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Some Important Aspects of Sustainable Portfolio Allocation

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Abstract

Leveraging stochastic analysis for defining continuous financial markets with a riskless asset, and based on earlier work by Platen and co-authors, we establish the Growth Optimal Portfolio (GOP) through unconstrained optimization. It is notable that any benchmarked portfolio, if the GOP is used as benchmark, exhibits a local nonnegative martingale property, rendering it a supermartingale. Expanding our analysis to continuous financial markets without the riskless asset reveals a duality in GOP formulations, consistently affirming the benchmarked portfolio's supermartingale property. In view of this, we are able to give a reasonable definition of approximating sequence for the so-called stock GOP. In parallel, we discuss the HWI, an index based on hierarchical diversification of stocks and the EWI, which is a trivial case of it. We then present and prove the Diversification Theorem, which immediately implies that the sequences of portfolios constructed with HWI and EWI serve as approximations to the GOP. At the end, we summarize some empirical evidence highlighting how portfolios based on a deeper hierarchical structure outperform in terms of optimal growth, opening to a possible discussion on various types of sustainable strategies.

1 Introduction

Climate protection has become an increasingly pervasive global concern, seeping into various aspects of societal discourse, including investment strategies. In recent years, there has been a discernible uptick in interest surrounding sustainable investment strategies, even within the sphere of retirement planning. It is important to underscore that the overarching objective of retirement savings is to ensure individuals accrue sufficient financial resources to sustain themselves in their later years. However, navigating this terrain is often fraught with challenges, as many investors are faced with financial constraints that curtail their flexibility in pursuing strategies that prioritize optimized growth. Despite these challenges, the importance of integrating sustainability considerations into investment decision-making is recognized by many people, reflecting a societal shift towards conscientious and responsible investing practices.

The landscape of sustainable investment strategies is vast and multifaceted, encompassing a diverse array of approaches; many of these approaches are founded on the notion that investments should shift focus from unsustainable branches to those that align better with sustainability goals. In this respect, the so-called best-in-class approaches operate, to a certain extent, differently. This strategy involves meticulously selecting companies within each sector based on their exemplary sustainability performance, thus incentivizing healthy competition and encouraging industry-wide improvements. However, detractors of this approach argue that its inclusivity, without excluding any sectors a priori, may inadvertently lead to investments flowing into industries considered less sustainable, diluting the overall impact of the strategy. Balancing the merits and drawbacks of various sustainable investment approaches is essential to put together a comprehensive and effective investment strategy that aligns with both, financial and sustainability objectives.

Conducting a comparative analysis of the performance of green portfolios versus traditional ones unveils a nuanced and complex landscape. While proponents of sustainable investing often posit that green portfolios should outperform traditional ones due to factors such as climate change mitigation efforts and associated socio-economic shifts, empirical evidence paints a more intricate picture. Indeed, even rudimentary comparisons of time series data over different intervals show disparities in the realized returns of green and non-green portfolios, underscoring the multifaceted nature of portfolio performance. Moreover, the challenges inherent in estimating future returns further compound the difficulty of making statistically robust conclusions regarding the relative performance of green investment portfolios versus traditional ones. Ultimately, navigating this terrain could require a nuanced understanding of the intricacies of portfolio dynamics and the broader socio-economic landscape.

Central to the discourse surrounding sustainable investment strategies is the need to understand the potential for growth losses associated with different approaches. Portfolio optimization, inherently an exercise in striking a delicate balance between constraints and objectives, necessitates a nuanced understanding of the trade-offs inherent in various investment strategies. Identifying which sustainable investment strategies may lead to higher or lower growth losses is essential for informed decision-making and effective risk management. By carefully navigating these considerations, investors can design investment portfolios that not only align with their sustainability goals but also effectively manage financial risks, thereby ensuring long-term financial resilience and sustainability.

With a view to these issues, this paper initially outlines key elements of the benchmark approach in quantitative finance within a continuous financial market, focusing on the so-called numéraire portfolio, often also called to be the growth optimal portfolio. The portfolio resulting from this approach frequently involves significant leverage, posing challenges in practical application. Building on the work of Filipović and Platen [1], this paper introduces theoretical extensions that serve as the foundation for approximation methodologies outlined by Platen and Rendeck in [3]. Subsequently, the methodological framework of these approaches is detailed. Finally, the paper delves into the potential implications of these findings and empirical results in [3] in relation to sustainable investment strategies.

2 Preliminaries

In this section we collect some results from stochastic analysis, which will be used throughout the rest of this report.

2.1 Stochastic Integral

Definition 2.1. We denote by $\mathbb{H}^2$ the space of L^2-bounded martingales, i.e. the space of $(\mathcal{F}_t, P)-$martingales M such that:

$$\sup_t \mathbb{E}[M_t^2] < +\infty.$$

Also, we denote by H^2 the subset of L^2-bounded continuous martingales, and H_0^2 the subset of elements in H^2 vanishing in zero.

In particular, by Doob's inequality ([4], p. 54) we know $M_\infty := \sup_t |M_t|$ is in L^2 if $M \in H^2$. In fact, the following result holds:

Proposition 2.2. *([4], p. 129). The space $\mathbb{H}^2$ is an Hilbert space with scalar product:*

$$(M, N)_{\mathbb{H}^2} := \lim_{t \to +\infty} \mathbb{E}[M_t N_t],$$

and it induces the norm:

$$\|M\|_{\mathbb{H}^2} = \lim_{t \to +\infty} \mathbb{E}[M_t^2]^{\frac{1}{2}} = \mathbb{E}[M_\infty^2]^{\frac{1}{2}} = \|M_\infty\|_{L^2(P)}.$$

Theorem 2.3. *([4], p. 125). If M and N are two continuous local martingales, there exists a unique continuous process $\langle M, N\rangle$ of bounded variation, vanishing at zero, predictable and such that $MN - \langle M, N\rangle$ is a local continuous martingale.*

Now, let us state an important criterion to see if a continuous local martingale belongs to H^2.

Proposition 2.4. *([4], p. 129). A continuous local martingale M is in H^2 if and only if the following two conditions hold:*

1. $M_0 \in L^2$.
2. *$\langle M\rangle$ is integrable i.e.* $\mathbb{E}[\langle M\rangle_\infty] < +\infty$.

In that case, $M^2 - \langle M\rangle$ is a uniformly integrable martingale and:

$$\|M\|_{\mathbb{H}^2}^2 = \mathbb{E}[M_0^2] + \mathbb{E}[\langle M\rangle_\infty].$$

Definition 2.5. If $M \in H^2$, we call $\mathcal{L}^2(M)$ the space of progressively measurable processes K such that:

$$\|K\|_M^2 := \mathbb{E}\left[\int_0^{+\infty} K_s^2 \, d\langle M\rangle_s\right] < +\infty.$$

We denote $L^2(M) = \mathcal{L}(M)/\sim$ with $\sim$ the equivalence relation given by the indistinguishability between processes.

Similarly to before, $L^2(M)$ is an Hilbert space for the norm $\|\cdot\|_M$.

Theorem 2.6. *([4], p. 137). Let $M \in H^2$; for each $K \in L^2(M)$, there is a unique element of H_0^2, denoted by $K \cdot M$, such that:*

$$\langle K \cdot M, N\rangle = K \cdot \langle M, N\rangle$$

for every $N \in H^2$. Furthermore, the map $K \to K \cdot M$ is an isometry from $L^2(M)$ into H_0^2.

Remark 2.7. Notice that the member $K \cdot \langle M, N\rangle$ is well-defined since $\langle M, N\rangle$ has a finite variation and so it is precisely the Stieltjes integral.

Definition 2.8. The martingale $K \cdot M$ is called the stochastic integral (or Itô integral) of K with respect to M and is also denoted:

$$(K \cdot M)_t =: \int_0^t K_s \, dM_s.$$

Let's state a sort of associativity property of Itô integral:

Proposition 2.9. *([4], p. 139). If $K \in L^2(M)$ and $H \in L^2(K \cdot M)$ then $HK \in L^2(M)$ and*

$$(HK) \cdot M = H \cdot (K \cdot M).$$

The following proposition shows how Itô integral commutes with optional stopping:

Proposition 2.10. *([4], p. 139). If T is a stopping time, $M \in H^2$ and $K \in L^2(M)$, then:*

$$K \cdot M^T = (K \cdot M)^T.$$

Given the close relationship between continuous martingales and continuous local martingales, we can proceed naturally as follows:

Definition 2.11. If M is a continuous local martingale, we call $L^2_{loc}(M)$ the space of classes of progressively measurable processes K for which there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times with $T_n \uparrow +\infty$ and such that

$$\mathbb{E}\left[\int_0^{T_n} K_s^2 \, d\langle M\rangle_s\right] < +\infty, \text{ for all } n \in \mathbb{N}.$$

Notice that, thanks to the continuity of the processes, this latter condition is equivalent to requiring

$$\int_0^t K_s^2 \, d\langle M\rangle_s < +\infty, \text{ for all } t \geq 0.$$

Theorem 2.12. *([4], p. 140). For any $K \in L^2_{loc}(M)$, there exists a unique continuous local martingale vanishing at 0 denoted $K \cdot M$ such that for any continuous local martingale N*

$$\langle K \cdot M, N\rangle = K \cdot \langle M, N\rangle.$$

Definition 2.13. The local continuous martingale $K \cdot M$ is called the stochastic integral (or Itô integral) of K with respect to M and is also denoted by:

$$(K \cdot M)_t =: \int_0^t K_s \, dM_s.$$

Keeping in mind the construction of the Stieltjes integral for processes of finite variation, it is not difficult to envision extending the construction of the stochastic integral to continuous semimartingales.

Definition 2.14. A progressively measurable process K is locally bounded if there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times with $T_n \uparrow +\infty$ and constants $(C_n)_{n\in\mathbb{N}}$ such that $|K^{T_n}| \leq C_n$.

Remark 2.15. Notice that all continuous adapted processes K are locally bounded by taking $T_n := \inf\{t : |K_t| \geq n\}$. Furthermore, locally bounded processes are a subset of $L^2_{loc}(M)$.

Definition 2.16. A $(\mathcal{F}_t, P)-$ semimartingale is a process X which admits a decomposition of the form:

$$X_t = X_0 + M_t + A_t \text{ for all } t \geq 0,$$

where M is a $(\mathcal{F}_t, P)-$local martingale and A an adapted process of bounded variation.
This process is a special semimartingale if A is also predictable. This is in particular the case for continuous semimartingale.

Definition 2.17. If K is a locally bounded processes and $X = M + A$ is a continuous semimartingale, the stochastic integral of K with respect to X is the continuous semimartingale

$$K \cdot X = K \cdot M + K \cdot A.$$

It is well-defined since $K \cdot M$ is the integral of Theorem 2.12 and $K \cdot A$ is the Stieltjes integral for processes of finite variation.

Proposition 2.18. *([4], p. 141). The map $K \to K \cdot X$ enjoys the following properties:*

1. *$(HK) \cdot M = H \cdot (K \cdot M)$ for any pair H,K of locally bounded processes;*
2. *$K \cdot M^T = (K \cdot M)^T$ for every stopping time T;*
3. *if X is a local martingale or a process of finite variation, so is $K \cdot X$.*

Let us now state and prove a fundamental result in the case where the integrator is the Brownian motion.

Theorem 2.19. *([4], p. 139). Let B be a $(\mathcal{F}_t, P) - BM$ standard, then:*

1. *if K is a progressively measurable process s.t.:*

$$\mathbb{E}\left[\int_0^t K_s^2 \, ds\right] < +\infty, \text{ for all } t \in [0, \infty),$$

then $K \cdot B$ is a martingale;

2. if K is progressively measurable with

$$\mathbb{E}\left[\int_0^{+\infty} K_s^2\, ds\right] < +\infty, \text{ for all } n \in \mathbb{N},$$

then $K \cdot B \in H_0^2$.

Proof. (1) Let $n \in \mathbb{N}$, then B^n is a martingale in H^2 and $K \in L^2(B^n)$, in fact:

$$\mathbb{E}\left[\int_0^{+\infty} K_s^2\, d\langle B^n\rangle_s\right] = \mathbb{E}\left[\int_0^{n} K_s^2\, ds\right] < +\infty.$$

So, $K \cdot B^n = (K \cdot B)^n \in H_0^2$ is a continuous martingale vanishing in 0. Given t, choose $n \geq t$, thus $(K \cdot B)_t = (K \cdot B)_t^n = (K \cdot B^n)_t$. That shows $(K \cdot B)_t$ is adapted and integrable. Furthermore, we have

$$\mathbb{E}[(K \cdot B)_t^n | \mathcal{F}_s] = (K \cdot B)_s^n$$

and so choosing $n \in \mathbb{N}$ such that $s \leq t \leq n$ we obtain

$$\mathbb{E}[(K \cdot B)_t | \mathcal{F}_s] = (K \cdot B)_s.$$

(2) It follows immediately from the Proposition 2.4 by observing that

$$\langle K \cdot B\rangle_t = \int_0^t K_s^2\, ds.$$

□

2.2 Itô's Formula

First of all, we state the integration by parts formula:

Proposition 2.20. *([4], p. 146). If X and Y are two continuous semimartingales, then*

$$X_tY_t = X_0Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t.$$

It can be written in "differential" form:

$$d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t.$$

Definition 2.21. A d-dimensional vector local martingale (vector continuous semimartingale) is a $\mathbb{R}^d$- valued process $X = (X^1, ..., X^d)$ such that each X^i is a local martingale (continuous semimartingale).

Theorem 2.22. *([4], p. 147). Let $X = (X^1, ..., X^d)$ be a continuous vector semimartingale and $F \in C^2(\mathbb{R}^d, \mathbb{R})$; then, $F(X)$ is a continuous semimartingale and*

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial F}{\partial x_i, x_j}\, d\langle X^i, X^j\rangle_s.$$

This equation is called Itô's Formula, and it can be stated also in "differential" form:

$$dF(X_t) = \sum_{i=1}^d \frac{\partial F}{\partial x_i}\, dX_t^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial F}{\partial x_i, x_j}\, d\langle X^i, Y^j\rangle_t.$$

Remark 2.23. The differentiability properties of F can be relaxed: if some X^i's are of finite variation, F needs only be of class C^1 in the corresponding coordinates.

2.3 Stochastic differential equation

In order to give a rigorous definition of a stochastic differential equation, we need to fix a precise setting.
We set $\mathbb{W}^d := C^0([0,+\infty), \mathbb{R}^d)$ and for all $s \geq 0$, $w_s : \mathbb{W}^d \to \mathbb{R}^d$ such that $w_s(\xi) = \xi(s)$. Hence, a function $f : [0,+\infty) \times \mathbb{W}^d \to \mathbb{R}^n$ is progressively measurable if $(f(t,\cdot))_{t\geq 0}$ is a stochastic process $(\mathcal{G}_t)$-progressively measurable where $\mathcal{G}_t := \sigma(w_s | s \leq t)$. Also, we will write $f(s, X(\omega))$ or $f(s, X)$ for the value taken by f at time s on the path $t \to X_t(\omega)$.

Definition 2.24. Given $f : [0,+\infty) \times \mathbb{W}^d \to \mathbb{R}^{d\times n}$ and $g : [0,+\infty) \times \mathbb{W}^d \to \mathbb{R}^d$ progressively measurable, a solution of the stochastic differential equation $e(f, g)$ is a pair (X, B) of adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that:

1. B is a standard $(\mathcal{F}_t, P)$-Brownian motion in $\mathbb{R}^n$;
2. for all $t \geq 0$:
$$\int_0^t |g_i(s,X)|\, ds < +\infty, \qquad \sum_{i,j=1}^{d} \int_0^t f_{i,j}^2(s,X)\, ds < +\infty \quad P - a.s.;$$
3. for all $i = 1, .., d$:
$$X_t^i = X_0^i + \int_0^t g_i(s,X)\, ds + \sum_{i=1}^{d} \int_0^t f_{i,j}(s,X)\, dB_s^j.$$

Furthermore we use the notation $e_x(f, g)$ if we impose the condition $X_0 = x$ P-a.s. on the solutions.

Definition 2.25. Given a stochastic differential equation $e(f, g)$, we say there is pathwise uniqueness if whenever (X, B) and (X', B') are two solutions defined on the same filtered space with $B = B'$ and $X_0 = X_0'$ *a.s.*, then X and X' are indistinguishable.
In addition, there is uniqueness in law if whenever (X, B) and (X', B') are two solutions defined possibly on different filtered spaces with $X_0 \overset{(d)}{=} X_0'$, then the laws of X and X' are equal.

Definition 2.26. Let (X, B) be a solution of $e(f, g)$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. It is said to be a strong solution if X is adapted to the filtration of B completed with respect to P. On the other hand, a solution which is not strong is termed a weak solution.

Now, we state an important result concerning the last two definitions, also known as "Yamada-Watanabe Theorem":

Theorem 2.27. *([4], p. 368). Let $e(f,g)$ be a stochastic differential equation. If pathwise uniqueness holds for $e(f,g)$, then:*

1. *uniqueness in law holds;*
2. *every solution to $e_x(f,g)$ is strong where $x \in \mathbb{R}$, i.e. the initial condition is deterministic.*

Definition 2.28. Let $e(f,g)$ be a stochastic differential equation. Let us say that f and g satisfy the hypotheses (IL) if there exists a $K > 0$ such that for all $\xi, \eta \in \mathbb{W}^d$ and for all $t \geq 0$:

$$\|f(t,\xi) - f(t,\eta)\| + \|g(t,\xi) - g(t,\eta)\| \leq K \sup_{s\in[0,t]} \|\xi(s) - \eta(s)\|,$$

and if $x \in \mathbb{R}^d$, denoting with $\eta^x \in \mathbb{W}^d$ the constant function in x, holds that $t \to f(t,\eta^x)$ and $t \to g(t,\eta^x)$ are locally bounded.

Remark 2.29. In the case $f(t,\eta) =: \sigma(\eta_t)$ and $g(t,\eta) =: b(\eta_t)$, the functions $t \to f(t,\eta^x) = \sigma(x)$ and $t \to g(t,\eta^x) = b(x)$ are constants and so locally bounded.

Theorem 2.30. *([4], p. 375). Let $x \in \mathbb{R}^d$ and $e_x(f,g)$ be fixed, and suppose that the hypotheses (IL) hold. Then, for every filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual hypotheses and for every B $(\mathcal{F}_t) - BM^n$ there exists a unique (up to indistinguishability) process X such that (X^x, B) is a strong solution of $e_x(f,g)$.*

Theorem 2.31. *([4], p. 379). In the same setting of Theorem 2.30, if f and g are also bounded, there exists a process X_t^x, $x \in \mathbb{R}^d$, $t \in [0,\infty)$, with continuous paths with respect to both variables t and x such that (X^x, B) is a strong solution of $e_x(f,g)$.*

3 Growth Optimal Portfolio (GOP)

3.1 Continuous Financial Markets

In this section, we will construct a model to describe a financial market over the interval $[0,+\infty)$. First of all, we fix an augmented filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We suppose $\mathcal{F}_0$ to be the trivial sigma algebra, and $\mathcal{F}_t$ represents the information we know about the market at time t. Furthermore, let $W^k = \{W_t^k, t \in [0,+\infty), k \in \{1,...,d\}\}$ be the independent standard $(\mathcal{F}_t, P)$-Wiener process which in our model will represent uncertainty.
Now, we introduce $d+1$ *primary security accounts*:

- *savings account*: it is a stochastic process $S^0 = \{S_t^0, t \in [0,+\infty)\}$ that represents the price of a risk-free asset in our market. Its behavior is described by the following equation:

$$S_t^0 = \exp\left(\int_0^t r_s\, ds\right) < +\infty, \tag{1}$$

  where $r = \{r_t, t \in [0,+\infty)\}$ is called the *short rate*.

- *risky accounts*: they are d nonnegative stochastic processes $S^j = \{S_t^j, t \in [0,+\infty)\}$ for $j \in \{1,...,d\}$ representing the value of the risky accounts. They satisfies the stochastic differential equation:

$$dS_t^j = S_t^j \left(a_t^j dt + \sum_{k=1}^d b_t^{j,k} dW_t^k\right) \tag{2}$$

  with $S_0^j > 0$ for $j \in \{1,...,d\}$. The process $a^j = \{a_t^j, t \in [0,+\infty)\}$ for $j \in \{1,...,d\}$ is called *appreciation rate*, while the process $b^{j,k} = \{b_t^{j,k}, t \in [0,+\infty)\}$ is the volatility of the j-th risky account with respect to the k-th Wiener process. Notice that, from Definition 2.24 perspective, for the SDE to be well-defined, we must have for $t \in [0,+\infty)$:

$$\sum_{j=1}^d \int_0^t |a_s^j|\, ds < +\infty, \qquad \sum_{j,k=1}^d \int_0^t (b_s^{j,k})^2\, ds < +\infty \quad P-a.s.,$$

  and they also must be predictable i.e. measurable with respect to the sigma algebra of subsets in $\Omega \times \mathbb{R}_+$ generated by all $\mathcal{F}_t-$adapted processes with left-continuous trajectories.

Proposition 3.1. *The Equation (2) has a unique solution up to indistinguishability which is:*

$$S_t^j = S_0^j \exp\left(\int_0^t a_s^j\, ds + \sum_{k=1}^d \int_0^t b_s^{j,k}\, dW_s^k - \frac{1}{2}\sum_{k=1}^d \int_0^t (b_s^{j,k})^2\, ds\right), \tag{3}$$

for all $j \in \{1,...,d\}$.

We will denote in a natural way:

- $\mathbf{S} := \{\mathbf{S}_t = (S_t^0, S_t^1, ..., S_t^d)^{\mathrm{T}}, t \in [0, +\infty)\}$;
- $\mathbf{a} := \{\mathbf{a}_t = (a_t^1, ..., a_t^d)^{\mathrm{T}}, t \in [0, +\infty)\}$;
- $\mathbf{b} := \{\mathbf{b}_t = [b_t^{j,k}]_{j,k=1}^d, t \in [0, +\infty)\}$.

Before defining a continuous financial market, it is useful to make a fundamental assumption throughout the discussion:

Assumption 3.2. *The covariance matrix* $\mathbf{b}_t^2$ *is symmetric positive definite for every* $t \in [0, +\infty)$*. In particular,* $\mathbf{b}_t = \sqrt{\mathbf{b}_t^2}$ *is symmetric positive definite hence invertible for every* $t \in [0, +\infty)$*.*

Definition 3.3. A continuous financial market (CFM) is a tuple $\mathcal{S}_{(d)}^C = (\mathbf{S}, \mathbf{a}, \mathbf{b}, r, (\mathcal{F}_t)_{t \geq 0}, P)$ that satisfies Assumption 3.2.

Definition 3.4. We denote by θ_t^k the k-th *market price of risk* with respect to W^k that is defined as:

$$\theta_t^k := \sum_{j=1}^d (b_t^{-1})^{k,j} \left(a_t^j - r_t \right). \tag{4}$$

Defining $\boldsymbol{\theta}_t := (\theta_t^1, ..., \theta_t^d)^{\mathrm{T}}$ and $\mathbf{1} := (1, ..., 1)^{\mathrm{T}}$, we have:

$$\boldsymbol{\theta}_t := (\mathbf{b}_t^{-1})(\mathbf{a}_t - r_t \mathbf{1}). \tag{5}$$

In particular, we obtain that $\mathbf{a}_t = \mathbf{b}_t \boldsymbol{\theta}_t + r_t \mathbf{1}$ and so $a_t^j = \sum_{k=1}^d b_t^{j,k} \theta_t^k + r_t$. Hence, we can rewrite (2) as:

$$dS_t^j = S_t^j \left(r_t dt + \sum_{k=1}^d b_t^{j,k} (\theta_t^k dt + dW_t^k) \right). \tag{6}$$

Definition 3.5. Let $\boldsymbol{\delta} := \{\boldsymbol{\delta}_t = (\delta_t^0, \delta_t^1, ..., \delta_t^d)^{\mathrm{T}}, t \in [0, +\infty)\}$ be a predictable stochastic process such that the stochastic integral:

$$\int_0^t \delta_s^j dS_s^j$$

is well-defined for every $j \in \{0, 1, ..., d\}$. A process of this kind is called *strategy* and its j-th component represents the quantity of the j-th primary security account held in the *portfolio* $V(\delta) := \{V_t(\delta), t \in [0, +\infty)\}$ at time t. In a completely natural way, we define the value of the portfolio at time t:

$$V_t(\delta) := \sum_{j=0}^d \delta_t^i S_t^j. \tag{7}$$

Now, we would like to narrow our study to portfolios whose value changes solely based on the performance of the primary security accounts; for this reason, we give the following definition and assumption:

Definition 3.6. A portfolio $V_t(\delta)$ is *self-financing* if:

$$dV_t(\delta) := \sum_{j=0}^{d} \delta_t^i dS_t^j. \tag{8}$$

Assumption 3.7. *All considered portfolios are self-financing.*

Proposition 3.8. *([2], p. 371). The value of a self-financing portfolio at time t is:*

$$dV_t(\delta) = V_t(\delta) r_t dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \delta_t^j S_t^j b_t^{j,k} (\theta_t^k dt + dW_t^k). \tag{9}$$

Furthermore, let us consider the set $\mathcal{V}^+ = \{V(\delta) : V_t(\delta) > 0 \text{ for all } t \in [0, +\infty)\}$. For a portfolio in $\mathcal{V}^+$, it makes sense to define the *fraction* $\pi_t^j(\delta)$ of $V_t(\delta)$ that is invested in S_t^j at time t. So, we have:

$$\pi_t^j(\delta) := \delta_t^j \frac{S_t^j}{V_t(\delta)}, \qquad j \in \{0, 1, ..., d\}. \tag{10}$$

Remark 3.9. From the definition, it holds:

$$\sum_{j=0}^{d} \pi_t^j(\delta) = 1. \tag{11}$$

Considering (9) and factoring out the term $V_t(\delta)$, we obtain the following corollary of the Proposition 3.8:

Corollary 3.10. *([2], p. 371). The value of a self-financing portfolio in $\mathcal{V}^+$ at time t satisfies:*

$$dV_t(\delta) = V_t(\delta) \left(r_t dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} (\theta_t^k dt + dW_t^k) \right). \tag{12}$$

Remark 3.11. From (12), we notice that a strictly positive self-financing portfolio is completely determined by its fractions and its initial value; therefore, we will also use the notation $V(\pi) := V(\delta)$.

3.2 Growth Optimal Portfolio in CFM

To arrive at the definition of *GOP*, we first need to define the *growth rate* of a portfolio:

Proposition 3.12. *([2], p. 372). Let $V(\pi)$ a self-financing portfolio in $\mathcal{V}^+$, then:*

$$d\ln(V_t(\pi)) = g_t(\pi) dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} dW_t^k, \tag{13}$$

with growth rate $g_t(\pi)$ that satisfies the equation:

$$g_t(\pi) = r_t + \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} \theta_t^k - \frac{1}{2} \left(\sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} \right)^2 \right). \tag{14}$$

Proof. We can rewrite Equation (12) as:

$$V_t(\pi) = X_t + \sum_{k=1}^{d} \int_0^t V_s(\pi) \sum_{j=1}^{d} \pi_s^j(\delta) b_s^{j,k} dW_s^k,$$

where:

$$X_t := V_0(\pi) + \int_0^t V_s(\pi) \left(r_s + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_s^j(\delta) b_s^{j,k} \theta_s^k \right) ds.$$

Hence:

$$\begin{aligned}
d\langle V(\pi)\rangle_t &= d\langle \sum_{k=1}^{d} \int_0^t V_s(\pi) \sum_{j=1}^{d} \pi_s^j(\delta) b_s^{j,k} dW_s^k \rangle_t \\
&= d \left(\sum_{k,i=1}^{d} \langle \int_0^t V_s(\pi) \sum_{j=1}^{d} \pi_s^j(\delta) b_s^{j,k} dW_s^k, \int_0^t V_s(\pi) \sum_{j=1}^{d} \pi_s^j(\delta) b_s^{j,i} dW_s^i \rangle_t \right) \\
&= d \left(\sum_{k,i=1}^{d} \left(\left(V_t(\pi) \sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} \right) \left(V_t(\pi) \sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,i} \right) \cdot \langle W^k, W^i \rangle_t \right) \right) \\
&= d \left(\sum_{k=1}^{d} \left(V_t(\pi)^2 \left(\sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} \right)^2 \cdot t \right) \right) \\
&= V_t(\pi)^2 \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} \right)^2 dt,
\end{aligned}$$

where we respectively used the fact that X is a finite variation process, the bilinearity of quadratic covariation, the characterization of stochastic integral, the fact that $\langle W^k, W^i \rangle_t = t\delta_{k,i}$ (Kronecker delta), and the associativity of Itô integral. Using Theorem 2.22 we obtain:

$$\begin{aligned}
d\ln(V_t(\pi)) &= \frac{1}{V_t(\pi)} dV_t(\pi) - \frac{1}{2V_t(\pi)^2} d\langle V(\pi)\rangle_t \\
&= r_t dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} (\theta_t^k dt + dW_t^k) - \frac{1}{2} \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} \right)^2 dt \\
&= g_t(\pi) dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_t^j(\delta) b_t^{j,k} dW_t^k.
\end{aligned}$$

□

Definition 3.13. In a CFM let $V(\pi^*) \in \mathcal{V}^+$ be a self-financing portfolio. $V(\pi^*)$ is said to be the *growth optimal portfolio* (or *GOP*) if for all $V(\pi) \in \mathcal{V}^+$ self-financing and for all $t \in [0, +\infty)$, it holds:

$$g_t(\pi^*) \geq g_t(\pi) \quad P - a.s. \tag{15}$$

Proposition 3.14. *([2], p. 373). $V(\pi)$ is a GOP if:*

$$\begin{cases} \pi_t^j = \sum_{k=1}^d (b_t^{-1})^{j,k} \theta_t^k, \quad \text{for } j = 1, ..., d \\ \pi_t^0 = 1 - \sum_{j=1}^d \pi_t^j. \end{cases} \tag{16}$$

Proof. In view of Remark 3.9 the second condition follows immediately from the first one. So, we are dealing with an unconstrained optimization problem where we want to find the maximum of $g_t(\pi)$ with respect to $\pi_t^1, ..., \pi_t^d$. Computing the first order conditions using Equation (14), we have:

$$\begin{aligned} \frac{\partial g_t(\pi)}{\partial \pi_t^j} &= \sum_{k=1}^d \left(b_t^{j,k} \theta_t^k - \left(\sum_{m=1}^d \pi_t^m b_t^{m,k} \right) b_t^{j,k} \right) \\ &= \sum_{k=1}^d b_t^{j,k} \left(\theta_t^k - \sum_{m=1}^d \pi_t^m b_t^{m,k} \right) = 0, \end{aligned}$$

for all $j \in \{1, .., d\}$. So we come to the most compact vector writing:

$$\mathbf{b}_t(\boldsymbol{\theta}_t - \mathbf{b}_t^{\mathrm{T}} \boldsymbol{\pi}_t) = \mathbf{0} \iff \boldsymbol{\theta}_t = \mathbf{b}_t^{\mathrm{T}} \boldsymbol{\pi}_t \iff \boldsymbol{\pi}_t = \mathbf{b}_t^{-\mathrm{T}} \boldsymbol{\theta}_t$$

that returning to scalar writing is precisely:

$$\pi_t^j = \sum_{k=1}^d (b_t^{-1})^{j,k} \theta_t^k,$$

for all $j \in \{1, .., d\}$. Now, let us calculate the Hessian matrix:

$$\frac{\partial g_t(\pi)}{\partial \pi_t^j, \pi_t^i} = - \sum_{k=1}^d b_t^{j,k} b_t^{i,k} = -(b_t^2)_{j,i}.$$

Hence, the Hessian is $-b_t^2$ and so is negative definite. In particular, it means that $g_t(\pi)$ is strictly concave with respect to π. Therefore, the first order conditions we have are sufficient (and necessary) for finding a global maximum point.

□

Corollary 3.15. *([2], p. 373). Let $V_t(\pi^*)$ the GOP. Then, its unique solution up to its initial value satisfies the SDE:*

$$dV_t(\pi^*) = V_t(\pi^*) \left(r_t dt + \sum_{k=1}^d \theta_t^k (\theta_t^k dt + dW_t^k) \right). \tag{17}$$

3.3 Numéraire portfolio and GOP

Definition 3.16. Let $V(\bar{\pi}) \in \mathcal{V}^+$ be a self-financing portfolio; it is a *numérarire portfolio* if the process:

$$\frac{V_t(\pi)}{V_t(\bar{\pi})}$$

is a supermartingale for any nonnegative portfolio $V_t(\pi)$.

Given the definition of GOP, it is natural to try to "evaluate" a portfolio $V(\pi)$ with respect to $V(\pi^*)$. This is what is called *benchmark approach.*

Definition 3.17. Let $V(\pi^*)$ be the GOP. We define the *benchmarked portfolio value* of a portfolio $V(\pi)$ the process:

$$\hat{V}_t(\pi) = \frac{V_t(\pi)}{V_t(\pi^*)}. \tag{18}$$

for $t \in [0, +\infty)$. In the same spirit, the *benchmarked price* process is:

$$\hat{S}_t^j = \frac{S_t^j}{V_t(\pi^*)} \tag{19}$$

for $j \in \{0, ...d\}$ and $t \in [0, +\infty)$.

Proposition 3.18. *([2], p. 375). The benchmarked portfolio value of a portfolio satisfies the SDE:*

$$d\hat{V}_t(\pi) = \hat{V}_t(\pi) \left(\sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_t^j b_t^{j,k} - \theta_t^k \right) dW_t^k \right). \tag{20}$$

Proof. Using the Itô's formula, we obtain:

$$d\left(\frac{V_t(\pi)}{V_t(\pi^*)}\right) = \frac{dV_t(\pi)}{V_t(\pi^*)} - \frac{V_t(\pi)}{(V_t(\pi^*))^2} dV_t(\pi^*) - \frac{d\langle V(\pi), V(\pi^*)\rangle_t}{(V_t(\pi^*))^2} + \frac{V_t(\pi)}{(V_t(\pi^*))^3} d\langle V(\pi^*)\rangle_t.$$

In the same spirit as in the proof of Proposition 3.12, in view of (12) and (17) we have:

$$d\langle V(\pi), V(\pi^*)\rangle_t = V_t(\pi)V_t(\pi^*) \sum_{k=1}^{d} \theta_t^k \left(\sum_{j=1}^{d} \pi_t^j b_t^{j,k} \right) dt;$$

$$d\langle V(\pi^*)\rangle_t = (V_t(\pi^*))^2 \sum_{k=1}^{d} (\theta_t^k)^2 dt.$$

Now, computing the four terms on the right-hand side in Itô's expansion:

$$\frac{dV_t(\pi)}{V_t(\pi^*)} = \frac{V_t(\pi)}{V_t(\pi^*)} \left(r_t dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_t^j b_t^{j,k} (\theta_t^k dt + dW_t^k) \right);$$

$$\frac{V_t(\pi)}{(V_t(\pi^*))^2} dV_t(\pi^*) = \frac{V_t(\pi)}{V_t(\pi^*)} \left(r_t dt + \sum_{k=1}^{d} \theta_t^k (\theta_t^k dt + dW_t^k) \right);$$

$$\frac{d\langle V(\pi), V(\pi^*)\rangle_t}{(V_t(\pi^*))^2} = \frac{V_t(\pi)}{V_t(\pi^*)} \sum_{k=1}^{d} \theta_t^k \left(\sum_{j=1}^{d} \pi_t^j b_t^{j,k} \right) dt;$$

$$\frac{V_t(\pi)}{(V_t(\pi^*))^3}d\langle V(\pi^*)\rangle_t = \frac{V_t(\pi)}{V_t(\pi^*)}\sum_{k=1}^{d}(\theta_t^k)^2 dt.$$

So,

$$d\hat{V}_t(\pi) = d\left(\frac{V_t(\pi)}{V_t(\pi^*)}\right) = \frac{V_t(\pi)}{V_t(\pi^*)}\left(\sum_{k=1}^{d}\left(\sum_{j=1}^{d}\pi_t^j b_t^{j,k} - \theta_t^k\right)dW_t^k\right).$$

□

Remark 3.19. Proposition 3.18 holds crucial importance in the course of the discussion: the SDE describing the behavior of the benchmarked portfolio is drift-less, and therefore, by Theorem 2.12, it follows that $\hat{V}_t(\pi)$ is a local martingale.

Proposition 3.20. *Let M be a non-negative local martingale, then it is a supermartingale.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of stopping time such that $T_n \uparrow +\infty$ and M^{T_n} is a martingale for each $n \in \mathbb{N}$. Since M is non-negative, we can use Fatou's Lemma:

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\liminf_{n\to\infty} M_{t\wedge T_n}|\mathcal{F}_s] \leq \liminf_{n\to\infty}\mathbb{E}[M_{t\wedge T_n}|\mathcal{F}_s] = \liminf_{n\to\infty} M_{s\wedge T_n} = M_s,$$

for every $s \leq t < +\infty$.

□

As an immediate corollary, we thus obtain the following proposition:

Proposition 3.21. *([2], p. 375). In a CFM, the GOP is a numéraire portfolio, i.e. the benchmarked portfolio value is a supermantigale for every self-financing portfolio in $\mathcal{V}^+$.*

3.4 GOP without a riskless asset

In this section, we will analyze how the conditions for the existence of the growth optimal portfolio change when the risk-free asset S_t^0 is absent from the market.

First of all, let us establish some notation to have a more compact expression for the d risky security accounts. In particular, we write

$$\frac{dS_t}{S_t} = a_t dt + b_t \cdot dW_t \tag{21}$$

for the d-vector of stochastic differentials $\left(\frac{dS_t^j}{S_t^j}\right)$ for $j = 1, ..., d$, using the "$\cdot$" to emphasize that we are performing a matrix-vector product.
Now, by naturally adapting the definitions from Section 3.1, we have:

- $V_t(\delta) = \sum_{j=1}^{d}\delta_t^j S_t^j$ (*value of a portfolio*);
- $dV_t(\delta) = \sum_{j=1}^{d}\delta_t^j dS_t^j$ (*self-financing condition*).

Thus:

$$\frac{dV_t(\delta)}{V_t(\delta)} = \sum_{j=1}^{d} \frac{\delta_t^j}{V_t(\delta)} dS_t^j = \sum_{j=1}^{d} \frac{\delta_t^j S_t^j}{V_t(\delta)} \frac{dS_t^j}{S_t^j} = \sum_{j=1}^{d} \pi_t^j \frac{dS_t^j}{S_t^j}.$$

Hence, in the same spirit of Equation 21 we have the equation for a self-financing portfolio in the form:

$$\frac{dV_t(\pi)}{V_t(\pi)} = \pi_t^{\mathrm{T}} \cdot \frac{dS_t}{S_t} = \pi_t^{\mathrm{T}} \cdot a_t dt + \pi_t^{\mathrm{T}} \cdot b_t \cdot dW_t. \tag{22}$$

Proposition 3.22. *([1], p. 43). Let $V(\pi) \in \mathcal{V}^+$ a self-financing portfolio in a CFM without a riskless asset, then:*

$$d\ln(V_t(\pi)) = g_t(\pi)dt + \pi_t^{\mathrm{T}} \cdot b_t \cdot dW_t \tag{23}$$

with growth rate $g_t(\pi)$ that satisfies the equation:

$$g_t(\pi) = \pi_t^{\mathrm{T}} \cdot a_t - \frac{1}{2}\pi_t^{\mathrm{T}} \cdot b_t^2 \cdot \pi_t. \tag{24}$$

Proof. By Itô's formula:

$$d\ln(V_t(\pi)) = \frac{1}{V_t(\pi)} dV_t(\pi) - \frac{1}{2V_t(\pi)^2} d\langle V(\pi)\rangle_t,$$

and using Equation 22 and the same computation in the proof of Proposition 3.12 for $d\langle V(\pi)\rangle_t$, we obtain:

$$d\ln(V_t(\pi)) = \pi_t^{\mathrm{T}} \cdot a_t + \pi_t^{\mathrm{T}} \cdot b_t \cdot dW_t - \frac{1}{2}\pi_t^{\mathrm{T}} \cdot b_t^2 \cdot \pi_t dt,$$

where we have used the symmetry of b_t to write:

$$\sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_t^j b_t^{j,k} \right)^2 = \pi_t^{\mathrm{T}} \cdot b_t \cdot b_t^{\mathrm{T}} \cdot \pi_t = \pi_t^{\mathrm{T}} \cdot b_t^2 \cdot \pi_t.$$

□

Definition 3.23. In a CFM without a riskless asset, let $V(\pi^*) \in \mathcal{V}^+$ be a self-financing portfolio. $V(\pi^*)$ is said to be the *growth optimal portfolio* (or *GOP*) if for all $V(\pi) \in \mathcal{V}^+$ self-financing and for all $t \in [0, +\infty)$, it holds:

$$g_t(\pi^*) \geq g_t(\pi) \quad P - a.s. \tag{25}$$

Lemma 3.24. *([1], p. 43) Let M be a $(d+1) \times (d+1)$ matrix of the form:*

$$M = \begin{pmatrix} N \cdot N^{\mathrm{T}} & \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} & 0 \end{pmatrix}.$$

It holds that $\ker(M) = (\ker(N \cdot N^{\mathrm{T}}) \cap \ker(\mathbf{1}^{\mathrm{T}})) \oplus \{0\} = (\ker(N^{\mathrm{T}}) \cap \ker(\mathbf{1}^{\mathrm{T}})) \oplus \{0\}$. *In particular, if $N \cdot N^{\mathrm{T}}$ is invertible then M is also non-singular.*

Proof. On one hand, let $x \in \ker(N \cdot N^{\mathrm{T}}) \cap \ker(\mathbf{1}^{\mathrm{T}})$, so $(x^{\mathrm{T}}, 0)^{\mathrm{T}} \in \ker(M)$. On the other hand, suppose $(x^{\mathrm{T}}, k)^{\mathrm{T}} \in \ker(M)$.
We obtain that $N \cdot N^{\mathrm{T}} \cdot x + k\mathbf{1} = 0$ and $\mathbf{1}^{\mathrm{T}} \cdot x = x^{\mathrm{T}} \cdot \mathbf{1} = 0$. So, multiplying the first equation by x^{T}, we have $0 = x^{\mathrm{T}} \cdot N \cdot N^{\mathrm{T}} \cdot x + kx^{\mathrm{T}} \cdot \mathbf{1} = \|N^{\mathrm{T}} \cdot x\|_2^2$. Hence $x \in \ker(N^{\mathrm{T}})$ and $k = 0$. From well-known results in linear algebra, we know that $\mathbb{R}^d = \ker(N^{\mathrm{T}}) \oplus \mathrm{Im}(N)$ and so $\ker(N^{\mathrm{T}}) = \ker(N \cdot N^{\mathrm{T}})$.

□

Theorem 3.25. *([1], p. 43) A portfolio $V(\pi)$ is a GOP in a CFM without riskless asset if and only if for a number λ_t holds:*

$$\begin{pmatrix} b_t^2 & \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \pi_t \\ \lambda_t \end{pmatrix} = \begin{pmatrix} a_t \\ 1 \end{pmatrix}, \tag{26}$$

for all $t \in [0, +\infty)$.

Proof. We have the following constrained problem:

$$\begin{cases} \max_{\pi_t \in \mathbb{R}^d} g_t(\pi) \\ \pi_t^{\mathrm{T}} \cdot \mathbf{1} = 1. \end{cases} \tag{27}$$

So, we can write the Lagrangian function:

$$\Lambda(\pi_t, \lambda_t) = g_t(\pi) + \lambda_t(1 - \pi_t^{\mathrm{T}} \cdot \mathbf{1}) = \pi_t^{\mathrm{T}} \cdot a_t - \frac{1}{2}\pi_t^{\mathrm{T}} \cdot b_t^2 \cdot \pi_t + \lambda_t(1 - \pi_t^{\mathrm{T}} \cdot \mathbf{1}).$$

Hence, we have the first order condition:

$$\nabla_{\pi_t}\Lambda(\pi_t, \lambda_t) = a_t - b_t^2 \cdot \pi_t - \lambda_t\mathbf{1} = 0,$$

This last equation and the constraint can be written in a compact manner as:

$$\begin{pmatrix} b_t^2 & \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \pi_t \\ \lambda_t \end{pmatrix} = \begin{pmatrix} a_t \\ 1 \end{pmatrix}. \tag{28}$$

Furthermore, we note that this is a quadratic problem in the variable π_t with negative coefficient $-\frac{1}{2}$, so the first order condition is necessary and sufficient to guarantee a maximum.

□

Remark 3.26. According to Lemma 3.24, under the Assumption 3.2, the square matrix b_t^2 is invertible and so the first order condition has a solution.

Even without the assumption regarding the invertibility of b_t^2, however, the value of the GOP portfolio is unique:

Proposition 3.27. *([1], p. 44) Let $((\pi_t^*)^{\mathrm{T}}, \lambda_t)^{\mathrm{T}}$ a solution of Equation 26. Given $\zeta_t := b_t \cdot \pi_t^*$, the value $V_t(\pi^*)$ of the GOP is unique, for some fixed initial value $V_0(\pi^*) > 0$, and satisfies the SDE:*

$$\frac{dV_t(\pi^*)}{V_t(\pi^*)} = \lambda_t dt + \zeta_t^{\mathrm{T}} \cdot (\zeta_t dt + dW_t). \tag{29}$$

Proof. Combining the Equations 22 and 26 together, we obtain:

$$\begin{aligned}\frac{dV_t(\pi^*)}{V_t(\pi^*)} &= (\pi_t^*)^{\mathrm{T}} \cdot (b_t^2 \cdot \pi_t^* + \lambda_t \mathbf{1})dt + (\pi_t^*)^{\mathrm{T}} \cdot b_t \cdot dW_t \\ &= (\pi_t^*)^{\mathrm{T}} \cdot b_t \cdot b_t \cdot \pi_t + \lambda_t (\pi_t^*)^{\mathrm{T}} \mathbf{1} dt + (\pi_t^*)^{\mathrm{T}} \cdot b_t \cdot dW_t \\ &= \zeta_t^{\mathrm{T}} \cdot \zeta_t dt + \lambda_t dt + \zeta_t^{\mathrm{T}} \cdot dW_t \\ &= \lambda_t dt + \zeta_t^{\mathrm{T}} \cdot (\zeta_t dt + dW_t).\end{aligned}$$

Now, let us suppose we have another solution of (26). It must be in the form:

$$\begin{pmatrix} \pi_t^* \\ \lambda_t \end{pmatrix} + \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \qquad \begin{pmatrix} x_t \\ y_t \end{pmatrix} \in \ker\left(\begin{pmatrix} b_t^2 & \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} & 0 \end{pmatrix}\right).$$

By Proposition 3.24, we obtain that $x_t \in \ker(b_t) \cap \ker(\mathbf{1}^{\mathrm{T}})$ and $y_t = 0$. In particular, $b_t \cdot (\pi_t^* + x_t) = b_t \cdot \pi_t^* + 0 = \zeta_t$. Hence, λ_t and ζ_t are unambiguously determined and for (29), $V_t(\pi^*)$ is also.

□

Remark 3.28. We have found a duality between the market with the riskless asset and the one without it. In fact, in the case where one of the assets in the previous proposition is risk-free, comparing the first equation in (16) and the definition of ζ_t, we have that $\zeta_t = \theta_t$. Hence, by (17) and (29) follows that $\lambda_t = r_t$.

Proposition 3.29. *([1], p. 44) Equation (22) can be written in the form:*

$$\frac{dV_t(\pi)}{V_t(\pi)} = \lambda_t dt + \pi_t^{\mathrm{T}} \cdot b_t \cdot (\zeta_t dt + dW_t). \tag{30}$$

Proof.

$$\begin{aligned}\frac{dV_t(\pi)}{V_t(\pi)} &= \pi_t^{\mathrm{T}} \cdot a_t dt + \pi_t^{\mathrm{T}} \cdot b_t \cdot dW_t = \pi_t^{\mathrm{T}} \cdot (\lambda_t \mathbf{1} + b_t \zeta_t)dt + \pi_t^{\mathrm{T}} \cdot b_t \cdot dW_t = \\ &= \lambda_t dt + \pi_t^{\mathrm{T}} \cdot b_t \cdot (\zeta_t dt + dW_t).\end{aligned}$$

□

As for the benchmark approach, even in this case, we find that the GOP portfolio is a numéraire portfolio. In fact, it holds:

Proposition 3.30. *([3], p. 25) The benchmarked portfolio value of a portfolio in a CFM without risk-free asset satisfies the SDE:*

$$\frac{d\hat{V}_t(\pi)}{\hat{V}_t(\pi)} = (\pi_t^{\mathrm{T}} \cdot b_t - \zeta_t^{\mathrm{T}}) \cdot dW_t. \tag{31}$$

Proof. From the Equations 29 and 22, it follows:

$$\begin{aligned} d\langle V(\pi), V(\pi^*)\rangle_t &= V_t(\pi) V_t(\pi^*) \zeta_t^{\mathrm{T}} \cdot (b_t \cdot \pi_t) dt; \\ d\langle V(\pi^*)\rangle_t &= (V_t(\pi^*))^2 \zeta_t^{\mathrm{T}} \cdot \zeta_t dt. \end{aligned} \tag{32}$$

So, by following the same steps in the proof of Proposition 20, using the Itô's formula, we obtain the result.

□

Proposition 3.31. *The benchmarked price process of the j-th stock satisfies the SDE:*

$$d\hat{S}_t^j = \hat{S}_t^j((b_t)_j - \zeta_t^{\mathrm{T}}) \cdot dW_t, \tag{33}$$

where $(b_t)_j$ is a row vector representing the j-th row of b_t.
Furthermore, we have the equality:

$$\frac{d\hat{V}_t(\pi)}{\hat{V}_t(\pi)} = \sum_{j=1}^{d} \pi_t^j \frac{d\hat{S}_t^j}{\hat{S}_t^j}. \tag{34}$$

Proof. By Equation 21, we obtain:

$$dS_t^j = S_t^j(a_t^j dt + (b_t)_j \cdot dW_t).$$

Hence, using Equation 29 we have that:

$$d\langle S_t^j, V(\pi^*)\rangle_t = S_t^j V_t(\pi^*)(b_t)_j \cdot \zeta_t dt.$$

In the same manner as the proof was conducted in Proposition 20, leveraging the equation $a_t^j = \lambda_t + (b_t)_j \cdot \zeta_t$ and the Equation 32 yield the first part of the proposition.

Concerning the second part, we arrive at the result through the following chain of equalities:

$$\begin{aligned}
\sum_{j=1}^{d} \pi_t^j \frac{d\hat{S}_t^j}{\hat{S}_t^j} &= \sum_{j=1}^{d} \pi_t^j((b_t)_j - \zeta_t^{\mathrm{T}}) \cdot dW_t \\
&= \sum_{j=1}^{d} \pi_t^j (b_t)_j \cdot dW_t - \sum_{j=1}^{d} \pi_t^j \zeta_t^{\mathrm{T}} \cdot dW_t \\
&= \pi_t^{\mathrm{T}} \cdot b_t \cdot dW_t - \zeta_t^{\mathrm{T}} \cdot dW_t \sum_{j=1}^{d} \pi_t^j \\
&= (\pi_t^{\mathrm{T}} \cdot b_t - \zeta_t^{\mathrm{T}}) \cdot dW_t = \frac{d\hat{V}_t(\pi)}{\hat{V}_t(\pi)}.
\end{aligned}$$

□

4 Diversification Theorem

4.1 Hierarchical diversification

In this section, we will describe the model on which the hierarchical diversification is based, as proposed in the article [3]. This model will then serve us to state the Diversification Theorem. Let us consider a CFM without risk-free assets, under the following assumptions:

- the stocks are classified into *hierarchical groupings* divided into $H \in \mathbb{N}^+$ *hierarchical levels*;
- let $0 < \underline{K} \leq \bar{K} < +\infty$ be fixed integers and M an integer striclty greater than one. Given a group within the hierarchy, we suppose there are at least $\underline{K}M$ and at most $\bar{K}M$ next lower level subgroups. In particular, this tells us that in our market there are at least $(\underline{K}M)^H$ and at most $(\bar{K}M)^H$ stocks;
- let $k_1 \in \{1, ..., \bar{K}M\}$, we set $W^{k_1} = \{W_t^{k_1}, t \in [0, +\infty)\}$ the k_1-th indipendent standard Brownian motion that represents uncertainty in the k_1-th group of the highest level of the hierarchy. In the same spirit, fixed $i \in \{1, ..., H-1\}$ and $k_1, ..., k_i \in \{1, ..., \bar{K}M\}$. Let $W^{k_1,...,k_i} = \{W_t^{k_1,...,k_i}, t \in [0, +\infty)\}$ be the $(k_1, ..., k_i)$-th indipendent standard Brownian motion that models the uncertainty driving the k_i-th group on the i-th highest level in the k_{i-1}-th group of the $(i-1)$-th highest level, et cetera. Lastly, let $k_1, ..., k_H \in \{1, ..., \bar{K}M\}$ and $W^{k_1,...,k_H} = \{W_t^{k_1,...,k_H}, t \in [0, +\infty)\}$ the $(k_1, ..., k_H)$-th indipendent standard Brownian motion that drives the k_H-th stock in the k_{H-1}-th group of the $(H-1)$-th highest level, et cetera.
- let $j = (j_1, ..., j_H) \in \Gamma_M = \{1, ..., \bar{K}M\}^H$ be the "label" identifying the benchmarked stock $\hat{S}_t^j$, i.e. the j_H-th benchmarked stock in the j_{H-1}-th group of the $(H-1)$-th highest level, et cetera. In order to capture the hierarchical structure of the market and in view of (33), we assume that $\hat{S}_t^j$ follows the SDE:

$$\begin{aligned} \frac{d\hat{S}_t^j}{\hat{S}_t^j} = \sum_{k_1=1}^{\bar{K}M} \Bigg(\psi_t^{j,k_1} dW_t^{k_1} + \sum_{k_2=1}^{\bar{K}M} \Bigg(\psi_t^{j,k_1,k_2} dW_t^{k_1,k_2} + \\ + ... + \sum_{k_H=1}^{\bar{K}M} \psi_t^{j,k_1,k_2,...,k_H} dW_t^{k_1,k_2,...,k_H} \Bigg)\Bigg). \end{aligned} \tag{35}$$

  In particular, if a group has fewer than $\bar{K}M$ elements in the subgroup at the lower level, we can simply set the respective volatility coefficients to zero. Furthermore, note that each stock is influenced by every uncertainty in our model, and it is the task of the volatility coefficients to describe the dependencies among the stocks.

Now, suppose that in the highest level of the hierarchy there are M_t groups. In the same way, let $M_t^{j_1}$ be the number of groups in the j_1-th group of the highest level; and so on. We set $M_t^{j_1,...,j_{H-1}}$ the number of stocks in the j_{H-1}-th groups of

the $(H-1)$-th highest level, et cetera. As we have previously stated, it must hold that $\underline{K}M \le M_t, M_t^{j_1}, ..., M_t^{j_1,...,j_{H-1}} \le \bar{K}M$.
The underlying idea of hierarchical diversification is inductive: to construct the $(j_1, ..., j_{i+1})$-th hierarchically weighted index we equal-weight the index we have formed in the previous step. Formalizing this idea, we arrive at the following definition:

Definition 4.1. We define recursively:

$$\begin{cases} \pi_t^{HWI,j_1} := \frac{1}{M_t} \\ \pi_t^{HWI,(j_1,...,j_{i+1})} := \frac{1}{M_t^{j_1,...,j_i}} \pi_t^{HWI,(j_1,...,j_i)} \end{cases} \tag{36}$$

for every $i \in \{1, ..., H-1\}$ and all $t \ge 0$.
So, for $i = H-1$, we have the *hierarchically weighted index* (HWI) of the $j = (j_1, ..., j_H)$-th stock:

$$\pi_t^{HWI,j} := \frac{1}{M_t} \frac{1}{M_t^{j_1}} \cdot ... \cdot \frac{1}{M_t^{j_1,...,j_{H-1}}} \tag{37}$$

for all $t \ge 0$.

Remark 4.2. In the case where there is no hierarchy, meaning each stock has the same weight, M_t becomes the total number of stocks N_t and so we obtain the *equally weighted index* (EWI) of the $j = (j_1, ..., j_H)$-th stock:

$$\pi_t^{EWI,j} := \frac{1}{N_t} \tag{38}$$

for all $t \ge 0$.

Remark 4.3. The main difference between HWI and EWI lies in the fact that the former utilizes the information that groups of stocks are driven by similar uncertainties, while the latter does not.

4.2 Diversification Theorem

First of all, denoting by $\pi_{M,t}^j$ the fraction invested in the j-th stock, from (34) we obtain for a benchmark portfolio:

$$\begin{aligned} \frac{d\hat{V}_t^{\pi_M}}{\hat{V}_t^{\pi_M}} &= \sum_{j\in\Gamma_M} \pi_{M,t}^j \frac{d\hat{S}_t^j}{\hat{S}_t^j} = \\ &= \sum_{k_1=1}^{\bar{K}M} \sum_{j\in\Gamma_M} \pi_{M,t}^j \psi_t^{j,k_1} dW_t^{k_1} + \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} \sum_{j\in\Gamma_M} \pi_{M,t}^j \psi_t^{j,k_1,k_2} dW_t^{k_1,k_2} + \\ &+ ... + \sum_{k_1=1}^{\bar{K}M} ... \sum_{k_H=1}^{\bar{K}M} \sum_{j\in\Gamma_M} \pi_{M,t}^j \psi_t^{j,k_1,k_2,...,k_H} dW_t^{k_1,k_2,...,k_H}. \end{aligned} \tag{39}$$

Proposition 4.4. *([3], p. 28) The logarithm of the benchmarked portfolio* $\hat{V}_t^{\pi_M}$ *follows the SDE:*

$$d\ln\Big(\hat{V}_t^{\pi_M}\Big) = \hat{g}_t^{\pi_M}\,dt + \frac{d\hat{V}_t^{\pi_M}}{\hat{V}_t^{\pi_M}}, \tag{40}$$

where $\hat{g}_t^{\pi_M}$ *is the expected instantaneous growth rate of* $\hat{V}_t^{\pi_M}$*:*

$$\begin{aligned}\hat{g}_t^{\pi_M} = -\frac{1}{2}\Bigg(&\sum_{k_1=1}^{\bar{K}M}\Big(\sum_{j\in\Gamma_M}\pi_{M,t}^{j}\psi_t^{j,k_1}\Big)^2 + \sum_{k_1=1}^{\bar{K}M}\sum_{k_2=1}^{\bar{K}M}\Big(\sum_{j\in\Gamma_M}\pi_{M,t}^{j}\psi_t^{j,k_1,k_2}\Big)^2 + \\ &+ ... + \sum_{k_1=1}^{\bar{K}M}...\sum_{k_H=1}^{\bar{K}M}\Big(\sum_{j\in\Gamma_M}\pi_{M,t}^{j}\psi_t^{j,k_1,k_2,...,k_H}\Big)^2\Bigg).\end{aligned} \tag{41}$$

Proof. By the Itô formula:

$$d\ln\Big(\hat{V}_t^{\pi_M}\Big) = \frac{d\hat{V}_t^{\pi_M}}{\hat{V}_t^{\pi_M}} - \frac{1}{2}\frac{d\langle\hat{V}^{\pi_M}\rangle_t}{(\hat{V}_t^{\pi_M})^2}.$$

Now, we set:

$$A_1 := \sum_{k_1=1}^{\bar{K}M}\sum_{j\in\Gamma_M}\pi_{M,t}^{j}\psi_t^{j,k_1}dW_t^{k_1};$$

$$\vdots$$

$$A_H := \sum_{k_1=1}^{\bar{K}M}...\sum_{k_H=1}^{\bar{K}M}\sum_{j\in\Gamma_M}\pi_{M,t}^{j}\psi_t^{j,k_1,k_2,...,k_H}dW_t^{k_1,k_2,...,k_H}.$$

To compute the covariation $d\langle\hat{V}^{\pi_M}\rangle_t$, we observe that:

$$d\langle A_i, A_j\rangle_t = 0,$$

for every $i,j\in\{1,...H\}$ with $i\neq j$; and:

$$d\langle A_i, A_i\rangle_t = \sum_{k_1=1}^{\bar{K}M}...\sum_{k_i=1}^{\bar{K}M}\Big(\sum_{j\in\Gamma_M}\pi_{M,t}^{j}\psi_t^{j,k_1,...,k_i}\Big)^2 dt,$$

for every $i\in\{1,...H\}$. Hence, for Equation 39 and bilinearity of the covariation we have the result.

□

Since the benchmarked GP portfolio $\hat{V}_t^{\pi_M^*} = 1$, its expected growth rate $\hat{g}_t^{\pi_M^*} = 0$. In this view, it makes sense to provide the following definition:

Definition 4.5. A sequence of benchmarked portfolio processes $(\hat{V}^{\pi_M})_{M\in\{2,3,..\}}$ is said to be a *sequence of benchmarked approximate GP processes* if:

$$\lim_{M\to+\infty}\hat{g}_t^{\pi_M} = 0 \tag{42}$$

almost surely for all $t\in[0,+\infty)$.

Now, in order to arrive at the main theorem of the section, we need to make an assumption about volatilities. Specifically, we must assume that the benchmarked stocks are not driven to a significant extent by the same uncertainty.

Assumption 4.6. *For given $k_1, k_2, ..., k_h \in \{1, 2, ..., \bar{K}M\}$ and for all $M \in \{2, 3, ...\}$ and all $h \in \{1, ..., H\}$, we assume:*

$$\sum_{j \in \Gamma_M} |\psi_t^{j,k_1,...,k_h}| \leq (\bar{K}M)^{H-h} \sigma_t, \tag{43}$$

where $\sigma = \{\sigma_t, t \in [0, +\infty)\}$ is an adapted stochastic process almost surely finite for all $t \in [0, +\infty)$.

Remark 4.7. The specific dynamics of the volatility processes do not need to be explicitly stated, we just want an upper bound for the sum of the absolute values of volatilities with respect to the same source of uncertainty.

At this point, we have all the ingredients to state and prove the Diversification Theorem:

Theorem 4.8 (Diversification Theorem)**.** *([3], p. 29) Under Assumption 4.6, let $(\hat{V}^{\pi_M})_{M \in \{2,3,..\}}$ be a sequence of benchmarked portfolio processes. It is a sequence of benchmarked approximate GP processes if for each $M \in \{2, 3, ..\}$ we have:*

$$\max_{j \in \Gamma_M} |\pi^j_{M,t}| \leq C M^{\xi - H} \tag{44}$$

for some $\xi \in [0, \frac{1}{2})$, $C \in (0, +\infty)$ and all $t \in [0, +\infty)$.

Proof. Starting from Equation (41), we obtain:

$$\begin{aligned}
\hat{g}_t^{\pi_M} &\geq -\frac{1}{2}\left(\max_{j\in\Gamma_M}|\pi^j_{M,t}|\right)^2 \Bigg(\sum_{k_1=1}^{\bar{K}M}\Bigg(\sum_{j\in\Gamma_M}|\psi_t^{j,k_1}|\Bigg)^2 + \sum_{k_1=1}^{\bar{K}M}\sum_{k_2=1}^{\bar{K}M}\Bigg(\sum_{j\in\Gamma_M}|\psi_t^{j,k_1,k_2}|\Bigg)^2 \\
&\quad + ... + \sum_{k_1=1}^{\bar{K}M} ... \sum_{k_H=1}^{\bar{K}M}\Bigg(\sum_{j\in\Gamma_M}|\psi_t^{j,k_1,k_2,...,k_H}|\Bigg)^2\Bigg) \\
&\geq -\frac{1}{2}\left(\max_{j\in\Gamma_M}|\pi^j_{M,t}|\right)^2 \sigma_t^2 \sum_{h=1}^{H}(\bar{K}M)^h((\bar{K}M)^{H-h})^2 \\
&\geq -\frac{1}{2}\left(\max_{j\in\Gamma_M}|\pi^j_{M,t}|\right)^2 \sigma_t^2 \bar{K}^{2H} \sum_{h=1}^{H}(M)^h((M)^{H-h})^2 \\
&= -\frac{1}{2}\left(\max_{j\in\Gamma_M}|\pi^j_{M,t}|\right)^2 \sigma_t^2 \bar{K}^{2H} M^{2H-1} \sum_{h=1}^{H}\left(\frac{1}{M}\right)^{h-1} \\
&\geq -\frac{1}{2}\left(\max_{j\in\Gamma_M}|\pi^j_{M,t}|\right)^2 \sigma_t^2 \bar{K}^{2H} \frac{M^{2H-1}}{1-\frac{1}{M}},
\end{aligned}$$

where in the various steps we have factored out the maximum of $|\pi^j_{M,t}|$; we have used Assumption 4.6 and that $-\frac{1}{2}\hat{K}^{2H-h} \geq -\frac{1}{2}\hat{K}^{2H}$ due to assumptions on $\hat{K}$ and

H. Furthermore, we have used the decomposition $2H - h = (2H - 1) - (h - 1)$ and the geometric series formula $\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r}$.
Now, we can use Equation 44:

$$\begin{aligned}
\hat{g}_t^{\pi_M} &\geq -\frac{1}{2} C^2 M^{2(\xi - H)} \sigma_t^2 \bar{K}^{2H} \frac{M^{2H-1}}{1 - \frac{1}{M}} \\
&= -\frac{1}{2} C^2 M^{2\xi - 1} \sigma_t^2 \bar{K}^{2H} \frac{1}{1 - \frac{1}{M}} \\
&\geq -C^2 M^{2\xi - 1} \sigma_t^2 \bar{K}^{2H}.
\end{aligned}$$

Since $\xi \in [0, \frac{1}{2})$, we obtain almost surely for all $t \in [0, +\infty)$ that

$$\lim_{M \to +\infty} \hat{g}_t^{\pi_M} \geq 0.$$

At the same time, however, due to (41), the expected instantaneous growth rate of a benchmarked (nonnegative) portfolio is at most zero. Hence, it must be:

$$\lim_{M \to +\infty} \hat{g}_t^{\pi_M} = 0.$$

□

Corollary 4.9. *([3], p. 29) The EWI and the HWI satisfy (44). Hence, both indexes form sequences of benchmarked approximate GP processes.*

Proof. For both indexes, we have that $\pi_{M,t}^j \leq \underline{K}^{-H} M^{-H}$ and so (44) holds with $C = K^{-H}$ and $\xi = 0$.

□

5 Discussion and Conclusion

5.1 Summary of Empirical Observations

In this paragraph, we will briefly summarize the empirical observations in [3], which will be useful for interpreting the theoretical results applied in a real-world context.

Specifically, Platen and Rendeck base their statistical tests on an HWI constructed from four hierarchical levels. Stocks are divided based on geographic region (Europe, Asia-Pacific, and North America), country, and industry type. At this point, portfolios are built using HWI, EWI, and the traditional market capitalization weighted index (MCI) are considered. For completeness, we remind the definition:

$$\pi_t^{MCI,j} := \frac{MV_t^j}{\sum_{k=1}^{N_t} MV_t^k},$$

where MV_t^j is the market value of the j-th stock at time t and N_t is the number of stocks. The rebalancing times $0 = t_0 < \cdots < t_i < t_{i+1} < \cdots$ are defined for a portfolio V_π (in the article they adopt a quarterly rebalancing frequency). Given an index π, the value of the portfolio at time t_i is given by:

$$V_{t_i}^{\pi} = V_{t_{i-1}}^{\pi} \left(1 + \sum_{j=1}^{N_{t_{i-1}}} \pi_{t_{i-1}}^{j} \frac{S_{t_i}^j - S_{t_{i-1}}^j}{S_{t_{i-1}}^j} \right)$$

for $i \in \{1, 2, \dots\}$ and $V_{t_0}^{\pi} = 1$.

Below are the key points of the statistical analysis:

- The graph in Fig. 1 immediately highlights how the value of the portfolio constructed with HWI consistently exceeds that of the other two, suggesting that HWI performs better than the other indices.

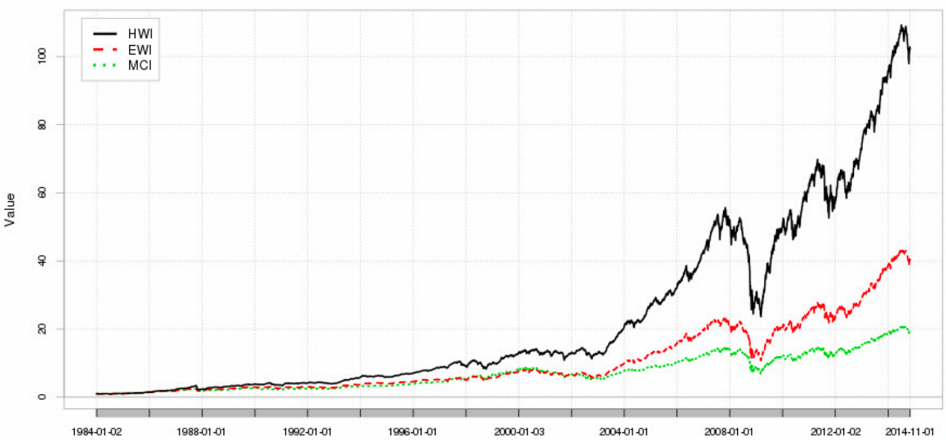

Figure 1: Values of HWI, EWI and MCI. *Source:* [3], p. 5.

- The analysis then moves to performance with respect to growth rate. Two new indices, HWI.c.r and HWI.r, are introduced; HWI.c.r is a hierarchically weighted index diversified by country and region, while HWI.r is diversified by region alone. Again, the results strengthen the idea that a deeper hierarchy yields better performance. In fact, as we can see in Fig. 2, the best index is HWI, followed by HWI.c.r, which in turn outperforms HWI.r, with EWI coming in last.

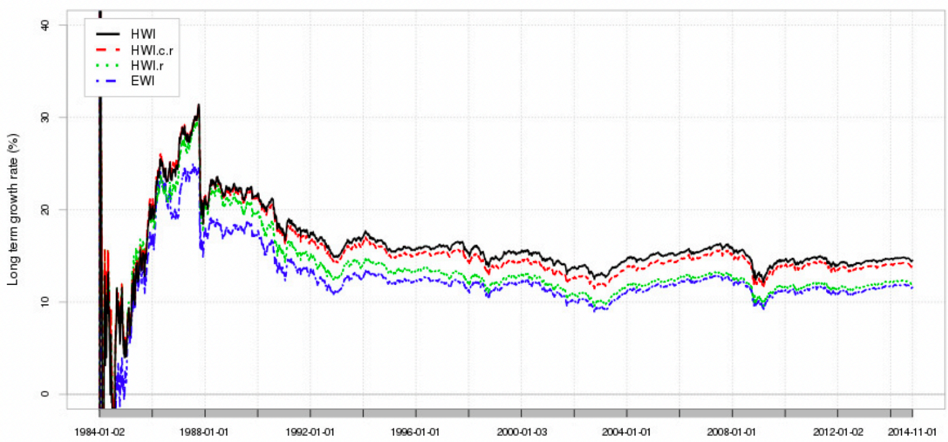

Figure 2: GRs in dependence of the final observation date for HWI, HWI.c.r., HWI.r., EWI. *Source:* [3], p. 12.

- We finally arrive at the most relevant statistical data of the section. Recall that the GOP is a numéraire portfolio, and thus from the definition of supermartingale, it follows that:

$$\mathbb{E}\left[\frac{\hat{V}_{t+h}(\pi) - \hat{V}_t(\pi)}{\hat{V}_t(\pi)}\right] \leq 0$$

for all $0 \leq t \leq t + h < \infty$. Thus, the expected return for a nonnegative benchmarked (if the GOP is used as benchmark) portfolio is never strictly positive. In view of that, the authors test the hypotheses:

$$H_0 : \mu \leq 0 \text{ versus } H_1 : \mu > 0$$

where μ is the "true" expected average of all daily returns of all benchmarked (a predefined portfolio is used as benchmark) stocks. Hence, they use a Z-test with a 1% significance level, varying the benchmark among the portfolios constructed with various indices (also considering transaction costs). In this way, they are able to reject the null hypothesis H_0 for the MCI and EWI. Furthermore, using the local martingale property for the GOP, they manage to reject the null hypothesis H_0 with a 99% confidence interval for all candidate proxies previously defined with the exception of the HWI.

5.2 Interpretation and Final Assessment

Drawing from the semester project's insights, we present an interpretation and summary of our findings.

The mathematical framework, especially the intricate nature of the model supporting the hierarchically weighted approximation of the numéraire portfolio, poses challenges for directly analyzing certain sustainable investment strategies. Furthermore, implicit assumptions about market actors' optimizing behavior removes the space for an explicit analysis of other behavior. The absence of exhaustive results on convergence rates of various approximation methods further complicate straightforward mathematical analysis.

However, empirical analyses, such as those discussed in Section 5.1, highlight the tangible impacts of sector diversification on portfolio growth rates over the medium to long term as the different weighting of sectors, which is implied by different hierarchical procedures, already leads to substantial differences in long term growth rates, even if none of the sectors are completely excluded. Avoiding diversification in specific sectors may result in substantial reductions in portfolio growth rates and consequential opportunity costs. In contrast, the best-in-class approach, which does not preclude sector diversification, clearly emerges as a more promising strategy in this context.

While the granularity of the mathematical model limits a direct mathematical analysis of the best-in-class approach, empirical evidence suggests its potential to reduce long-term opportunity costs compared to other sustainable investment strategies. Therefore, despite the mathematical challenge associated with the considered model, empirical evidence supports use of the best-in-class approach as a means to limit unutilized optimized asset growth potential in favor of long-term portfolio performance in cases where sustainability has to be incorporated into design of portfolios.

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