}y + 2a_{30}z + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + a_{11}x^2 + a_{22}y^2 + a_{33}z^2 = 0$$

at some of its point  $A_0(x_0, y_0, z_0) \in Q$  is

$$T_{A_0}(Q) a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{30}(z + z_0) + a_{12}(x_0y + xy_0) + a_{13}(z_0x + zx_0) + 2a_{23}(y_0z + yz_0) + a_{11}x_0x + a_{22}y_0y + a_{33}z_0z = 0.$$

and can be obtained by polarizing the quadric's equation, i.e. by replacing

1.  $x^2$  with  $x_0x$
2.  $y^2$  with  $y_0y$
3.  $z^2$  with  $z_0z$
4.  $2x$  with  $x + x_0$
5.  $2y$  with  $y + y_0$
6.  $2z$  with  $z + z_0$
7.  $2xy$  with  $x_0y + xy_0$
8.  $2yz$  with  $y_0z + yz_0$
9.  $2zx$  with  $z_0x + zx_0$ .

## 8.5 Problems

1. (2p) Show that the angle between the tangent of the circular helix

$$\begin{cases} x = a \cos t \\ y = a \sin t \\ z = bt \end{cases}$$

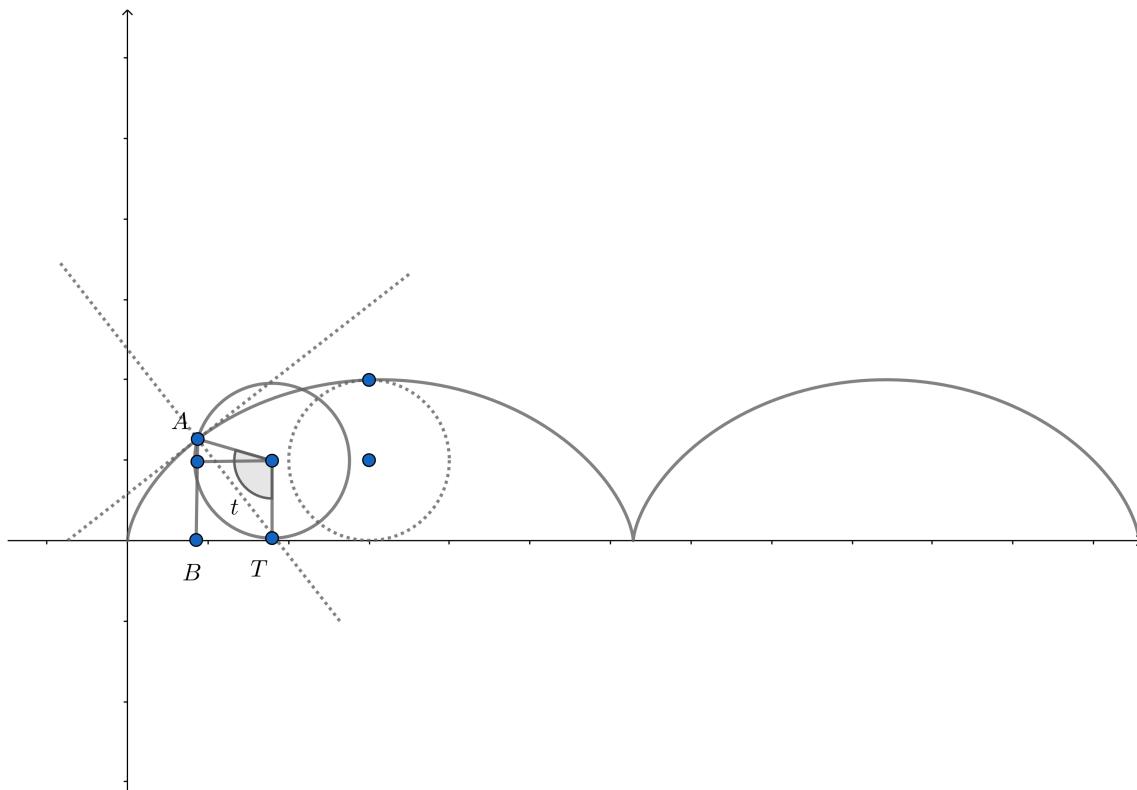
and the  $z$ -axis is constant.

*Solution.*

2. (3p) A *cycloid* is the curve traced by a chosen point on the circumference of a circle which rolls along a straight line without slipping. Show that the parametric equations of the are:

$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t) \end{cases}, t \in \mathbb{R}.$$

*Solution.*



3. **(2p)** Show that the normal line to the cycloid at a certain point passes through the tangency point between the generating circle and the line along which the generating circle rolls on.

*Solution.*

4. An *epicycloid* is a plane curve traced by a chosen point on the circumference of a circle which rolls without slipping around a fixed circle. Find the equations of the epicycloid.

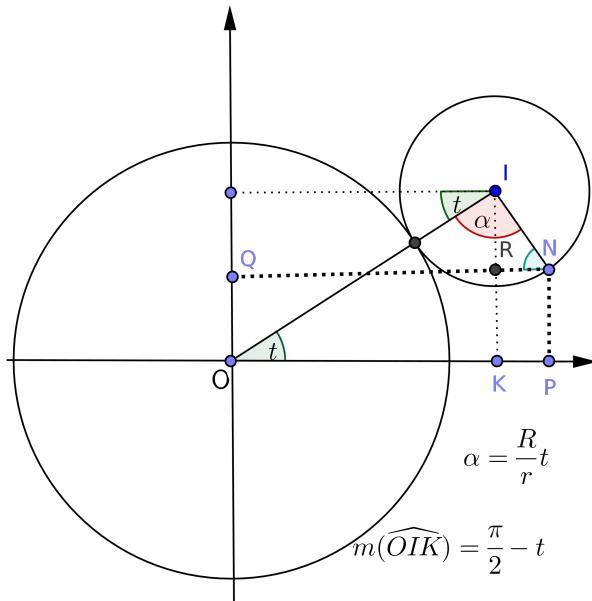
*Solution.* The equations of the epicycloid are

$$\begin{cases} x = (R + r) \cos t - r \cos \left( \frac{R+r}{r} t \right) \\ y = (R + r) \sin t - r \sin \left( \frac{R+r}{r} t \right) \end{cases}, t \in \mathbb{R},$$

or

$$\begin{cases} x = r(k+1) \cos t - r \cos((k+1)t) \\ y = r(k+1) \sin t - r \sin((k+1)t) \end{cases}, t \in \mathbb{R},$$

where  $k = \frac{R}{r}$ . If  $k$  is an integer, then the epicycloid is a closed curve.



$$m(\widehat{NIR}) = \alpha - m(\widehat{OIK}) = -\frac{\pi}{2} + \left( \frac{R}{r} + 1 \right) t$$

$$m(\widehat{INR}) = \frac{\pi}{2} - m(\widehat{NIR}) = \pi - \frac{R+r}{r} t$$

$$IR = r \sin \frac{R+r}{r} t, RN = -r \cos \frac{R+r}{2} t$$

5. A *hypocycloid* is a plane curve traced by a chosen point on a small circle that rolls without slipping within a larger circle. Find the equations of the hypocycloid.

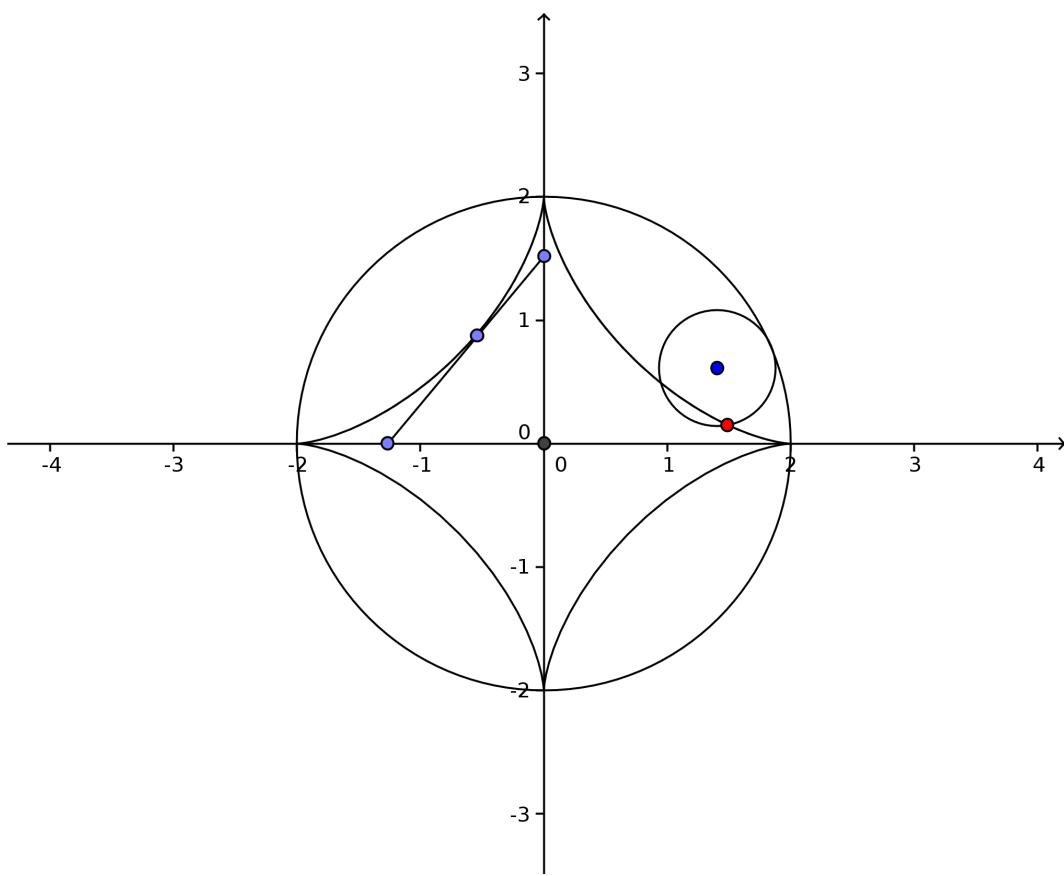
*Answer:* The equations of the hypocycloid are:

$$\begin{cases} x = (R - r) \cos t + r \cos \left( \frac{R-r}{r} t \right) \\ y = (R - r) \sin t - r \sin \left( \frac{R-r}{r} t \right) \end{cases}, t \in \mathbb{R},$$

or

$$\begin{cases} x = r(k-1) \cos t + r \cos((k-1)t) \\ y = r(k-1) \sin t - r \sin((k-1)t) \end{cases}, t \in \mathbb{R},$$

where  $k = \frac{R}{r}$ . If  $k$  is an integer, then the hypocycloid is a closed curve. In particular, for  $k = 4$  the hypocycloid is called *astroid*.

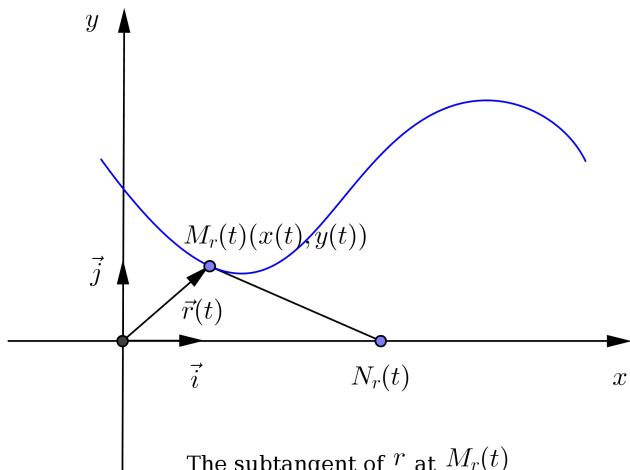


6. The *subtangent* of a planar parametrized differentiable curve is the segment which unify the tangency point between the tangent and the curve with the intersection point between the tangent and the  $x$ -axis. Show that the length of the subtangent of the planar parametrized differentiable curve

$$r : (0, \pi) \rightarrow \mathbb{R}^2, r(t) = a(\ln \tan(t/2) + \cos t, \sin t),$$

called the *tractrix* is constant and equal to  $a$ .

*Solution.*



The parametric equations of the tractrix are

$$\begin{cases} x = a \log \tan(t/2) + a \cos t \\ y = a \sin t \end{cases}, t \in (0, \pi)$$

and its vector equation is

$$\vec{r}(t) = (a \ln \tan(t/2) + a \cos t) \vec{i} + (a \sin t) \vec{j}.$$

and its tangent vector

$$\begin{aligned} \vec{r}'(t) &= \left( a \frac{1}{\tan(t/2)} \frac{1}{\cos^2(t/2)} \frac{1}{2} - a \sin t \right) \vec{i} + (a \cos t) \vec{j} \\ &= \left( \frac{a}{\sin t} - a \sin t \right) \vec{i} + (a \cos t) \vec{j} \\ &= \frac{a \cos^2 t}{\sin t} \vec{i} + (a \cos t) \vec{j} = a \cos t (\cot t \vec{i} + \vec{j}). \end{aligned}$$

Thus, the equation of the tangent line to the tractrix at the regular points  $M_r(t)$ , i.e.  $t \in (0, \pi) \setminus \{0\}$  is

$$(T_r)(t) : \frac{X - x(t)}{x'(t)} = \frac{Y - y(t)}{y'(t)} \iff \frac{X - a \log \tan(t/2) - a \cos t}{a \cos t \cot t} = \frac{Y - a \sin t}{a \cos t}. \quad (8.12)$$

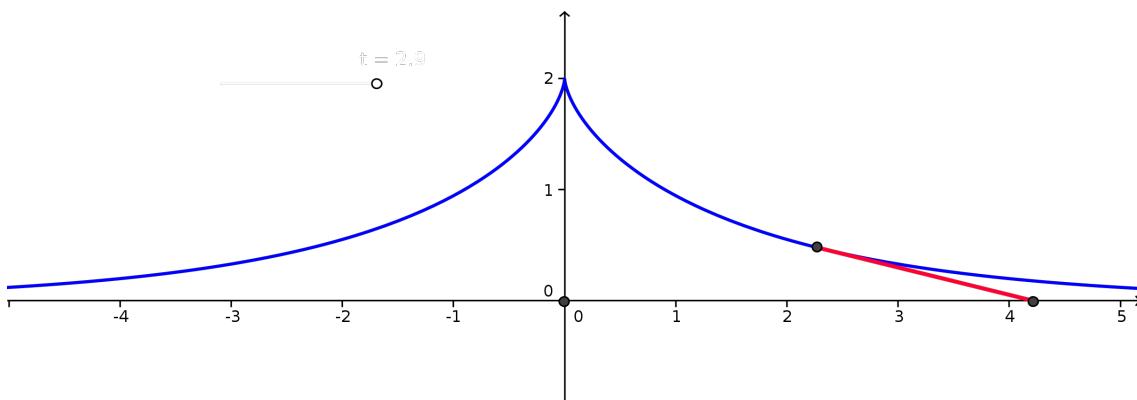
The coordinates of the intersection point  $N_r(t)$  of the tangent  $T_r(t)$  to the tractrix at  $M_r(t)$  with the  $x$ -axis can be obtained by taking  $Y = 0$  in (8.12), which implies  $X = a \log \tan(t/2)$ , i.e.  $N_r(t)(a \log \tan(t/2), 0)$ . The distance between

$$M_r(t)(a \log \tan(t/2) + a \cos t, a \sin t) \text{ and } N_r(t)(a \log \tan(t/2), 0)$$

is

$$\sqrt{(a \log \tan(t/2) + a \cos t - a \cos t)^2 + (a \sin t - 0)^2} = \sqrt{a^2} = |a| = a.$$

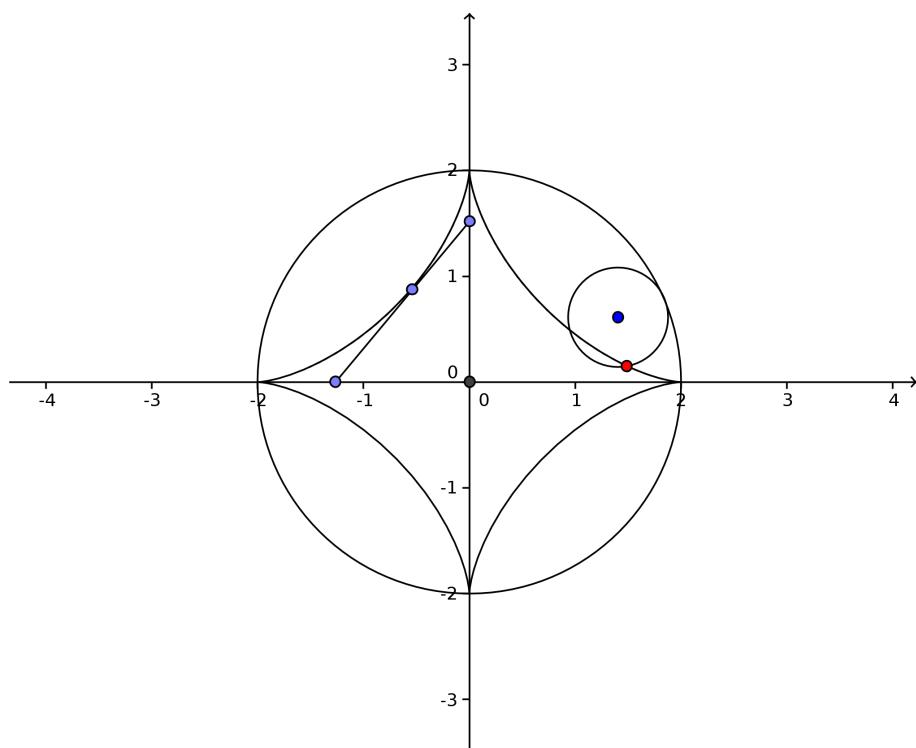
Note that  $t = \pi/2$  is the only singular point of  $\vec{r}$ . Since  $\vec{r}''(\pi/2) = a \vec{j}$ , it follows that  $t = \pi/2$  is a singular point of order two for  $\vec{r}$ , i.e.  $\vec{r}''(\pi/2)$  is a director vector of the tangent line of  $r$  at  $t = \pi/2$ . In other words the  $y$ -axis is the tangent line to  $r$  at  $t = \pi/2$ . Note that  $M_r(\pi/2)(0, a)$  and  $N_r(\pi/2)$  is the origin  $O(0, 0)$ . Thus the distance between  $M_r(\pi/2)(0, a)$  and  $N_r(\pi/2)$  is  $a$ .



7. (2p) Show that the tangents of the astroid

$$\begin{cases} x = r \cos^3 t \\ y = r \sin^3 t \end{cases}$$

determines on the coordinate axes segments of constant length.



*Solution.*

8. Write the equations of the tangent line and the normal plane for the following curves, whenever these associated objects are well-determined:

(a) (2p)

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases} \quad \text{at the point corresponding to the value } t = 0 \text{ of the parameter}$$

(b) (2p)

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases} \quad \text{at the point corresponding to the value } t = \frac{\pi}{4} \text{ of the parameter}$$

*Solution.*

9. **(2p)** Write the equations of the tangent planes of the hyperboloid of one sheet  $x^2 + y^2 - z^2 = 1$  at the points of the form  $(x_0, y_0, 0)$  and show that these are parallel to the  $z$ -axis.

*Solution.*

10. **(2p)** Show that the trace of the parametrized differentiable curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$ ,  $\alpha(t) = (e^t \cos t, e^t \sin t, 2t)$  is contained in the regular surface of equation  $z = \ln(x^2 + y^2)$  and write the equation of the tangent plane of the surface at the points  $\alpha(t)$ ,  $t \in \mathbb{R}$ .

*Solution.*

11. **(3p)** Show that the tangent planes of the surface of equation  $z = xf\left(\frac{y}{x}\right)$ , where  $f$  is a differentiable function, are passing through the origin.

*Solution.*

12. **(3p)** Show that the set  $S = \{(x, y, z) \in \mathbf{R}^3 \mid xyz = a^3\}$ ,  $a \neq 0$  is a regular surface and the its tangent plane at an arbitrary point  $p \in S$  determines on the coordinate axes three points which form, together with the origin a tetrahedron of constant volume (independent of  $p$ ).

*Solution.*

## 9 Week 9:Conics

### 9.1 The Ellipse

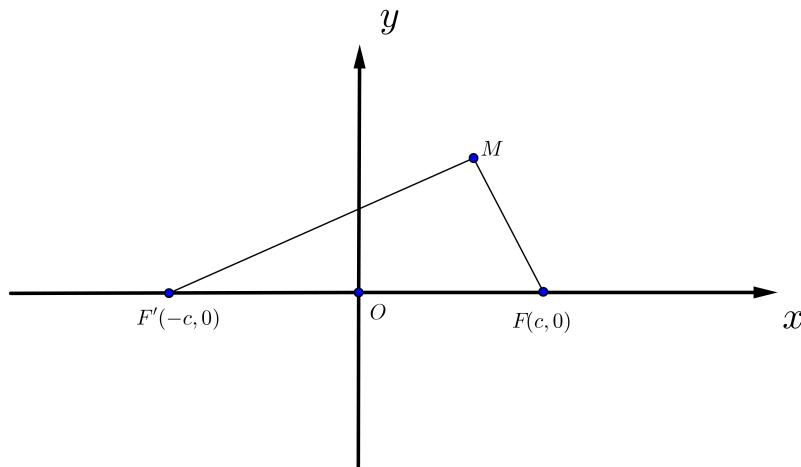
**Definition 9.1.** An *ellipse* is the locus of points in a plane, the sum of whose distances from two fixed points, say  $F$  and  $F'$ , called *foci* is constant.

The distance between the two fixed points is called the *focal distance*

Let  $F$  and  $F'$  be the two foci of an ellipse and let  $|FF'| = 2c$  be the focal distance. Suppose that the constant in the definition of the ellipse is  $2a$ . If  $M$  is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment  $[F'F]$ , so that  $F(c, 0)$  and  $F'(-c, 0)$ .



**Remark 9.1.** In  $\Delta MFF'$  the following inequality  $|MF| + |MF'| > |FF'|$  holds. Hence  $2a > 2c$ . Thus, the constants  $a$  and  $c$  must verify  $a > c$ .

Thus, for the generic point  $M(x, y)$  of the ellipse we have succesively:

$$\begin{aligned} |MF| + |MF'| = 2a &\Leftrightarrow \sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a \\ \sqrt{(x - c)^2 + y^2} &= 2a - \sqrt{(x + c)^2 + y^2} \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2 \\ a\sqrt{(x + c)^2 + y^2} &= cx + a^2 \\ a^2(x^2 + 2xc + c^2) + a^2y^2 &= c^2x^2 + 2a^2cx + a^2 \\ (a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) &= 0. \end{aligned}$$

Denote  $a^2 - c^2$  by  $b^2$ , as ( $a > c$ ). Thus  $b^2x^2 + a^2y^2 - a^2b^2 = 0$ , i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{9.1}$$

**Remark 9.2.** The ellipse

$$(E) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is a regular curve and the equation of its tangent line  $T_{P_0}(E)$  at some point  $P_0(x_0, y_0) \in E$  is

$$T_{P_0}(E) \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1. \tag{9.2}$$

**Remark 9.3.** The equation (9.1) is equivalent to

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}; \quad x = \pm \frac{a}{b} \sqrt{b^2 - y^2},$$

which means that the ellipse is symmetric with respect to both the  $x$  and the  $y$  axes. In fact, the line  $FF'$ , determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment  $[FF']$  are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of  $[FF']$ , is the center of symmetry of the ellipse, or, simply, its *center*.

**Remark 9.4.** In order to sketch the graph of the ellipse, observe that it is enough to represent the function

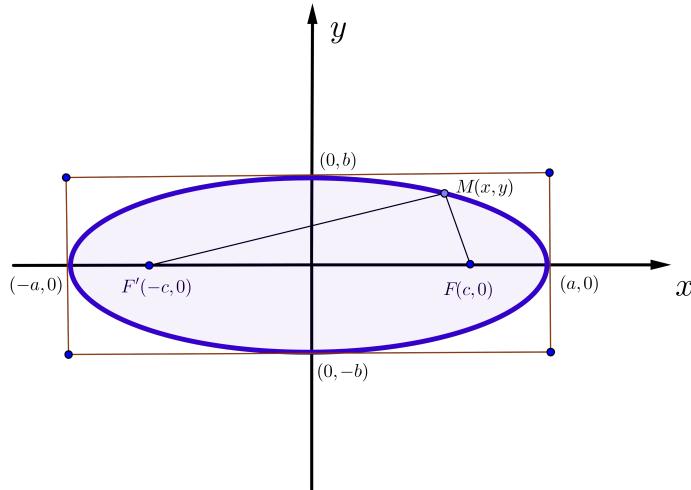
$$f : [-a, a] \rightarrow \mathbb{R}, \quad f(x) = \frac{b}{a} \sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to the  $x$ -axis.

One has

$$f'(x) = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}}, \quad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

$x$	$-a$	$0$	$a$
$f'(x)$	+ + + 0	— — —	
$f(x)$	0 ↗ b ↘ 0		
$f''(x)$	— — — — — —		



## 9.2 The Hyperbola

**Definition 9.2.** The *hyperbola* is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say  $F$  and  $F'$  is constant.

The two fixed points are called the *foci* of the hyperbola, and the distance  $|FF'| = 2c$  between the foci is the *focal distance*.

Suppose that the constant in the definition is  $2a$ . If  $M(x, y)$  is an arbitrary point of the hyperbola, then

$$||MF| - |MF'||| = 2a.$$

Choose a Cartesian system of coordinates, having the origin at the midpoint of the segment  $[FF']$  and such that  $F(c, 0), F'(-c, 0)$ .

**Remark 9.5.** In the triangle  $\Delta MFF'$ ,  $||MF| - |MF'|| < |FF'|$ , so that  $a < c$ .

Let us determine the equation of a hyperbola. By using the definition we get  $|MF| - |MF'| = \pm 2a$ , namely

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

We therefore have successively

$$\begin{aligned} x^2 - 2cx + c^2 + y^2 &= 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ cx + a^2 &= \pm a\sqrt{(x+c)^2 + y^2} \\ c^2x^2 + 2a^2cx + a^4 &= a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \\ (c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) &= 0. \end{aligned}$$

By using the notation  $c^2 - a^2 = b^2$  ( $c > a$ ) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \quad (9.3)$$

**Remark 9.6.** The hyperbola

$$(H) \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is a regular curve and the equation of its tangent line  $T_{P_0}(H)$  at some point  $P_0(x_0, y_0) \in H$  is

$$T_{P_0}(H) \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1. \quad (9.4)$$

The equation (9.3) is equivalent to

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}; \quad x = \pm \frac{a}{b} \sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

**Remark 9.7.** To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \rightarrow \mathbb{R}, \quad f(x) = \frac{b}{a} \sqrt{x^2 - a^2},$$

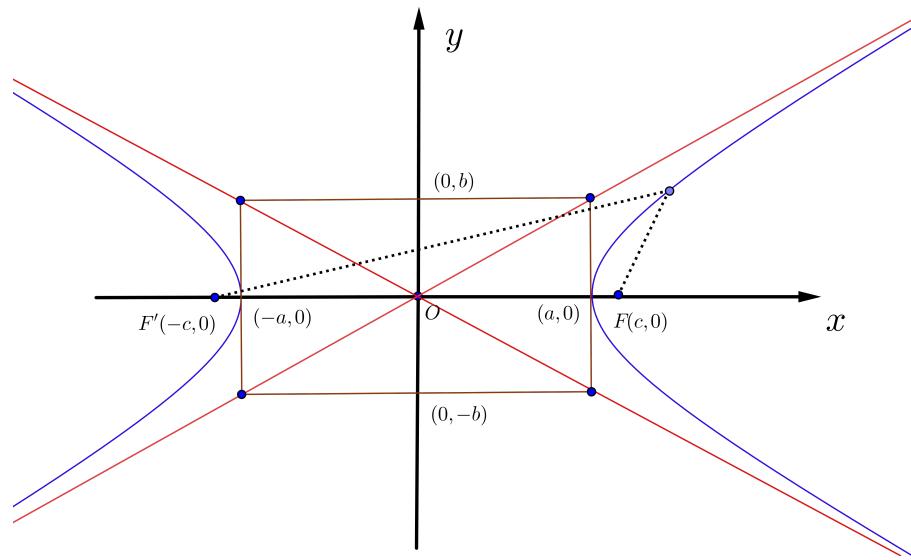
by taking into account that the hyperbola is symmetric with respect to the  $x$ -axis.

Since  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \frac{b}{a}$  and  $\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = -\frac{b}{a}$ , it follows that  $y = \frac{b}{a}x$  and  $y = -\frac{b}{a}x$  are asymptotes of  $f$ .

One has, also

$$f'(x) = \frac{b}{a} \frac{x}{\sqrt{x^2 - a^2}}, \quad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

$x$	$-\infty$	$-a$	$a$	$\infty$
$f'(x)$	- - - -   / / / /   + + + +			
$f(x)$	$\infty$ ↘ 0   / / /   0 ↗ $\infty$			
$f''(x)$	- - - -   / / / /   - - - -			



### 9.3 The Parabola

**Definition 9.3.** The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line  $d$  is equal to its distance to a fixed point  $F$ .

The line  $d$  is the *director line* and the point  $F$  is the *focus*. The distance between the focus and the director line is denoted by  $p$  and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates  $xOy$ , in which  $F\left(\frac{p}{2}, 0\right)$  and  $d : x = -\frac{p}{2}$ . If  $M(x, y)$  is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where  $N$  is the orthogonal projection of  $M$  on  $d$ .

Thus, the coordinates of a point of the parabola verify

$$\begin{aligned} \sqrt{\left(x + \frac{p}{2}\right)^2 + 0} &= \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} \\ \left(x + \frac{p}{2}\right)^2 &= \left(x - \frac{p}{2}\right)^2 = y^2 \\ x^2 + px + \frac{p^2}{4} &= x^2 - px + \frac{p^2}{4} + y^2, \end{aligned}$$

and the equation of the parabola is

$$y^2 = 2px. \quad (9.5)$$

**Remark 9.8.** The parabola

$$(P) y^2 = 2px$$

is a regular curve and the equation of its tangent line  $T_{P_0}(P)$  at some point  $Q_0(x_0, y_0) \in P$  is

$$T_{Q_0}(P) y_0 y = p(x + x_0). \quad (9.6)$$

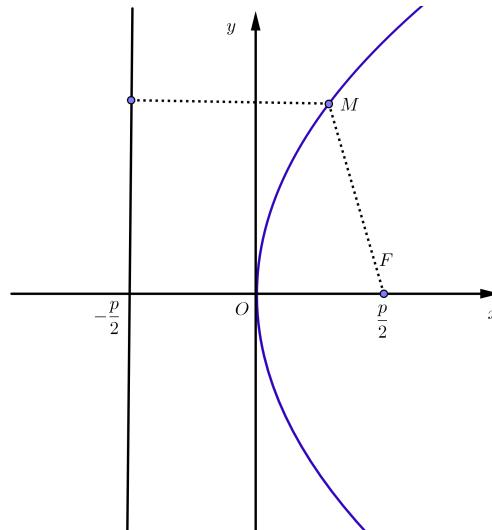
**Remark 9.9.** The equation (9.5) is equivalent to  $y = \pm\sqrt{2px}$ , so that the parabola is symmetric with respect to the  $x$ -axis.

Representing the graph of the function  $f : [0, \infty) \rightarrow [0, \infty)$  and using the symmetry of the curve with respect to the  $x$ -axis, one obtains the graph of the parabola.

One has

$$f'(x) = \frac{p}{\sqrt{2px_0}}; \quad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

$x$	0	$\infty$
$f'(x)$	+	+
$f(x)$	0	$\nearrow$
$f''(x)$	-	-



## 9.4 Problems

1. Find the equations of the tangent lines to the ellipse  $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$  having a given angular coefficient  $m \in \mathbb{R}$ . (see [1, p. 110]).

*Solution.* We are looking for the lines  $d : y = mx + n$ , which are tangent to the ellipse, i.e. each of them has one single common point with the ellipse. Their intersection is given by the solutions of the system of equations

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases},$$

or, by replacing  $y$  in the equation of the ellipse,

$$(a^2m^2 + b^2)x^2 + 2a^2mnx + a^2(n^2 - b^2) = 0.$$

The discriminant  $\Delta$  of the last equation is given by

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 + b^2)(n^2 - b^2)]$$

and the line  $(d)$  and the ellipse  $(E)$  have one single common point if and only if  $a^4m^2n^2 - a^2(a^2m^2 + b^2)(n^2 - b^2) = 0$ , i.e.  $n = \pm\sqrt{a^2m^2 + b^2}$ . The equations of the tangent lines of direction  $m$  are therefore

$$y = mx \pm \sqrt{a^2m^2 + b^2}. \quad (9.7)$$

2. (2p) Find the equations of the tangent lines to the ellipse  $E : x^2 + 4y^2 - 20 = 0$  which are orthogonal to the line  $d : 2x - 2y - 13 = 0$ .

*Solution.*

3. **(2p)** Find the equations of the tangent lines to the ellipse  $\mathcal{E} : \frac{x^2}{25} + \frac{y^2}{16} - 1 = 0$ , passing through  $P_0(10, -8)$ .

*Solution.*

4. If  $M(x, y)$  is a point of the tangent line  $T_{M_0}(E)$  of the ellipse  $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at one of its points  $M_0(x_0, y_0) \in \mathcal{E}$ , show that  $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$ .

*Solution 1.* Every director vector of the tangent line  $T_{M_0}(E) : \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$  is orthogonal to the normal vector  $\vec{n} = \left( \frac{x_0}{a^2}, \frac{y_0}{b^2} \right)$  of the tangent line  $T_{M_0}(E)$ . Such an orthogonal vector is  $\vec{v} = \left( \frac{y_0}{b^2}, -\frac{x_0}{b^2} \right)$ . Thus, the parametric equations of the tangent line are

$$T_{M_0}(E) : \begin{cases} x = x_0 + \frac{y_0}{b^2}t \\ y = y_0 - \frac{x_0}{b^2}t \end{cases}, t \in \mathbb{R},$$

i.e. the coordinates of  $M$  are of this form. In order to completely solve the question, we only

need to show that  $\varphi \geq 1$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\varphi(t) = \frac{\left( x_0 + \frac{y_0}{b^2}t \right)^2}{a^2} + \frac{\left( y_0 - \frac{x_0}{b^2}t \right)^2}{b^2}$ . This is actually the case as

$$\begin{aligned} \varphi(t) &= \frac{x_0^2 + 2\frac{x_0y_0}{b^2}t + \frac{y_0^2}{b^4}t^2}{a^2} + \frac{y_0^2 - 2\frac{x_0y_0}{b^2}t + \frac{x_0^2}{b^4}t^2}{b^2} \\ &= \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{1}{a^2b^2} \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} \right) t^2 = 1 + \frac{t^2}{a^2b^2} \geq 1, \forall t \in \mathbb{R}. \end{aligned}$$

*Solution 2.* Since  $M(x, y) \in T_{M_0}(E) : \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$ , it follows via the C-B-S inequality that

$$\begin{aligned} 1 &= \frac{x_0}{a} \cdot \frac{x}{a} + \frac{y_0}{b} \cdot \frac{y}{b} \leq \left| \frac{x_0}{a} \cdot \frac{x}{a} + \frac{y_0}{b} \cdot \frac{y}{b} \right| \\ &\leq \sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}. \end{aligned}$$

5. Find the equations of the tangent lines to the hyperbola  $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$  having a given angular coefficient  $m \in \mathbb{R}$ . (see [1, p. 115]).

*Solution.* The intersection of the hyperbola ( $\mathcal{H}$ ) with the line ( $d$ )  $y = mx + n$  is given by the solution of the system

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

By substituting  $y$  in the first equation, one obtains

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0. \quad (9.8)$$

- If  $a^2m^2 - b^2 = 0$ , (or  $m = \pm \frac{b}{a}$ ), then the equation (9.8) becomes

$$\pm 2bnx + a(n^2 + b^2) = 0.$$

- If  $n = 0$ , there are no solutions (this means, geometrically, that the two asymptotes do not intersect the hyperbola);
- If  $n \neq 0$ , there exists a unique solution (geometrically, a line  $d$ , which is parallel to one of the asymptotes, intersects the hyperbola at exactly one point);
- If  $a^2m^2 - b^2 \neq 0$ , then the discriminant of the equation (9.8) is

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 - b^2)(n^2 + b^2)].$$

The line  $d : y = mx + n$  is tangent to the hyperbola  $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$  if the discriminant  $\Delta$  of the equation (9.8) is zero, i.e.  $a^2m^2 - n^2 - b^2 = 0$ .

- If  $a^2m^2 - b^2 \geq 0$ , i.e.  $m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$ , then  $n = \pm\sqrt{a^2m^2 - b^2}$ . The equations of the tangent lines to  $\mathcal{H}$ , having the angular coefficient  $m$  are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \quad (9.9)$$

- If  $a^2m^2 - b^2 < 0$ , there are no tangent lines to  $\mathcal{H}$ , of angular coefficient  $m$ .

6. (2p) Find the equations of the tangent lines to the hyperbola  $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$  which are orthogonal to the line  $d : 4x + 3y - 7 = 0$ .

*Solution.*

7. Find the equations of the tangent lines to the parabola  $\mathcal{P} : y^2 = 2px$  having a given angular coefficient  $m \in \mathbb{R}$ . (see [1, p. 119]).

*Solution.* The intersection between the parabola  $(P)$  and the line  $(d)y = mx + n$  is given by the solution of the system

$$\begin{cases} y^2 = 2px \\ y = mx + n. \end{cases}$$

This leads to a second degree equation

$$m^2x^2 + 2(mn - p)x + n^2 = 0,$$

having the discriminant

$$\Delta = 4p(2mn - p) \quad (9.10)$$

The line  $d : y = mx + n$  (with  $m \neq 0$ ) is tangent to the parabola  $\mathcal{P} : y^2 = 2px$  if the discriminant  $\Delta$  which appears in (9.10) is zero, i.e.  $2mn = p$ . Then, the equation of the tangent line to  $\mathcal{P}$ , having the angular coefficient  $m$ , is

$$y = mx + \frac{p}{2m}. \quad (9.11)$$

8. **(2p)** Find the equation of the tangent line to the parabola  $\mathcal{P} : y^2 - 8x = 0$ , parallel to  $d : 2x + 2y - 3 = 0$ .

*Solution.*

9. **(2p)** Find the equation of the tangent line to the parabola  $\mathcal{P} : y^2 - 36x = 0$ , passing through  $P(2, 9)$ .

*Solution.*

10. (3p) Find the locus of the orthogonal projections of the center  $O(0,0)$  of the ellipse

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

on its tangents.

*Solution.*

11. (3p) Find the locus of the orthogonal projections of the center  $O(0,0)$  of the hyperbola

$$H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

on its tangents.

*Solution.*

12. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

*Solution.* Let  $F_1(-c, 0)$ ,  $F_2(c, 0)$  be the foci of the ellipse  $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . Recall that the gradient  $\text{grad}(f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$  is a normal vector of the ellipse  $\mathcal{E}$  to its point  $M_0(x_0, y_0)$ , where

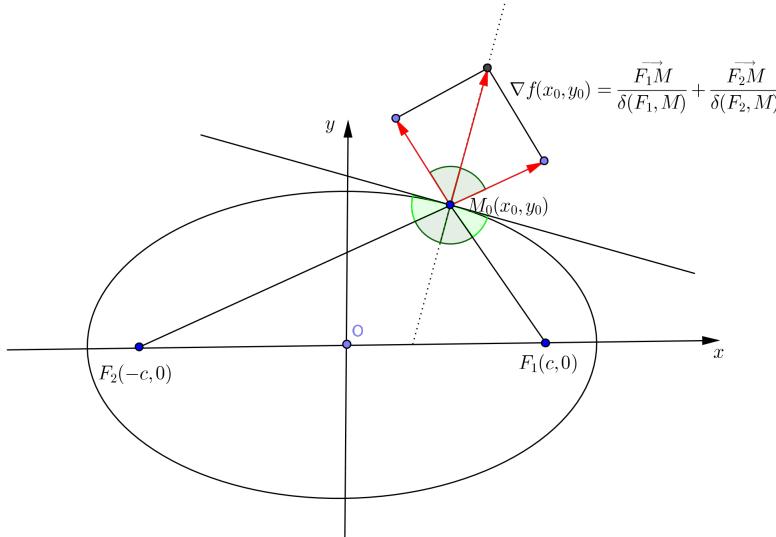
$$f(x, y) = \delta(F_1, M) + \delta(F_2, M) = \sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2}$$

and  $M(x, y)$ . Note that

$$f_x(x_0, y_0) = \frac{x_0 - c}{\delta(F_1, M_0)} + \frac{x_0 + c}{\delta(F_2, M_0)} \text{ and } f_y(x_0, y_0) = \frac{y}{\delta(F_1, M_0)} + \frac{y}{\delta(F_2, M_0)},$$

and shows that

$$\begin{aligned} \text{grad}(f) &= (f_x(x_0, y_0), f_y(x_0, y_0)) = \left( \frac{x_0 - c}{\delta(F_1, M_0)} + \frac{x_0 + c}{\delta(F_2, M_0)}, \frac{y}{\delta(F_1, M_0)} + \frac{y}{\delta(F_2, M_0)} \right) \\ &= \frac{(x_0 - c, y)}{\delta(F_1, M_0)} + \frac{(x_0 + c, y)}{\delta(F_2, M_0)} = \frac{\vec{F_1 M_0}}{\delta(F_1, M_0)} + \frac{\vec{F_2 M_0}}{\delta(F_2, M_0)}. \end{aligned}$$



The versors  $\frac{\vec{F_1 M_0}}{\delta(F_1, M_0)}$  and  $\frac{\vec{F_2 M_0}}{\delta(F_2, M_0)}$  point towards the exterior of the ellipse  $\mathcal{E}$  and their sum make obviously equal angles with the directions of the vectors  $\vec{F_1 M_0}$  and  $\vec{F_2 M_0}$  and (the sum) is also orthogonal to the tangent  $T_{M_0}(\mathcal{E})$  of the ellipse at  $M_0(x_0, y_0)$ . This shows that the angle between the ray  $F_1 M$  and the tangent  $T_{M_0}(\mathcal{E})$  equals the angle between the ray  $F_2 M$  and the tangent  $T_{M_0}(\mathcal{E})$ .

13. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).

*Solution.* Let  $F_1(-c, 0)$ ,  $F_2(c, 0)$  be the foci of the hyperbola  $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . Recall that the gradient  $\text{grad}(f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$  is a normal vector of the hyperbola  $\mathcal{H}$  to its point  $M_0(x_0, y_0)$ , where

$$f(x, y) = \delta(F_2, M) - \delta(F_1, M) = \sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} \quad (9.12)$$

on the left hand side branch of  $\mathcal{H}$  and

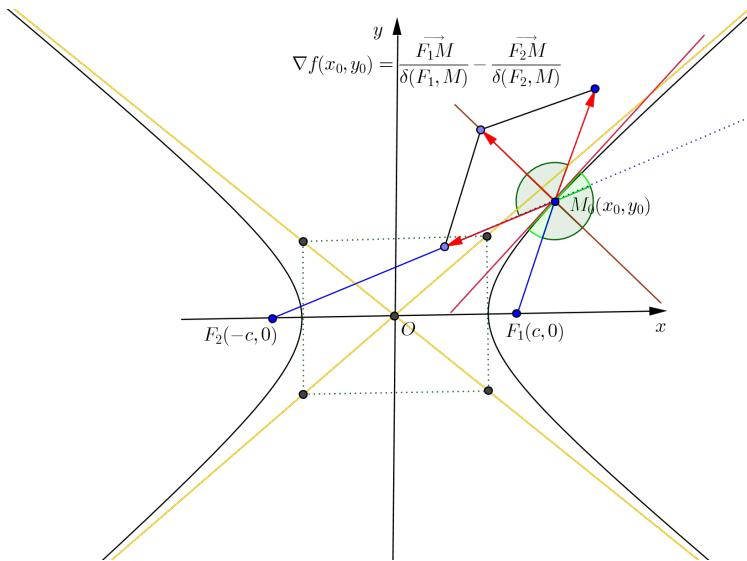
$$f(x, y) = \delta(F_1, M) - \delta(F_2, M) = \sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} \quad (9.13)$$

on the right hand side branch of  $\mathcal{H}$  and  $M(x, y)$ . We shall only use the version (9.12) of  $f$ , as judgement for the version (9.13) works in a similar way. Note that

$$f_x(x_0, y_0) = \frac{x_0 - c}{\delta(F_1, M_0)} - \frac{x_0 + c}{\delta(F_2, M_0)} \text{ and } f_y(x_0, y_0) = \frac{y}{\delta(F_1, M_0)} - \frac{y}{\delta(F_2, M_0)},$$

and shows that

$$\begin{aligned} \text{grad}(f) &= (f_x(x_0, y_0), f_y(x_0, y_0)) = \left( \frac{x_0 - c}{\delta(F_1, M_0)} - \frac{x_0 + c}{\delta(F_2, M_0)}, \frac{y_0}{\delta(F_1, M_0)} - \frac{y_0}{\delta(F_2, M_0)} \right) \\ &= \frac{(x_0 - c, y)}{\delta(F_1, M_0)} - \frac{(x_0 + c, y)}{\delta(F_2, M_0)} = \frac{\vec{F_1 M_0}}{\delta(F_1, M_0)} - \frac{\vec{F_2 M_0}}{\delta(F_2, M_0)}. \end{aligned}$$



The versors  $\frac{\vec{F_1 M_0}}{\delta(F_1, M_0)}$  and  $-\frac{\vec{F_2 M_0}}{\delta(F_2, M_0)}$  point towards the 'exterior' of the hyperbola  $\mathcal{H}$ <sup>3</sup> and their sum make obviously equal angles with the directions of the vectors  $\vec{F_1 M_0}$  and  $\vec{F_2 M_0}$  and (the sum) is also orthogonal to the tangent  $T_{M_0}(\mathcal{H})$  of the hyperbola at  $M_0(x_0, y_0)$ . This shows that the angle between the ray  $F_1 M$  and the tangent  $T_{M_0}(\mathcal{H})$  equals the angle between the ray  $F_2 M$  and the tangent  $T_{M_0}(\mathcal{H})$ .

14. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).

*Solution.* Let  $F(\frac{p}{2}, 0)$  be the focus of the parabola  $\mathcal{P} : y^2 = 2px$  and  $d : x = -\frac{p}{2}$  be its director line. Recall that the gradient  $\text{grad}(f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$  is a normal vector of parabola  $\mathcal{P}$  to its point  $M_0(x_0, y_0)$ , where

$$f(x, y) = \delta(F, M) - \delta(M, d) = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} - \left(x + \frac{p}{2}\right)$$

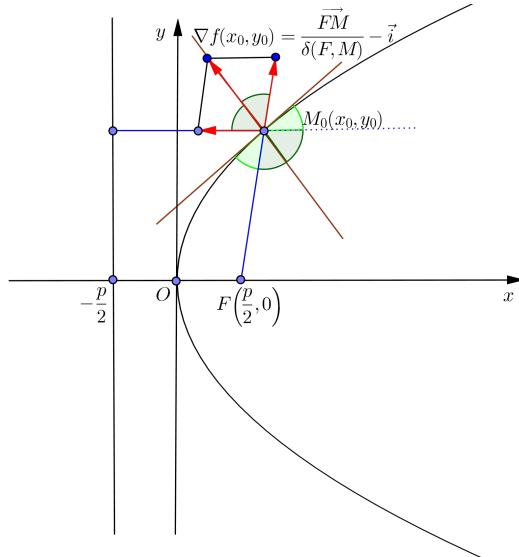
and  $M(x, y)$ . Note that

$$f_x(x_0, y_0) = \frac{x_0 - \frac{p}{2}}{\delta(F, M_0)} - 1 \text{ and } f_y(x_0, y_0) = \frac{y_0}{\delta(F, M_0)},$$

<sup>3</sup>The exterior of a hyperbola is the nonconvex component of its complement

and shows that

$$\begin{aligned}\text{grad}(f) &= (f_x(x_0, y_0), f_y(x_0, y_0)) = \left( \frac{x_0 - \frac{p}{2}}{\delta(F, M_0)} - 1, \frac{y_0}{\delta(F, M_0)} \right) \\ &= \left( \frac{x_0 - \frac{p}{2}}{\delta(F, M_0)}, \frac{y_0}{\delta(F, M_0)} \right) - (1, 0) = \frac{\vec{FM}_0}{\delta(F, M_0)} - \vec{i}.\end{aligned}$$



The versors  $\frac{\vec{FM}_0}{\delta(F, M_0)}$  and  $-\vec{i}$  point towards the 'exterior' of the parabola  $\mathcal{P}$ <sup>4</sup> and their sum make obviously equal angles with the directions of the vectors  $\vec{FM}_0$  and  $\vec{i}$  and (the sum) is also orthogonal to the tangent line  $T_{M_0}(\mathcal{P})$  of the parabola at  $M_0(x_0, y_0)$ . This shows that the angle between the ray  $FM$  and the tangent line  $T_{M_0}(\mathcal{P})$  equals the angle between  $Ox$  and the tangent  $T_{M_0}(\mathcal{E})$ .

<sup>4</sup>The exterior of a parabola is the nonconvex component of its complement

## 10 Week 10: Quadrics

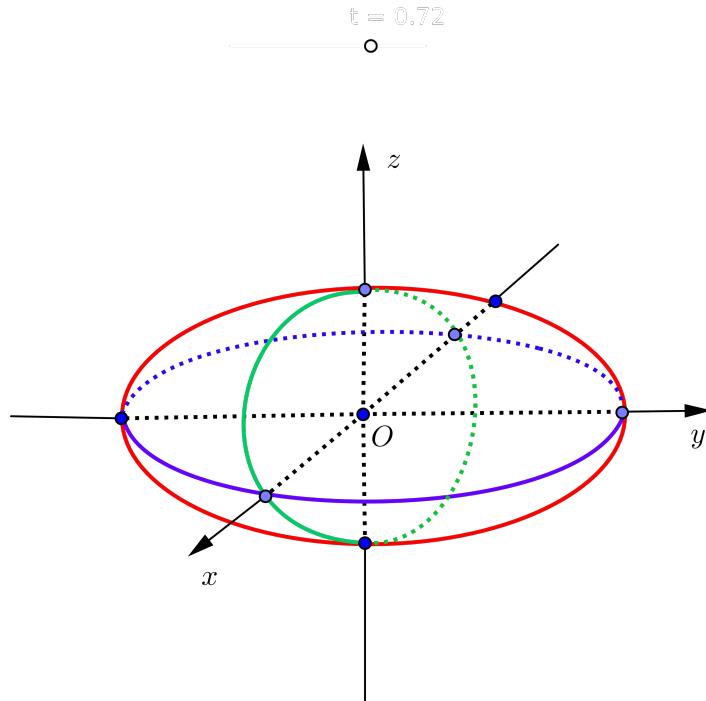
### 10.1 The ellipsoid

The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a, b, c \in \mathbb{R}_+^*. \quad (10.1)$$

- The coordinate planes are all planes of symmetry of  $\mathcal{E}$  since, for an arbitrary point  $M(x, y, z) \in \mathcal{E}$ , its symmetric points with respect to these planes,  $M_1(-x, y, z)$ ,  $M_2(x, -y, z)$  and  $M_3(x, y, -z)$  belong to  $\mathcal{E}$ ; therefore, the coordinate axes are axes of symmetry for  $\mathcal{E}$  and the origin  $O$  is the center of symmetry of the ellipsoid (10.1);
- The traces in the coordinates planes are ellipses of equations

$$\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0. \end{cases}$$



- The sections with planes parallel to  $xOy$  are given by setting  $z = \lambda$  in (10.1). Then, a section is of equations  $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$ .
- If  $|\lambda| < c$ , the section is an ellipse

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1-\frac{\lambda^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1-\frac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases};$$

- If  $|\lambda| = c$ , the intersection is reduced to one (tangency) point  $(0, 0, \lambda)$ ;
- If  $|\lambda| > c$ , the plane  $z = \lambda$  does not intersect the ellipsoid  $\mathcal{E}$ .

The sections with planes parallel to  $xOz$  or  $yOz$  are obtained in a similar way.

## 10.2 The hyperboloid of one sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \quad a, b, c \in \mathbb{R}_+^*, \quad (10.2)$$

is called *hyperboloid of one sheet*.

- The coordinate planes are planes of symmetry for  $\mathcal{H}_1$ ; hence, the coordinate axes are axes of symmetry and the origin  $O$  is the center of symmetry of  $\mathcal{H}_1$ ;
- The intersections with the coordinates planes are, respectively, of equations

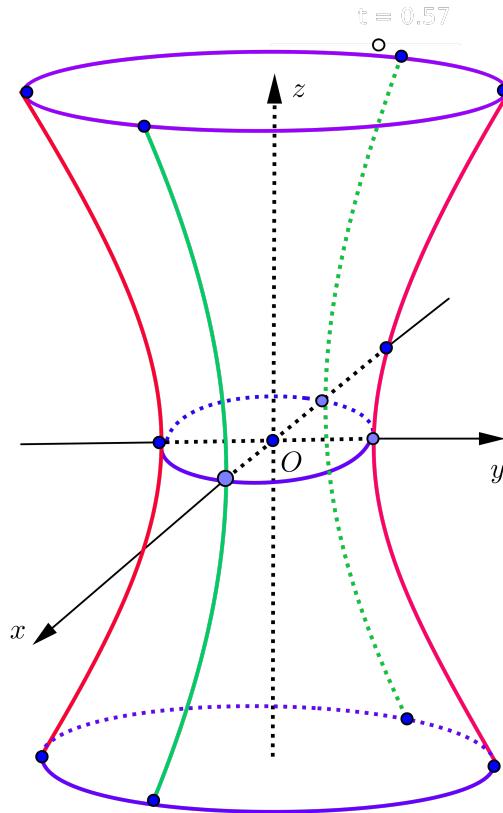
$$\left\{ \begin{array}{l} \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{array} \right. ; \left\{ \begin{array}{l} \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{array} \right. ; \left\{ \begin{array}{l} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0 \end{array} \right. ;$$

a hyperbola                    a hyperbola                    an ellipse

- The intersections with planes parallel to the coordinate planes are

$$\left\{ \begin{array}{l} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{\lambda^2}{a^2} \\ x = \lambda \end{array} \right. ; \left\{ \begin{array}{l} \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{\lambda^2}{b^2} \\ y = \lambda \end{array} \right. ; \left\{ \begin{array}{l} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{\lambda^2}{c^2} \\ z = \lambda \end{array} \right. ;$$

hyperbolas                    hyperbolas                    ellipses



*Remark:* The surface  $\mathcal{H}_1$  contains two families of lines, as

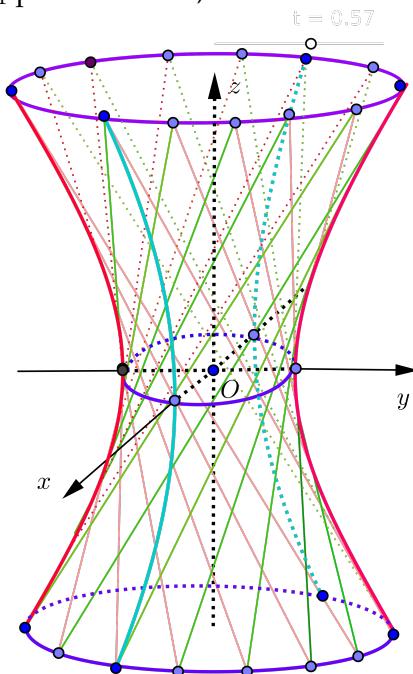
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \Leftrightarrow \left( \frac{x}{a} + \frac{z}{c} \right) \left( \frac{x}{a} - \frac{z}{c} \right) = \left( 1 + \frac{y}{b} \right) \left( 1 - \frac{y}{b} \right).$$

The equations of the two families of lines are:

$$d_\lambda : \begin{cases} \lambda \left( \frac{x}{a} + \frac{z}{c} \right) = 1 + \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} = \lambda \left( 1 - \frac{y}{b} \right) \end{cases}, \lambda \in \mathbb{R},$$

$$d'_\mu : \begin{cases} \mu \left( \frac{x}{a} + \frac{z}{c} \right) = 1 - \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} = \mu \left( 1 + \frac{y}{b} \right) \end{cases}, \mu \in \mathbb{R}.$$

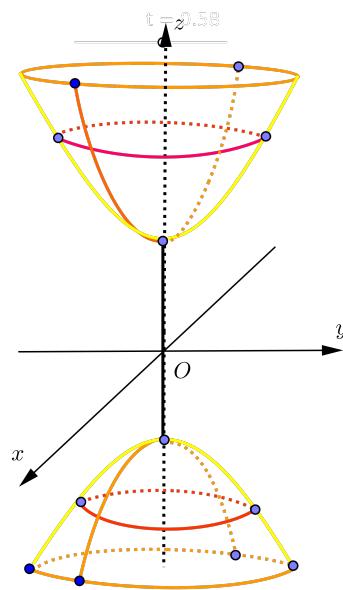
Through any point on  $\mathcal{H}_1$  pass two lines, one line from each family.



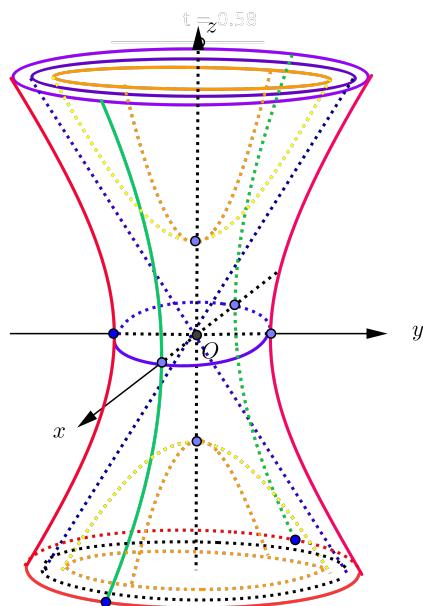
### 10.3 Th hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \quad a, b, c \in \mathbb{R}_+^*. \quad (10.3)$$



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

- The coordinate planes are planes of symmetry for  $\mathcal{H}_1$ , the coordinate axes are axes of symmetry and the origin  $O$  is the center of symmetry of  $\mathcal{H}_1$ ;
- The intersections with the coordinate planes are, respectively,

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \\ x = 0 \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0 \\ y = 0 \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0 \\ z = 0 \end{cases};$$

a hyperbola;      a hyperbola      the empty set

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{a^2} \\ x = \lambda \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{b^2} \\ y = \lambda \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{\lambda^2}{c^2} \\ z = \lambda. \end{cases}$$

hyperbolas      hyperbolas

- If  $|\lambda| > c$ , the section is an ellipse;
- If  $|\lambda| = c$ , the intersection reduces to the point of coordinates  $(0, 0, \lambda)$ ;
- If  $|\lambda| < c$ , one obtains the empty set.

## 10.4 Elliptic Cones

The surface of equation

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \quad a, b, c \in \mathbb{R}_+^*, \quad (10.4)$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for  $\mathcal{C}$ , the coordinate axes are axes of symmetry and the origin  $O$  is the center of symmetry of  $\mathcal{C}$ ;
- The intersections with the coordinate planes are

$$\left\{ \begin{array}{l} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \\ x = 0 \end{array} \right. , \left\{ \begin{array}{l} \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \\ y = 0 \end{array} \right. , \left\{ \begin{array}{l} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \\ z = 0 \end{array} \right. , \text{ the point } O(0, 0, 0).$$

- The intersections with planes parallel to the coordinate planes are

$$\left\{ \begin{array}{l} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -\frac{\lambda^2}{a^2} \\ x = \lambda \end{array} \right. , \text{ hyperbolas} ; \quad \left\{ \begin{array}{l} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -\frac{\lambda^2}{b^2} \\ y = \lambda \end{array} \right. , \text{ hyperbolas.} ; \quad \left\{ \begin{array}{l} \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\lambda^2}{c^2} \\ z = \lambda. \end{array} \right. , \text{ ellipses}$$

## 10.5 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \quad p, q \in \mathbb{R}_+^*, \quad (10.5)$$

is called *elliptic paraboloid*.

- The planes  $xOz$  and  $yOz$  are planes of symmetry;
- The traces in the coordinate planes are

$$\left\{ \begin{array}{l} \frac{y^2}{q} = 2z \\ x = 0 \end{array} \right. , \text{ a parabola} ; \quad \left\{ \begin{array}{l} \frac{x^2}{p} = 2z \\ y = 0 \end{array} \right. , \text{ a parabola} ; \quad \left\{ \begin{array}{l} \frac{x^2}{p} + \frac{y^2}{q} = 0 \\ z = 0 \end{array} \right. , \text{ the point } O(0, 0, 0).$$

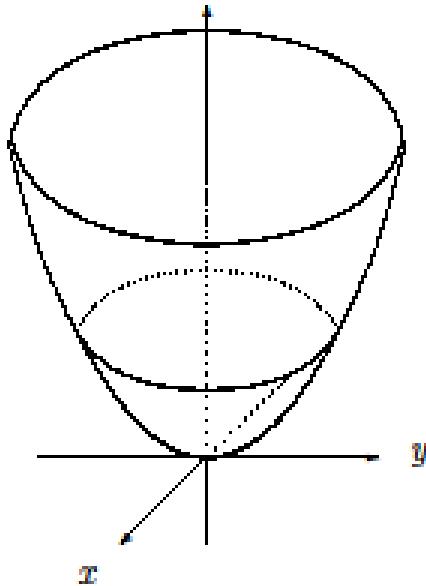
- The intersection with the planes parallel to the coordinate planes are  $\left\{ \begin{array}{l} \frac{x^2}{p} + \frac{y^2}{q} = 2\lambda \\ z = \lambda \end{array} \right. ,$
- If  $\lambda > 0$ , the section is an ellipse;
- If  $\lambda = 0$ , the intersection reduces to the origin;
- If  $\lambda < 0$ , one has the empty set;

and

$$\begin{cases} \frac{y^2}{q} = 2z - \frac{\lambda^2}{p} \\ x = \lambda \end{cases}; \quad \begin{cases} \frac{x^2}{p} = 2z - \frac{\lambda^2}{q} \\ y = \lambda \end{cases};$$

parabolas                                    parabolas

**2**



## 10.6 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \quad p, q > 0. \quad (10.6)$$

- The planes  $xOz$  and  $yOz$  are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\begin{cases} -\frac{y^2}{q} = 2z \\ x = 0 \end{cases}; \quad \begin{cases} \frac{x^2}{p} = 2z \\ y = 0 \end{cases}; \quad \begin{cases} \frac{x^2}{p} - \frac{y^2}{q} = 0 \\ z = 0 \end{cases};$$

a parabola                                    a parabola                                    two lines.

- The intersection with the planes parallel to the coordinate planes are

$$\begin{cases} \frac{y^2}{q} = -2z + \frac{\lambda^2}{p} \\ x = \lambda \end{cases}; \quad \begin{cases} \frac{x^2}{p} = 2z + \frac{\lambda^2}{q} \\ y = \lambda \end{cases}$$

parabolas                                    parabolas.

$$\begin{cases} \frac{x^2}{p} - \frac{y^2}{q} = 2\lambda \\ z = \lambda \end{cases}$$

hyperbolas

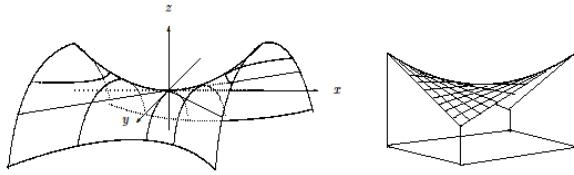
*Remark:* The hyperbolic paraboloid contains two families of lines. Since

$$\left( \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} \right) \left( \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} \right) = 2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = \lambda \\ \lambda \left( \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} \right) = 2z \end{cases}, \lambda \in \mathbb{R} \text{ and}$$

$$d'_\mu : \begin{cases} \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = \mu \\ \mu \left( \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} \right) = 2z \end{cases}, \mu \in \mathbb{R}.$$



## 10.7 Singular Quadrics

### Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

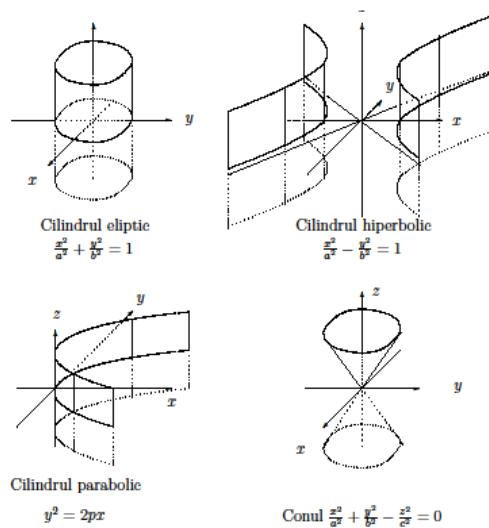
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0 \text{ or } \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0, \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0. \quad (10.7)$$

- The *hyperbolic cylinder* is the surface of equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0 \text{ or } \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0. \quad (10.8)$$

- The *parabolic cylinder* is the surface of equation

$$y^2 = 2px, \quad p > 0, \quad (\text{or an alternative equation}). \quad (10.9)$$



## 10.8 Problems

- Find the intersection points of the ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$$

with the line

$$\frac{x-4}{2} = \frac{y+6}{-3} = \frac{z+2}{-2}$$

and write the equations of the tangent planes as well as the equations of the normal lines to the ellipsoid at the intersection points.

*Solution.*

2. Find the rectilinear generatrices of the quadric  $4x^2 - 9y^2 = 36z$  which passes through the point  $P(3\sqrt{2}, 2, 1)$ .

*Solution.*

3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1) \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

which are parallel to the plane ( $\pi$ )  $x + y + z = 0$ .

*Solution.*

4. Find the locus of points on the hyperbolic paraboloid ( $\mathcal{P}_h$ )  $y^2 - z^2 = 2x$  through which the rectilinear generatrices are perpendicular.

*Solution.*

5. Compute the distance from  $O(0,0,0)$  to the tangent plane  $T_M(\mathcal{E})$  of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at some of its point  $M(x, y, z) \in \mathcal{E}$ .

*Solution.*

6. Show that the intersection between a straight line  $d$  and the sphere  $S(O, r)$  is a singleton if and only if  $\text{dist}(O, d) = r$ .

*Solution.*

## 11 Week 11: Generated Surfaces

Consider the 3-dimensional Euclidean space  $\mathcal{E}_3$ , together with a Cartesian system of coordinates  $Oxyz$ . Generally, the set

$$S = \{M(x, y, z) : F(x, y, z) = 0\},$$

where  $F : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$  is a real function and  $D$  is a domain, is called *surface* of implicit equation  $F(x, y, z) = 0$ . For example the quadric surfaces, defined in the previous chapter for  $F$  a polynomial of degree two, are such of surfaces. On the other hand, the set

$$S_1 = \{M(x, y, z) : x = x(u, v), y = y(u, v), z = z(u, v)\},$$

where  $x, y, z : D_1 \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ , is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x, y, z) : F(x, y, z) = 0, G(x, y, z) = 0\},$$

where  $F, G : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ , is the curve of *implicit* equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x, y, z) : x = x(t), y = y(t), z = z(t)\},$$

where  $x, y, z : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $I$  is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter  $\lambda$ ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x, y, z; \lambda) = 0 \\ F_2(x, y, z; \lambda) = 0 \end{cases}.$$

In general, the family  $\mathcal{C}_\lambda$  does not cover the entire space. By eliminating the parameter  $\lambda$  between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters  $\lambda, \mu$ ,

$$\mathcal{C}_{\lambda, \mu} : \begin{cases} F_1(x, y, z; \lambda, \mu) = 0 \\ F_2(x, y, z; \lambda, \mu) = 0 \end{cases},$$

and that the parameters are related through  $\varphi(\lambda, \mu) = 0$ . If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

## 11.1 Cylindrical Surfaces

**Definition 11.1.** The surface generated by a variable line, called *generatrix*, which remains parallel to a fixed line  $d$  and intersects a given curve  $\mathcal{C}$ , is called *cylindrical surface*. The curve  $\mathcal{C}$  is called the *director curve* of the cylindrical surface.

**Theorem 11.1.** *The cylindrical surface, with the generatrix parallel to the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

which has the director curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

( $d$  and  $\mathcal{C}$  are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \quad (11.1)$$

*Proof.* The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x, y, z) = 0 \\ \pi_2(x, y, z) = 0 \end{cases}, \text{ are } d_{\lambda, \mu} : \begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}.$$

Not every line from the family  $d_{\lambda, \mu}$  intersects the curve  $\mathcal{C}$ . This happens only when the system of equations

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}$$

is compatible. By eliminating  $\lambda$  and  $\mu$  between four equations of the system, one obtains a *necessary condition*  $\varphi(\lambda, \mu) = 0$  for the parameters  $\lambda$  and  $\mu$  in order to nonempty intersection between the line  $d_{\lambda, \mu}$ . The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

and it is immediate that  $\varphi(\pi_1, \pi_2) = 0$ . □

**Remark 11.1.** Any equation of the form (11.1), where  $\pi_1$  and  $\pi_2$  are linear function of  $x$ ,  $y$  and  $z$ , represents a cylindrical surface, having the generatrices parallel to  $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$ .

**Example 11.1.** Let us find the equation of the cylindrical surface having the generatrices parallel to

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices  $d$  are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases}.$$

They must intersect the curve  $\mathcal{C}$ , i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

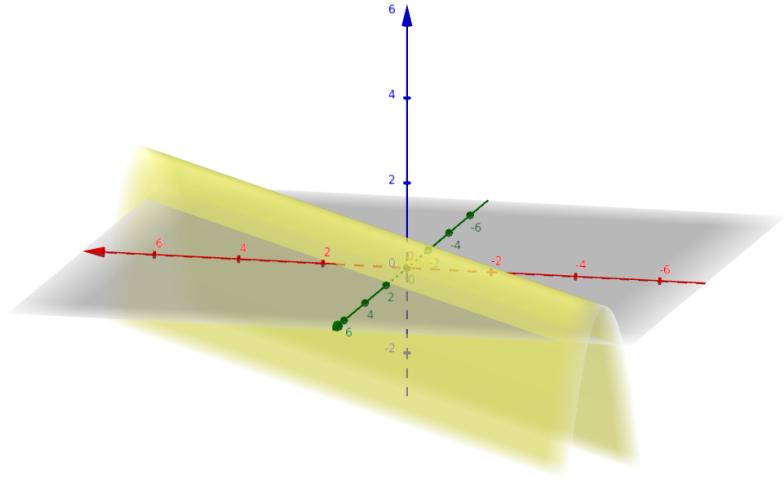
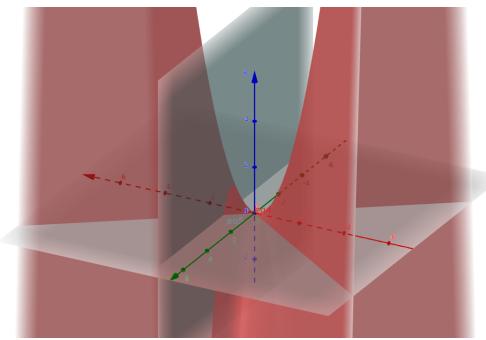
$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

Thus, the equation of the required cylindrical surface is

$$2(x + y - 1)^2 + z - 1 = 0.$$



## 11.2 Conical Surfaces

**Definition 11.2.** The surface generated by a variable line, called *generatrix*, which passes through a fixed point  $V$  and intersects a given curve  $\mathcal{C}$ , is called *conical surface*. The point  $V$  is called the *vertex* of the surface and the curve  $\mathcal{C}$  *director curve*.

**Theorem 11.2.** The conical surface, of vertex  $V(x_0, y_0, z_0)$  and director curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

( $V$  and  $\mathcal{C}$  are not coplanar), is characterized by an equation of the form

$$\varphi \left( \frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0} \right) = 0. \quad (11.2)$$

*Proof.* The equations of an arbitrary line through  $V(x_0, y_0, z_0)$  are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases} .$$

A generatrix has to intersect the curve  $\mathcal{C}$ , hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters  $\lambda$  and  $\mu$ , which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating  $x$ ,  $y$  and  $z$  in the previous system of equations. In these conditions, the equation of the conical surface rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases} ,$$

i.e.

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

□

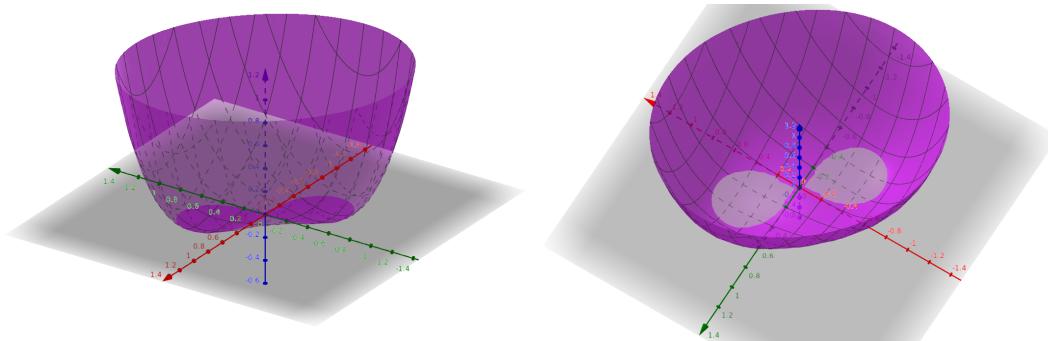
**Remark 11.2.** If  $\varphi$  is a polynomial function, then the equation (11.2) can be written in the form

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where  $\phi$  is homogeneous with respect to  $x - x_0$ ,  $y - y_0$  and  $z - z_0$ . If  $\varphi$  is polynomial and  $V$  is the origin of the system of coordinates, then the equation of the conical surface is  $\phi(x, y, z) = 0$ , with  $\phi$  a homogeneous polynomial. Conversely, an algebraic homogeneous equation in  $x$ ,  $y$  and  $z$  represents a conical surface with the vertex at the origin.

**Example 11.2.** Let us determine the equation of the conical surface, having the vertex  $V(1, 1, 1)$  and the director curve

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases} .$$



The family of lines passing through  $V$  has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases} .$$

The system of equations

$$\begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \\ x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1 - \lambda \\ y = 1 - \mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1 - \lambda)^2 + (1 - \mu)^2]^2 - (1 - \lambda)(1 - \mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters  $\lambda$  and  $\mu$  in

$$\begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \\ ((1 - \lambda)^2 + (1 - \mu)^2)^2 - (1 - \lambda)(1 - \mu) = 0 \end{cases}.$$

Expressing  $\lambda = \frac{x-1}{z-1}$  and  $\mu = \frac{y-1}{z-1}$  and replacing in the compatibility condition, one obtains

$$\left[ \left( \frac{z-x}{z-1} \right)^2 + \left( \frac{z-y}{z-1} \right)^2 \right]^2 - \left( \frac{z-x}{z-1} \right) \left( \frac{z-y}{z-1} \right) = 0,$$

or

$$[(z-x)^2 + (z-y)^2]^2 - (z-x)(z-y)(z-1)^2 = 0.$$

### 11.3 Conoidal Surfaces

**Definition 11.3.** The surface generated by a variable line, which intersects a given line  $d$  and a given curve  $C$ , and remains parallel to a given plane  $\pi$ , is called *conoidal surface*. The curve  $C$  is the *director curve* and the plane  $\pi$  is the *director plane* of the conoidal surface.

**Theorem 11.3.** *The conoidal surface whose generatrix intersects the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

*and the curve*

$$C : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

*and has the director plane  $\pi = 0$ , ( $\pi$  is not parallel to  $d$  and that  $C$  is not contained into  $\pi$ ), is characterized by an equation of the form*

$$\varphi \left( \pi, \frac{\pi_1}{\pi_2} \right) = 0. \quad (11.3)$$

*Proof.* An arbitrary generatrix of the conoidal surface is contained into a plane parallel to  $\pi$  and, on the other hand, comes from the bundle of planes containing  $d$ . Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases}.$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

By expressing  $\lambda$  and  $\mu$ , one obtains (11.3).  $\square$

**Example 11.3.** Let us find the equation of the conoidal surface, whose generatrices are parallel to  $xOy$  and intersect  $Oz$  and the curve

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of  $xOy$  and  $Oz$  are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda, \mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}.$$

From the compatibility of the system of equations

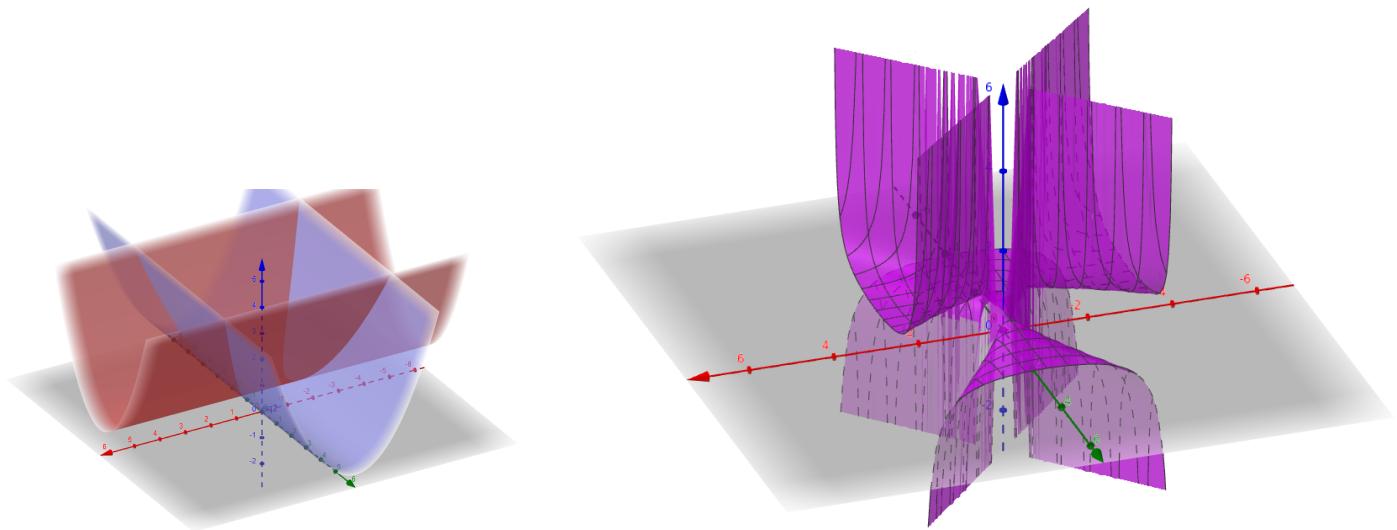
$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

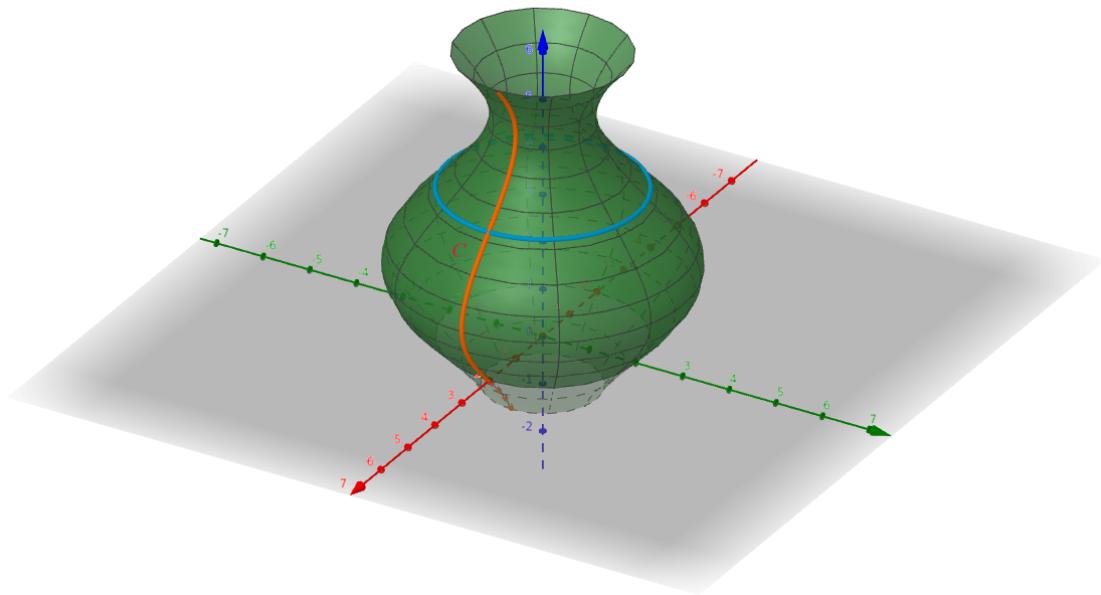
and, replacing  $\lambda = \frac{x}{y}$  and  $\mu = z$ , the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0. \quad (11.4)$$



## 11.4 Revolution Surfaces

**Definition 11.4.** The surface generated by rotating of a given curve  $\mathcal{C}$  around a given line  $d$  is said to be a *revolution surface*.



**Theorem 11.4.** The equation of the revolution surface generated by the curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

in its rotation around the line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

is of the form

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \quad (11.5)$$

*Proof.* An arbitrary point on the curve  $\mathcal{C}$  will describe, in its rotation around  $d$ , a circle situated into a plane orthogonal on  $d$  and having the center on the line  $d$ . This circle can be seen as the intersection between a sphere, having the center on  $d$  and of variable radius, and a plane, orthogonal on  $d$ , so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

The circle has to intersect the curve  $\mathcal{C}$ , therefore the system

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (11.5).  $\square$

## 11.5 Problems

- Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C) \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve.

*Solution.*

2. A disk of radius 1 is centered at the point  $A(1, 0, 2)$  and is parallel to the plane  $yOz$ . A source of light is placed at the point  $P(0, 0, 3)$ . Characterize analitically the shadow of the disk rushed over the plane  $xOy$ .

*Solution.* Consider the conical surface of vertex  $P$  whose director curve is the circle of radius 1 which is centered at the point  $A(1, 0, 2)$  and is parallel to the plane  $yOz$ . The shadow of the disk rushed over the plane  $xOy$  is the convex component of the complement, in the plane  $xOy$ , of the intersection curve between the plane  $xOy$  and the described conical surface.

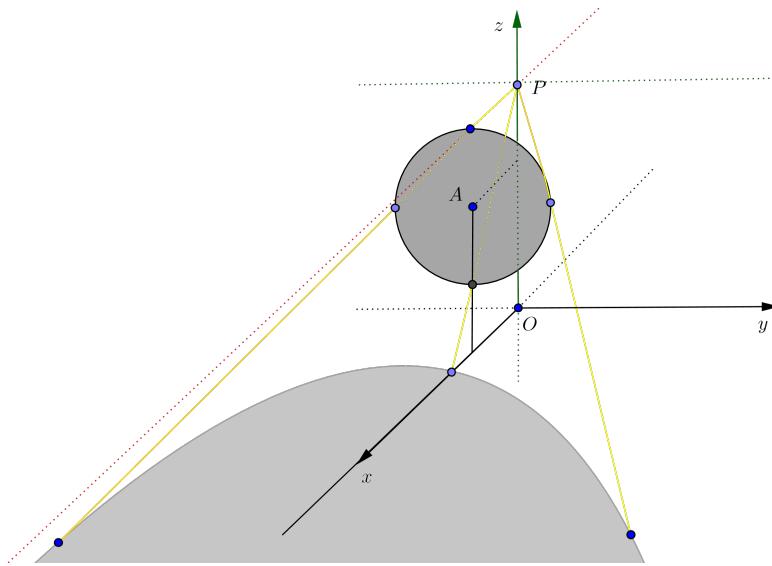
In order to find the equation of the conical surface we consider the lines

$$(Oz) \quad \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{and} \quad (d) \quad \begin{cases} x = 0 \\ z = 3 \end{cases}$$

as well as the family of lines

$$(\Delta_{\lambda\mu}) \quad \begin{cases} y - \lambda x = 0 \\ z - 3 - \mu x = 0 \end{cases}$$

depending on the parameters  $\lambda$  and  $\mu$  of the reduced pencils of lines  $x - \lambda y = 0$  through  $Oz$  and  $z - 3 - \mu z = 0$  through  $d$ .



The circle  $C$  which borders the disk is given by the equations

$$(C) \begin{cases} (x-1)^2 + y^2 + (z-2)^2 = 1 \\ x = 1. \end{cases}$$

The intersection point of the line  $\Delta_{\lambda\mu}$  with the plane of the circle is described by the system

$$\begin{cases} x = 1 \\ y - \lambda x = 0 \\ z - 3 - \mu x = 0 \end{cases}$$

which has the solution

$$(\Delta_{\lambda\mu} \cap (x = 1)) \begin{cases} x = 1 \\ y = \lambda \\ z = 3 + \mu. \end{cases} \quad (11.6)$$

By imposing the condition on the intersection point (11.6) to belong the other surface which defines  $C$ , namely the sphere  $(x-1)^2 + y^2 + (z-2)^2 = 1$ , we obtain the relation  $\lambda^2 + (\mu+1)^2 = 1$ , between  $\lambda$  and  $\mu$ , in order to have concurrence between  $\Delta_{\lambda\mu}$  and  $C$ . The equation of the conical surface is

$$\left(\frac{y}{x}\right)^2 + \left(\frac{z-3}{x} + 1\right)^2 = 1, \text{ or } y^2 + (x+z-3)^2 = x^2.$$

The latter equation is equivalent with

$$y^2 + z^2 + 2xz - 6x - 6z + 9 = 0.$$

Its intersection curve with the plane  $xOy$  is the parabola

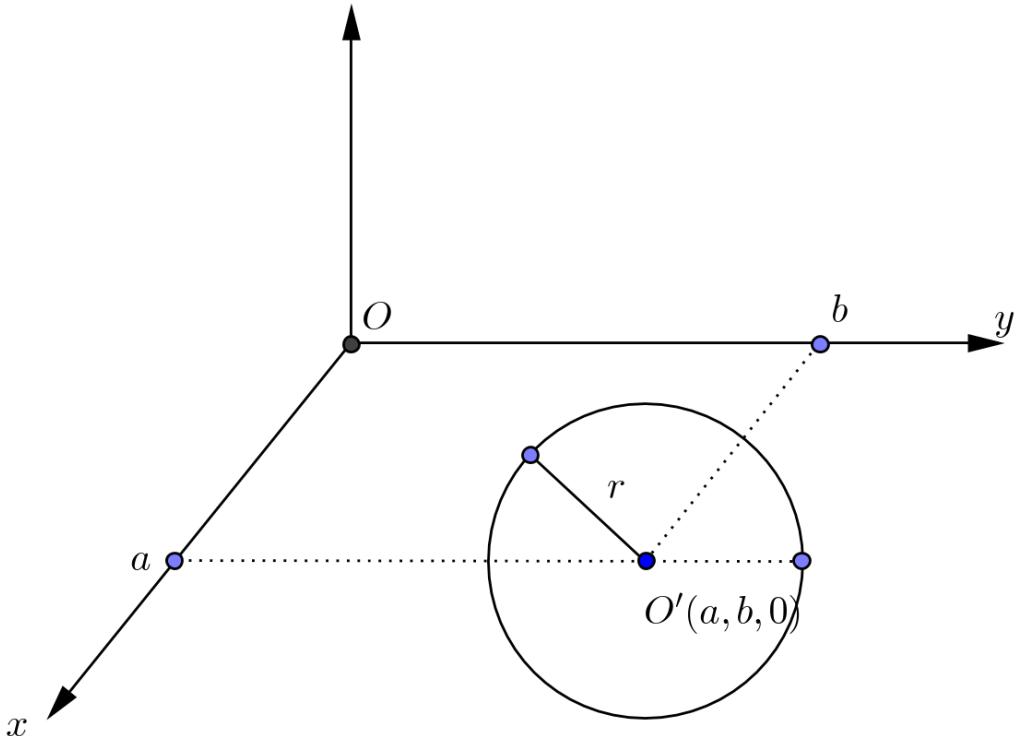
$$(\mathcal{P}) \begin{cases} z = 0 \\ y^2 - 6x + 9 = 0. \end{cases}$$

The convex component of the complement  $xOy \setminus \mathcal{P}$  coincides with the required shadow and is characterized by the following system

$$\begin{cases} y^2 - 6x + 9 \leq 0 \\ z = 0. \end{cases}$$

3. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface generated by a variable line which intersects the line ( $d$ ) and the circle ( $C$ ) and remains orthogonal to ( $d$ ). (The Willis conoid)

*Solution.* We have the freedom to choose the orthonormal Cartesian reference system, as the involved geometrical objects are not given through any equations. In this respect we are going to consider the given line  $d$  as the  $z$ -axis and the  $xOy$ -plane as the orthogonal plane on  $d$  through the center  $O'$  of the given circle, where  $O$  is the intersection point of this plane with the line  $d$ , i.e. the  $z$ -axis. Within this plane we shall choose the  $y$ -axis to be the one through the point  $O$  which is parallel to the plane of the circle and the remaining  $x$ -axis is now completely determined.



The equations of  $d = Oz$  with respect to this reference system are

$$(d = Oz) \quad \begin{cases} x = 0 \\ y = 0 \end{cases}$$

and the equations of the given circle are

$$(C) \quad \begin{cases} (x - a)^2 + (y - b)^2 + z^2 = r^2 \\ x = a \end{cases},$$

where  $(a, b, 0)$  are the coordinates of the center  $O'$  and  $r$  is the radius of the given circle.

We next consider the family lines

$$d_{\lambda\mu} \quad \begin{cases} y = \lambda x \\ z = \mu \end{cases}$$

which are all parallel to the  $xOy$  plane, i.e. perpendicular to  $d = Oz$ . Out of this family we need to select those lines having some nonempty intersection with the circle  $C$ . In this respect we solve the system

$$\begin{cases} x = a \\ y = \lambda x \\ z = \mu \end{cases}$$

whose solution

$$\begin{cases} x = a \\ y = \lambda a \\ z = \mu \end{cases}$$

represent the coordinates of the intersection point between the line  $d_{\lambda\mu}$  and the plane  $x = a$  of the circle. The connection relation between the parameters  $\lambda$  and  $\mu$  in order for the line  $d_{\lambda\mu}$  to intersect the circle is  $(\lambda a - b)^2 + \mu^2 = r^2$ . Thus, the remaining steps towards the equation of the Willis connoid are

$$\left(\frac{y}{x}a - b\right)^2 + z^2 = r^2 \implies (ya - b)^2 + x^2z^2 = r^2x^2.$$

4. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

*Solution.* We are going to split the solution in two cases depending on whether the two given lines are coplanar or not. The coplanar case will be split again in two subcases depending on whether the two lines are parallel or concurrent. In order to simplify the calculations, a suitable orthonormal coordinate system will be considered in each situation.

(a) *The two lines are coplanar.*

- i. *The two lines are parallel.* We choose the coordinate reference system to have the fixed line as the  $z$ -axis and a common perpendicular line as the  $x$ -axis. The  $y$ -axis is now completely determined. The equations of the two lines are

$$(d_1 = Oz) \quad \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (d_2) \quad \begin{cases} x = a \\ y = 0 \end{cases}$$

Consider the family of circles

$$(C_{\lambda\mu}) \quad \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

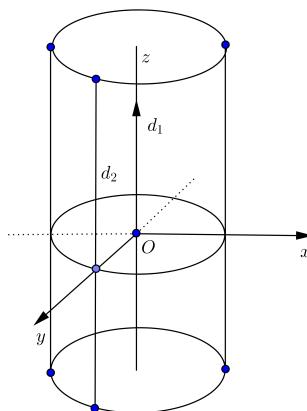
and determine its intersection point with the halfplane of  $xOz$  in which the line  $d_2$  lies, by solving the system

$$\begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \\ y = 0 \end{cases}$$

for  $x, y$  and  $z$  in terms of  $\lambda$  and  $\mu$ . This solution is

$$x = \sqrt{\lambda - \mu^2}, \quad y = 0, \quad z = \mu$$

and this intersection point meets the line  $d_2$  when  $x = a$ , namely  $\lambda - \mu^2 = a^2$ . Thus, the equation of the revolution surface, in this case, is  $x^2 + y^2 = a^2$  and represents a circular cylinder.



ii. *The two lines are concurrent.* We choose the coordinate reference system to have the fixed line as the  $z$ -axis and their common perpendicular line as the  $x$ -axis. The  $y$ -axis is now completely determined. The equations of the two lines are

$$(d_1 = Oz) \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (d_2) \begin{cases} x = 0 \\ z = my. \end{cases}$$

Consider the family of circles

$$(C_{\lambda\mu}) \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

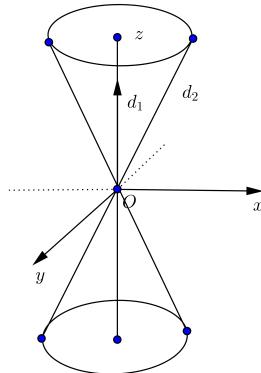
and determine its intersection points with the coordinate plane  $yOz$ , by solving the system

$$\begin{cases} x = 0 \\ x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

for  $x, y$  and  $z$  in terms of  $\lambda$  and  $\mu$ . This solution is

$$x = 0, y = \pm\sqrt{\lambda - \mu^2}, z = \mu$$

and this intersection points meets the line  $d_2$  when  $z = my$ , namely  $\mu^2 = \lambda - \mu^2$ . Thus, the equation of the revolution surface, in this case, is  $x^2 + y^2 = z^2$  and represents a circular cone.



(b) *The two lines are non-coplanar.* We choose the coordinate reference system to have the fixed line as the  $z$ -axis and their common perpendicular line as the  $x$ -axis. The  $y$ -axis is now completely determined. The equations of the two lines are

$$(d_1 = Oz) \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (d_2) \begin{cases} x = a \\ z = my. \end{cases}$$

Consider the family of circles

$$(C_{\lambda\mu}) \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

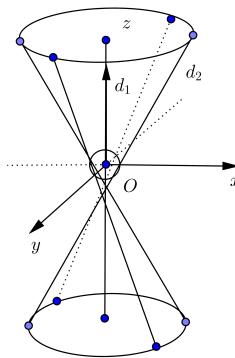
and determine its intersection points with the plane  $x = a$ , by solving the system

$$\begin{cases} x = a \\ x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

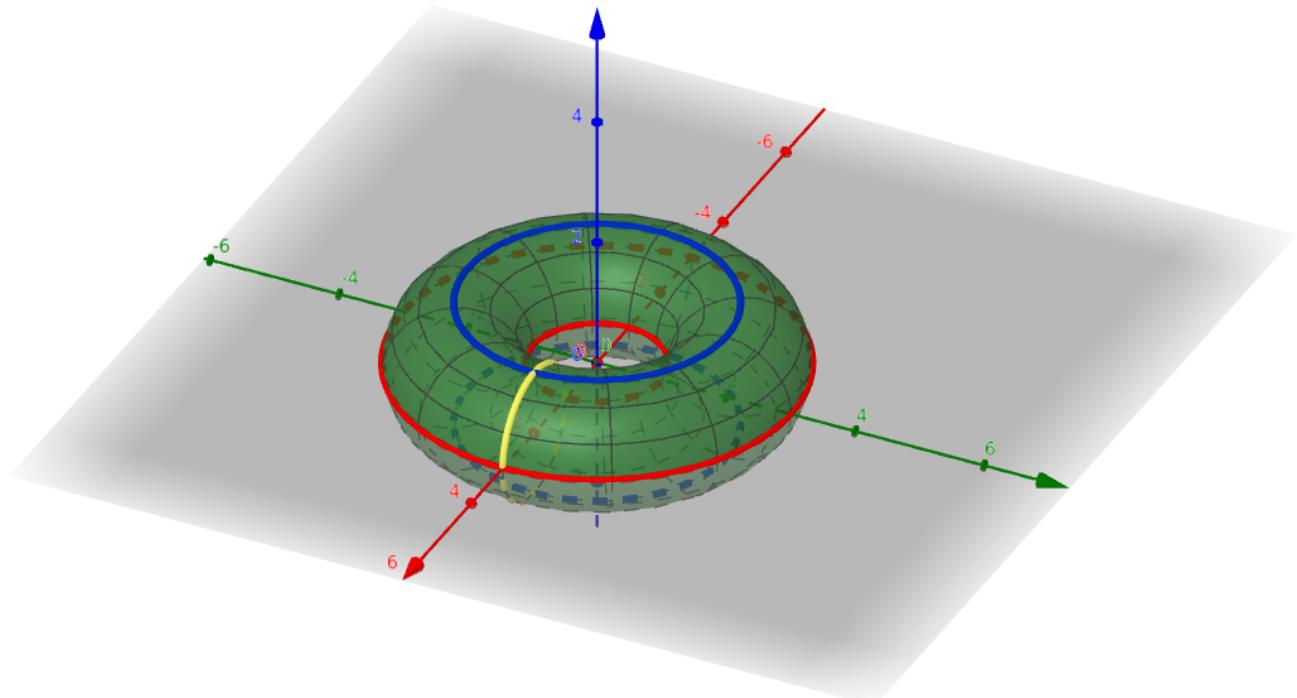
for  $x, y$  and  $z$  in terms of  $\lambda$  and  $\mu$ . This solution is

$$x = a, y = \pm\sqrt{\lambda - \mu^2 - a^2}, z = \mu$$

and this intersection points meets the line  $d_2$  when  $z = my$ , namely  $\mu^2 = \lambda - \mu^2 - a^2$ . Thus, the equation of the revolution surface, in this case, is  $x^2 + y^2 - z^2 = a^2$  and represents a circular hyperboloid of one sheet.



5. The *torus* is the revolution surface obtained by the rotation of a circle  $C$  about a fixed line  $(d)$  within the plane of the circle such that  $d \cap C = \emptyset$ . Find the equation of the torus<sup>5</sup>



*Solution.*

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<sup>5</sup>The torus is a regular surface.

## References

- [1] Andrica, D., Țopan, L., Analytic geometry, Cluj University Press, 2004.
- [2] Galbură Gh., Radó, F., Geometrie, Editura didactică și pedagogică-București, 1979.
- [3] Pintea, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferențială a curbelor și suprafețelor, Presa Universitară Clujeană, 2001.
- [4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. "Babeş-Bolyai", Cluj-Napoca, 1979.
- [5] Tabachnikov, S., Geometry and Billiards, American Mathematical Society, 2005.