Exercises of Martingales in Financial Mathematics

Exercises Martingales in Financial Mathematics: Viable Financial Markets

Week 1, 2024

√ Exercise 1: Cash & Carry and Arbitrage

Consider the price S_t of an asset at time t for $t = 0,\ T > 0$. The risk-free interest rate (here percentage of the initial price) for the time interval $[0, T]$ is R_T. Derive the arbitrage-free price (forward price) F_0 for buying the asset at T if the contract is fixed at $t = 0$.

(a) If the price is paid at $t = 0$ (show that there is an arbitrage opportunity for $F_0 \neq S_0$ and conclude that $F_0 = S_0$).

(b) How should the price be modified if it will be paid at T?

(c) Assume that the asset will generate an income being given by a (positive) percentage R_d of the initial price and paid at T. What is the price under this assumption and if the price is paid at $t = 0$? How could a negative R_d be interpreted?

√ Exercise 2: Calls, Puts, and Arbitrage

Denote the risk-free interest rate for the time interval $[0, T]$ by R_T (as above), the price of the European call at $t = 0$ with maturity T and strike K by C and the price of the analogously specified European put by P, respectively.

(a) Show with the help of an arbitrage argument that

$$
\begin{aligned}
\max\left(0,\ S_0 - K\,(1 + R_T)^{-1}\right) &\le C \le \ S_0\,, \\
\max\left(0,\ K\,(1 + R_T)^{-1} - S_0\right) &\le P \le \ K(1 + R_T)^{-1}\,.
\end{aligned}
$$

(b) Derive the European put-call parity being given by $C - P = S_0 - K(1 + R_T)^{-1}$.

√ Exercise 3: European and American options

Assume that the price of the risk-free asset is given by $(1+r)^n$, where $r > 0$ is the corresponding strictly positive interest rate. Let C_n and C_n^A be the prices at time n of the European call and the American call on a non-dividend paying stock, respectively, with strike K and maturity N, $n \in \{1, \ldots, N\}$. Furthermore, denote the prices at time n of the analogously specified puts by P_n and P_n^A, respectively.

(a) Show with the help of an arbitrage argument that

$$\begin{aligned} C_n &\leq C_n^A, \\ P_n &\leq P_n^A. \end{aligned}$$

(b) Show that in an arbitrage-free market the equality $C_n = C_n^A$ holds (Hint. use the European put-call parity, i.e. $C_n + K/(1+r)^{N-n} = S_n + P_n$, cf. Ex. 2, and later Week 2, Ex. 2.1). Is the same true for the put options?

√ Exercise 4: Change of numéraire

Consider a financial market with (strictly) positive asset prices $S_n = (S_n^0, S_n^1, \ldots, S_n^d)$ at time n, where the filtration of the model is given by $\{\mathcal{F}_n\}_{n=0,\ldots,N}$. A sequence of positive random variables is called numéraire if

(i) the sequence $\{X_n\}_{n=0,\ldots,N}$ is adapted to the filtration $\{\mathcal{F}_n\}_{n=0,\ldots,N}$;

(ii) $X_0 = 1$ and $X_n > 0$, for all n;

(iii) $X_n = V_n(\theta)$, i.e. X_n is the value of a self-financing portfolio θ.

Denote the prices expressed in quantities of X_n by $S_n^X = S_n/X_n$.

(a) Describe the economical meaning of S_n^X.

(b) Can $Y_n = S_n^i/S_0^i$ for a fixed i be used as a numéraire?

(c) Show that the market extended by the asset $\{X_n\}_n$ is viable if and only if the initial market is viable.

(d) Consider a predictable strategy $\phi = \{\phi_n, n = 0, \ldots, N\}$. Show that this strategy is self-financing if and only if for all n:

$$V_n^X(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j, \cdot \left(S_j^X - S_{j-1}^X\right) .$$

(e) Show that there is at most one deterministic numéraire in a viable market.

√Exercise 1: Cash & Carry and Arbitrage

Consider the price S_t of an asset at time t for $t = 0, T > 0$. The risk-free interest rate (here percentage of the initial price) for the time interval $[0, T]$ is R_T. Derive the arbitrage-free price (forward price) F_0 for buying the asset at T if the contract is fixed at $t = 0$.

~~(a)~~ If the price is paid at $t = 0$ (show that there is an arbitrage opportunity for $F_0 \neq S_0$ and conclude that $F_0 = S_0$). ← We want to prove something obvious

~~(b)~~ How should the price be modified if it will be paid at T? → $S_0 R_d$

~~(c)~~ Assume that the asset will generate an income being given by a (positive) percentage R_d of the initial price and paid at T. What is the price under this assumption and if the price is paid at $t = 0$? How could a negative R_d be interpreted?

(a) Suppose $F_0 < S_0$: consider the following strategy:

Time $t=0$	CF
• Sell S_0	$+S_0$
• Buy S_0 for F_0	$-F_0$
• Invest $\mathcal{E} := S_0 - F_0$	$-(S_0 - F_0)$
TOT	0

At time $t=T$	CF
• I have sell S_0 so I have to buy S_T	$-S_T$
• I have S_T and I sell it	$+S_T$
• Close the investment	$+\mathcal{E}(1+R_T)$
TOT	$\mathcal{E}(1+R_T)$

Suppose $F_0 > S_0$: consider the following strategy:

Time $t=0$	CF
• Sell S_0 for F_0	$+F_0$
• Buy S_0	$-S_0$
• Invest $\mathcal{E} := F_0 - S_0$ in risk-free rate	$-\mathcal{E}$
TOT	0

Time $t=T$	CF
• Buy and sell S_T to close the open posit.	$-S_T + S_T$
• Take money from the bank	$\mathcal{E}(1+R_T)$
TOT	$\mathcal{E}(1+R_T)$

Observe that the portfolio is <u>self-financed</u> and the strategy is predictable (we decide everything at time $t=0$).

↓ we can also see it in this way:

$$V_T = \mathcal{E}(1+R_T) = \underbrace{(1 \cdot S_0 - 1 \cdot F_0 - 1 \cdot \mathcal{E})}_{V_0} + (-1)\underbrace{(S_T - S_0)}_{\Delta S} + (1)\underbrace{(S_T - S_0)}_{\Delta S} + \mathcal{E}\underbrace{((1+R_T) - 1)}_{\Delta \text{Bond}}$$

(b) To give an idea of what the answer is: if I'm selling the S_t asset and I'll have the payment at time T, the price has to be $S_0(1+R_T)$ because if I had the money S_0 at time $t=0$ I would have invest in the risk-free rate and I would have $S_0(1+R_T)$ at time T.

So, let's prove it:

$F_T < S_0(1+R_T)$: consider this strategy:

At time $t=0$	CF
• Short the asset	$+S_0$
• Buy S_0 for F_T	0 (I will pay at T)
• Invest S_0 in the rate	$-S_0$
TOT	0

At time $t=T$	CF
• Close the short pos.	$-S_T$
• Close the contract	$S_T - F_T$
• Take the money back for the rate	$S_0(1+R_T)$
TOT	$S_0(1+R_T) - F_T$

$F_T > S_0(1+R_T)$: consider this strategy:

At time $t=0$	CF
• Borrow S_0	$+S_0$
• Buy the asset	$-S_0$
• Short the forw. contract	0 (payment at T)
TOT	0

At time $t=T$	CF
• Close the forw. contract	$F_T - S_T$
• Sell asset	$+S_T$
• Give the money back	$-S_0(1+R_T)$
TOT	$F_T - S_0(1+R_T)$

$\tilde{c}$ Firstly we suppose an income given by R_d and the price will be paid at T.

We show that $F_T^d = S_0(1+R_T-R_d)$.

If $F_T^d > S_0(1+R_T-R_d)$:

At time $t=0$	CF
Buy S_0	$-S_0$
Borrow money	$+S_0$
Short the forward	0 ← will be pay at T
TOT	0

At time $t=T$	CF
Sell +income	$S_T + S_0 R_d$
Pay money back	$-S_0(1+R_T)$
Close the short position	$F_T^d - S_T$
	$F_T^d - S_0(1+R_T-R_d) > 0$

If $F_T^d < S_0(1+R_T-R_d)$:

At time $t=0$	CF
Sell S_0	$+S_0$
Pay S_0 in account	$-S_0$
Long pos. forward	0
TOT	0

At time $t=T$	CF
Close short posit.	$-S_T - S_0 R_d$
Take money back	$+S_0(1+R_T)$
Close long pos.	$S_T - F_T^d$
TOT	>0

c If we pay the contract at $t=0$ we have to discount F_T^d so we expect

$$F_0^d = \frac{S_0(1+R_T-R_d)}{(1+R_T)}$$

Suppose $F_0^d(1+R_T) < S_0(1+R_T-R_d)$, so:

At time $t=0$	CF
Sell S_0	$+S_0$
Buy F_0^d	$-F_0^d$
Pay $S_0-F_0^d$ into an account	$-(S_0-F_0^d)$
TOT	0

$\rightarrow S_0-F_0^d>0$

At time $t=T$	CF
Close short S_0	$-S_T-R_dS_0$
Close long F_0^d	S_T
Take money back	$+(S_0-F_0^d)(1+R_T)$
	>0

Suppose $F_0^d(1+R_T) > S_0(1+R_T-R_d)$, so:

At time $t=0$	CF
Buy S_0	$-S_0$
Sell F_0^d	$+F_0^d$
Put $F_0^d-S_0$ into account	$-(F_0^d-S_0)$
	0

At time $t=T$	CF
Close long	$+S_T+R_dS_0$
Close short	$-S_T$
Take money back	$+(F_0^d-S_0)(1+R_T)$
	>0

If $R_d<0$ it means that could be storage or transportation cost.

Exercise 2: Calls, Puts, and Arbitrage

Denote the risk-free interest rate for the time interval $[0,T]$ by R_T (as above), the price of the European call at $t=0$ with maturity T and strike K by C and the price of the analogously specified European put by P, respectively.

~~(a)~~ Show with the help of an arbitrage argument that

$$\max\left(0,\ S_0 - K\,(1+R_T)^{-1}\right) \le C \le S_0\,,$$
$$\max\left(0,\ K\,(1+R_T)^{-1} - S_0\right) \le P \le K(1+R_T)^{-1}\,.$$

~~(b)~~ Derive the European put-call parity being given by $\boxed{C - P = S_0 - K(1+R_T)^{-1}}$. Parity Call-Option

a) • $C \geq \left(S_0 - K\,(1+R_T)^{-1}\right)^+$: since the call gives a payoff ≥ 0, C has to be ≥ 0. Now let's see $C \geq S_0 - K(1+R_T)^{-1}$. If it is not $C < S_0 - K\,(1+R_T)^{-1}$, so:

At time $t=0$	CF
• Buy the call	$-C$
• Short the asset	$+S_0$
• Invest $K(1+R_T)^{-1}$	$-K\,(1+R_T)^{-1}$
TOT	$S_0 - C - K(1+R_T)^{-1} > 0$

At time $t=T$	CF
• Close the short	$-S_T$
• Take the money back	$+K$
• Payoff Call	$(S_T - K)^+$
TOT	$(S_T-K)^+ + K - S_T \geq 0$ (if $S_T \geq K$ is 0, if $S_T < K$ is $K - S_T$ that is positive)

• If $C > S_0$:

At time $t=0$	CF
• Sell the Call	$+C$
• Buy S_0	$-S_0$
TOT	$C - S_0 > 0$

we could also invest $C - S_0$ because we don't want cash

At time $t=T$	CF
• Close the long	S_T
• Close the short	$-(S_T - K)^+$
TOT	$S_T - (S_T - K)^+ \geq 0$

• If $P < K\,(1+R_T)^{-1} - S_0$:

At time $t=0$	CF
• Buy a Put	$-P$
• Buy an asset	$-S_0$
• Borrow $\frac{K}{(1+R_T)}$	$+\frac{K}{(1+R_T)}$
TOT	$\frac{K}{1+R_T} - S_0 - P > 0$

At time $t=T$	CF
• Close the put	$(K - S_T)^+$
• Close the long	$+S_T$
• Give the money back	$-K$
TOT	$(K - S_T)^+ + S_T - K \geq 0$

• If $P > K(1+R_T)^{-1}$:

At time $t=0$	CF
• Sell a Put	$+P$
• Invest $\frac{K}{(1+R_T)}$	$-\frac{K}{(1+R_T)}$
TOT	$P - \frac{K}{1+R_T}$

At time $t=T$	CF
• Close the put	$-(K-S_T)^+$
• Take the money invested	K
TOT	$K-(K-S_T)^+ \geq 0$

(b) • If $C - P < S_0 - \frac{K}{1+R_T}$:

At time $t=0$	CF
• Sell Put	$+P$
• Buy Call	$-C$
• Sell asset	S_0
• Invest $\frac{K}{1+R_T}$	$-\frac{K}{1+R_T}$
TOT	>0

At time $t=T$	CF
• Close the Put	$-(K-S_T)^+$
• Close the Call	$(S_T-K)^+$
• Close the short on the asset	$-S_T$
• Take the money invested back	$+K$
TOT	$(S_T-K)^+ - (K-S_T)^+ - S_T + K = 0$

If $C - P > S_0 - \frac{K}{1+R_T}$ the same switching the long positions with the short positions and viceversa.

• A smoother way to do this is using this "obvious" fact: "if two portfolio are equals at time $t=T$ $\Rightarrow$ they have to be equal at time $t=0$ (if it is not I buy the cheap one, sell the expensive and pay the relsulting cash in account)". So Consider:

Portfolio A: • long call
• $\frac{K}{1+R_T}$ in a bond

Portfolio B: • long put
• long asset

at $t=T$ $\quad V_{P.A.}^T = (S_T-K)_+ + K = \max(S_T, K) \qquad V_{P.B}^T = (K-S_T)_+ + S_T = \max(S_T, K)$

So at time $t=0$ $\quad V_{P.A.}^0 = V_{P.B.}^0 \Rightarrow C + \frac{K}{1+R_T} = P + S_0$

Exercise 3: European and American options

Assume that the price of the risk-free asset is given by $(1+r)^n$, where $r > 0$ is the corresponding strictly positive interest rate. Let C_n and C_n^A be the prices at time n of the European call and the American call on a non-dividend paying stock, respectively, with strike K and maturity N, $n \in \{1, \ldots, N\}$. Furthermore, denote the prices at time n of the analogously specified puts by P_n and P_n^A, respectively.

(a) Show with the help of an arbitrage argument that

$$C_n \leq C_n^A, \quad P_n \leq P_n^A.$$

← it makes sense because with American option you can close the position at any time

(b) Show that in an arbitrage-free market the equality $C_n = C_n^A$ holds (Hint. use the European put-call parity, i.e. $C_n + K/(1+r)^{N-n} = S_n + P_n$, cf. Ex. 2, and later Week 2, Ex. 2.1). Is the same true for the put options?

(a) If $C_n > C_n^A$ at time n:

At $t=n$	CF
Buy C_n^A	$-C_n^A$
Sell C_n	$+C_n$
Put in an account	$-(C_n - C_n^A)$
TOT	0

At $t=N$	CF
Close C_n	$-(S_T-K)^+$
Close C_n^A (you choose to not close earlier)	$(S_T-K)^+$
Take money back	$(C_n - C_n^A)(1+r)^{N-n}$
TOT	>0

The same for the put option.

(b) • $C_n^A \geq C_n \overset{\text{Pull-Call parity}}{=} S_n - \frac{K}{(1+r)^{N-n}} + \underbrace{P_n}_{\geq 0} \geq S_n - \frac{K}{(1+r)^{N-n}} \underset{r\geq 0,\ n<N}{>} S_n - K.$

So at every time $n<N$ the cost of the call is strictly greater than what we would obtain if we exercise (and if we do not exercise is even worse). So it is convenient to wait the maturity when $(1+r)^{N-N} = 1$.

• It is not true for the put: suppose $S_0 = 0$, if we exercise immediately we obtain K that at maturity will be $K(1+r)^N$ while the european put at maturity yields $(K-S_N)_+ \leq K < K(1+r)^N$ so there is no hope $P_0^A = P_0$.

√ **Exercise 4: Change of numéraire**

Consider a financial market with (strictly) positive asset prices $S_n = (S_n^0, S_n^1, ..., S_n^d)$ at time n, where the filtration of the model is given by $\{\mathcal{F}_n\}_{n=0,...,N}$. A sequence of positive random variables is called numéraire if

(i) the sequence $\{X_n\}_{n=0,...,N}$ is adapted to the filtration $\{\mathcal{F}_n\}_{n=0,...,N}$;

(ii) $X_0 = 1$ and $X_n > 0$, for all n;

(iii) $X_n = V_n(\theta)$, i.e. X_n is the value of a self-financing portfolio θ.

Denote the prices expressed in quantities of X_n by $S_n^X = S_n/X_n$.

~~(a)~~ Describe the economical meaning of S_n^X.

~~(b)~~ Can $Y_n = S_n^i/S_0^i$ for a fixed i be used as a numéraire?

~~(c)~~ Show that the market extended by the asset $\{X_n\}_n$ is viable if and only if the initial market is viable.

~~(d)~~ Consider a predictable strategy $\phi = \{\phi_n, n = 0, ..., N\}$. Show that this strategy is self-financing if and only if for all n:

$$V_n^X(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j, \cdot (S_j^X - S_{j-1}^X) .$$

~~(e)~~ Show that there is at most one deterministic numéraire in a viable market.

(a) S_n^X is the vetor of the asset prices discounted respect to X_n, so we have to think X_n as a sort measure for the values.

(b) Y_n is adapted to $\mathcal{F}_n$ by definition, $Y_0 = S_0^i/S_0^i = 1$ and $Y_n > 0$ $\forall n$; consider the portfolio where you buy $\frac{1}{S_0^i}$ unit of S_n^i.

(c) $\boxed{\Rightarrow}$ Suppose ϕ_n^{d+1} is the quantity we invest on X_n at time n. So if $\forall\ (\phi_n^0, \_\_, \phi_n^{d+1})$ strategy there is no arbitrage $\Rightarrow$ in particular $\forall\ (\phi_n^0, \_\_, \phi_n^d, 0)$ strategy of the original market there is no strategy.

$\boxed{\Leftarrow}$ X_n is a numeraire so $X_n = V_n(\theta) = \theta_n \cdot S_n = \sum_{i=1}^{d} \theta_n^i S_n^i$ By contraddiction assume $\exists\ (\bar{\phi}^0, \_\_, \bar{\phi}^{d+1})$ in the enlarged market that leads to arbitrage. So:

$$V_n(\bar{\phi}) = \sum_{i=1}^{d} \bar{\phi}_n^i S_n^i + \bar{\phi}_n^{d+1} X_n = \sum_{i=1}^{d} \bar{\phi}_n^i S_n^i + \bar{\phi}_n^{d+1} \left(\sum_{i=1}^{d} \theta_n^i S_n^i\right) =$$

$$= \sum_{i=1}^{d} (\bar{\phi}_n^i\ \overline{\phi_n^{d+1} \theta_n^i})\, S_n^i$$

So $(\phi_n^0 \phi_n^{d+1} \theta_n^0, \_\_, \phi_n^d \phi_n^{d+1} \theta_n^d)$ is an arbitrage strategy in the non extended market.

(d) First, we prove $V_n^X(\phi) + \phi_{n+1} \cdot (S_{n+1}^X - S_n^X) = V_{n+1}^X(\phi)$ (it is $\Leftrightarrow$ self financing):

$$\phi_{n+1} \cdot S_n = \phi_n \cdot S_n \iff \phi_{n+1} \cdot S_n^X = \phi_n \cdot S_n^X \quad \text{so} \quad V_{n+1}^X(\phi) = \phi_{n+1} \cdot S_{n+1}^X = \phi_{n+1} \cdot (S_{n+1}^X - S_n^X) +$$

$$+ \phi_{n+1} \cdot S_n^X = \phi_{n+1} (S_{n+1}^X - S_n^X) + \phi_n \cdot S_n^X = \phi_{n+1} (S_{n+1}^X - S_n^X) + V_n^X.$$

$$\boxed{\Rightarrow} \quad V_n^X(\phi) = V_{n-1}^X + \phi_n (S_n^X - S_{n-1}^X) \overset{\text{iterate}}{=} V_0 + \sum_{i=1}^{n} \phi_i (S_i^X - S_{i-1}^X)$$

$\Leftarrow$ $V_n^X(\phi) - V_{n-1}^X(\phi) = \phi_n(S_n^X - S_{n-1}^X)$.

(e) We can suppose X, Y are in the market thanks to (c).

Suppose there are X, Y 2 deterministic numeraires. Now, consider $\frac{X}{Y}$ and we prove $\frac{X_n}{Y_n} = 1$ $\forall n$. First we prove $\frac{X_n}{Y_n} = \text{constant}$ and since $\frac{X_0}{Y_0} = \frac{1}{1} = 1$ we would have the thesis.

If $\frac{X_n}{Y_n}$ is not constant, $\exists\, i < j$ s.t. $\frac{X_i}{Y_i} < \frac{X_j}{Y_j}$ (or $\frac{X_i}{Y_i} > \frac{X_j}{Y_j}$, it's the same proof).

it's cheap!

We buy quantity "1" of X_i and sell quantity "$\frac{X_i}{Y_i}$" Y_i, so $V_i(\phi) = 0$. Now at time j $V(\phi) = 1\, X_j + \left(-\frac{X_i}{Y_i}\right) Y_j = \left(\frac{X_j}{Y_j} - \frac{X_i}{Y_i}\right) Y_j > 0$. (the opposite if $\frac{X_i}{Y_i} > \frac{X_j}{Y_j}$)

Exercises Martingales in Financial Mathematics: The CRR model

Week 2, 2024

√ Exercise 1: Cox Ross Rubinstein model

There is only one risky asset in the CRR model with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. The risky asset is modelled as follows. Between two consecutive periods the price changes by a factor $1+a$ or $1+b$

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}$$

where $-1 < a < b$.

Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+a,\, 1+b\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. For $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). We define the random variables $T_n = S_n/S_{n-1}$, with possible values $1+a$ and $1+b$, respectively.

1. Show that in order to end up with a viable market it is necessary that $r \in]a, b[$.
2. Find examples for violation of the assumption of absence of arbitrage if $r \notin]a, b[$.
3. Now let $r \in]a, b[$ and denote $p^* = (b-r)/(b-a)$. Show that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$ if and only if the random variables $T_1, T_2, \ldots, T_N$ are i.i.d. and $\mathbb{Q}[T_1 = 1+a] = p^* = 1 - \mathbb{Q}(T_1 = 1+b)$.
4. Derive that the viable market obtained in 3 is complete (see Slides 52 and 53) and give a formula for the price of a claim with payoff H in the form of a conditional expectation with respect to $\mathbb{Q}$.

√ Exercise 2: Pricing of options

Continue with the notation and assumptions in the previous exercise. Furthermore, denote by C_n (P_n, respectively) the value at n of a European call (put) option with strike K and maturity N (both being written on the risky asset).

1. "Rediscover" the European put-call parity based on Point 4 of the previous exercise, i.e. derive
$$C_n - P_n = S_n - K\,(1+r)^{-(N-n)}\,.$$

2. Show that $C_n = c(n,\, S_n)$, where c is a function which can be expressed with the help of K, a, b, r and p^*.

3. Show that

$$c(n,\, x) = \frac{p^*}{1+r} c(n+1,\, x\,(1+a)) + \frac{1-p^*}{1+r} c(n+1,\, x\,(1+b)) \;\; n = 0, \ldots, N-1\,.$$

4. Show that the perfect hedging strategy of a European call at n is defined by a quantity $H_n = \Delta(n, S_{n-1})$ representing the investment in the risky asset, where the Δ is a function, which can be expressed in terms of the function c.

$\sqrt{}$ Exercise 1: Cox Ross Rubinstein model

There is only one risky asset in the CRR model with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. The risky asset is modelled as follows. Between two consecutive periods the price changes by a factor $1+a$ or $1+b$

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}$$

where $-1 < a < b$.

Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+a,\, 1+b\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. For $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). We define the random variables $T_n = S_n/S_{n-1}$, with possible values $1+a$ and $1+b$, respectively.

1. Show that in order to end up with a viable market it is necessary that $r \in]a, b[$.
2. Find examples for violation of the assumption of absence of arbitrage if $r \notin]a, b[$.
3. Now let $r \in]a, b[$ and denote $p^* = (b-r)/(b-a)$. Show that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$ if and only if the random variables $T_1, T_2, \ldots, T_N$ are i.i.d. and $\mathbb{Q}[T_1 = 1+a] = p^* = 1 - \mathbb{Q}(T_1 = 1+b)$.
4. Derive that the viable market obtained in 3 is complete (see Slides 52 and 53) and give a formula for the price of a claim with payoff H in the form of a conditional expectation with respect to $\mathbb{Q}$.

(1) We know a market is viable $\iff \exists\, \mathbb{Q}$ "martingale probability", i.e. $\mathbb{Q} \sim \mathbb{P}$ and $\frac{S_n}{S_n^0}$ is a $\mathbb{Q}$-martingale. So, suppose $\exists\, \mathbb{Q}$. Now:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_n}{S_n^0} \,\Big|\, \mathcal{F}_{n-1}\right] = \frac{S_{n-1}}{S_{n-1}^0} \iff \frac{1}{(1+r)^{\cancel{n}}}\, \mathbb{E}^{\mathbb{Q}}\left[S_n \mid \mathcal{F}_{n-1}\right] = \frac{S_{n-1}}{\cancel{(1+r)^{n-1}}}$$

$$\iff \mathbb{E}^{\mathbb{Q}}\left[S_n \mid \mathcal{F}_{n-1}\right] = (1+r)\, S_{n-1} \iff$$

$$\iff \mathbb{E}^{\mathbb{Q}}\left[S_{n-1}(1+a)\mathbb{1}_{\{S_n = S_{n-1}(1+a)\}} + S_{n-1}(1+b)\mathbb{1}_{\{S_n = S_{n-1}(1+b)\}} \,\Big|\, \mathcal{F}_{n-1}\right] = (1+r)\, S_{n-1}$$

$$\iff \cancel{S_{n-1}}(1+a)\, \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_n = S_{n-1}(1+a)\}} \mid \mathcal{F}_{n-1}\right] + \cancel{S_{n-1}}(1+b)\, \mathbb{E}\left[\mathbb{1}_{\{S_n = S_{n-1}(1+b)} \mid \mathcal{F}_{n-1}\right] = (1+r)\cancel{S_{n-1}}$$

but $\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_n = S_{n-1}(1+a)\}} \mid \mathcal{F}_{n-1}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_n = S_{n-1}(1+b)\}} \mid \mathcal{F}_{n-1}\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_n = S_{n-1}(1+a)\}} + \mathbb{1}_{\{S_n = S_{n-1}(1+b)\}} \mid \mathcal{F}_{n-1}\right] =$

$= \mathbb{E}^{\mathbb{Q}}\left[1 \mid \mathcal{F}_{n-1}\right] = 1$. So, $(1+a)\cdot t + (1+b)\cdot(1-t) = 1+r \quad t \in (0,1) \iff a < r < b.$

T_n è a valori in $\{(1+a), (1+b)\}$ dunque non valgono gli estremi

(2) If $r \leq a$:

$t=0$	CF
Borrow money	$+S_0$
Buy asset	$-S_0$
TOT	0

$t=N$	CF
Pay the bank	$-S_0(1+r)^N$
Sell asset	at least $S_0(1+a)^N$
TOT	$\geq S_0(1+a)^N - S_0(1+r)^N \geq 0$

Furthermore, if $a=r$ it is still an arbitrage opportunity because $\mathbb{P}\left(S_N = S_0(1+b)^N\right) > 0$ and remember that there is arbitrage if $\mathbb{P}(V_N > 0) > 0$.

If $r \geq b$ I lend money and short the asset.

3 $\boxed{\Rightarrow}$ Let $\mathbb{Q}$ be the martingale probability :

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}}{(1+r)^{n+1}} \,\Big|\, \mathcal{F}_n\right] = \frac{S_n}{(1+r)^n} \iff \mathbb{E}^{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] = (1+r) S_n \overset{S_n\ \mathcal{F}_n\text{-measurable}}{\iff} \mathbb{E}^{\mathbb{Q}}\left[T_{n+1} \mid \mathcal{F}_n\right] = 1+r$$

$$\iff \mathbb{E}^{\mathbb{Q}}\left[(1+a)\mathbb{1}_{\{T_{n+1}=1+a\}} + (1+b)\mathbb{1}_{\{T_{n+1}=1+b\}} \mid \mathcal{F}_n\right] = 1+r \iff$$

$$\iff (1+a)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right] + (1+b)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+b\}} \mid \mathcal{F}_n\right] = 1+r \iff$$

$$\iff \left(\text{as in } 1\ \ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+b\}} \mid \mathcal{F}_n\right] = 1\right)$$

$$(1+a)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right] + (1+b)\left(1 - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right]\right) = 1+r$$

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right] = \frac{r-b}{a-b} = \frac{b-r}{b-a} = p^*$$

But, $\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right] = \mathbb{Q}\left[T_{n+1}=1+a \mid \mathcal{F}_n\right] = p^* \quad \forall\, n=0,\dots,N-1.$

$$\mathbb{Q}(T_1=t_1,\dots,T_n=t_n) \overset{\text{Bayes}}{=} \underbrace{\mathbb{Q}\left[T_n=t_n \mid T_1=t_1,\dots,T_{n-1}=t_{n-1}\right]}_{\sigma(T_1,\dots,T_{n-1})=\sigma(S_1,\dots,S_{n-1})} \cdot \mathbb{Q}\left[T_1=t_1,\dots,T_{n-1}=t_{n-1}\right] =$$

$$\underset{\text{Bayes}}{=} p_n^* \cdot \mathbb{Q}\left[T_{n-1}=t_{n-1} \mid T_{n-2}=t_{n-2},\dots,T_1=t_1\right] \cdot \mathbb{Q}\left[T_{n-2}=t_{n-2},\dots,T_1=t_1\right] \underset{\text{iterate Bayes}}{=}$$

$= p_n^* \cdot p_{n-1}^* \cdot \dots \cdot p_1^*$ (*) where $p_j^* = p^*$ if $t_j = 1+a$ or $p_j^* = 1-p^*$ if $t_j = 1+b$. In fact:

$p^* = \mathbb{Q}\left[T_{n+1}=1+a \mid \mathcal{F}_n\right] = \mathbb{E}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right]$ so apply the expected value and

$p^* = \mathbb{E}\left[\mathbb{1}_{\{T_{n+1}=1+a\}}\right] = \mathbb{Q}\left[T_{n+1}=1+a\right]$. So (*) is the factorization of joint distribution

$\Rightarrow$ T_i are indipendent and identically distibuted.

$\boxed{\Leftarrow}$ T_n are Bernoulli i.i.d. , $\mathbb{E}^{\mathbb{Q}}\left[\widetilde{S_{n+1}} \mid \mathcal{F}_n\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}}{(1+r)^{n+1}} \cdot \frac{S_n}{S_n} \mid \mathcal{F}_n\right] =$

$$= \frac{S_n}{(1+r)^{n+1}} \mathbb{E}^{\mathbb{Q}}\left[T_{n+1} \mid \mathcal{F}_n\right] \underset{\text{indipendent}}{=} \frac{S_n}{(1+r)^{n+1}} \mathbb{E}^{\mathbb{Q}}\left[T_{n+1}\right] = \frac{S_n}{(1+r)^{n+1}} \left((1+a)p^* + (1+b)(1-p^*)\right) =$$

$$= \frac{S_n}{(1+r)^{n+1}} \left((1+a)\left(\frac{b-r}{b-a}\right) + (1+b)\left(1 - \frac{b-r}{b-a}\right)\right) = \frac{S_n}{(1+r)^{n+1}} \left(\frac{b-r+ab-ar}{b-a} + \frac{(1+b)(\cancel{b}-a-\cancel{b}+r)}{b-a}\right) =$$

$$= \frac{S_n}{(1+r)^{n+1}} \left(\frac{b-\cancel{r}+\cancel{ab}-ar-a+\cancel{r}-\cancel{ab}+rb}{b-a}\right) = \frac{S_n}{(1+r)^{\cancel{n+1}}} \left(\frac{\cancel{(b-a)}\cancel{(1+r)}}{\cancel{(b-a)}}\right) = \widetilde{S_n}$$

(4) $\mathbb{Q}$ is unique because determines $T_1, \ldots, T_N$ hence $S_1, \ldots, S_n$ (S_0 is deterministic).

Forthermore the value of a contingent claim at time n is:

$$\tilde{V}_n(H) = \mathbb{E}^{\mathbb{Q}}\left[\tilde{H} \mid \mathcal{F}_n\right] \Rightarrow V_n(H) = \frac{1}{(1+r)^{N-n}} \mathbb{E}^{\mathbb{Q}}\left[H \mid \mathcal{F}_n\right]$$

Exercise 2: Pricing of options

Continue with the notation and assumptions in the previous exercise. Furthermore, denote by C_n (P_n, respectively) the value at n of a European call (put) option with strike K and maturity N (both being written on the risky asset).

1. "Rediscover" the European put-call parity based on Point 4 of the previous exercise, i.e. derive
$$C_n - P_n = S_n - K\,(1+r)^{-(N-n)}\,.$$

2. Show that $C_n = c(n,\,S_n)$, where c is a function which can be expressed with the help of K, a, b, r and p^*.

3. Show that
$$c(n,\,x) = \frac{p^*}{1+r}c(n+1,\,x\,(1+a)) + \frac{1-p^*}{1+r}c(n+1,\,x\,(1+b))\;\; n = 0,\ldots,N-1\,.$$

4. Show that the perfect hedging strategy of a European call at n is defined by a quantity $H_n = \Delta(n,\,S_{n-1})$ representing the investment in the risky asset, where the Δ is a function, which can be expressed in terms of the function c.

① $$C_n - P_n = \frac{1}{(1+r)^{N-n}} \mathbb{E}^Q\Big[\underbrace{(S_N-K)^+ - (K-S_N)^+}_{\text{payoff of } C_n - S_n} \,\Big|\, \mathcal{F}_n\Big] = \frac{1}{(1+r)^{N-n}}\mathbb{E}^Q[S_N - K \mid \mathcal{F}_n] =$$

$\tilde{S}_n$ Q-mart.

$$= \frac{1}{(1+r)^{N-n}}\left(\mathbb{E}^Q[S_N \mid \mathcal{F}_n] - K\right) = \frac{1}{(1+r)^{N-n}}\left((1+r)^N\,\mathbb{E}^Q[\tilde{S}_N \mid \mathcal{F}_n] - K\right) =$$

$$= \frac{1}{(1+r)^{N-n}}\left((1+r)^N\cdot \tilde{S}_n - K\right) = \frac{1}{(1+r)^{N-n}}\left((1+r)^{N-n} S_n - K\right) = S_n - \frac{K}{(1+r)^{N-n}}$$

② $$C_n = \frac{1}{(1+r)^{N-n}}\mathbb{E}^Q[(S_N - K)^+ \mid \mathcal{F}_n] =$$ $S_N = S_n T_{n+1}\cdot \ldots \cdot T_N$

S_n $\mathcal{F}_n$-meas.

$$= \frac{1}{(1+r)^{N-n}}\mathbb{E}^Q[(S_n T_{n+1}\cdot\ldots\cdot T_N - K)^+ \mid \mathcal{F}_n] =$$ $T_{n+1}\cdot\ldots\cdot T_N$ $\mathcal{F}_n$-indipendent

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $f(x\,y) = (x\cdot y - K)^+$

$\Rightarrow$ Freezing Lemma

$$= \frac{1}{(1+r)^{N-n}}\varphi(S_n)$$

where $$\varphi(x) = \mathbb{E}^Q[(x\,T_{n+1}\cdot\ldots\cdot T_N - K)^+] =$$ $T_i \in \{1+a,\,1+b\} \Rightarrow T_{n+1}\cdot\ldots\cdot T_N \in \{(1+a)^i(1+b)^{N-n-i}\}$

and T_i properties

$$= \sum_{i=0}^{N-n} \mathbb{P}(T_{j_1},\ldots,T_{j_i} = 1+a \,\wedge\, T_{j_{i+1}},\ldots,T_{j_{N-n-i}} = 1+b)\,(x\,(1+a)^i(1+b)^{N-n-i} - K)^+ =$$

$$= \sum_{i=0}^{N-n}\binom{N-n}{i}(p^*)^i(1-p^*)^{N-n-i}\,(x\,(1+a)^i(1+b)^{N-n-i} - K)^+$$

③ $\varphi(x)$ found in ② can be seen as $\varphi(n,x)$.

$$\varphi(n+1,\,x(1+a)) = \mathbb{E}^Q[(x(1+a)\,T_{n+2}\cdot\ldots\cdot T_N - K)^+]$$ and

$$\varphi(n+1,\,x(1+b)) = \mathbb{E}^Q[(x(1+b)\,T_{n+2}\cdot\ldots\cdot T_N - K)^+]\,.$$

We compute:

$$\varphi(n,x) = \mathbb{E}^{Q}\left[\left(x\, T_{n-1} \cdot \ldots \cdot T_N - K\right)^+\right] = \mathbb{E}^{Q}\left[\left(x\,(1+a)\, T_{n-2} \cdot \ldots \cdot T_N - K\right)^+ \mathbb{1}_{\{T_{n-1}=1+a\}}\right] +$$

$$+ \mathbb{E}^{Q}\left[\left(x(1+b)\, T_{n-2} \cdot \ldots \cdot T_N - K\right)^+ \mathbb{1}_{\{T_{n-1}=1+b\}}\right] =$$

T_{n-1} indipendent from $T_{n-2}, \ldots, T_N$

$$= \mathbb{E}^{Q}\left[\left(x\,(1+a)\, T_{n-2} \cdot \ldots \cdot T_N - K\right)^+\right] \mathbb{E}^{Q}\left[\mathbb{1}_{\{T_{n-1}=1+a\}}\right] +$$

$$+ \mathbb{E}^{Q}\left[\left(x(1+b)\, T_{n-2} \cdot \ldots \cdot T_N - K\right)^+\right] \mathbb{E}^{Q}\left[\mathbb{1}_{\{T_{n-1}=1+b\}}\right] =$$

$$= \varphi(n+1, x(1+a))\, Q\left[T_{n-1}=1+a\right] + \varphi(n+1, x(1+b))\, Q\left[T_{n-1}=1+b\right] =$$

$$= p^* \varphi(n+1, x(1+a)) + (1-p^*)\, \varphi(n+1, x(1+b)).$$

Now, by definition $c(n,x) = \dfrac{\varphi(n,x)}{(1+r)^{N-n}}$ so:

$$c(n,x) = \frac{\varphi(n,x)}{(1+r)^{N-n}} = \frac{p^* \varphi(n+1, x(1+a)) + (1-p^*)\, \varphi(n+1, x(1+b))}{(1+r)^{N-n}} =$$

$$= \frac{p^*}{1+r}\, c(n+1, x(1+a)) + \frac{(1-p^*)}{1+r}\, c(n+1, x(1+b))$$

4 (H_n^0, H_n) is the strategy at time n. We want a replicant portfolio so:

$$H_n^0 (1+r)^n + H_n S_n = c(n, S_n)$$

So, if $S_n = S_{n-1}(1+a) \Rightarrow H_n^0 (1+r)^n + H_n S_{n-1}(1+a) = c(n, S_{n-1}(1+a))$, if $S_n = S_{n-1}(1+b) \Rightarrow H_n^0 (1+r)^n + H_n S_{n-1}(1+b) = c(n, S_{n-1}(1+b))$. And so:

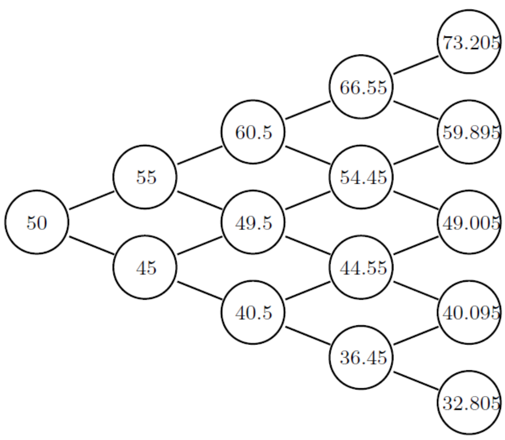
$$\Delta(n,x) = \frac{c(n, S_{n-1}(1+a)) - c(n, S_{n-1}(1+b))}{x(a-b)}$$

Exercises Martingales in Financial Mathematics: The complete CRR in action / on incomplete markets

Week 3, 2024

Exercise 1: Example

Consider the following binomial-tree of possible realizations for a price process, with time tick 1 month and interest rate $r_c = 0.2$ (continuous compounding), i.e. $N = 4$, $T = N\Delta t = 1/3$, $S_n^0 = e^{r_c n \Delta t}$:



~~(a)~~ Calculate the value of a call with strike \$50 at time $t = 0$.

~~(b)~~ Describe the replication strategy for the following scenarios:

(i) 1. move up, 2. move down, 3. move up, 4. move up;

(ii) 1. move down, 2. move down, 3. mote up, 4. move down.

Hint.: Use the following table:

n	move	S_n	V_n	ϕ_n	ϕ_n^0	$\tilde{V}_n$	$\tilde{S}_n$
0	-	50	?	-	-	?	50

Exercise 2: Trinomial model

The trinomial model can be considered as being an extension of the Cox Ross Rubinstein model (binomial model). There is again only one risky asset with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. However, between two consecutive periods the price changes here by a factor $1+d$ or $1+m$ or $1+u$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+d) \\ S_n\,(1+m) \\ S_n\,(1+u) \end{cases},$$

where $-1 < d < m < u$. Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+d, 1+m, 1+u\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ being a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. Furthermore, again for $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). Finally, we define $T_n = S_n/S_{n-1}$, with possible values $1+d$, $1+m$ and $1+u$, and we assume that the T_i are i.i.d.

1. Show that in order to end up with a viable market it is necessary that $r \in]d, u[$.
2. Derive conditions for martingale measures $\mathbb{Q}$.
3. Derive that a viable market in this model is not complete.

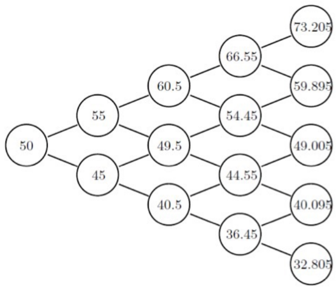
√ Exercise 3: Incomplete markets

Denote by $(S_n)_{n=0,\ldots,N}$ the price vector in a viable (but not necessarily complete) market defined on a finite probability space where each element is an atom. Suppose that the random variable h defined on the same space is attainable (recall that this means that it can be replicated by an admissible strategy).

1. Show that the price V_n at n of a derivative with payoff h can be calculated uniquely by
$$V_n = S_n^0 \, \mathbb{E}^* \left[\frac{h}{S_N^0} \,|\, \mathcal{F}_n \right]$$
where $\mathbb{E}^*$ is the expectation with respect to any measure $\widetilde{\mathbb{P}}$ under which $(\widetilde{S}_n)$ is a martingale.
2. Give an example of an incomplete market and of an attainable product in this market.

Exercise 1: Example

Consider the following binomial-tree of possible realizations for a price process, with time tick 1 month and interest rate $r_c = 0.2$ (continuous compounding), i.e. $N = 4$, $T = N\Delta t = 1/3$, $S_n^0 = e^{r_c n \Delta t}$:



(a) Calculate the value of a call with strike \$50 at time $t = 0$.

(b) Describe the replication strategy for the following scenarios:

(i) 1. move up, 2. move down, 3. move up, 4. move up;

(ii) 1. move down, 2. move down, 3. mote up, 4. move down.

Hint.: Use the following table:

n	move	S_n	V_n	ϕ_n	ϕ_n^0	$\tilde{V}_n$	$\tilde{S}_n$
0	-	50	?	-	-	?	50

(a) We refer to notation of Exercise Sheet 2:

Now the interest is continuous compounded so if (3) of Exercise 1:

$$(1+a)\, \mathbb{E}^{Q}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right] + (1+b)\left(1 - \mathbb{E}^{Q}\left[\mathbb{1}_{\{T_{n+1}=1+a\}} \mid \mathcal{F}_n\right]\right) = e^{r_c \Delta t}$$

$$\Rightarrow p^* = \frac{e^{r_c \Delta t} - (1+b)}{(1+a)-(1+b)} = \frac{(1+b) - e^{r_c\Delta t}}{b-a}$$. Now, by hypothesis $b > a$ so:

$$\Rightarrow a = \frac{45}{50} - 1 = -0{,}1 \qquad b = \frac{55}{50} - 1 = 0{,}1 \quad \Rightarrow p^* = \frac{1{,}1 - e^{0{,}2\cdot\frac{1}{12}}}{0{,}1-(-0{,}1)} = 0{,}416$$

In the same way we have:

$$C(n, S_n) = \frac{1}{e^{r_c (N-n)\Delta t}} \sum_{i=0}^{N-n} \binom{N-n}{i} (p^*)^i (1-p^*)^{N-n-i} \left(S_n (1+a)^i (1+b)^{N-n-i} - K\right)^+$$

$$C(0, S_0) = \frac{1}{e^{r_c N \Delta t}} \sum_{i=0}^{N} \binom{N}{i} (p^*)^i (1-p^*)^{N-i} \left(S_0 (1+a)^i (1+b)^{N-i} - K\right)^+ =$$

$$= \frac{1}{e^{0{,}2\cdot\frac{1}{3}}} \sum_{i=1}^{4} \binom{4}{i} (0{,}416)(1-0{,}416)^{4-i} \left(50 \cdot 0{,}9^i \cdot 1{,}1^{4-i} - 50\right)^+ = 5{,}594$$

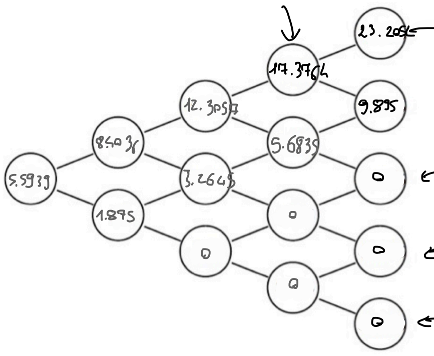
For computing $C(n, S_n)$ we can do faster using Exercise 2 Sheet 2 Point 3:

$$C(n, S_n) = e^{-r_c \Delta t}\left(p^*\, C(n+1, S_n(1+a)) + (1-p^*)\, C(n+1, S_n(1+b))\right)$$

and remembering that $C(N, S_N) = (S_N - K)^+$

So:

$*$



$(S_N - K)^+ = (73{,}205 - 50)^+$

"

"

"

"

So for example, to compute $* = C(3, 66{,}55) = e^{-\frac{0{,}2}{12}} \left(0{,}416 \cdot 9.895 + (1 - 0{,}416) \cdot 23.205 \right) =$

$= 17.3764$. And so on the other in grey.

(b) As in Ex. 2 Sheet 2 Point 4 we do the computation to find the quantity ϕ_n of S_n

$$\phi_n S_n + \phi_n^0 e^{r_c n \Delta t} = c(n, S_n) \qquad \text{if} \quad S_n = S_{n-1}(1+a) \quad \text{we have:}$$

$$\phi_n S_{n-1}(1+a) + \phi_n^0 e^{r_c n \Delta t} = c(n, S_{n-1}(1+a)) \; ; \quad \text{if} \quad S_n = S_{n-1}(1+b):$$

$$\phi_n S_{n-1}(1+b) + \phi_n^0 e^{r_c n \Delta t} = c(n, S_{n-1}(1+b)) \; ; \quad \text{so:}$$

$$\phi_n = \frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}$$

To compute the table at time n we proceed in this order:

- $\tilde{S}_n = \dfrac{S_n}{S_n^0}$, $\tilde{V}_n \underset{\text{Self financing}}{=} V_0 + \sum_{k=1}^{n} \phi_k \Delta \tilde{S}_k$
- $\phi_{n+1} = \dfrac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}$
- $\phi_{n+1}^0 = \tilde{V}_n - \phi_{n+1} S_n e^{-r_c n \Delta t}$

at time n I have to choose how manage the asset for time $n+1$

So we have the tables in the following page.

(i)

time n	move	S_n	V_n	ϕ_n	ϕ_n^0	$\tilde{V}_n$	$\tilde{S}_n$
0	—	50	5,594	–	–	5,594	50
1	Up	55	8,4036	0,65286	-27,0491	8,2647	54,0909
2	Down	49.5	3,2645	0,82193	-36,1943	3,1575	47,8772
3	Up	54,45	5,6835	0,57409	-24,3283	5,4063	51,7944
4	Up	59,895	9,8950	0,90863	-44,6556	9,2568	56,0322

(ii)

time n	move	S_n	V_n	ϕ_n	ϕ_n^0	$\tilde{V}_n$	$\tilde{S}_n$
0	—	50	5,594	–	–	5,594	50
1	Down	45	1,8750	0,65286	-27,0491	1,8450	44,2562
2	Down	40,5	0	0,36272	-14,2086	0	39,1723
3	Up	44,55	0	0	0	0	42,3773
4	Down	40,095	0	0	0	0	37,5092

Observation: when it goes down twice in 1 and 2, there the probability under $\mathbb{P}$ that $S_N \geq 50$ is 0 and so under $\mathbb{Q}$ too ($\mathbb{P}$ and $\mathbb{Q}$ are equivalent). So the call is worthess and so the replicant portfolio too.

Exercise 2: Trinomial model

The trinomial model can be considered as being an extension of the Cox Ross Rubinstein model (binomial model). There is again only one risky asset with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. However, between two consecutive periods the price changes here by a factor $1+d$ or $1+m$ or $1+u$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+d) \\ S_n\,(1+m) \\ S_n\,(1+u) \end{cases},$$

where $-1 < d < m < u$. Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+d, 1+m, 1+u\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ being a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. Furthermore, again for $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). Finally, we define $T_n = S_n/S_{n-1}$, with possible values $1+d$, $1+m$ and $1+u$, and we assume that the T_i are i.i.d.

1. Show that in order to end up with a viable market it is necessary that $r \in]d, u[$.
2. Derive conditions for martingale measures $\mathbb{Q}$.
3. Derive that a viable market in this model is not complete.

(1) Viable market $\iff \exists\, \mathbb{Q}$ s.t. $\mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_{n+1} \mid \mathcal{F}_n\right] = \tilde{S}_n$. So:

$\mathbb{E}^{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] = (1+r)\, S_n$ but

$$\mathbb{E}^{\mathbb{Q}}\left[S_{n+1} \mid \mathcal{F}_n\right] = \mathbb{E}^{\mathbb{Q}}\left[S_n(1+d)\,\mathbb{1}_{\{S_{n+1} = S_n(1+d)\}} + S_n(1+m)\,\mathbb{1}_{\{S_{n+1} = S_n(1+m)\}} + S_n(1+u)\,\mathbb{1}_{\{S_{n+1} = S_n(1+u)\}} \mid \mathcal{F}_n\right]$$

$$= S_n(1+d)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_{n+1} = S_n(1+d)\}}\right] + S_n(1+m)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_{n+1} = S_n(1+m)\}} \mid \mathcal{F}_n\right] + S_n(1+u)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_{n+1} = S_n(1+u)\}} \mid \mathcal{F}_n\right]$$

$$\Rightarrow \cancel{S_n}(1+d)\,\mathbb{Q}[\underbrace{S_{n+1} = S_n(1+d) \mid \mathcal{F}_n}_{D}] + \cancel{S_n}(1+m)\,\mathbb{Q}[\underbrace{S_{n+1} = S_n(1+m) \mid \mathcal{F}_n}_{M}] + \cancel{S_n}(1+u)\,\mathbb{Q}[\underbrace{S_{n+1} = S_n(1+u) \mid \mathcal{F}_n}_{U}] = \cancel{S_n}(1+r)$$

$(1+d)\,\mathbb{Q}[D] + (1+m)\,\mathbb{Q}[M] + (1+u)\,\mathbb{Q}[U] = 1+r$ and $\mathbb{Q}[D] + \mathbb{Q}[M] + \mathbb{Q}[U] = 1$.

Now $d < m < u \Rightarrow 1+r > (1+d)\left(\mathbb{Q}[D] + \mathbb{Q}[M] + \mathbb{Q}[U]\right) = 1+d \Rightarrow r > d$.

$\Rightarrow 1+r < (1+u)\left(\quad // \quad\right) = 1+u \Rightarrow u > r$.

(2) Now, T_i are iid so $\mathbb{Q}[D] = \mathbb{Q}[T_{n+1} = 1+d \mid \mathcal{F}_n] = \mathbb{Q}[T_{n+1} = 1+d] =: x$, and it's the same for $\mathbb{Q}[U] = \mathbb{Q}[T_{n+1} = 1+u] =: y$ and $\mathbb{Q}[M] = \mathbb{Q}[T_{n+1} = 1+m] =: z$. So:

($\mathcal{F}_n = \sigma(T_1, \ldots, T_n)$)

$(1+d)\,x + (1+m)\,y + (1+u)\,z = 1+r$

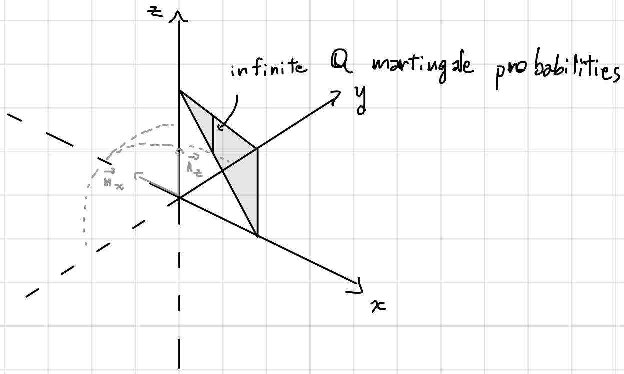
$x+y+z = 1$

$(1+d)\,x + (1+m)\,y + (1+u)\,z = (1+r)(x+y+z)$

$(1+d-1-r)\,x + (1+m-1-r)\,y + (1+u-1-r)\,z = 0$

$$\begin{cases} (d-r)\,x + (m-r)\,y + (u-r)\,z = 0 \\ x+y+z = 1 \\ 0 < x, y, z < 1 \end{cases}$$

(3) $x+y+z=1$ is the 2-simplex, and $(d-r)x+(m-r)y+(u-r)z=0$ is linear plane where $\vec{n}=(\underbrace{d-r}_{<0},\ m-r,\ \underbrace{u-r}_{>0})$ is the normal vector, so:



√ ## Exercise 3: Incomplete markets

Denote by $(S_n)_{n=0,\dots,N}$ the price vector in a viable (but not necessarily complete) market defined on a finite probability space where each element is an atom. Suppose that the random variable h defined on the same space is attainable (recall that this means that it can be replicated by an admissible strategy).

1. Show that the price V_n at n of a derivative with payoff h can be calculated uniquely by

$$V_n = S_n^0\, \mathbb{E}^*\left[\frac{h}{S_N^0} \mid \mathcal{F}_n\right]$$

they all must be V_n

If it is not complete the Q-prob. martingale is not unique but if S_n is attainable his price is unique!

where $\mathbb{E}^*$ is the expectation with respect to any measure $\widetilde{\mathbb{P}}$ under which $(\widetilde{S}_n)$ is a martingale.

2. Give an example of an incomplete market and of an attainable product in this market.

(1) $\tilde{V}_n$ is a martingale under * so:

$$\tilde{V}_n = \mathbb{E}^*[\tilde{V}_N = \tilde{h} \mid \mathcal{F}_n] \Rightarrow \frac{V_n}{S_n^0} = \mathbb{E}^*\left[\frac{h}{S_N^0} \mid \mathcal{F}_n\right] \Rightarrow V_n = S_n^0\, \mathbb{E}^*\left[\frac{h}{S_N^0} \mid \mathcal{F}_n\right].$$

If U_n is a replicant portfolio we have $U_n = S_n^0\, \mathbb{E}^*\left[\frac{h}{S_N^0} \mid \mathcal{F}_n\right]$ but

$U_n = U_0 + \sum_{i=1}^{n} \phi_i \Delta S_i$ so it doesn't depend on the choice of * and so V_n is unique

(2) The trinomial model is incomplete and, for example, the forward contract is atteinable

(I go long at time 0 in the forward contract).

Exercises Martingales in Financial Mathematics: Further aspects of discrete time models

Week 4, 2024

Exercise 1: An example for an incomplete model

Consider a one-period model with one risky asset. The spot price of this risky asset is 10. Assume that after the period, i.e. at $n = 1$, $S_1 \in \{9, 10, 12\}$ holds. Furthermore, assume that the risk-free interest rate is vanishing. Finally, let $H(S_1)$ be a European claim written on the risky asset S.

1. Is this market viable?
2. Describe the risk-neutral probabilities.
3. Describe a potentially existing replicating strategy for the claim being defined by H and conclude that the strategy exists if and only if $H(12) - 3H(10) + 2H(9) = 0$.
4. Show that the value of an option admitting a replication strategy does not depend on the choice of the risk-neutral probability measure.
5. Can the claims satisfying $H(12) - 3H(10) + 2H(9) = 0$ be described?

Exercise 2: Doob decomposition

Consider our finite probability space setting. Show that supermartingale (X_n) has the following decomposition

$$X_n = X_0 + M_n - A_n$$

where (M_n) is a martingale null at 0 and (A_n) is a non-decreasing predictable process, null at 0. Furthermore, show that this decomposition is unique.

Exercise 3: American call

Let C_n^A, C_n be the price of an American, European option, respectively, at n with payoff being determined by $\{Z_n\}_{0\leq n\leq N}$, Z_N, respectively.

- Show that for all n $C_n^A \geq C_n$.
- Show that $Z_n \leq C_n$ for all n implies $C_n^A = C_n$ for all n.
- (Application: American Call) Assume that the risk-free interest rate is positive. Show that the price of an American call is equal to the price of an analogously specified European Call.

Exercise 1: An example for an incomplete model

Consider a one-period model with one risky asset. The spot price of this risky asset is 10. $\nearrow S_0$ Assume that after the period, i.e. at $n = 1$, $S_1 \in \{9, 10, 12\}$ holds. Furthermore, assume that the risk-free interest rate is vanishing. Finally, let $H(S_1)$ be a European claim written on the risky asset S.

1. Is this market viable?
2. Describe the risk-neutral probabilities.
3. Describe a potentially existing replicating strategy for the claim being defined by H and conclude that the strategy exists if and only if $H(12) - 3H(10) + 2H(9) = 0$.
4. Show that the value of an option admitting a replication strategy does not depend on the choice of the risk-neutral probability measure.
5. Can the claims satisfying $H(12) - 3H(10) + 2H(9) = 0$ be described?

risk-less asset $S^0 = 1$ constant

risk free interest rate is vanishing

(1) A market is viable $\iff \exists \underset{\sim}{\mathbb{Q}} \overset{\mathbb{P}}{}$ martingale probability i.e. $\tilde{S}_1 = S_1$ is a $\mathbb{Q}$-martingale.

So: $\mathbb{E}^{\mathbb{Q}}[S_1 \mid \mathcal{F}_0] = \mathbb{E}^{\mathbb{Q}}[S_1] = 9\, \mathbb{Q}(S_1 = 9) + 10\, \mathbb{Q}(S_1 = 10) + 12\, \mathbb{Q}(S_1 = 12) =$

$\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\underset{\text{martingale}}{=} S_0 = 10$. Furthermore, $\overbrace{\mathbb{Q}(S_1=9)}^{\alpha} + \overbrace{\mathbb{Q}(S_1=10)}^{\beta} + \overbrace{\mathbb{Q}(S_1=12)}^{\gamma} = 1$ and $\mathbb{Q}(S_1=9)$, $\mathbb{Q}(S_1=10)$, $\mathbb{Q}(S_1=12) > 0$ ($\mathbb{Q} \sim \mathbb{P}$ supposing $\mathbb{P}$ atom-prob.).

So:
$$\begin{cases} 9\alpha + 10\beta + 12\gamma = 10 \\ \alpha + \beta + \gamma = 1 \implies \alpha = 1 - \beta - \gamma \\ 0 < \alpha, \beta, \gamma < 1 \end{cases} \implies 9 - 9\beta - 9\gamma + 10\beta + 12\gamma = 10 \implies \beta + 3\gamma = 1 \implies \beta = 1 - 3\gamma$$

$\implies$ for example $\gamma = \frac{1}{6} \implies \beta = \frac{1}{2} \implies \alpha = \frac{1}{3}$.

(2) Using equations in (1) $\gamma = \frac{1-\beta}{3}$ and $\alpha = 2\left(\frac{1-\beta}{3}\right) \implies \begin{cases} \beta \in (0,1) \\ \alpha = 2\left(\frac{1-\beta}{3}\right) \\ \gamma = \frac{1-\beta}{3} \end{cases}$ $\leftarrow$ note that in this form $\alpha, \beta, \gamma \in (0,1)$

(3) A strategy ϕ is admissible if it is admissable $(V_n(\phi) \geq 0\ \forall n)$ and self-financing; we want to replicate $H(S_1)$ so $V_1(\phi) = H(S_1)$. But $V_1(\phi) = \phi_1^0 \overset{1}{\overset{\shortparallel}{S_1^0}} + \phi_1 S_1$ so:

$$\begin{cases} \phi_1^0 + \phi_1 9 = H(9) & (S_1 = 9) \\ \phi_1^0 + \phi_1 \cdot 10 = H(10) & (S_1 = 10) \\ \phi_1^0 + \phi_1 \cdot 12 = H(12) & (S_1 = 12) \end{cases} \implies \begin{cases} \phi_1^0 = H(9) - 9\phi_1 \\ H(9) - 9\phi_1 + 10\phi_1 = H(10) \implies \phi_1 = H(10) - H(9) \\ H(9) - 9H(10) + 9H(9) + 12H(10) - 12H(9) = H(12) \end{cases}$$

$\iff -2H(9) + 3H(10) = H(12) \iff H(12) - 3H(10) + 2H(9) = 0$. So this is a necessary condition to replicate $H(S_1)$. Let's see that this is sufficient i.e. exist ϕ admissible and self-financing but this follows immediatly since there is only one time so only one portfolio.

(4) The value of an option attenaible: follows immediatly from Point 1 Exercise 3 Sheet 3.

We can also do the direct computation, $\mathbb{E}^{Q_1}[H \mid \mathcal{F}_0] = \mathbb{E}^{Q_2}[H \mid \mathcal{F}_0] \iff$ $\leftarrow \mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathbb{E}^{Q_1}[H] = \mathbb{E}^{Q_2}[H] \iff$

$$\underbrace{Q_1(S_1=9)}_{\alpha_1}H(9)+\underbrace{Q_1(S_1=10)}_{\beta_1}H(10)+\underbrace{Q_1(S_1=12)}_{\gamma_1}H(12)=\underbrace{Q_2(S_1=9)}_{\alpha_2}H(9)+\underbrace{Q_2(S_1=10)}_{\beta_2}H(10)+\underbrace{Q_2(S_1=12)}_{\gamma_2}H(12)$$

$$LHS = \alpha_1 H(9) + \beta_1 H(10) + \gamma_1 H(12) \underset{\uparrow \text{viable}}{=} 2\left(\frac{1-\beta_1}{3}\right)H(9) + \beta_1 H(10) + \frac{1-\beta_1}{3} H(12) =$$

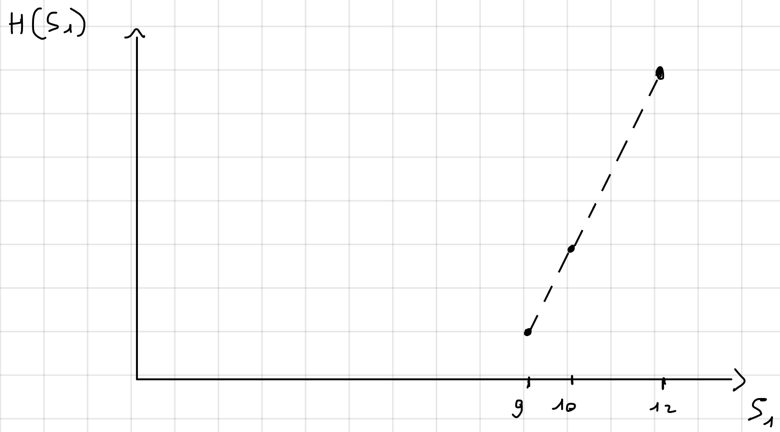
$$= \frac{2H(9) - 2\beta_1 H(9) + 3\beta_1 H(10) + H(12) - \beta_1 H(12)}{3} \underset{\uparrow \text{attainable}}{=} 2H(9) + H(12).$$

So it doesn't depend on Q_1, so the same for Q_2.

⑤ $H: \mathbb{R} \longrightarrow \mathbb{R}$ $S_1 \longmapsto H(S_1)$. Let's see the increments

$$\frac{H(12)-H(10)}{12-10} = \frac{H(12)-H(10)}{2} = \frac{3H(10)-2H(9) - H(10)}{2} = \frac{2H(10)-2H(9)}{2} = \frac{H(10)-H(9)}{10-9}$$

So $(9, H(9))$, $(10, H(10))$ and $(12, H(12))$ lies in an affine straight line.



✓Exercise 2: Doob decomposition

Consider our finite probability space setting. Show that supermartingale (X_n) has the following decomposition

$$X_n = X_0 + M_n - A_n$$

where (M_n) is a martingale null at 0 and (A_n) is a non-decreasing predictable process, null at 0. Furthermore, show that this decomposition is unique.

<u>Existence:</u> by induction :

$n = 0$: $M_0 = A_0 = 0 \Rightarrow X_0 = X_0$ ok.

$n \to n+1$: $M_{n+1} := X_{n+1} - X_0 + A_{n+1}$ with $A_{n+1} := X_n - \mathbb{E}[X_{n+1} | \mathcal{F}_n] + A_n$

(from this we have automatically the decomposition). Let's see :

- A_{n+1} predictable: sum of $\mathcal{F}_n$-measurable function

- A_{n+1} non decreasing: $A_{n+1} - A_n = X_n - \mathbb{E}[X_{n+1} | \mathcal{F}_n] + \cancel{A_n} - \cancel{A_n} \geq 0$ (X_n supermartingale)

- Martingale: $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_{n+1} | \mathcal{F}_n] - \mathbb{E}[X_0 | \mathcal{F}_n] + \mathbb{E}[A_{n+1} | \mathcal{F}_n] \underset{A_{n+1}\ \mathcal{F}_n\text{-meas.}}{=}$

$$= \cancel{\mathbb{E}[X_{n+1} | \mathcal{F}_n]} - X_0 + X_n - \cancel{\mathbb{E}[X_{n+1} | \mathcal{F}_n]} + A_n =$$

$$= X_n - X_0 + A_n \underset{\text{induction hyp.}}{=} M_n$$

<u>Uniqueness</u> : Suppose $X_n = X_0 + M_n - A_n = X_0 + M_n' - A_n'$. By induction we prove

$M_n = M_n'$ $A_n = A_n'$ $\forall n$:

$n = 0$: $M_0 = M_0' = A_0 = A_0' = 0$

$n \to n+1$: $X_{n+1} = X_0 + M_{n+1} - A_{n+1} = X_0 + M_{n+1}' - A_{n+1}' \Rightarrow M_{n+1} - M_{n+1}' = A_{n+1} - A_{n+1}'$

Now : $0 \underset{\text{induction}}{=} M_n - M_n' \underset{\text{martingale}}{=} \mathbb{E}[M_{n+1} - M_{n+1}' | \mathcal{F}_n] = \mathbb{E}[A_{n+1} - A_{n+1}' | \mathcal{F}_n] \underset{\text{predic.}}{=} A_{n+1} - A_{n+1}'$

$\Rightarrow A_{n+1} = A_{n+1}' \Rightarrow M_{n+1} = M_{n+1}'$.

Exercise 3: American call

Let C_n^A, C_n be the price of an American, European option, respectively, at n with payoff being determined by $\{Z_n\}_{0\le n\le N}$, Z_N, respectively.

1. Show that for all n $C_n^A \ge C_n$.
2. Show that $Z_n \le C_n$ for all n implies $C_n^A = C_n$ for all n.
3. (Application: American Call) Assume that the risk-free interest rate is positive. Show that the price of an American call is equal to the price of an analogously specified European Call.

1 Backward induction:

$$N: \quad C_N^A = Z_N = (S_N - K)^+ = C_N.$$

$$n+1 \longrightarrow n: \quad C_n^A = \max\left\{ Z_n, \frac{1}{1+r} E^Q\left[C_{n+1}^A \mid \mathcal{F}_n\right]\right\} \underset{\text{def of max}}{\ge} \frac{1}{1+r} E^Q\left[C_{n+1}^A \mid \mathcal{F}_n\right] \underset{C_{n+1}^A \ge C_{n+1}}{\ge}$$

$$\ge \frac{1}{1+r} E^Q\left[C_{n+1} \mid \mathcal{F}_n\right] \underset{\text{viable}}{=} C_n$$

2 The market is complete so $\tilde{C}_N$ is replicable in particular:

$$E^Q\left[\tilde{C}_N \mid \mathcal{F}_n\right] = E^Q\left[\tilde{V}_N \mid \mathcal{F}_n\right] = \tilde{V}_n = \tilde{C}_n$$

$\tilde{C}_N = \tilde{V}_N \Rightarrow$ they have the same value at time n

And so $\tilde{C}_n$ is a Q-martingale $\Rightarrow$ is a super martingale under Q that dominate $\tilde{Z}_n$

$\Rightarrow$ for a proposition seen in class $\tilde{C}_n \ge \tilde{C}_n^A$. So $\tilde{C}_n = \tilde{C}_n^A$

3 We want to use 2, so we want to show that $C_n \ge Z_n$:

$$C_n = (1+r)^{N-n} E^Q\left[(S_N - K)^+ \mid \mathcal{F}_n\right] \ge (1+r)^{N-n} E^Q\left[S_N - K \mid \mathcal{F}_n\right] =$$

$$= \frac{(1+r)^{N-n}}{(1+r)^N} E^Q\left[\tilde{S}_N - \frac{K}{(1+r)^N} \mid \mathcal{F}_n\right] \underset{\text{martingale}}{=} (1+r)^{-n}\left(\tilde{S}_n - \frac{K}{(1+r)^N}\right) =$$

$$= (1+r)^{-n}\left(\frac{S_n}{(1+r)^n} - \frac{K}{(1+r)^N}\right) = S_n - \frac{K}{(1+r)^{N-n}} \underset{r>0}{\ge} S_n - K$$

But, also $C_n \ge 0$ so $C_n \ge (S_n - K)^+ = Z_n$.

Exercises Martingales in Financial Mathematics: Model CRR and towards Black Scholes

Week 5, 2024

Exercise 1: Model CRR

The Cox Ross Rubinstein (binomial) model is sometimes used in order to price American options. Recall that there is only one risky asset with price S_n at n for all $n \in \{0, 1, \ldots, N\}$ along with a risk-less asset. The risk-free interest rate for each time period is given by r. The price process (S_n) can be modeled by the relative variations of the quotes over the time subperiods, being denoted by a and b with $-1 < a < b$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}.$$

For our example we fix $N = 3$, $r = 0.02$, $b = 0.1$, $a = -0.1$, and $S_0 = 100$.

1. Is this particular market viable? Is it complete?
2. Describe the martingale measure $\mathbb{Q}$.
3. Draw a graph of the tree representing the possible values of S_n.
4. We intend to derive the price of an American put, where the maturity is represented by N and where the strike is given by $K = 95$. Draw a graph of the tree with the possible payoffs at n, i.e. with the used notation visualize Z_n for each possible value of S_n.
5. Starting at $n = N$, construct the Snell envelope of Z_n.

Exercise 2: The SDE of a GBM

We are interested in the following equation

$$S_t = x_0 + \mu \int_0^t S_s\,ds + \sigma \int_0^t S_s\,dW_s\,, \quad t \in [0, T]. \tag{1}$$

1. Assume
$$S_t = x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t}.$$
By using the Itô formula, show that (S_t) is a solution of (1).
2. Show that (Z_t) defined by $Z_t = x_0/S_t$ solves
$$Z_t = 1 + \mu' \int_0^t Z_s ds + \sigma' \int_0^t Z_s dW_s,$$
where μ' and σ' are constants, which have to be identified.

3. Assume that (Y_t) is another solution of the SDE (1). Show that $d(Y_t Z_t) = 0$. Derive the uniqueness of the solution of (1). In finance, the unique solution of this equation is called Black and Scholes model.

Exercise 1: Model CRR

The Cox Ross Rubinstein (binomial) model is sometimes used in order to price American options. Recall that there is only one risky asset with price S_n at n for all $n \in \{0, 1, \dots, N\}$ along with a risk-less asset. The risk-free interest rate for each time period is given by r. The price process (S_n) can be modeled by the relative variations of the quotes over the time subperiods, being denoted by a and b with $-1 < a < b$, i.e.

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5. Starting at $n = N$, construct the Snell envelope of Z_n.

Exercise 1: Cox Ross Rubinstein model

There is only one risky asset in the CRR model with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. The risky asset is modelled as follows. Between two consecutive periods the price changes by a factor $1+a$ or $1+b$

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}$$

where $-1 < a < b$.

Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+a, 1+b\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. For $n = 1, \dots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \dots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). We define the random variables $T_n = S_n/S_{n-1}$, with possible values $1+a$ and $1+b$, respectively.

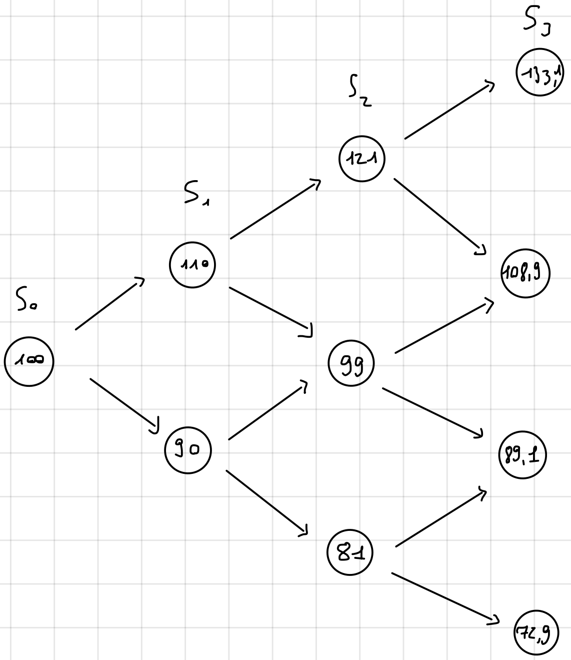
1. Show that in order to end up with a viable market it is necessary that $r \in]a, b[$.
2. Find examples for violation of the assumption of absence of arbitrage if $r \notin]a, b[$.
3. Now let $r \in]a, b[$ and denote $p^* = (b-r)/(b-a)$. Show that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$ if and only if the random variables $T_1, T_2, \dots, T_N$ are i.i.d. and $\mathbb{Q}[T_1 = 1+a] = p^* = 1 - \mathbb{Q}(T_1 = 1+b)$.
4. Derive that the viable market obtained in 3 is complete (see Slides 52 and 53) and give a formula for the price of a claim with payoff H in the form of a conditional expectation with respect to $\mathbb{Q}$.

① Viable $\iff r \in (a,b)$, $-0,1 < 0,02 < 0,1$. We know also that in this case, a viable market implies that is complete (See Exercise 1 Sheet 2).

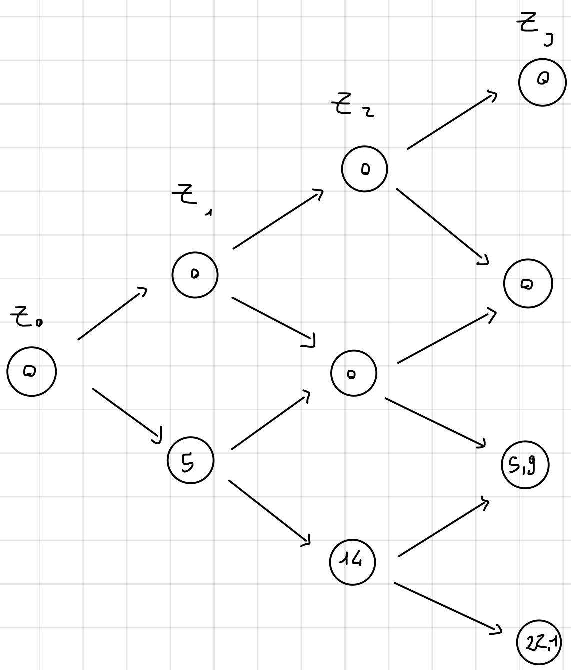
② Using always Exercise 1 Sheet 2:

$$\mathbb{Q}\left[\frac{S_{n+1}}{S_n} = 0,9\right] = p^* = \frac{0,1-0,02}{0,2} = \frac{8}{100}\cdot\frac{10}{2} = 0,4 \quad \text{and} \quad \mathbb{Q}\left[\frac{S_{n+1}}{S_n} = 1,1\right] = 1-0,4 = 0,6$$

③ $S_n = S_{n-1}\cdot 0,9$ or $S_n = S_{n-1}\cdot 1,1$



④ $Z_n = (K - S_n) = (95 - S_n)^+$ (Put)



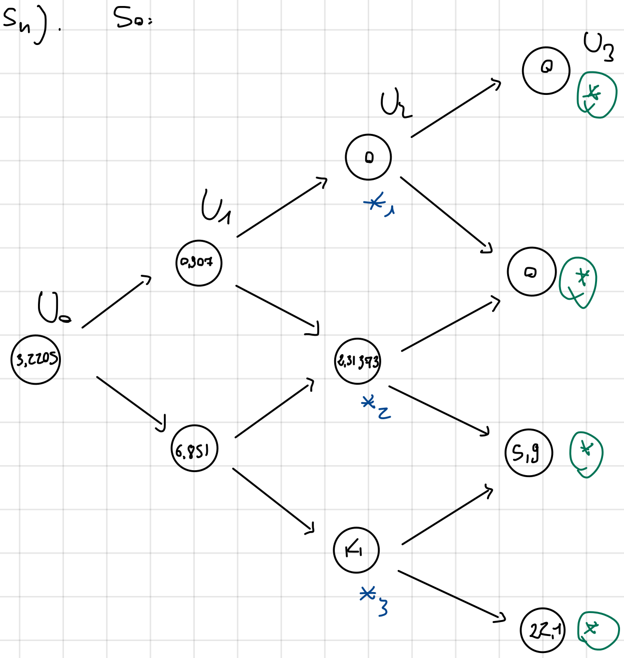
⑤ $$\begin{cases} U_N = Z_N \\ U_n = \max\left\{ Z_n ,\ S_n^0\, \mathbb{E}^{Q}\left[\dfrac{U_{n+1}}{S^0_{n+1}} \,\middle|\, \mathcal{F}_n \right] \right\} \quad \forall n \le N-1 \end{cases}$$

Q martingale measure $\Rightarrow$ T_n i.i.d. (for Ex 1 Sheet 2)

$\Rightarrow$ $S_n^0 = (1+r)^n$ $\quad S^0_{n+1} = (1+r)^{n+1}$

$$\Rightarrow X_n := S_n^0\, \mathbb{E}^{Q}\left[\frac{U_{n+1}}{S^0_{n+1}} \,\middle|\, \mathcal{F}_n \right] = \frac{1}{1+r}\, \mathbb{E}^{Q}\left[U_{n+1} \,|\, \mathcal{F}_n \right].$$

Remember, in this model, $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$. So:



$n = 3$: $U_3 = Z_3$ $\Rightarrow$ we have the last "column": (*)

$n = 2$: $$X_2 = \frac{1}{1+r}\, \mathbb{E}^{Q}\left[U_3 \,|\, \mathcal{F}_2 \right] = \frac{1}{1+r}\, \mathbb{E}^{Q}\left[Z_3 \,|\, \mathcal{F}_2 \right] = \frac{1}{1{,}02}\Big(0 \cdot Q\left[Z_3 = 0 \,|\, \mathcal{F}_2 \right] +$$

$$+\ 5{,}9 \cdot Q\left[Z_3 = 5{,}9 \,|\, \mathcal{F}_2 \right] + 22{,}1\ Q\left[Z_3 = 22{,}1 \,|\, \mathcal{F}_2 \right] \Big)$$

So we have 3 cases:

can't reach 5.9 from 5.9.

- $*_1$: $\mathbb{Q}[Z_3 = 5.9 \mid \mathcal{F}_2] = 0$, idem $\mathbb{Q}[Z_3 = 22.1 \mid \mathcal{F}_2] = 0 \implies X_2 = 0 \implies U_2 = \max\{0, 0\} = 0$

- $*_2$: $\mathbb{Q}[S_3 = 83,1 \mid \mathcal{F}_2] = \mathbb{Q}[\text{down}] = 0,4$ and $\mathbb{Q}[S_3 = 42,9 \mid \mathcal{F}_2] = 0 \implies$

X_2 in $*_2$ is $\frac{1}{1,02} \cdot 5,9 \cdot 0,4 = 2,31373 \implies U_2 = \max\{0, 2,31373\} = 2.31373$

- $*_3$: $\frac{1}{1,02} \cdot (5,9 \cdot 0,6 + 22,1 \cdot 0,4) = 12,1373 \implies U_2 = \max\{14, 12.1373\} = 14$

And so on.

Exercise 2: The SDE of a GBM

We are interested in the following equation

$$S_t = x_0 + \mu \int_0^t S_s\, ds + \sigma \int_0^t S_s\, dW_s, \quad t \in [0,T].$$

1. Assume
$$S_t = x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}.$$
By using the Itô formula, show that (S_t) is a solution of (1).

2. Show that (Z_t) defined by $Z_t = x_0/S_t$ solves
$$Z_t = 1 + \mu' \int_0^t Z_s ds + \sigma' \int_0^t Z_s dW_s,$$
where μ' and σ' are constants, which have to be identified.

3. Assume that (Y_t) is another solution of the SDE (1). Show that $d(Y_t Z_t) = 0$. Derive the uniqueness of the solution of (1). In finance, the unique solution of this equation is called Black and Scholes model.

① $S_t = f(t, W_t)$ where $f(t,x) = x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x}$. So

the other terms are 0

$$S_t = f(0,0) + \int_0^t \frac{\partial f(s,W_s)}{\partial s} ds + \int_0^t \frac{\partial f(s,W_s)}{\partial x} dW_s + \frac{1}{2}\left(\int_0^t \frac{\partial^2 f(s,W_s)}{\partial x} d\langle W\rangle_s\right) =$$

$$= x_0 + \int_0^t x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)s + \sigma W_s} \cdot \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)s + \sigma W_s} \cdot \sigma\, dW_s +$$

$$+ \frac{1}{2}\int_0^t \sigma x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)s + \sigma W_s} \cdot \sigma\, ds =$$

$$= x_0 + \int_0^t S_s\left(\mu - \cancel{\frac{\sigma^2}{2}} + \cancel{\frac{\sigma^2}{2}}\right) ds + \int_0^t S_s \cdot \sigma\, dW_s =$$

$$= x_0 + \mu \int_0^t S_s\, ds + \sigma \int_0^t S_s\, dW_s.$$

② $Z_t = \dfrac{x_0}{S_t}$. $f(x) = x_0 \cdot x^{-1}$

$$Z_t = \frac{x_0}{x_0} + \int_0^t x_0\left(-\frac{1}{S_s^2}\right) dS_s + \frac{1}{\cancel{2}}\int_0^t x_0\left(\frac{\cancel{2}}{S_s^3}\right) d\langle S\rangle_s =$$

$$= 1 - \int_0^t Z_s \frac{dS_s}{S_s} + \int_0^t Z_s \frac{d\langle S\rangle_s}{S_s^2}.$$

Now: $dS_s = \mu S_s\, ds + \sigma S_s\, dW_s = S_s(\mu\, ds + \sigma\, dW_s) \Rightarrow \dfrac{dS_s}{S_s} = \mu\, ds + \sigma\, dW_s$

and $d\langle S\rangle_s = \sigma^2 S_s^2\, ds \Rightarrow \dfrac{d\langle S\rangle_s}{S_s^2} = \sigma^2 ds$ And so:

$$Z_t = 1 - \int_0^t Z_s(\mu\, ds + \sigma\, dW_s) + \int_0^t Z_s\, \sigma^2 ds =$$

$$= 1 + \int_0^t Z_s\left(\sigma^2 - \mu\right) ds - \int_0^t Z_s \sigma \, dW_s = 1 + \left(\sigma^2 - \mu\right)\int_0^t Z_s \, ds - \sigma \int_0^t Z_s \, dW_s .$$

$$\langle Z, Y\rangle_t = \langle -\sigma Z_t \cdot W_t , \sigma Y_t \cdot W_t\rangle = -\sigma^2 Z_t Y_t \cdot t$$

3 $d\left(Y_t Z_t\right) = Y_t \, dZ_t + Z_t \, dY_t + d\langle Z, Y\rangle_t =$

$$= Y_t\left(\left(\sigma^2-\mu\right) Z_t \, dt - \sigma Z_t \, dW_t\right) + Z_t\left(\mu Y_t \, dt + \sigma Y_t \, dW_t\right) - \sigma^2 Z_t Y_t \, dt =$$

$$= \cancel{Y_t Z_t \sigma^2 dt} - \cancel{\mu Y_t Z_t dt} - \cancel{\sigma Y_t Z_t dW_t} + \cancel{\mu Z_t Y_t dt} + \cancel{\sigma Z_t Y_t dW_t} - \cancel{\sigma^2 Z_t Y_t dt} =$$

$= 0 \implies Y_t Z_t =$ constant but $Y_0 Z_0 = x_0 \cdot 1 = x_0 \implies Y_t Z_t = x_0 \implies$

$$Y_t \cdot \frac{\cancel{x_0}}{S_t} = \cancel{x_0} \implies Y_t = S_t .$$

$\swarrow$ Ex. 5 Sheet 6

Alternatively from theory of SDE we know:

- $|\mu x - \mu y| + |\sigma x - \sigma y| = |\mu|\,|x-y| + |\sigma|\,|x-y| \leq 2\max\left(|\mu|, |\sigma|\right)|x-y|$
- $|\mu x| + |\sigma x| = |\mu|\,|x| + |\sigma|\,|x| \leq 2\max\left(|\mu|, |\sigma|\right)|x| \leq 2\max\left(|\mu|, |\sigma|\right)\left(1+|x|\right)$

$\implies$ unique solution

Exercises Martingales in Financial Mathematics: Borwnian motion driven models

Week 6, 2024

Exercise 1: Black-Scholes and European put-call parity

Assume that $(S_t, t \geq 0)$ solves

$$dS_t = S_t \left(\mu dt + \sigma dW_t\right), \quad S_0 = x\,,$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion. Furthermore, r stands for the risk-free interest rate (with continuous compounding). We analyse some aspects of

$$\mathbb{E}\left(e^{-rT} f(S_T)\right)\,, \tag{1}$$

as being a somehow (on the first glance) a "potential candidate" for the price of a European derivative being defined by the payoff function f.

1. Write S_T as a function of W_T.
2. Compute the discounted expected payoffs of a European call (denoted by c) and of a European put (denoted by p) expressed as a function of $\mathcal{N}(d) = \int_{-\infty}^{d} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$.
3. Show that the European put-call parity only holds for $\mu = r$ (for the discounted expected payoffs considered as candidates for prices).

Exercise 2: Martingale

Let $(M_t)_{t\geq 0}$ be a non-negative martingale, which satisfies $\mathbb{E}(M_t^2) < \infty$ for all $t \in [0,T]$. Show that for $s \leq t$

$$\mathbb{E}[(M_t - M_s)^2 | \mathcal{F}_s] = \mathbb{E}[M_t^2 - M_s^2 | \mathcal{F}_s]$$

holds.

Exercise 3: Repetition of Itô

Compute

$$d \sin W_t\,, \quad de^{W_t^2}\,, \quad de^{tW_t}\,,$$

and verify that (X_t)

$$X_t = \frac{\exp(\sigma W_t)}{1+t}$$

solves

$$dX_t = X_t \left(\frac{1}{2}\sigma^2 - \frac{1}{1+t} \right) dt + \sigma X_t dW_t$$

((W_t) is a standard Brownian motion).

√ Exercise 4: European power call

Assume that $(S_t, t \geq 0)$ solves

$$dS_t = S_t \left(r dt + \sigma dW_t \right), \quad S_0 = x \,,$$

where (W_t) is a standard Brownian motion with respect to $\mathbb{Q}$. Compute

$$\mathbb{E}_{\mathbb{Q}}(S_T - k)_+^n \,.$$

Exercice 5: Ornstein-Uhlenbeck process

Consider

$$dX_t = -cX_t dt + \sigma dW_t \,, \; X_0 = x. \tag{2}$$

1. Does the SDE (2) admit a unique solution?
2. Show that the solution of (2) is given by

$$X_t = e^{-ct}x + \sigma e^{-ct} \int_0^t e^{cs} dW_s.$$

3. Compute the expectation and variance of the variables X_t (Remark: The process (X_t) is a gaussian process).
4. Which is the law of X_{t+s} given $X_s = x$? Derive a simulation method $(X_{kh}, 1 \leq k \leq N)$.
5. We now assume that X_0 is a centered gaussian random variable with variance σ_0^2 being independent of (W_s). For which σ_0 we have that the law of X_t does not depend on t?

Exercise 1: Black-Scholes and European put-call parity

Assume that $(S_t, t \geq 0)$ solves

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = x,$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion. Furthermore, r stands for the risk-free interest rate (with continuous compounding). We analyse some aspects of

$$\mathbb{E}\left(e^{-rT} f(S_T)\right), \tag{1}$$

as being a somehow (on the first glance) a "potential candidate" for the price of a European derivative being defined by the payoff function f.

1. Write S_T as a function of W_T.
2. Compute the discounted expected payoffs of a European call (denoted by c) and of a European put (denoted by p) expressed as a function of $\mathcal{N}(d) = \int_{-\infty}^{d} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$.
3. Show that the European put-call parity only holds for $\mu = r$ (for the discounted expected payoffs considered as candidates for prices).

① We know that $S_t = x \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$ so $S_T = x\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}$.

② $\mathbb{E}\left[e^{-rT}(S_T - K)_+\right] = e^{-rT}\,\mathbb{E}\left[(S_T - K)_+\right] =$

$\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T =: Z + \mu T$ with $Z \sim N\left(-\frac{\sigma^2}{2}T, \sigma^2 T\right)$

$$= e^{-rT}\,\mathbb{E}\left[\left(x\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T} - K\right)_+\right] =$$

$$= e^{-rT} \int_{\mathbb{R}} \left(x\, e^{z + \mu T} - K\right)_+ \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{\left(z + \frac{\sigma^2}{2}T\right)^2}{2\sigma^2 T}\right) dz =$$

$x\, e^{z + \mu T} \geq K$

$\iff z \geq \ln\left(\frac{K}{x}\right) - \mu T$

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\ln\left(\frac{K}{x}\right) - \mu T}^{+\infty} \left(x\, e^{z + (\mu - r)T} - K e^{-rT}\right) \exp\left(-\frac{\left(z + \frac{\sigma^2}{2}T\right)^2}{2\sigma^2 T}\right) dz =$$

$v := \dfrac{-\left(z + \frac{1}{2}\sigma^2 T\right)}{\sigma\sqrt{T}}$

$z = -\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T}$

$dv = -\frac{1}{\sigma\sqrt{T}} dz$

$z \longrightarrow +\infty \quad v \longrightarrow -\infty$

$z = \ln\left(\frac{K}{x}\right) - \mu T \implies$

$v = \dfrac{\ln\left(\frac{x}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =: d_-(\mu)$

$$= \frac{\sigma\sqrt{T}}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{d_-(\mu)} \left(x\, e^{-\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T} + (\mu - r)T} - K e^{-rT}\right) \exp\left(-\frac{1}{2}v^2\right) dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\mu)} x\, e^{-\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T} + (\mu - r)T - \frac{1}{2}v^2} dv - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\mu)} K e^{-rT - \frac{1}{2}v^2} dv =$$

$$= \frac{e^{(\mu - r)T}}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\mu)} x\, e^{-\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T} - \frac{1}{2}v^2} dv - \frac{e^{-rT} K}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\mu)} e^{-\frac{1}{2}v^2} dv$$

$-\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T} - \frac{1}{2}v^2 =$

$= -\frac{1}{2}\left(\sigma\sqrt{T} + v\right)^2$

$w := v + \sigma\sqrt{T} \quad d_+ := d_- + \sigma\sqrt{T}$

$$= \frac{e^{(\mu - r)T} \cdot x}{\sqrt{2\pi}} \int_{-\infty}^{d_+(\mu)} e^{-\frac{1}{2}w^2} dw - e^{-rT} \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\mu)} e^{-\frac{1}{2}v^2} dv$$

$$= x \cdot e^{(\mu - r)T} N\left(d_+(\mu)\right) - e^{-rT} K\, N\left(d_-(\mu)\right).$$

To derive the price of a put:

$$f-g=(f-g)_+-(f-g)_-$$

$$\mathbb{E}\left[e^{-rT}(S_T-K)\right]=\underbrace{\mathbb{E}\left[e^{-rT}(S_T-K)_+\right]}_{\text{Price of a Call}}-\underbrace{\mathbb{E}\left[e^{-rT}(K-S_T)_+\right]}_{\text{Price of a put}}\ .\ \text{So}$$

$$\mathbb{E}\left[e^{-rT}(K-S_T)_+\right]=\mathbb{E}\left[e^{-rT}(S_T-K)_+\right]-\mathbb{E}\left[e^{-rT}(S_T-K)\right]=$$

$$\mathbb{E}\left[e^{-rT}(S_T-K)\right]=$$

$$=e^{-rT}\mathbb{E}[S_T]-Ke^{-rT}=$$

$$=e^{-rT}xe^{\mu T}-Ke^{-rT}$$

theory

$$=x\cdot e^{(\mu-r)T}N(d_+(\mu))-e^{rT}K\,N(d_-(\mu))-\mathbb{E}\left[e^{-rT}(S_T-K)\right]=$$

$$=x\cdot e^{(\mu-r)T}N(d_+(\mu))-e^{rT}K\,N(d_-(\mu))+Ke^{-rT}-x\,e^{(\mu-r)T}=$$

$$=-x\,e^{(\mu-r)T}\left(1-N(d_+(\mu))\right)+K\,e^{-rT}\left(1-N(d_-(\mu))\right)=$$

$$=-x\,e^{(\mu-r)T}N(-d_+(\mu))+Ke^{-rT}N(-d_-(\mu)).$$

3 From the previous point $P-C=S_0\,e^{T(\mu-r)}-Ke^{-rT}\overset{\text{Call - Put parity}}{=}S_0-Ke^{-rT}$. So there is the "Call - Put parity" $\Leftrightarrow$ $\mu=r$.

√Exercise 2: Martingale

Let $(M_t)_{t\geq 0}$ be a non-negative martingale, which satisfies $\mathbb{E}(M_t^2)<\infty$ for all $t\in[0,T]$. Show that for $s\leq t$

$$\mathbb{E}[(M_t-M_s)^2|\mathcal{F}_s]=\mathbb{E}[M_t^2-M_s^2|\mathcal{F}_s]$$

holds.

$$\mathbb{E}\left[(M_t-M_s)^2\,|\mathcal{F}_s\right]=\mathbb{E}\left[M_t^2+M_s^2-2M_tM_s\,|\mathcal{F}_s\right]=\mathbb{E}\left[M_t^2|\mathcal{F}_s\right]+M_s^2-2M_s^2=\mathbb{E}\left[M_t^2-M_s^2\,|\mathcal{F}_s\right].$$

$$\mathbb{E}[M_tM_s|\mathcal{F}_s]=M_s\,\mathbb{E}[M_t|\mathcal{F}_s]=M_s^2$$

√Exercise 3: Repetition of Itô

Compute

$$d\sin W_t,\quad de^{W_t^2},\quad de^{tW_t},$$

and verify that (X_t)

$$X_t=\frac{\exp(\sigma W_t)}{1+t}$$

solves

$$dX_t=X_t\left(\frac{1}{2}\sigma^2-\frac{1}{1+t}\right)dt+\sigma X_t dW_t$$

((W_t) is a standard Brownian motion).

$$d \sin W_t = \cos W_t \, dW_t - \frac{1}{2} \sin W_t \, dt \, ;$$

$$d e^{W_t^2} = e^{W_t^2} \cdot 2W_t \, dW_t + \frac{1}{2}\left(e^{W_t^2} \cdot 2W_t \cdot 2W_t + e^{W_t^2} 2 \right) dt =$$

$$= 2 e^{W_t^2} W_t^2 \, dt + e^{W_t^2} dt + 2 e^{W_t^2} W_t \, dW_t = e^{W_t^2}\left((2W_t^2 + 1) dt + 2W_t \, dW_t \right);$$

$$d e^{tW_t} = e^{tW_t} t \, dW_t + e^{tW_t} W_t \, dt + \frac{1}{2}\left(t e^{tW_t} \cdot t \right) dt + 0 + 0 =$$

$$= e^{tW_t}\left(t \, dW_t + \left(W_t + \frac{1}{2} t^2 \right) dt \right).$$

$$dX_t = \frac{\exp(\sigma W_t)}{1+t} \cdot \sigma \cdot dW_t + \exp(\sigma W_t)\left(- \frac{1}{(1+t)^2} \right) dt + \frac{1}{2} \exp(\sigma W_t) \cdot \sigma \cdot \frac{\sigma}{1+t} \, dt =$$

$$= X_t \cdot \sigma \, dW_t - \frac{X_t}{1+t} dt + \sigma^2 X_t \, dt = \sigma X_t \, dW_t + X_t \left(\frac{1}{2}\sigma^2 - \frac{1}{1+t} \right) dt$$

√ Exercise 4: European power call

Assume that $(S_t, t \geq 0)$ solves

$$dS_t = S_t \left(r dt + \sigma dW_t \right), \quad S_0 = x \, ,$$

where (W_t) is a standard Brownian motion with respect to $\mathbb{Q}$. Compute

$$\mathbb{E}_{\mathbb{Q}} (S_T - k)_+^n \, .$$

We know $S_t = S_0 \, e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$. So:

$$\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)_+^n \right] = \mathbb{E}_{\mathbb{Q}}\left[\left(S_T \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T} \right)_+^n \right] =$$

$(r - \frac{1}{2}\sigma^2)T + \sigma W_T =: Z + Tr$

with $Z \sim N(-\frac{1}{2}\sigma^2, \sigma^2 T)$

$$= \int_{\mathbb{R}} \left(S_0 e^{z + Tr} - K \right)_+^n \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{1}{2} \frac{(z + \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T} \right) dz =$$

$S_0 e^{z + Tr} - K \geq 0$

$\iff z \geq \ln\left(\frac{K}{S_0}\right) - Tr$

$$\rightarrow = \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\ln\left(\frac{K}{S_0}\right) - Tr}^{+\infty} \left(S_0 e^{z + Tr} - K \right)^n \exp\left(-\frac{1}{2} \frac{(z + \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T} \right) dz =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\ln\left(\frac{K}{S_0}\right) - Tr}^{+\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} S_0^{n-j} e^{(n-j)(z + Tr)} K^j \exp\left(-\frac{1}{2} \frac{(z + \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T} \right) dz =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \sum_{j=0}^{n} (-1)^j S_0^{n-j} \binom{n}{j} e^{(n-j)Tr} K^j \int_{\ln\left(\frac{K}{S_0}\right) - Tr}^{+\infty} e^{(n-j)z} \exp\left(-\frac{1}{2} \frac{(z + \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T} \right) dz =$$

$$v := -\frac{z+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \rightarrow \quad = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} (-1)^j S_0^{n-j} \binom{n}{j} e^{(n-j)Tr} K^j \int_{-\infty}^{d} e^{(n-j)\left(-\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T}\right) - \frac{1}{2}v^2} dv =$$

$$z = -\frac{1}{2}\sigma^2 T - v\sigma\sqrt{T}$$

$$z \longrightarrow +\infty \quad v \longrightarrow -\infty$$

$$dz = -\frac{dv}{\sigma\sqrt{T}}$$

$$z = \ln\left(\frac{K}{S_0}\right) - Tr \Rightarrow$$

$$v = \frac{\ln\left(\frac{S_0}{K}\right) + T\left(r - \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}} =: d$$

$$= \sum_{j=0}^{n} (-1)^j S_0^{n-j} e^{(n-j)\left(Tr + \frac{\sigma^2 T}{2}(n-j-1)\right)} K^j \underbrace{\frac{1}{2\pi}\int_{-\infty}^{d+\sigma\sqrt{T}(n-j)} e^{-\frac{1}{2}w^2} dw}_{= N\left(d+\sigma\sqrt{T}(n-j)\right)}$$

$$-(n-j)\left(\frac{1}{2}\sigma^2 T\right) - (n-j)v\sigma\sqrt{T} - \frac{1}{2}v^2 =$$

$$= -\frac{1}{2}\left(v^2 + 2(n-j)v\sigma\sqrt{T} + \sigma^2 T(n-j)^2\right) + \frac{\sigma^2 T}{2}\left((n-j)^2 - (n-j)\right) =$$

add and subtract $\frac{\sigma^2 T}{2}(n-j)^2$ and factor out

$$= -\frac{1}{2}\left(v + \sigma\sqrt{T}(n-j)\right)^2 + \frac{\sigma^2 T}{2}(n-j)(n-j-1)$$

$$w := v + \sigma\sqrt{T}(n-j)$$

Exercice 5: Ornstein-Uhlenbeck process

Consider

$$dX_t = -cX_t dt + \sigma dW_t, \; X_0 = x. \tag{2}$$

1. Does the SDE (2) admit a unique solution?
2. Show that the solution of (2) is given by
$$X_t = e^{-ct}x + \sigma e^{-ct}\int_0^t e^{cs} dW_s.$$
3. Compute the expectation and variance of the variables X_t (Remark: The process (X_t) is a gaussian process).
4. Which is the law of X_{t+s} given $X_s = x$? Derive a simulation method $(X_{kh}, 1 \leq k \leq N)$.
5. We now assume that X_0 is a centered gaussian random variable with variance σ_0^2 being independent of (W_s). For which σ_0 we have that the law of X_t does not depend on t?

1 IL Condition: given $e(f,g)$, we say that f,g satisfy the IL hypothesis if $\exists K > 0$ s.t.

- $\|f(t,\xi) - f(t,\eta)\| + \|g(t,\xi) - g(t,\eta)\| \leq K \sup_{s\in[0,t]} \|\xi(s) - \eta(s)\| \quad \forall \xi,\eta \in W^d \; \forall t \geq 0$
- $\forall x \in \mathbb{R}^d$, $t \longmapsto f(t,\varepsilon^x)$ and $t \longmapsto g(t,\varepsilon^x)$ are locally bounded with ε^x the constant function in x. Instead of this the professor use

$$\boxed{\exists K < +\infty \quad |f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq K|x-y| \;,\; |f(t,x)| + |g(t,x)| \leq K(1+|x|)}$$

If $e_y(f,g)_{y\in\mathbb{R}^d}$ satisfy the IL hypothesis, the solution is strong and unique.

Here the function are $g(t,x) = -cx$ and $f(t,x) = \sigma$. So:

$$\left.\begin{array}{l} |\sigma-\sigma| + |-cx+cy| = |c||x-y| \quad \text{and} \\ |\sigma| + |-cx| \leq |\sigma| + |c||x| \leq \max(|\sigma|,|c|)(1+|x|) \end{array}\right] \to K = \max(|\sigma|,|c|) < +\infty$$

(2) $$dX_t = d(e^{-ct}x) + d\left(\sigma e^{-ct} \int_0^t e^{cs}\, dW_s\right) =$$

$$= -xce^{-ct} + \sigma\left(e^{-ct} e^{ct} dW_t - \int_0^t e^{cs} dW_s \, e^{-ct} \cdot c + 0\right) =$$

$$= -xce^{-ct} + \sigma dW_t - \sigma c e^{-ct} \int_0^t e^{cs} dW_s =$$

$$= -c\left(xe^{-ct} + \sigma e^{-ct} \int_0^t e^{cs} dW_s\right) + \sigma dW_t = -cX_t + \sigma dW_t$$

(3) $f \in L^2_{loc}(B)$ deterministic, $g \in C^0 \Rightarrow X_t := g(t) + \int_0^t f(s)\, dB_s$ is a gaussian r.v. $\forall t \in [0,T]$

$\uparrow$ we just need continuous since we are in compact set $[0,T]$

Proof

We just need to prove that $\int_0^t f(s)\, dB_s$ is gaussian since $g(t)$ is deterministic for X_t.

Now, using Novikov condition for the function $\underset{\in \mathbb{R}}{u} \cdot f(t)$ $\left(\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2 f(s)^2 ds\right)\right] < +\infty \right)$

we have $\left(\exp\left(\int_0^t u f(s)\, dB_s - \frac{1}{2}\int_0^t (u f(s))^2 ds\right)\right)_{t \in [0,T]}$ is a true martingale $\forall u \in \mathbb{R}$

Setting $I_t = \int_0^t f(s)\, dB_s$ we have:

$$1 \underset{\substack{\uparrow \\ \text{martingale}}}{=} \mathbb{E}\left[\exp\left(u I(t) - \frac{1}{2}\int_0^t (u f(s))^2 ds\right)\right]$$

$$\Rightarrow \mathbb{E}[\exp(u I(t))] = \exp\left(\frac{1}{2} u^2 \int_0^t f(s)^2 ds\right) \quad (u = it \Rightarrow \text{characteristic function})$$

So the characteristic function of $I(t)$ is equal to a characteristic function of a $N\left(0, \int_0^t f(s)^2 ds\right) \Rightarrow I(t) \sim N\left(0, \int_0^t f(s)^2 ds\right)$. $\square$

Corollary: $(X_t)_t$ as above is a gaussian process

Proof $\lambda_1 X_{t_1} + \dots + \lambda_n X_{t_n} = \sum_{i=1}^n g(t_i) + \int_0^T \sum_{i=1}^n \mathbb{1}_{\{s \leq t_i\}} f(s)\, ds$ that for the previous prop. is a gaussian variable. $\square$

So the process is gaussian.

Now:

$$\mathbb{E}[X_t] = \mathbb{E}\left[e^{-ct}x + \sigma e^{-ct}\int_0^t e^{cs}\, dW_s \right] =$$

$$= e^{-ct}x + \sigma e^{-ct}\,\mathbb{E}\left[\int_0^t e^{cs}\, dW_s\right] = e^{-ct}x + 0 = e^{-ct}x$$

$K \in L^2(B) \Rightarrow K \cdot B$ is a martingale

$\Rightarrow \mathbb{E}[(K\cdot B)_t] = \mathbb{E}[(K\cdot B)_0]$

$$\mathrm{Var}(X_t) = \mathbb{E}\left[(X_t - \mathbb{E}[X_t])^2\right] = \mathbb{E}\left[\left(\cancel{e^{-ct}x} + \sigma e^{-ct}\int_0^t e^{cs}\, dW_s - \cancel{e^{-ct}x}\right)^2\right] =$$

$$= \sigma^2 e^{-2ct}\,\mathbb{E}\left[\left(\int_0^t e^{cs}\, dW_s\right)^2\right] = \sigma^2 e^{-2ct}\,\mathbb{E}\left[\int_0^t e^{2cs}\, ds\right] =$$

Itō isometry for brownian motion

$$= \sigma^2 e^{-2ct}\int_0^t e^{2cs}\, ds = \sigma^2 e^{-2ct}\left[\frac{e^{2cs}}{2c}\right]_0^t = \sigma^2 e^{-2ct}\left(\frac{e^{2ct}}{2c} - \frac{1}{2c}\right) =$$

$$= \frac{\sigma^2}{2c}\left(1 - \frac{1}{e^{2ct}}\right)$$

We also compute the covariance to know everything about the gaussian process:

$$\mathrm{Cov}(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}[X_t]\,\mathbb{E}[X_s]$$

$$= \mathbb{E}\left[\left(e^{-ct}x + \sigma e^{-ct}\int_0^t e^{cu}\, dW_u\right)\left(e^{-cs}x + \sigma e^{-cs}\int_0^s e^{cu}\, dW_u\right)\right] - e^{-ct}x\, e^{-cs}x =$$

martingale $\Rightarrow = 0$

$$= \cancel{e^{-ct}x\, e^{-cs}x} + e^{-ct}x\sigma e^{-cs}\,\mathbb{E}\left[\int_0^s e^{cu}\, dW_u\right] + \sigma e^{-ct} e^{-cs}x\,\mathbb{E}\left[\int_0^t e^{cu}\, dW_u\right] +$$

$$+ \sigma^2 e^{-ct-cs}\,\mathbb{E}\left[\int_0^t e^{cu}\, dW_u \int_0^s e^{cu}\, dW_u\right] - \cancel{e^{-ct}x\, e^{-cs}x}$$

$$= \sigma^2 e^{-c(t+s)}\,\mathbb{E}\left[\int_0^s e^{2cu}\, du\right] + \mathbb{E}\left[\int_s^t e^{cu}\, dW_u \int_0^s e^{cu}\, dW_u\right] =$$

if $s \le t$ — indipendent $\Rightarrow$ split $\mathbb{E}$ and is 0

$$= \sigma^2 e^{-c(t+s)}\left[\frac{e^{2cu}}{2c}\right]_0^s + 0 =$$

$$= \frac{\sigma^2 e^{-c(t+s)}}{2c}\left(e^{2cs} - 1\right)$$. So in general is:

$$\mathrm{Cov}(X_t, X_s) = \frac{\sigma^2 e^{-c(t+s)}}{2c}\left(e^{2c\min(t,s)} - 1\right)$$

(4) Remark: the $\mathcal{F}_t$- adapted process (X_t) satisfies the Markov property if for any bounded Borel function f and for any s and t such that $s \leq t$, we have:

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid \sigma(X_s)]$$

SDE with with "sufficiently regularit" parameters are Markov processes.

Furthermore X_t does not depend directly on $t \Rightarrow X_{t+s} \mid X_s = x \sim X_t \mid X_0 = x$

For the simulation X_{kh} with $k \in \{1, \dots, N\}$:

- fix the starting point $X_0 = x$ and the time step h.
- The solution is $X_h = x e^{-ch} + \sigma e^{-ch} \underbrace{\int_0^h e^{cs} \, dW_s}_{?} =$

$$N\left(0, \int_0^h (e^{cs})^2 ds\right) = N\left(0, \left[\frac{e^{2cs}}{2c}\right]_0^h\right) = N\left(0, \frac{e^{2ch}-1}{2c}\right)$$

$$= x e^{-ch} + \sigma e^{-ch} N(0,1)$$

$$\sigma e^{-ch} N\left(0, \frac{e^{2ch}-1}{2c}\right) \sim \sigma N\left(0, \frac{1-e^{-2ch}}{2c}\right) \sim \sigma \sqrt{\frac{1-e^{-2ch}}{2c}} N(0,1)$$

- Indeterevely by Markov $X_{hk} = X_{(k-1)h} e^{-ch} + \sigma \sqrt{\frac{1-e^{-2ch}}{2c}} N(0,1)$

(5) $X_t = \underbrace{X_0 e^{-ct}}_{\text{gaussian}} + \underbrace{\sigma e^{-ct} \int_0^t e^{cs} dW_s}_{\text{gaussian}}$, X_t is a sum of two indipendent gaussian

$\Rightarrow$ it's gaussian $\Rightarrow$ its law is detrmined by $\mathbb{E}[X_t]$ and $Var(X_t)$.

Now:

$\mathbb{E}[X_t] = 0 + 0 = 0$ $\quad$ $Var(X_t) \underset{\text{indipendent}}{=} Var(X_0 e^{-ct}) + Var\left(\sigma e^{-ct} \int_0^t e^{cs} dW_s\right) =$

$$= e^{-2ct} \sigma_0^2 + \sigma^2 e^{-2ct} \left(\left[\frac{e^{2cs}}{2c}\right]_0^t\right) =$$

$$= e^{-2ct} \sigma_0^2 + \sigma^2 e^{-2ct} \left[\frac{e^{2ct}-1}{2c}\right] =$$

$$= e^{-2ct} \sigma_0^2 + \sigma^2 \left(\frac{1-e^{-2ct}}{2c}\right) = e^{-2ct}\left(\sigma_0^2 - \frac{\sigma^2}{2c}\right) + \frac{\sigma^2}{2c}$$

$\Rightarrow$ we want to eliminate $e^{-2ct} \Rightarrow \sigma_0^2 = \frac{\sigma^2}{2c}$

Exercises Martingales in Financial Mathematics: Aspects of Brownian Motion and Barrier Options

Week 7, 2024

Exercise 1: Reflection principle

Let (W_t) be a standard Brownian motion and $(\mathcal{F}_t)$ the corresponding Brownian filtration (as introduced in the lecture course). Let M_t be the corresponding running maximum, i.e. $M_t = \sup_{s\leq t} W_s$. Derive by a heuristic argument or by a proof that for $w \leq m$, $m > 0$

$$\mathbb{P}(M_t \geq m, W_t \leq w) = \mathbb{P}(W_t \geq 2m - w)\,.$$ [1]

Exercise 2: Joint distribution of W_t and M_t

For a $t > 0$, find the joint probability density function of $M_t = \sup_{s\in[0,t]} W_s$ and W_t for $w \leq m$, $m > 0$.
Hint: Use Exercise 1.

Exercise 3: Brownian motion with drift

Let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion defined on $(\Omega\,, \mathcal{F}, \mathbb{Q})$ and let $\hat{W} = (\alpha t + W_t)_{t\in[0,T]}$ for a given real α, i.e. the Brownian motion $(\hat{W}_t)$ has drift α under $\mathbb{Q}$. We further define $\hat{M}_T = \sup_{0\leq t\leq T} \hat{W}_t$. Show that for $m > 0$, $w \leq m$, the joint density function of $(\hat{M}_T, \hat{W}_T)$ under $\mathbb{Q}$ is given by

$$\tilde{f}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\,. \tag{1}$$

Hint: Use Girsanov for the density $\hat{Z}_t = e^{-\alpha W_t - \frac{1}{2}\alpha t}$ and the result from Exercise 2.

[1] Hints for a proof: For a proof use the fact that for a stopping time τ with $\mathbb{P}(\tau < \infty) > 0$ we have that conditionally on $\{\tau < \infty\}$ the process $(W_{t+\tau} - W_\tau, t \geq 0)$ is a $(\mathcal{F}_{\tau+t})$-Brownian motion independent of $\mathcal{F}_\tau$ and that for positive m $S_m := \inf\{t : W_t \geq m\} = \inf\{t : W_t = m\} =: T_m$ is/are a.s. finite stopping time(s) (you are not expected to proof that, the interested student is referred e.g. to Revuz and Yor, Continuous Martingales and Brownain Motion, Sec. 3, Ch. II and Sec. 3, Ch. III; or for a rather elementary proof of the strong Markov Property to Th. 32 in P. E. Protter, Stochastic Integral and Differential Equations (Version 2.1) and for T_m to be a stopping time to Th. 4 in the same book).

$\sqrt{}$ Exercise 4: Value of a up-and-out call

With the usual notation, price (at $t = 0$) the following so-called up-and-out call being defined by

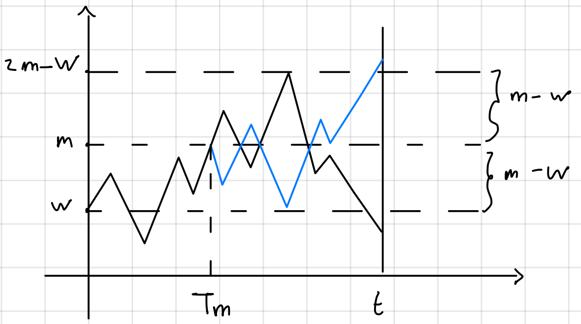
$$h = (S_T - k)_+ \mathbb{1}_{S_t < b\,, \forall t \in [0,T]}$$

where we assume that $S_0 < b$ and $0 < k < b$ (otherwise, the option must be knocked out in order to be in the money and hence, could only pay off zero).

Exercise 1: Reflection principle

Let (W_t) be a standard Brownian motion and $(\mathcal{F}_t)$ the corresponding Brownian filtration (as introduced in the lecture course). Let M_t be the corresponding running maximum, i.e. $M_t = \sup_{s\le t} W_s$. Derive by a heuristic argument or by a proof that for $w \le m$, $m > 0$

$$\mathbb{P}(M_t \ge m, W_t \le w) = \mathbb{P}(W_t \ge 2m - w)\,.^1$$



Set $T_m = \inf\{t : W_t = m\} \overset{\text{continuous path}}{=} \inf\{t : W_t \ge m\}$ is a stopping time $\big([m, +\infty)$ closed subset of $\mathbb{R}\big)$

Notice that $\{T_m \le t\} = \{M_t \ge m\}$ $\big(T_m(\omega) \le t \iff \inf\{s : W_s(\omega) \ge m\} \le t \iff \sup_{s\le t} W_s(\omega) \ge m$

$\iff M_t(\omega) \ge m\big)$ so the thesis is:

$$\mathbb{P}(T_m \le t, W_t \le w) = \mathbb{P}(W_t \ge 2m - w)$$

furthermore $\mathbb{P}(W_t \ge 2m-w) = \mathbb{P}(W_t \ge 2m - w, T_m \le t)$ since $2m - w > m$ so

the idea is that for each path that at time t is at or below w (and has reached m before or $\overbrace{\text{at } t}^{\leftarrow \text{has vanishing probability}}$) there is a path that at time t is at or above $2m-w$ (and it is free that has reached m before or $\overbrace{\text{at } m}^{\text{has vanishing prob.}}$). This path is constructed as in the figure: reflecting with respect to $y = m$ the original path from T_m.

Formally: for what we have said we want to prove

$$\mathbb{P}(T_m \le t, W_t \le w) = \mathbb{P}(T_m \le t, W_t \ge 2m-w)$$

So:

$$\mathbb{P}(T_m \le t, W_t \le w) \overset{W_{T_m} = m}{=} \mathbb{P}(T_m \le t, W_t - W_{T_m} \le w - m) =$$

$$= \mathbb{E}\left[\mathbb{1}_{\{T_m \le t\}} \mathbb{1}_{\{W_t - W_{T_m} \le w - m\}}\right] =$$

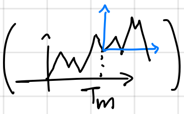
X $\mathcal{F}$-mea:

$\mathbb{E}[XY] \overset{\text{def.}}{=} \mathbb{E}[\mathbb{E}[XY|\mathcal{F}]] = \mathbb{E}[X\,\mathbb{E}[Y|\mathcal{F}]]$ (X $\mathcal{F}$-mea.)

$$= \mathbb{E}\Big[\underbrace{\mathbb{1}_{\{T_m \le t\}}}_{T_m\text{-meas.}} \mathbb{E}\left[\mathbb{1}_{\{W_t - W_{T_m} \le w - m\}} \,\middle|\, T_m\right]\Big] =$$

$$\tilde{W}_s := (W_{T_m+s} - W_{T_m})_{s\ge 0} \quad = \mathbb{E}\left[\mathbb{1}_{\{T_m\le t\}}\, \mathbb{E}\left[\mathbb{1}_{\{\tilde{W}_{t-T_m} \le w-m\}}\right]\right] =$$

is a brownian motion ind.

from T_m ()

$$\text{also } -\tilde{W}_s \text{ is a BM and} \quad = \mathbb{E}\left[\mathbb{1}_{\{T_m\le t\}}\, \mathbb{E}\left[\mathbb{1}_{\{-\tilde{W}_{t-T_m} \le w-m\}}\right]\right] =$$

so $\tilde{W}_{t-T_m}$ and $-\tilde{W}_{t-T_m}$

are both $\sim N(0, |t-T_m|)$

$$= \mathbb{E}\left[\mathbb{1}_{\{T_m\le t\}}\, \mathbb{E}\left[\mathbb{1}_{\{\tilde{W}_{t-T_m} \ge m-w\}}\right]\right] =$$

$$= \mathbb{P}\left(T_m \le t,\ W_t - W_{T_m} \ge m-w\right) =$$

$$W_{T_m} = m \quad = \mathbb{P}\left(T_m\le t,\ W_t \ge 2m-w\right)$$

Exercise 2: Joint distribution of W_t and M_t

For a $t>0$, find the joint probability density function of $M_t = \sup_{s\in[0,t]} W_s$ and W_t for $w \le n$, $n>0$.
Hint: Use Exercise 1.

From Ex. 1 we know $P(M_t \ge n, W_t \le \omega) = P(W_t \ge 2n-\omega)$, $n>0$, $\omega\le n$. So:

$$\int_n^{+\infty}\int_{-\infty}^{\omega} f(x,y)\,dy\,dx = \frac{1}{\sqrt{2\pi t}}\int_{2n-\omega}^{+\infty} e^{-\frac{z^2}{2t}}\,dz$$

differentiable in n ← differentiable in n

$$\frac{\partial}{\partial n}(LHS) = -\int_{-\infty}^{\omega} f(n,y)\,dy \qquad \left(\frac{\partial}{\partial x}\int_x^b f(y)\,dy = -f(x)\right) \leftarrow \text{differentiable in } \omega$$

$$\frac{\partial}{\partial n}(RHS) = \frac{1}{\sqrt{2\pi t}}\left(-e^{-(2n-\omega)^2/2t}\cdot 2\right) = -\frac{2}{\sqrt{2\pi t}}\, e^{-(2n-\omega)^2/2t} \leftarrow \text{differentiable in } \omega$$

$$\frac{\partial}{\partial\omega}\left(\frac{\partial}{\partial n}(LHS)\right) = -f(n,\omega)$$

$$\frac{\partial}{\partial\omega}\left(\frac{\partial}{\partial n}(RHS)\right) = -\frac{2}{\sqrt{2\pi t}}\, e^{-(2n-\omega)^2/2t}\cdot\left(-\frac{1}{2t}\right)\left(2(2n-\omega)\cdot(-1)\right) = -\frac{2(2n-\omega)}{t\sqrt{2\pi t}}\, e^{-(2n-\omega)^2/2t}$$

$$\Longrightarrow f(n,\omega) = \frac{2(2n-\omega)}{t\sqrt{2\pi t}}\, e^{-(2n-\omega)^2/2t} \quad \text{for } n>0,\ \omega\le n.$$

But, $W_t \le M_t$ by definition $\Longrightarrow$ $f(n,\omega)=0$ $\forall \omega > n$; $M_t \ge B_0 = 0 \Longrightarrow f(n,\omega)=0$ $\forall n<0$ ($\forall n \le 0$ it's the same). So the density $\forall n, \omega$:

$$f(n,\omega) = \frac{2(2n-\omega)}{t\sqrt{2\pi t}} e^{-(2n-\omega)^2/2t} \mathbb{1}_{\{n>0\}} \mathbb{1}_{\{\omega \le n\}}.$$

So we have found the density, we can write the cumulative distribution function:

Furthermore, note $P(A \cap B) = P(A \setminus (A \cap B^c)) = P(A) - P(A \cap B^c)$, $A := \{W_t \le \omega\}$ $B = \{M_t \le n\}$

① $n > 0$, $\omega \le n$:

refl. principle $(P(M_t > n) = P(M_t \ge n))$

$$P(M_t \le n, W_t \le \omega) = P(W_t \le \omega) - P(M_t > n, W_t \le \omega) =$$

$$= P(W_t \le \omega) - P(W_t \ge 2n-\omega) = N\left(\frac{\omega}{\sqrt{t}}\right) - \left(1 - N\left(\frac{2n-\omega}{\sqrt{t}}\right)\right) =$$

$W_t \sim N(0,t)$

$$= N\left(\frac{\omega}{\sqrt{t}}\right) - N\left(\frac{\omega - 2n}{\sqrt{t}}\right)$$

② $0 < n \le \omega$:

$$P(M_t \le n, W_t \le \omega) \overset{M_t \ge W_t}{=} P(M_t \le n, W_t \le n) \overset{\text{as above}}{=} N\left(\frac{n}{\sqrt{t}}\right) - N\left(\frac{z-2n}{\sqrt{t}}\right) = N\left(\frac{\omega}{\sqrt{t}}\right) - N\left(-\frac{n}{\sqrt{t}}\right)$$

③ $n \le 0$: $M_t \ge M_0 = 0 \Rightarrow P(M_t \le n, W_t \le \omega) = 0$

<u>Remarks</u>:

- $P(M_t \le n) = P(M_t \le n, W_t \le n) \underset{②}{=} N\left(\frac{n}{\sqrt{t}}\right) - N\left(-\frac{n}{\sqrt{t}}\right) = P(W_t \le n) - P(W_t \le -n) =$

$$= P(W_t \le n) - (1 - P(W_t \ge -n)) = P(W_t \le n) - 1 + P(W_t \ge -n) =$$

$$= P(-n \le W_t \le n) = P(|W_t| \le n)$$

$$1 = P(W_t \le n) + P(-n \le W_t \le n) + P(W_t \ge n)$$

Furthermore $\frac{\partial}{\partial n}(P(M_t \le n)) = \frac{\partial}{\partial n} N\left(\frac{n}{\sqrt{t}}\right) - \frac{\partial}{\partial n} N\left(\frac{-n}{\sqrt{t}}\right) = \frac{\partial}{\partial n}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\frac{n}{\sqrt{t}}} \exp\left(-\frac{z^2}{2}\right)dz\right)$

$$- \frac{\partial}{\partial n}\left(\int_{-\infty}^{-\frac{n}{\sqrt{t}}} \exp\left(\frac{z^2}{2}\right)dz\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{n^2}{t}\right) \cdot \frac{1}{\sqrt{t}} - \exp\left(-\frac{1}{2}\frac{n^2}{t}\right)\left(-\frac{1}{\sqrt{t}}\right) =$$

$$= \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{n^2}{t}\right), \quad n > 0.$$

- Density of $(M_t, M_t - W_t)$: we know the density of (M_t, W_t), so the function of the change of variables is $\varphi(x,y) = (x, x-y) =: (u, v)$

$$J_\varphi = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \Rightarrow \left| \det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right| = 1$$

$$f_{(M_t, M_t - W_t)}(u, v) = f_{(M_t, W_t)}(\varphi^{-1}(u,v)) \cdot 1 = \quad \leftarrow \varphi^{-1}(u,v) = (u, u-v)$$

$n > 0 \Leftrightarrow u > 0$

$w \le n \Leftrightarrow u - v \le u \Leftrightarrow v \geq 0$

$$= \frac{2\,(2u - u + v)}{t\sqrt{2\pi t}} \, e^{-(2u-u+v)^2/2t} \, \mathbb{1}_{\{n>0\}} \mathbb{1}_{\{v \geq 0\}} =$$

$$= \frac{2\,(u+v)}{t\sqrt{2\pi t}} \, e^{-(u+v)^2/2t} \, \mathbb{1}_{\{n>0\}} \mathbb{1}_{\{v \geq 0\}}$$

In particular M_t, $M_t - W_t$ are exchangeable (and so $M_t \overset{D}{\sim} M_t - W_t$).

Exercise 3: Brownian motion with drift

Let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{Q})$ and let $\hat{W} = (\alpha t + W_t)_{t\in[0,T]}$ for a given real α, i.e. the Brownian motion $(\hat{W}_t)$ has drift α under $\mathbb{Q}$. We further define $\hat{M}_T = \sup_{0\le t\le T} \hat{W}_t$. Show that for $m > 0$, $w \le m$, the joint density function of $(\hat{M}_T, \hat{W}_T)$ under $\mathbb{Q}$ is given by

$$\tilde{f}(m, w) = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}. \tag{1}$$

Hint: Use Girsanov for the density $\hat{Z}_t = e^{-\alpha W_t - \frac{1}{2}\alpha t}$ and the result from Exercise 2.

$X_t := \alpha \Rightarrow \mathbb{E}\left[e^{\frac{1}{2}\int_0^T \alpha^2 dt}\right] = \mathbb{E}\left[e^{\frac{1}{2}\alpha^2 T}\right] = e^{\frac{1}{2}\alpha^2 T} \Rightarrow$ the Novikov condition is satisfied for $X_t = \alpha \Rightarrow e^{-\alpha W_t - \frac{1}{2}\alpha^2 t}$ is a true martingale. $\hat{Z}_t := e^{-\alpha W_t - \frac{1}{2}\alpha^2 t} = e^{-\alpha \hat{W}_t + \frac{1}{2}\alpha^2 t}$

, for Girsanov we know $\hat{W}_t = (\alpha t + W_t)_{t\in[0,T]}$ is a standard BM w.r.t. P where:

$$\frac{dP}{dQ} = \hat{Z}_T = e^{-\alpha \hat{W}_T + \frac{1}{2}\alpha^2 T}$$

So, under P we know $(\hat{M}_T, \hat{W}_T)$ has density $\hat{f}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}$, $w \le m$, $n > 0$.

$$\hat{f} = \frac{dP}{d\lambda} \Rightarrow \text{I want to find } \tilde{f} = \frac{dQ}{d\lambda} = \frac{dQ}{dP}\frac{dP}{d\lambda} = \frac{dQ}{dP}\hat{f}$$

$\uparrow$ Lebesgue measure

$$\Rightarrow \frac{dQ}{d\lambda}(m, w) = e^{\alpha w - \frac{1}{2}\alpha^2 T} \hat{f}(n, w)$$

√ Exercise 4: Value of a up-and-out call

With the usual notation, price (at $t = 0$) the following so-called up-and-out call being defined by

$$h = (S_T - k)_+ \mathbb{1}_{S_t<b, \forall t\in[0,T]}$$

where we assume that $S_0 < b$ and $0 < k < b$ (otherwise, the option must be knocked out in order to be in the money and hence, could only pay off zero).

First, let's prove that h is a claim:

- square integrable: $\mathbb{E}_Q\left[(S_T-K)_+ \mathbb{1}_{\{\dots\}}\right] \le \mathbb{E}_Q\left[(S_T-K)_+^2\right] \le \mathbb{E}_Q\left[S_T^2\right] < +\infty$
- $\mathcal{F}_T$-measurable: $(S_T-K)_+$ $\mathcal{F}_T$-measurable, $\mathbb{1}_{\{S_t<b\}}$ $\mathcal{F}_t$-meas. but $\mathcal{F}_t \subseteq \mathcal{F}_T \Rightarrow \mathcal{F}_T$-meas.

  product of $\mathcal{F}_T$-measurable is measurable
- h nonnegative: evident.

So, we know:

$$V_0 = e^{-rT}\mathbb{E}_Q\left[(S_T-K)_+ \mathbb{1}_{\{S_t<b,\ \forall t\in[0,T]\}}\right] =$$

$$= e^{-rT}\mathbb{E}_Q\left[\left(S_0 e^{(r-\frac12\sigma^2)T+\sigma W_T}-K\right)_+ \mathbb{1}_{\{S_0 e^{(r-\frac12\sigma^2)t+\sigma W_t}<b,\ \forall t\in[0,T]\}}\right] =$$

$\hat W_t = \alpha t + W_t$

if $\alpha := \frac{1}{\sigma}\left(r-\frac12\sigma^2\right)$

$\Rightarrow \left(r-\frac12\sigma^2\right)T + \sigma W_T =$

$= \sigma\left(W_T + \frac{(r-\frac12\sigma^2)}{\sigma}T\right) =$

$= \sigma \hat W_T$

there is not "+" but the indicator

$$= e^{-rT}\mathbb{E}_Q\left[\left(S_0 e^{\sigma\hat W_T}-K\right)\mathbb{1}_{\{S_0e^{\sigma \hat W_T}>K\}}\mathbb{1}_{\{S_0e^{\sigma\hat W_t}<b,\ \forall t\in[0,T]\}}\right]$$

comes from here

$$= e^{-rT}\mathbb{E}_Q\left[\left(S_0e^{\sigma\hat W_T}-K\right)\mathbb{1}_{\{K<S_0e^{\sigma\hat W_T}<b\}}\mathbb{1}_{\{S_0e^{\sigma\hat W_t}<b\ \forall t\in[0,T]\}}\right]$$

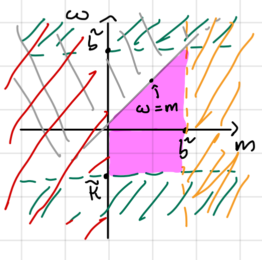
def $\hat M_t$

$$= e^{-rT}\mathbb{E}_Q\left[\left(S_0e^{\sigma\hat W_T}-K\right)\mathbb{1}_{\left\{\underbrace{\frac{\log(\frac{K}{S_0})}{\sigma}}_{=:\tilde K}<\hat W_T<\underbrace{\frac{\log(\frac{b}{S_0})}{\sigma}}_{=:\tilde b}\right\}}\mathbb{1}_{\left\{\hat M_T<\underbrace{\frac{\log(\frac{b}{S_0})}{\sigma}}_{=:\tilde b}\right\}}\right] =$$

we use density of $(\hat W_T, \hat M_T)$

$$= e^{-rT}\int_{\mathbb R}\int_{\mathbb R}\left(S_0e^{\sigma\omega}-K\right)\mathbb{1}_{\{\tilde K<\omega<\tilde b\}}\mathbb{1}_{\{m<\tilde b\}}\mathbb{1}_{\{m\ge0\}}\mathbb{1}_{\{\omega\le m\}}\frac{2(2m-\omega)}{T\sqrt{2\pi T}}e^{\alpha\omega-\frac12\alpha^2T-\frac{1}{2T}(2m-\omega)^2}\,dm\,d\omega =$$

$\tilde b>0$ $\tilde K<0$ by def.



$$= e^{-rT}\int_{\tilde K}^{\tilde b}\int_{\omega_+}^{\tilde b} \text{''} = e^{-rT}\int_{\tilde K}^{\tilde b}\left(S_0e^{\sigma\omega}-K\right)\frac{e^{\alpha\omega-\frac12\alpha^2T}}{\sqrt{2\pi T}}\int_{\omega_+}^{\tilde b}\frac{2(2m-\omega)}{T}e^{-\frac{1}{2T}(2m-\omega)^2}\,dm\,d\omega =$$

positive part

without indicator functions

$$= e^{-rT}\int_{\tilde K}^{\tilde b}\left(S_0e^{\sigma\omega}-K\right)\frac{e^{\alpha\omega-\frac12\alpha^2T}}{\sqrt{2\pi T}}\int_{\omega^2}^{(2\tilde b-\omega)^2}\frac{1}{2T}e^{-\frac{1}{2T}z}\,dz\,d\omega =$$

$z:=(2m-\omega)^2$

$\frac{dz}{dm} = 2(2m-\omega)2$

$m=\omega_+ \Rightarrow z = \begin{cases}(2\omega-\omega)^2=\omega^2, & \omega\ge0\\ (0-\omega)^2=\omega^2, & \omega<0\end{cases} = \omega^2$

$m=\tilde b \Rightarrow z=(2\tilde b-\omega)^2$

$$\int_{\omega^2}^{2\tilde{b}-\omega} \frac{1}{2T} e^{-\frac{1}{2T}z}\, dz = \left[-e^{-\frac{1}{2T}z}\right]_{\omega^2}^{2\tilde{b}-\omega}$$

$$= \int_{\tilde{K}}^{\tilde{b}} \left(S_0 e^{\sigma\omega} - K\right) \frac{e^{-rT+\alpha\omega-\frac{1}{2}\alpha^2 T - \frac{1}{2T}\omega^2}}{\sqrt{2\pi T}}\, d\omega \; -$$

$$- \int_{\tilde{K}}^{\tilde{b}} \left(S_0 e^{\sigma\omega} - K\right) \frac{e^{-rT+\alpha\omega-\frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\tilde{b}-\omega)^2}}{\sqrt{2\pi T}}\, d\omega =$$

$$= S_0 \Bigg(\overbrace{\frac{1}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{\sigma\omega - rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\omega^2}\, d\omega}^{I_1} \Bigg) - K \Bigg(\overbrace{\frac{1}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{-rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\omega^2}\, d\omega}^{I_2} \Bigg) -$$

$$- S_0 \Bigg(\overbrace{\frac{1}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{\sigma\omega - rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}\omega}{T} - \frac{\omega^2}{2T}}\, d\omega}^{I_3} \Bigg) +$$

$$+ K \Bigg(\overbrace{\frac{1}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{-rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}\omega}{T} - \frac{\omega^2}{2T}}\, d\omega}^{I_4} \Bigg) = S_0 I_1 - K I_2 - S_0 I_3 + K I_4$$

Note that each of these integrals are of the form:

$$\frac{1}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{\beta + \gamma\omega - \frac{\omega^2}{2T}}\, d\omega \underset{\substack{\uparrow \\ \text{complet} \\ \text{the square:}}}{=} \frac{1}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{\beta - \frac{\gamma^2 T}{2} - \frac{1}{2T}(\omega - \gamma T)^2}\, d\omega =$$

$$\beta + \gamma\omega - \frac{\omega^2}{2T} = \beta - \frac{1}{2T}\left(-2\gamma T\omega + \omega^2\right) = \beta + \frac{\gamma^2 T}{2} - \frac{1}{2T}\left(\omega - \gamma T\right)^2$$

$$= \frac{e^{\beta + \frac{\gamma^2 T}{2}}}{\sqrt{2\pi T}} \int_{\tilde{K}}^{\tilde{b}} e^{-\frac{1}{2T}(\omega-\gamma T)^2}\, d\omega \underset{\uparrow}{=}$$

$$y := \frac{\omega - \gamma T}{\sqrt{T}} \qquad dy = \frac{1}{\sqrt{T}}\, d\omega \; ; \quad \text{if } \omega = \tilde{b} \Rightarrow y = \frac{\tilde{b} - \gamma T}{\sqrt{T}}$$

$$\text{if } \omega = \tilde{K} \Rightarrow y = \frac{\tilde{K} - \gamma T}{\sqrt{T}}$$

$$= \frac{e^{\beta + \frac{\gamma^2 T}{2}}}{\sqrt{2\pi \cancel{T}}} \cancel{\sqrt{T}} \int_{\frac{\tilde{K}-\gamma T}{\sqrt{T}}}^{\frac{\tilde{b}-\gamma T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\, dy =$$

$$= e^{\beta + \frac{\gamma^2 T}{2}} \left(N\left(\frac{\tilde{b} - \gamma T}{\sqrt{T}}\right) - N\left(\frac{\tilde{K} - \gamma T}{\sqrt{T}}\right) \right) = e^{\beta + \frac{\gamma^2 T}{2}} \left[1 - N\left(\frac{\gamma T - \tilde{b}}{\sqrt{T}}\right) - \left(1 - N\left(\frac{\gamma T - \tilde{K}}{\sqrt{T}}\right)\right) \right] =$$

$$\underset{\substack{\uparrow \\ \text{def } \tilde{b}, \tilde{K}}}{=} e^{\beta + \frac{\gamma^2 T}{2}} \left[N\left(\frac{-\log\left(\frac{K}{S_0}\right) + \gamma\sigma T}{\sigma\sqrt{T}} \right) - N\left(\frac{-\log\left(\frac{b}{S_0}\right) + \gamma\sigma T}{\sigma\sqrt{T}} \right) \right] =$$

$$= e^{\beta + \frac{\gamma^2 T}{2}} \left[N\left(\frac{\log\left(\frac{S_0}{K}\right) + \gamma\sigma T}{\sigma\sqrt{T}} \right) - N\left(\frac{\log\left(\frac{S_0}{b}\right) + \gamma\sigma T}{\sigma\sqrt{T}} \right) \right]$$

Let's set $\delta_{\pm}(s) = \dfrac{\log(s) + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$

So, to compute the integrals we have to compute $e^{\beta + \frac{\gamma^2 T}{2}}$ and $\gamma\sigma$:

- I_1: $\beta = -rT - \frac{1}{2}\alpha^2 T$, $\gamma = \alpha + \sigma$ $\Rightarrow$ calculus $\Rightarrow$ $\frac{1}{2}\gamma^2 T + \beta = 0$ and $\gamma\sigma = r + \frac{1}{2}\sigma^2$

$$\Rightarrow I_1 = N\left(\frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\log\left(\frac{S_0}{b}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) = N\left(\delta_+\left(\frac{S_0}{K}\right)\right) - N\left(\delta_+\left(\frac{S_0}{b}\right)\right)$$

- I_2: $\beta = -rT - \frac{1}{2}\alpha^2 T$, $\gamma = \alpha$ $\Rightarrow$ calculus: $\frac{1}{2}\gamma^2 T + \beta = -rT$ and $\gamma\sigma = r - \frac{1}{2}\sigma^2$

$$I_2 = e^{-rT}\left(N\left(\frac{\log\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\log\left(\frac{S_0}{b}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)\right) = e^{-rT}\left(N\left(\delta_-\left(\frac{S_0}{K}\right)\right) - N\left(\delta_-\left(\frac{S_0}{b}\right)\right)\right)$$

- I_3 ... computation

Exercises Martingales in Financial Mathematics: Black–Scholes and Brownian Motion

Week 8, 2024

Recall that in a Black–Scholes setting, the price of a European call is given by

$$V_t = C(T-t,\, S_t,\, K,\, r,\, \sigma) = S_t\, \mathcal{N}(d_+) - K\, e^{-r(T-t)}\, \mathcal{N}(d_-)$$

with

$$d_\pm = \frac{\log \frac{S_t}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\,(T-t)}{\sigma\sqrt{T-t}}\,.$$

√Exercise 1: Black–Scholes model and Greeks

We are interested in the prices of European call and put options. In particular in the partial derivatives of the pricing function with respect to model parameters.

1. Use the European put-call parity in order to (re)derive the pricing formula for European puts.
2. Compute the deltas

$$\begin{aligned}
\Delta_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial C}{\partial x}(T-t,\, x,\, K,\, r,\, \sigma)\bigg|_{x=S_t} \\
\Delta_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P}{\partial x}(T-t,\, x,\, K,\, r,\, \sigma)\bigg|_{x=S_t}
\end{aligned}$$

3. Compute the gammas

$$\begin{aligned}
\Gamma_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial^2 C}{\partial x^2}(T-t,\, x,\, K,\, r,\, \sigma)\bigg|_{x=S_t} \\
\Gamma_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P^2}{\partial x^2}(T-t,\, x,\, K,\, r,\, \sigma)\bigg|_{x=S_t}
\end{aligned}$$

4. Compute the vegas

$$\begin{aligned}
\mathrm{Vega}_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial C}{\partial x}(T-t,\, S_t,\, K,\, r,\, x)\bigg|_{x=\sigma} \\
\mathrm{Vega}_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P}{\partial x}(T-t,\, S_t,\, K,\, r,\, x)\bigg|_{x=\sigma}
\end{aligned}$$

5. Compute the thetas

$$\begin{aligned}\Theta_C(T-t,\,S_t,\,K,\,r,\,\sigma) &= -\frac{\partial C}{\partial s}(s,\,S_t,\,K,\,r,\,\sigma)\Big|_{s=T-t}\\ \Theta_P(T-t,\,S_t,\,K,\,r,\,\sigma) &= -\frac{\partial P}{\partial s}(s,\,S_t,\,K,\,r,\,\sigma)\Big|_{s=T-t}\end{aligned}$$

6. Compute the rhos

$$\begin{aligned}\rho_C(T-t,\,S_t,\,K,\,r,\,\sigma) &= \frac{\partial C}{\partial p}(T-t,\,S_t,\,K,\,p,\,\sigma)\Big|_{p=r}\\ \rho_P(T-t,\,S_t,\,K,\,r,\,\sigma) &= \frac{\partial P}{\partial p}(T-t,\,S_t,\,K,\,p,\,\sigma)\Big|_{p=r}\end{aligned}$$

7. Give some remarks on Δ, Γ, Vega, and Θ.

Exercise 2: An numerical example of an application of the Black–Scholes model

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is $38.
(a) What is the probability that a European call option with strike price of $40 maturing in three months will be exercised? What is the value of this option at maturity (assume 10%p.a. risk-free interest rate)?
(b) Answer the same questions for a European put option?

Exercise 3: Black–Scholes Model: Another financial derivative

Compute the price and describe the replication strategy at $t = 0$ of the European derivative being defined by the following payoff function (where we assume that the asset price process follows a geometric Brownian motion and where $k > 0$ is a positive constant).

$$f(S_T) = \max(S_T, k)\,.$$

Exercise 4: Brownian motion

Let (W_t) be a standard Brownian motion. Which one of the following processes are standard Brownian motions as well (justify your answers)?

1. The process (X_t), being defined by $X_t = 2(W_{1+\frac{t}{4}} - W_1)$.
2. The process (Y_t), being defined by $Y_t = \sqrt{t}W_1$.
3. The process (Z_t), being defined by $Z_t = W_{2t} - W_t$.

Exercise 1: Black–Scholes model and Greeks

We are interested in the prices of European call and put options. In particular in the partial derivatives of the pricing function with respect to model parameters.

1. Use the European put-call parity in order to (re)derive the pricing formula for European puts.
2. Compute the deltas

$$\Delta_C(T-t, S_t, K, r, \sigma) = \frac{\partial C}{\partial x}(T-t, x, K, r, \sigma)\Big|_{x=S_t}$$
$$\Delta_P(T-t, S_t, K, r, \sigma) = \frac{\partial P}{\partial x}(T-t, x, K, r, \sigma)\Big|_{x=S_t}$$

3. Compute the gammas

$$\Gamma_C(T-t, S_t, K, r, \sigma) = \frac{\partial^2 C}{\partial x^2}(T-t, x, K, r, \sigma)\Big|_{x=S_t}$$
$$\Gamma_P(T-t, S_t, K, r, \sigma) = \frac{\partial P^2}{\partial x^2}(T-t, x, K, r, \sigma)\Big|_{x=S_t}$$

4. Compute the vegas

$$\text{Vega}_C(T-t, S_t, K, r, \sigma) = \frac{\partial C}{\partial x}(T-t, S_t, K, r, x)\Big|_{x=\sigma}$$
$$\text{Vega}_P(T-t, S_t, K, r, \sigma) = \frac{\partial P}{\partial x}(T-t, S_t, K, r, x)\Big|_{x=\sigma}$$

5. Compute the thetas

$$\Theta_C(T-t, S_t, K, r, \sigma) = -\frac{\partial C}{\partial s}(s, S_t, K, r, \sigma)\Big|_{s=T-t}$$
$$\Theta_P(T-t, S_t, K, r, \sigma) = -\frac{\partial P}{\partial s}(s, S_t, K, r, \sigma)\Big|_{s=T-t}$$

6. Compute the rhos

$$\rho_C(T-t, S_t, K, r, \sigma) = \frac{\partial C}{\partial p}(T-t, S_t, K, p, \sigma)\Big|_{p=r}$$
$$\rho_P(T-t, S_t, K, r, \sigma) = \frac{\partial P}{\partial p}(T-t, S_t, K, p, \sigma)\Big|_{p=r}$$

7. Give some remarks on Δ, Γ, Vega, and Θ.

(1) $$C_t - P_t = S_t - Ke^{-r(T-t)} \Rightarrow$$

$$P_t = C_t - S_t + Ke^{-r(T-t)} = S_t N(d_+) - Ke^{-r(T-t)} N(d_-) - S_t + Ke^{-r(T-t)} =$$
$$= Ke^{-r(T-t)}(1 - N(d_-)) - S_t(1 - N(d_+))$$

(2) <u>Technical lemma:</u> $\tau := T-t$. $Ke^{-r\tau}\varphi(d_-) = S_t\varphi(d_+)$ where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

<u>Proof</u> $$\varphi(d_-) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right) =$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\underbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}_{=:\, d_+} - \sigma\sqrt{\tau}\right)^2\right) =$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}d_+^2 - \frac{1}{2}\sigma^2\tau + d_+\sigma\sqrt{\tau}\right) =$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}d_+^2 \cancel{- \frac{1}{2}\sigma^2\tau} + \log\left(\frac{S_t}{K}\right) + r\tau \cancel{+ \frac{1}{2}\sigma^2\tau}\right) =$$

$$= \varphi(d_+)\frac{S_t}{K} e^{r\tau}.$$

□

Now:

$$\Delta_C = \frac{\partial C_t}{\partial S_t} = \frac{\partial}{\partial S_t}\left(S_t N\Bigg(\overbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}^{d_+}\Bigg) - Ke^{-r\tau} N\Bigg(\overbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}^{d_-}\Bigg)\right) =$$

$$= N(d_+) + S_t\,\varphi(d_+)\cdot\left(\frac{\cancel{K}}{S_t}\cdot\frac{1}{\cancel{K}}\right)\frac{1}{\sigma\sqrt{\tau}} - \underbrace{Ke^{-r\tau}\varphi(d_-)}_{\text{Lemma} = S_t\varphi(d_+)}\frac{\cancel{K}}{S_t}\cdot\frac{1}{\cancel{K}}\cdot\frac{1}{\sigma\sqrt{\tau}} = N(d_+)$$

$$\Delta_p = \frac{\partial P_t}{\partial S_t} = \frac{\partial}{\partial S_t}\left(K e^{-r(T-t)}\left(1-N(d_-)\right) - S_t\left(1-N(d_+)\right)\right) =$$

$$= \frac{\partial}{\partial S_t}\left(K e^{-r(T-t)} - K e^{-r(T-t)} N(d_-) - S_t + S_t N(d_+)\right) =$$

$$= \frac{\partial}{\partial S_t}\left(K e^{-r(T-t)} - S_t + C_t\right) =$$

$$= -1 + N(d_+)$$

(3) $$\Gamma_c = \frac{\partial^2 C_t}{\partial^2 S_t} = \frac{\partial}{\partial S_t}\left(N(d_+)\right) = \varphi(d_+)\frac{1}{\sigma\sqrt{\tau}}\frac{\cancel{K}}{S_t}\frac{1}{\cancel{K}} = \frac{\varphi(d_+)}{\sigma\sqrt{\tau}\, S_t}$$

$$\Gamma_p = \frac{\partial^2 P_t}{\partial^2 S_t} = \frac{\partial}{\partial S_t}\left(N(d_+)-1\right) = \frac{\varphi(d_+)}{\sigma\sqrt{\tau}\, S_t}$$

(4) $$\nu_c = \frac{\partial C_t}{\partial\sigma} = \frac{\partial}{\partial \sigma}\left(S_t N\Big(\overbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r+\frac12\sigma^2\right)\tau}{\sigma\sqrt\tau}}^{d_+}\Big) - Ke^{-r\tau}N\Big(\overbrace{\frac{\log\left(\frac{S_t}{K}\right)+\left(r-\frac12\sigma^2\right)\tau}{\sigma\sqrt\tau}}^{d_-}\Big)\right) =$$

$$= S_t\,\varphi(d_+)\left(\frac{\tau\sigma\sigma\sqrt\tau - \sqrt\tau\left(\log\left(\frac{S_t}{K}\right) + \left(r+\frac12\sigma^2\right)\tau\right)}{\sigma^2\tau}\right) - \underbrace{Ke^{-r\tau}\varphi(d_-)}_{S_t\varphi(d_+)}\left(\frac{-\tau\sigma\sigma\sqrt\tau - \sqrt\tau\left(\log\left(\frac{S_t}{K}\right)+\left(r-\frac12\sigma^2\right)\tau\right)}{\sigma^2\tau}\right)$$

$$= S_t\,\varphi(d_+)\left(2\sqrt\tau - \sqrt\tau\right) = S_t\,\varphi(d_+)\sqrt\tau$$

$$\nu_p = \frac{\partial P_t}{\partial \sigma} = \frac{\partial}{\partial\sigma}\left(Ke^{-r(T-t)} - S_t + C_t\right) = \frac{\partial}{\partial\sigma}(C_t) = S_t\,\varphi(d_+)\sqrt\tau$$

(5) $$\Theta_c = -\frac{\partial C_t}{\partial\tau} = -\frac{\partial}{\partial\tau}\left(S_t N\Big(\overbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r+\frac12\sigma^2\right)\tau}{\sigma\sqrt\tau}}^{d_+}\Big) - Ke^{-r\tau}N\Big(\overbrace{\frac{\log\left(\frac{S_t}{K}\right)+\left(r-\frac12\sigma^2\right)\tau}{\sigma\sqrt\tau}}^{d_-}\Big)\right) =$$

$$= -\left(S_t\,\varphi(d_+)\left(\frac{\log(S_t)}{K}\left(-\frac{1}{2\tau^{\frac32}}\right) + \frac{\left(r+\frac12\sigma^2\right)}{\sigma}\frac12\frac{1}{\sqrt\tau}\right) - \underbrace{Ke^{-r\tau}\varphi(d_-)}_{S_t\varphi(d_+)}\left(\frac{\log\left(\frac{S_t}{K}\right)}{\sigma}\left(-\frac{1}{2\tau^{\frac32}}\right) + \frac{\left(r-\frac12\sigma^2\right)}{\sigma}\frac{1}{2\sqrt\tau}\right)\right)$$

$$+ Ke^{-r\tau}(-r)N(d_-) =$$

$$= -\left(S_t\,\varphi(d_+)\left(\frac{1}{\cancel{4}\,2}\frac{\sigma}{\sqrt\tau}\cdot\cancel{2}\right)\right) + Ke^{-r\tau}(-r)N(d_-) = -rKe^{-r\tau}N(d_-) - \frac{S_t\sigma}{2\sqrt\tau}\varphi(d_+)$$

$$\Theta_p = -\frac{\partial}{\partial\tau}\left(Ke^{-r\tau} - S_t + C_t\right) = +Ke^{-r\tau}(-r) - Ke^{-r\tau}N(d_-) - \frac{S_t\sigma}{2\sqrt\tau}\varphi(d_+) =$$

$$= e^{-r\tau}Kr\left(1-N(d_-)\right) - \frac{S_t\sigma}{2\sqrt\tau}\varphi(d_+)$$

(6) $\rho_c = \frac{\partial C_t}{\partial r} = \frac{\partial}{\partial r}\left(S_t\, N\Big(\underbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r+\frac{1}{2}\sigma^2\right)\tilde{\tau}}{\sigma\sqrt{\tilde{\tau}}}}_{d_+}\Big) - Ke^{-r\tilde{\tau}}\, N\Big(\underbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r-\frac{1}{2}\sigma^2\right)\tilde{\tau}}{\sigma\sqrt{\tilde{\tau}}}}_{d_-}\Big)\right) =$

$$= S_t\, \varphi(d_+)\frac{\sqrt{\tilde{\tau}}}{\sigma} - \left(Ke^{-r\tilde{\tau}}(-\tilde{\tau})\, N(d_-) + \underbrace{Ke^{-r\tilde{\tau}}\varphi(d_-)}_{\varphi(d_+)}\frac{\sqrt{\tilde{\tau}}}{\sigma}\right) =$$

$$= Ke^{-r\tilde{\tau}}\, N(d_-)$$

$$\rho_p = \frac{\partial P_t}{\partial r}\left(Ke^{-r\tilde{\tau}} - S_t + C_t\right) = Ke^{-r\tilde{\tau}}(-\tilde{\tau}) + Ke^{-r\tilde{\tau}}\, N(d_-) = Ke^{-r\tilde{\tau}}\left(N(d_-) - 1\right)$$

(7)
- $\Delta_c = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_+} e^{-\frac{1}{2}y^2}\,dy \Rightarrow 0 < \Delta_c < 1$ and when $S_t \uparrow +\infty \Rightarrow d_+ \uparrow +\infty \Rightarrow$

  $\Delta_c \uparrow 1$; when $S_t \downarrow 0 \Rightarrow d_- \downarrow -\infty \Rightarrow \Delta_c \downarrow 0$

- $-1 < \Delta_p < 0$ is the shift of Δ_c
- $\Gamma_c = \Gamma_p > 0$
- $\mathcal{V}_c = \mathcal{V}_p > 0$

Exercise 2: An numerical example of an application of the Black–Scholes model

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.
(a) What is the probability that a European call option with strike price of \$40 maturing in three months will be exercised? What is the value of this option at maturity (assume 10%p.a. risk-free interest rate)?
(b) Answer the same questions for a European put option?

$$S_T = S_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T} \qquad (B_t) - \mathbb{P}\text{ s BM}$$

$$S_T = S_0\, e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma B_T} \qquad (B_t) - \mathbb{Q}\text{ s BM}$$

$\mu = 0{,}16 \quad \sigma = 0{,}35 \;,\; S_0 = 38 \;,\; K = 40 \;,\; T = \frac{3}{12} = \frac{1}{4} \;,\; r = 0{,}1$

(a) $\cdot\ \mathbb{P}\left(S_T > 40\right) =$

$$S_T = S_0\, e^{N\left(\left(\mu-\frac{1}{2}\sigma^2\right)T,\ \sigma^2 T\right)}$$

$$= \mathbb{P}\Big(\underbrace{N\left(\left(\mu-\frac{1}{2}\sigma^2\right)T,\ \sigma^2 T\right)}_{} > \ln\left(\frac{40}{38}\right)\Big) = \qquad \left(\mu-\frac{1}{2}\sigma^2\right)T = 0{,}16 - \frac{1}{2}0{,}35^2 = 0{,}0247 \qquad \sigma^2 T = 0{,}0306$$

$$= \mathbb{P}\left(\frac{* - 0{,}0247}{\sqrt{0{,}0306}} > \frac{0{,}0513 - 0{,}0247}{\sqrt{0{,}0306}}\right) =$$

$$= \mathbb{P}\left(N(0,1) > 0{,}152\right) = 1 - N\left(0{,}152\right)$$

- Value at maturity of the Call:

$$d_+ = \frac{\log\left(\frac{38}{40}\right) + \left(0{,}1 + \frac{1}{2}\,0{,}35^2\right)\cdot 0{,}25}{0{,}35\sqrt{0{,}25}} = -0{,}06275$$

$$d_- = -0{,}06275 - 0{,}35\sqrt{0{,}25} = -0{,}23475$$

$$C_T = 38\, N(-0{,}06275) - 40\, e^{-0{,}1\cdot 0{,}25}\, N(-0{,}23475) \approx 2{,}209$$

(b) $\mathbb{P}(S_T < 40) = 1 - \mathbb{P}(S_T > 40) \approx 1 - 0{,}4396 = 0{,}5604$

$$P_t = 40\, e^{-0{,}1\cdot 0{,}25}\, N(0{,}23475) - 38\, N(0{,}0675) = 3{,}22$$

$\checkmark$ Exercise 3: Black–Scholes Model: Another financial derivative

Compute the price and describe the replication strategy at $t = 0$ of the European derivative being defined by the following payoff function (where we assume that the asset price process follows a geometric Brownian motion and where $k > 0$ is a positive constant).

$$f(S_T) = \max(S_T, k)\,.$$

- Price:

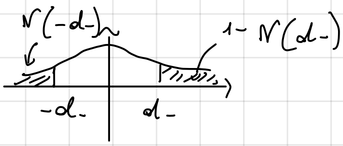
$$V_0 = e^{-rT}\,\mathbb{E}_\mathbb{Q}\left[\max(S_T, K)\right] = e^{-rT}\,\mathbb{E}_\mathbb{Q}\left[(S_T - K)^+ + K\right] =$$

$$= \underbrace{e^{-rT}\,\mathbb{E}_\mathbb{Q}\left[(S_T - K)^+\right]}_{C_0} + e^{-rT}\,\mathbb{E}_\mathbb{Q}[K] =$$

$$= S_0 N(d_+) - K e^{-rT} N(d_-) + e^{-rT} K =$$

$$= S_0 N(d_+) + e^{-rT} K\left(1 - N(d_-)\right) =$$

$$= S_0 N(d_+) + e^{-rT} K\, N(-d_-)$$



simmetry of gaussian density

(For the min would be $V_0 = e^{-rT}\,\mathbb{E}_\mathbb{Q}\left[\min(S_T, K)\right] = e^{-rT}\,\mathbb{E}_\mathbb{Q}\left[-(K - S_T)^+ + K\right] =$

$$= -\underbrace{e^{-rT}\,\mathbb{E}_\mathbb{Q}\left[(K - S_T)^+\right]}_{P_0} + e^{-rT}[K] =$$

$$= -K e^{-rT}\left(1 - N(d_-)\right) + S_0\left(1 - N(d_+)\right) + e^{-rT} K =$$

$$= +K e^{-rT} N(d_-) + S_0\left(1 - N(d_+)\right)$$)

- $H_0 = \frac{\partial V_0}{\partial S_0} = \underbrace{\frac{\partial}{S_0}(C_0)}_{\Delta_c} + \frac{\partial}{\partial S_0}\left(e^{-rT}\mathbb{E}_Q[K]\right) =$

$= N(d_+) + 0 = N(d_+)$

$H_0^0 = V_0 - H_0 S_0 = S_0 \cancel{N(d_+)} + e^{-rT} K N(-d_-) - \cancel{N(d_+) S_0} = K e^{-rT} N(-d_-)$

Exercise 4: Brownian motion

Let (W_t) be a standard Brownian motion. Which one of the following processes are standard Brownian motions as well (justify your answers)?

1. The process (X_t), being defined by $X_t = 2(W_{1+\frac{t}{4}} - W_1)$.
2. The process (Y_t), being defined by $Y_t = \sqrt{t}W_1$.
3. The process (Z_t), being defined by $Z_t = W_{2t} - W_t$.

$X_t = 2\left(W_{1+\frac{t}{4}} - W_1\right)$

- Continuous: is continuous since $W_{1+\frac{t}{4}}$ is composition of two continuous function $W_\cdot$, $1+\frac{\cdot}{4}$ and addition / multiplication of continuous function

- Independent increments:

$$X_t - X_s = 2\left(W_{1+\frac{t}{4}} - W_1\right) - 2\left(W_{1+\frac{s}{4}} - W_1\right) =$$

$$= \underbrace{2\left(W_{1+\frac{t}{4}} - W_{1+\frac{s}{4}}\right)}_{\text{indipendent from } \mathcal{F}_{1+\frac{s}{4}}} \Rightarrow \text{I take the } \sigma\text{-algebra}$$

$$\left(\tilde{\mathcal{F}}_t\right)_{t\geq 0} := \left(\mathcal{F}_{1+\frac{t}{4}}\right)_{t\geq 0}.$$

- Gaussian: we just need to verify that X_t is gaussian $\forall t$ and the increments $X_{t_1} - X_{t_2}$ are gaussian $\forall t_1, t_2$. Indeed for every vector:

note that $X_{t_i} = X_{t_1} + \sum_{j=1}^{i-1}\left(X_{j+1} - X_j\right) \quad i \geq 2$

$$\sum_{i=1}^{n} \alpha_i X_{t_i} = \alpha_1 X_{t_1} + \sum_{i=2}^{n} \alpha_i \left(X_{t_1} + \sum_{j=1}^{i-1}\left(X_{j+1} - X_j\right)\right) =$$

$$= \underbrace{\sum_{i=1}^{n}\left(\alpha_i \underbrace{X_{t_i}}\right) + \sum_{i=2}^{n}\left(\alpha_i \sum_{j=1}^{i-1}\right.}_{\text{indipendent}} \underbrace{\left(X_{j+1} - X_j\right)}_{\text{indipendent varying } j}\Bigg)$$

$\Rightarrow$ linear combintion of indipendent gaussian $\Rightarrow$ gaussian.

So, $X_t = 2\left(W_{1+\frac{t}{4}} - W_1\right) = 2\left(\underbrace{W_{1+\frac{t}{4}} - W_{1+\frac{0}{4}}}_{N\left(0,\, 1+\frac{t}{4} - 1 - \frac{0}{4}\right) = N\left(0, \frac{t}{4}\right)}\right) \sim N(0,t)$ and

$$X_t - X_s = 2\left(\underbrace{W_{1+\frac{t}{4}} - W_{1+\frac{s}{4}}}_{N\left(0,\, \frac{t-s}{4}\right)}\right) \sim N(0, t-s)$$

It's a sBM w.r.t. $\tilde{\mathcal{F}}_t$.

2 $Y_t = \sqrt{t}\, W_1$

$$\left.\begin{aligned} Y_1 - Y_0 &= W_1 \\ Y_4 - Y_1 &= 2W_1 - W_1 = W_1 \end{aligned}\right] \rightarrow Y_1 - Y_0 = Y_4 - Y_1 \Rightarrow \text{not indipendent} \Rightarrow \text{No sBM}$$

3 $Z_t = W_{2t} - W_t$. Take s,t s.t. $s < t < 2s$:

$$Z_t - Z_s = W_{2t} - W_t - W_{2s} + W_s = \underbrace{W_{2t} - W_{2s}}_{\text{indipendent of } \mathcal{F}_{2s}} - (\underbrace{W_t - W_s}_{\mathcal{F}_{2s} \text{ measurable } (\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_{2s})})$$

If $Z_t - Z_s$ was indipendent of $\mathcal{F}_{2s}$ $\Rightarrow$ $-(W_t - W_s)$ indipendent of $\mathcal{F}_{2s}$

(sum of r.v. indipendent of σ is indipendent of σ) but indipendent + measurable

$\Rightarrow$ $-(W_t - W_s)$ should be constant (deterministic), contraddiction.

Martingales in Financial Mathematics: Static and semi-static hedging

Week 9, 2024

We assume a risk-neutral Black–Scholes setting, i.e. we have a risk-less bond with price process $B_t = e^{rt}$, $t \in [0, T]$, $r > 0$ and a risky asset with a price process satisfying

$$dS_t = rS_t dt + \sigma S_t dW_t \,,$$

where $\sigma > 0$ and $(W_t)_{t\in[0,T]}$ is a standard Brownian motion (with respect to $\mathbb{Q}$).

√ Exercise 1: Forward price

Recall that a forward contract is an agreement to pay a specified delivery price k at a maturity date $T \geq 0$ for the asset whose price at time t is S_t.

The T-forward price $F_{t,T}$ of this asset at time $t \in [0, T]$ is the value of k that makes the forward contract have no-arbitrage price zero at time t. Describe the process $F_{t,T}$, $t \in [0, T]$.

Exercise 2: European put-call symmetry

Show that in the Black–Scholes case the European put-call symmetry holds, which can be expressed by the property that for arbitrary $k \geq 0$

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(F_{t,T}\eta_{t,T} - k)_+|\mathcal{F}_t] &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(F_{t,T} - k\eta_{t,T})_+|\mathcal{F}_t] \\ &= e^{-r(T-t)}\frac{k}{F_{t,T}}\mathbb{E}_{\mathbb{Q}}\Big[\big(\frac{F_{t,T}^2}{k} - F_{t,T}\eta_{t,T}\big)_+|\mathcal{F}_t\Big] \,, \end{aligned}$$

where $\eta_{t,T} = e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T - W_t)}$, $t \in [0, T]$, so that $F_{t,T}\eta_t = S_T$ under $\mathbb{Q}$, i.e. we obtain a relation between European call and put prices (often this relation is only formulated for $t = 0$, i.e. for the random variable S_T).
Hint: Write the Balck–Scholes formulas in terms of $F_{t,T}$.

Exercise 3: Semi-static hedge of a down-and-out call

Semi-static hedging strategies are often defined to be replicating strategies where trading is no more needed than two times after inception. Assume that there is a barrier H, a strike k satisfying $H < k$, where $S_0 > H$, and assume that there are no carrying costs. The down-and-out call is knocked-out if H is hit any time before maturity. Otherwise pays $(S_T - k)_+$, i.e.

$$X_{\text{doc}} = (S_T - k)_+ \mathbb{1}_{S_t > H\,, \forall t\in[0,T]} \,.$$

Assume that European calls and puts are available for arbitrary strikes. Use the European put-call symmetry in order to derive a semi-static hedging strategy (use without proof that the put-call symmetry from Exercise 2 also holds for $[0,T]$-valued stopping times).

Exercice 4: Decomposition of European payoff functions

Assume that a payoff function $f:\mathbb{R}_+ \to \mathbb{R}$ is two times *continuously* differentiable. Show that for $a \in \mathbb{R}_+$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+ dk + \int_0^a f''(k)(k-x)_+ dk\,,$$

and give an economic interpretation.

Exercise 5: Implicit distribution

Consider a risk-neutral setting where $\tilde{S}_T$ is sampled from a martingale and assume for simplicity that the strictly positive random variable S_T is absolutely continuous with continuous density q. Show that either the prices of European calls or European puts for arbitrary strikes uniquely determine the distribution of S_T.[1]

[1] This result is obviously interesting in its own right but has also some consequences for the caracterisation of random variables satisfying European put-call symmetry.

$\sqrt{}$ Exercise 1: Forward price

Recall that a forward contract is an agreement to pay a specified delivery price k at a maturity date $T \geq 0$ for the asset whose price at time t is S_t.

The T-forward price $F_{t,T}$ of this asset at time $t \in [0,T]$ is the value of k that makes the forward contract have no-arbitrage price zero at time t. Describe the process $F_{t,T}$, $t \in [0,T]$.

Note that the payoff at time T is $S_T - K = (S_T - K)^+ - (K - S_T)^+$ that are the payoffs of a call and a put option at time T. So it is a claim (replicable and squared integrable).

So, we can use the formula to have a fair price:

$$V_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[S_T - K \mid \mathcal{F}_t] =$$

$$= e^{-r(T-t)} \cdot \mathbb{E}_{\mathbb{Q}}(S_T \mid \mathcal{F}_t] - Ke^{-r(T-t)} \stackrel{(\tilde{S}_t)_t \ \mathbb{Q}\text{-mart.}}{=}$$

$$= e^{rt} \tilde{S}_t - Ke^{-r(T-t)} = S_t - Ke^{-r(T-t)} \stackrel{\text{we want to have value } 0}{=} 0$$

$$\Rightarrow F_{t,T} = K = S_t e^{r(T-t)} \quad t \in [0,T]$$

Furthermore, we can also write the SDE $F_{t,T}$ satisfies:

$$dF_{t,T} = e^{r(T-t)} dS_t + S_t \, d(e^{r(T-t)}) + 0 \stackrel{dS_t = rS_t dt + \sigma S_t dW_t}{=}$$

$$= e^{r(T-t)}(rS_t dt + \sigma S_t dW_t) + S_t e^{r(T-t)} \cdot (-r) dt$$

$$= \cancel{e^{r(T-t)} r S_t dt} + e^{r(T-t)} \sigma S_t dW_t - \cancel{e^{r(T-t)} r S_t dt} =$$

$$= F_t \sigma dW_t \quad (\text{under } \mathbb{Q})$$

Exercise 2: European put-call symmetry

Show that in the Black–Scholes case the European put-call symmetry holds, which can be expressed by the property that for arbitrary $k \geq 0$

$$e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[(F_{t,T}\eta_{t,T} - k)_+ | \mathcal{F}_t] = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[(F_{t,T} - k\eta_{t,T})_+ | \mathcal{F}_t]$$
$$= e^{-r(T-t)} \frac{k}{F_{t,T}} \mathbb{E}_{\mathbb{Q}}\Big[\Big(\frac{F_{t,T}^2}{k} - F_{t,T}\eta_{t,T}\Big)_+ | \mathcal{F}_t\Big],$$

where $\eta_{t,T} = e^{-\frac{1}{2}\sigma^2(T-t) + \sigma(W_T - W_t)}$, $t \in [0,T]$, so that $F_{t,T}\eta_t = S_T$ under $\mathbb{Q}$, i.e. we obtain a relation between European call and put prices (often this relation is only formulated for $t = 0$, i.e. for the random variable S_T).

Hint: Write the Balck–Scholes formulas in terms of $F_{t,T}$.

Remember the formulas:

$$C_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left[(S_T - K)_+ \mid \mathcal{F}_t\right] = S_t N(d_+) - K e^{-r(T-t)} N(d_-)$$

$$P_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left[(K - S_T)_+ \mid \mathcal{F}_t\right] = K e^{-r(T-t)} N(-d_-) - S_t N(-d_+)$$

with $$d_\pm = \frac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\log\left(\frac{S_t e^{r(T-t)}}{K}\right) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} =$$

$$= \frac{\log(F_{t,T}) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

But $$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right) =$$

$$= \underbrace{S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t + r(T-t)\right)}_{F_{t,T}} \underbrace{\exp\left(\sigma(W_T - W_t) - \frac{1}{2}\sigma^2(T-t)\right)}_{\eta_{t,T}}$$

So $$C_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left[(S_T - K)_+ \mid \mathcal{F}_t\right] = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left[\left(\underset{\text{measurable}}{F_{t,T}}\ \overset{\text{indipendent}}{\eta_{t,T}} - K\right)_+ \mid \mathcal{F}_t\right] =$$

$$= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\Big[$$

$$\mathbb{E}\left[(\overset{\text{meas.}}{X}\ \overset{\text{ind.}}{Y} - K)_+ \mid \mathcal{F}_t\right] \qquad \mathbb{E}\left[(x Y - K)_+\right] = \int_{\mathbb{R}} (xy - K)_+ f_Y(y)\,dy = \int_{\frac{K}{x}}^{+\infty} (xy - K) f_Y(y)\,dy =$$

$$\mathbb{E}\left[(X - KY)_+ \mid \mathcal{F}_t\right] \qquad \mathbb{E}\Big[$$

$$C_t = S_t N\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - K e^{-r(T-t)} N\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) =$$

$$=$$

$$F_{t,T} = S_t e^{r(T-t)}$$