Proof of the irrationality of $\sqrt{2}$

David

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1 Proof using contradiction

Let us assume $\sqrt{2}$ is a rational number, where a rational number is defined as follows:

$$\text{A number r is rational if } r = \frac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0 \tag{1}$$

Now, assuming $\sqrt{2}$ is rational, we can write the following: $\sqrt{2} = \frac{p}{q}$ squaring both sides leads to: $2 = \frac{p^2}{q^2} \therefore p^2 = 2q^2$

This is very important, as only an even number squared can give another even number, this leads to the following about p:

$$\text{p even, so let } p = 2r \therefore (2r)^2 = 2q^2 \therefore 2r^2 = q^2 \tag{2}$$

we now reach the same conclusion for q as for p, if q^2 is even, q is even as well, but if both p and q are even, both have a factor of 2, so $\frac{p}{q}$ cannot be in lowest terms, this leads to the contradiction proving $\sqrt{2}$ is an irrational number.