Net present value (NPV)

It's an approach for valuating, it is based on discount future cash flows:

- We do an investment that generates cash flows: C_0 at time $t=0$, C_1 at time $t=1$ and C_n at time n;
- i_t interest rate for $t=0,\dots,n$; reflects the cost of tying up capital / risk:
- we assume that the discount factor is $\frac{1}{(1+i_t)^t}$

We want to understand what is a "fair" price for this investement:

C_0 discounted is C_0; C_1 discounted is $\frac{C_1}{(1+i_1)}$; C_2 discounted is $\frac{C_2}{(1+i_2)^2}$ ...

$C_n = \frac{C_n}{(1+i_n)^n}$. So:

$$\boxed{NPV = \sum_{t=0}^{n} \frac{C_t}{(1+i_t)^t}}$$

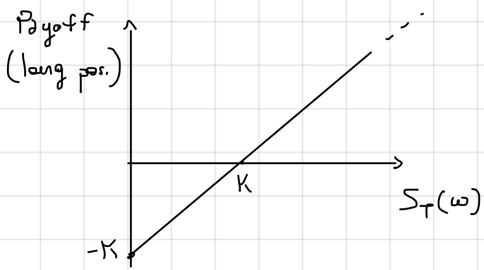
Financial derivatives: is an instrument whose value is derived from the value of another asset called underlings (ex. of underlings are bonds, interest rates ...).

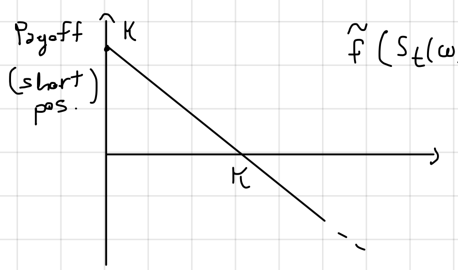
↓

Formalization: $(S_t)_{t\in[0,T]}$ stochastic process that represents the price of a quote and $f: \mathbb{R}^+ \to \mathbb{R}$ ($\mathbb{R}^+$ due to the fact the price will be always positive). The payoff of the derivative at maturity $T>0$ is $f(S_T)$ (it is a random variable so to make sense it is $f(S_T(\omega))$ for $\omega\in\Omega$).

Examples of derivatives

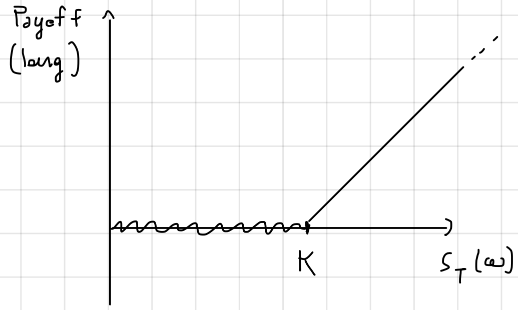
↳ **Forward contract**: the right and the obligation to buy the asset (S_t) at maturity T for the strike price K. Usually there isn't a premium to pay at time 0 (you have the obligation): $\boxed{f(S_T(\omega)) = S_T(\omega) - K}$

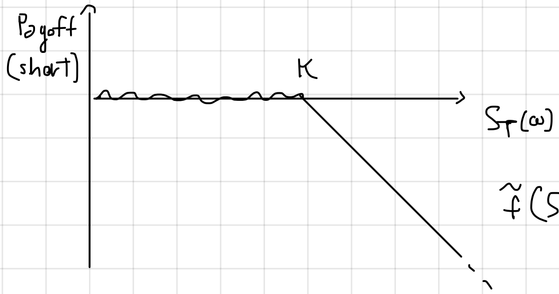




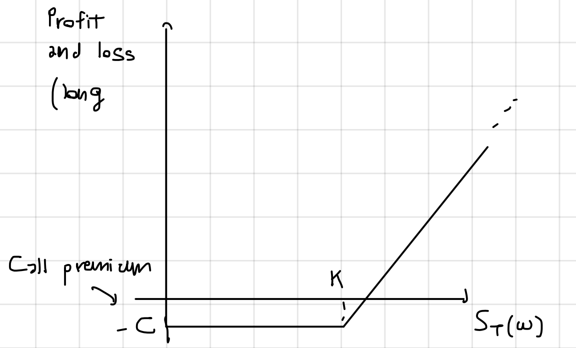
$$\tilde{f}(S_t(\omega)) = -f(S_T(\omega)) = K - S_T(\omega)$$

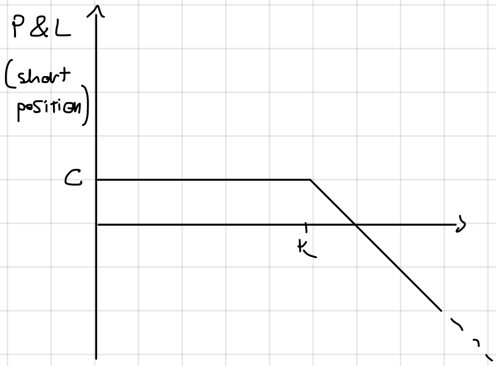
↳ European Call option: the right but not the obligation to buy at time T for the strike price T. There is a premium to pay at time 0 (the cost of the option) and obviously only who buy (go long) has the right, who sells (go short) hasn't this right, indeed he gets the premium at time $t=0$. So: $\boxed{f(S_T(\omega)) = (S_T(\omega) - K)^+}$



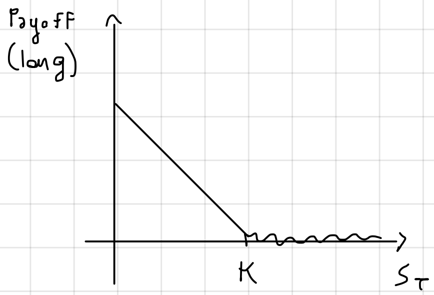


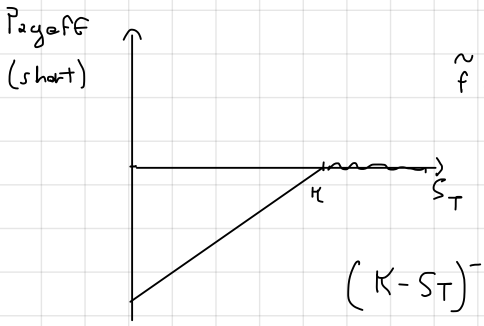
$\tilde{f}(S_T(\omega)) = -f(S_T(\omega))$



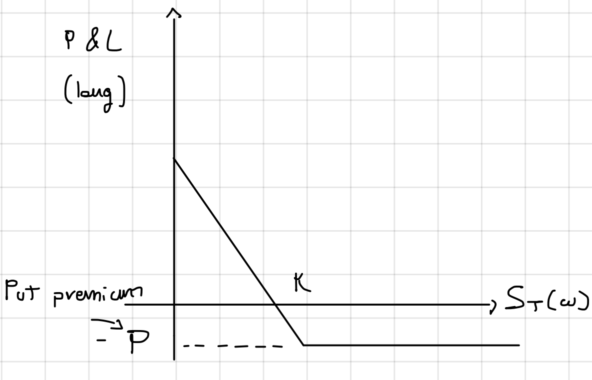


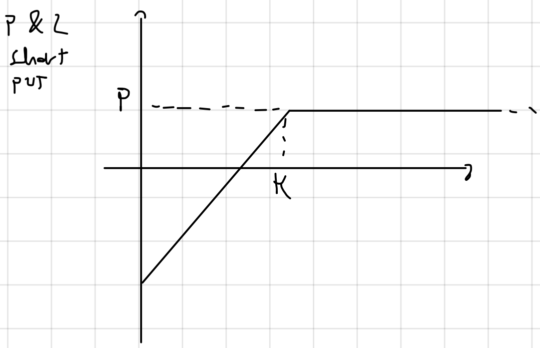
↳ European put option: the right but not the obligation to sell the asset (S_t) at maturity T for the price K: $\boxed{f(S_T(\omega)) = (K - S_T(\omega))^+}$





$\tilde{f}(S_T(\omega)) = -(K - S_T(\omega))^+$





↳ <u>American option</u>: similar to European but they can be exercised until the time T (so at any time between 0 and T, instead of European that can be exercised only at time T).

<u>Princing a derivative</u>: could we use discounting cash flows for deriving values of financial derivatives?
$S_T(\omega)$ and $f(S_T(\omega))$ depend on the same $\omega \in \Omega$, where the payoff of the derivative is given by a deterministic function on S_T $\Rightarrow$ the price of the derivative and the price of the underlying asset should be in a certain relation.

↓

So an approach to financial derivatives is to exclude <u>arbitrage opportunity</u> (<u>NA assumption</u>) where arbitrage opportunity means "to make money from nothing without risks".

↓

<u>Example: Forward contract</u>: we want to proof the strike price K under NA assumption is $S_0 e^{rT}$ (continous compounding and r constant).

<u>Proof</u>: • if $K > S_0 e^{rT}$ the following strategy gives arbitrage:

At $t=0$	CF
1 Borrow money	$+S_0$
2 Buy S_0	$-S_0$
3 Short the forward	0

At $t=T$	CF
4 I have S_T (from 2) and I have to sell it at price K (I went short)	K
5 Pay back the money credit with interest	$-S_0 e^{rT}$

• if $S_0 e^{rT} > K$ the following strategy gives arbitrage:

At $t=0$	CF
1 Sell S_0	$+S_0$
2 Pay the money in bank	$-S_0$
3 Long the forward	0

At $t=T$	CF
4 Close bank account	$S_0 e^{rT}$
5 I have to buy what I have sold (1) but I can buy it at K	$-K$

$\square$

<u>Discrete time model</u>: $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_n\}_{n=0,\dots,N}$ $N \in \mathbb{N}^*$. N will be the maturity. $\mathcal{F}_n$ is the information available at n. We also assume:

- Standard filtration: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$
- Atoms $\mathbb{P}(\omega) > 0 \quad \forall \omega \in \Omega$
- Assets: The market consists of $d+1$ assets with price at time n being given by $S_n = (S_n^0, \dots, S_n^d)$ and S_n is $\mathcal{F}_n$-adapted.
- Risk-less asset: $S_0^0 = 1$ and $S_n^0 = (1+r)^n$ with $r > 0$ the risk-less interest rate.

<u>Strategy</u>: A stochastic process $\phi = (\phi_n^0, \dots, \phi_n^d)_{n=1,\dots,N}$ with values in $\mathbb{R}^{d+1}$ where ϕ_n^i denotes the number of shares of asset i held in the portfolio between $n-1$ and n. ϕ has to be predictable (ϕ_n $\mathcal{F}_{n-1}$-measurable, ϕ_0 $\mathcal{F}_0$-measurable).

<u>Value of the portfolio</u>: $\boxed{\begin{array}{l} V_n(\phi) = \sum_{i=0}^{n} \phi_n^i S_n^i = \phi_n \cdot S_n \\ \tilde{V}_n(\phi) = V_n(\phi)(S_n^0)^{-1} \quad (\text{discounted}) \end{array}}$, $\boxed{\tilde{S}_n = S_n (S_n^0)^{-1} \quad (\text{discounted})}$

<u>Self-financing strategies</u>: $\boxed{\phi_{n+1} \cdot S_n = \phi_n \cdot S_n}$ (the investor readjust his position without consuming/bringing money)

<u>Prop.</u>: $\phi_{n+1} \cdot S_n = \phi_n \cdot S_n \iff V_{n+1}(\phi) = V_n(\phi) + \phi_{n+1} \cdot \Delta S_{n+1}$ $\quad (\Delta S_{n+1} = S_{n+1} - S_n)$

<u>Proof</u> $\boxed{\Longrightarrow}$ $V_{n+1}(\phi) = \phi_{n+1} \cdot S_{n+1} = \phi_{n+1} \cdot \Delta S_{n+1} + \phi_{n+1} \cdot S_n \underset{hp.}{=} \phi_{n+1} \cdot \Delta S_{n+1} + \phi_n \cdot S_n$

$\boxed{\Longleftarrow}$ Same as $\Rightarrow$.

$\square$

<u>Prop.</u>: $\phi_{n+1} \cdot S_n = \phi_n \cdot S_n \iff \tilde{V}_{n+1}(\phi) = \tilde{V}_n(\phi) + \phi_{n+1} \cdot \Delta \tilde{S}_{n+1}$

<u>Proof.</u> $\phi_{n+1} \cdot \frac{S_n}{S_n^0} = \phi_n \cdot \frac{S_n}{S_n^0} \Rightarrow \phi_{n+1} \cdot \tilde{S}_n = \phi_n \cdot \tilde{S}_n$. So, now, it's basically the same as earlier. $\square$

<u>Prop.</u>: The following properties are equivalent:

① ϕ is self-financing ② $V_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta S_i$ ③ $\tilde{V}_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta \tilde{S}_i$ (Note $\Delta \tilde{S}_i^0 = 0$)

<u>Proof</u> ① $\to$ ②: For the previous proposition iterated:

$$V_n(\phi) = V_{n-1}(\phi) + \phi_n \cdot \Delta S_n = \left(V_{n-2}(\phi) + \phi_{n-1} \cdot \Delta S_{n-1}\right) + \phi_n \cdot \Delta S_n = \dots = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta S_i$$

② ⟶ ①: Immediatly with induction + previous proposition

① ⟷ ③: as $1 \longleftrightarrow 2$ using the other proposition with discounted objects.

$\square$

Prop.: For any predictable process $(\phi_n^1, \dots, \phi_n^d)_{n=0,\dots,N}$ ← risk and for any V_0, $\exists!$ predictable process $(\phi_n^0)_{n=0,\dots,N}$ ← risk free such that the strategy $\phi = (\phi_n^0, \dots, \phi_n^d)$ is self-financing and its initial value is V_0.

Proof

By definition $\tilde{V}_n(\phi) = \phi_n^0 \cdot 1 + \dots + \phi_n^d \tilde{S}_n^d$, and by self-financing $\tilde{V}_n(\phi) = V_0(\phi) + \sum_{i=1}^n \phi_i \cdot \Delta \tilde{S}_i$.

So
$$\phi_n^0 = V_0(\phi) + \sum_{i=1}^n \phi_i \cdot \Delta \tilde{S}_i - \phi_n^1 \tilde{S}_n^1 - \dots - \phi_n^d \tilde{S}_n^d = V_0(\phi) + \sum_{i=1}^{n-1} \phi_i \cdot \Delta \tilde{S}_i + \phi_n \cdot \Delta \tilde{S}_n -$$
$$- \phi_n^1 \tilde{S}_n^1 - \dots - \phi_n^d \tilde{S}_n^d = V_0(\phi) + \sum_{i=1}^{n-1} \phi_i \cdot \Delta \tilde{S}_i + \sum_{i=1}^{d} \phi_n^i \Delta \tilde{S}_n^i - \phi_n^1 \tilde{S}_n^1 - \dots - \phi_n^d \tilde{S}_n^d =$$
$$= V_0(\phi) + \underbrace{\sum_{i=1}^{n-1} \phi_i \cdot \Delta \tilde{S}_i}_{\mathcal{F}_{n-1}\text{-meas.}} - \sum_{i=1}^{d} \underbrace{\phi_n^i}_{\text{predict.}} \underbrace{\tilde{S}_{n-1}^i}_{\mathcal{F}_{n-1}\text{-meas.}}.$$

Now, check $n=0$:
$$\tilde{V}_0(\phi) = V_0(\phi) \Rightarrow \phi_0^0 = V_0 - \phi_0^1 S_0^1 - \dots + \phi_0^d S_0^d \Rightarrow \mathcal{F}_0\text{-meas.}$$
□

Admissible strategy: A strategy ϕ is admissible if it is self-financing and if $V_n(\phi) \geq 0 \quad \forall n \in \{0, \dots, N\}$ (we avoid bankrupt).

Arbitrage: an arbitrage strategy is an admissible strategy with zero value and non-vanishing probability of an non-vanishing final value, i.e. $V_0 = 0$ and $\mathbb{P}\{V_N > 0\} > 0$. (We have relaxed the previous definition where $\mathbb{P}\{V_N > 0\} = 1$).

Martingale: $\{M_n\}_{n=0,\dots,N}$ of real valued random variable is a martingale process if:

1. M_n is integrable $\forall n$ (in this case is free due to $|\Omega| < +\infty$)
2. $\{M_n\}$ is adapted to $\{\mathcal{F}_n\}_{n=0,\dots,N}$.
3. $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$ a.s. $\forall n$. (i.e. orthogonal increments $\mathbb{E}[(M_{n+1} - M_n) \mathcal{F}_n] = 0$)

If $M_n \leq \mathbb{E}[M_{n+1} | \mathcal{F}_n]$ is a submartingale, if $M_n \geq \mathbb{E}[M_{n+1} | \mathcal{F}_n]$ is a supermartingale.

Properties of martingales

1. $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n \ \forall n \iff \mathbb{E}[M_{n+1} | \mathcal{F}_j] = M_j \quad \forall j \leq n$
2. $\mathbb{E}[M_n] = \mathbb{E}[M_0] \quad \forall n$
3. {Martingales} is a vector space

Proof. ① $\boxed{\Leftarrow}$ obvious $\boxed{\Rightarrow}$ $E[M_{n+1} | \mathcal{F}_j] = E[E[M_{n+1} | \mathcal{F}_n] | \mathcal{F}_j] = E[M_n | \mathcal{F}_j] =$

$= E[E[M_n | \mathcal{F}_{n-1}] | \mathcal{F}_j] = E[M_{n-1} | \mathcal{F}_j] = \dots = M_j$

② $E[M_n | \mathcal{F}_0] = M_0 \Rightarrow E[E[M_n | \mathcal{F}_0]] = E[M_0] \Rightarrow E[M_n] = E[M_0]$.

③ $E[\lambda M_n + \mu N_n | \mathcal{F}_{n-1}] = \lambda E[M_n | \mathcal{F}_{n-1}] + \mu E[N_n | \mathcal{F}_{n-1}] = \lambda M_{n-1} + \mu N_{n-1}$.

□

Sthochastic integral (finite time): let $(M_n)_{n=0,\dots,N}$ a martingale and $(H_n)_{n=0,\dots,N}$ ($|H_n| \le C_n \ \forall n$) a predictable process with respect to $(\mathcal{F}_n)_{n=0,\dots,N}$. We define:

$$X_n := \begin{cases} H_0 M_0 & , n=0 \\ H_0 M_0 + \sum_{i=1}^{n} H_i (M_i - M_{i-1}) & , n \ge 1 \end{cases}$$

Prop. The stochastic integral X_n is a martingale.

Proof: By definition is $\mathcal{F}_n$-adapted (and integrable if H_n is as well). Now:

$$E[X_n - X_{n-1} | \mathcal{F}_{n-1}] = E[H_n (M_n - M_{n-1}) | \mathcal{F}_{n-1}] \underset{H_n\text{-predictable}}{=} H_n E[M_n - M_{n-1} | \mathcal{F}_{n-1}] \underset{M_n \text{ martingale}}{=} 0.$$

□

Theorem (Characterisation of martingales): An adapted sequence of real-valued variables (M_n) is a martingale if and only if for any predictable sequence (H_n), we have: $E\left(\sum_{n=1}^{N} H_n \Delta M_n\right) = 0$

Proof $\boxed{\Rightarrow}$ For the previous prop. $(H_0 M_0 = 0)$ we have $\sum_{n=1}^{N} H_n \Delta M_n$ is a martingale so

$$E\left[\sum_{n=1}^{N} H_n \Delta M_n\right] = E[X_0] = E[H_0 M_0] = 0.$$

$\boxed{\Leftarrow}$ We want to show that $E[\mathbb{1}_A (M_{n+1} - M_n)] = 0 \ \forall A \in \mathcal{F}_{n-1}$ $(\forall n)$ and by definition of expected value we have $E[M_{n+1} - M_n | \mathcal{F}_{n-1}] = 0$, so the thesis. To prove the first equality, set

$H_j = \begin{cases} 0, & j \ne n+1 \\ \mathbb{1}_A, & j = n+1 \end{cases}$. So H_j is predictable and $\sum_{j=1}^{N} H_j \Delta M_j = \mathbb{1}_A \Delta M_{n+1}$ for hypothesis:

$$E[\mathbb{1}_A (M_{n+1} - M_n)] = 0.$$

□

<u>Viable market</u>: A market is viable if there is no arbitrage opportunity.

<u>Theorem</u> (<u>Fundamental theorem of Asset Pricing</u>): The market is viable $\iff \exists\, \mathbb{Q}$ probability measure s.t. $\mathbb{Q} \sim \mathbb{P}$ and $\tilde{S}_n^j$ are $\mathbb{Q}$-martingales $\forall j$. <u>N.B.</u> $\mathbb{Q}$ is not unique.

<u>Proof</u> $\boxed{\Leftarrow}$ Suppose we have ϕ admissible s.t. $\tilde{V}_0(\phi) = V_0(\phi) = 0$. Now, $\tilde{V}_n \overset{\text{self-financing}}{=} \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$

$\Rightarrow \tilde{V}_n$ is a $\mathbb{Q}$-martingale $\Rightarrow \mathbb{E}^{\mathbb{Q}}[\tilde{V}_N(\phi)] = \mathbb{E}^{\mathbb{Q}}[\tilde{V}_0(\phi)] = 0$, but $\tilde{V}_n(\phi) \geq 0\ \forall n$ (ϕ admissible), in particular $\tilde{V}_N(\phi) \geq 0 \Rightarrow V_N(\phi) = 0$ $\mathbb{Q}$-a.s.; but $\mathbb{Q} \sim \mathbb{P} \Rightarrow V_N(\phi) = 0$ $\mathbb{P}$-a.s. and so arbitrage is excluded.

$\boxed{\Longrightarrow}$ We do step by step:

1. $\Gamma := \{ X$ nonnegative r.v. s.t. $\mathbb{P}(X>0)>0 \}$, so the market is viable $\iff \forall \phi$ admissible with $V_0(\phi) = 0 \Rightarrow \tilde{V}_N(\phi) \notin \Gamma$. Let's see Γ is a convex cone:

$$\alpha, \beta > 0\ ;\ X, Y \in \Gamma \Rightarrow \mathbb{P}(\alpha X + \beta Y > 0) \underset{\beta Y \geq 0}{\geq} \mathbb{P}(\alpha X > 0) = \mathbb{P}(X > 0) \underset{\alpha \in \Gamma}{>} 0$$

2. ϕ predictable and admissible, $\tilde{G}_n := \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$ (cumulative discounted gained process).

We have $(V_0 = 0)$ $\tilde{G}_n(\phi) \overset{\text{self-fin.}}{=} \tilde{V}_n(\phi)$ and since it is admissible $\tilde{G}_n(\phi) \geq 0$ and $\tilde{G}_N(\phi) = 0$ $\mathbb{P}$-a.s.

so $\tilde{G}_N(\phi) \notin \Gamma$

3. We get rid of admissability:

<u>Lemma</u>: if a market is viable any predictable process $(\phi^1, \ldots, \phi^d)$ satisfies $\tilde{G}_N(\phi) \notin \Gamma$.
from 1 to d (without 0 to have self-financing free)

<u>Proof</u>: Assume by contraddiction $\tilde{G}_N(\phi) \in \Gamma$. If $\tilde{G}_n \geq 0\ \forall n$ we back in admissability hypothesis and so $\tilde{G}_N(\phi) \notin \Gamma$, $\lightning$. So $\bar{n} := \max\{n : \mathbb{P}(\tilde{G}_n < 0) > 0\}$ ($\bar{n} \leq N$ since $\tilde{G}_N(\phi) \in \Gamma$) hence $\mathbb{P}(\tilde{G}_{\bar{n}}(\phi) < 0) > 0$ and $\tilde{G}_m(\phi) \geq 0$ $\mathbb{P}$-q.c. $\forall m > n$. We define the process:

$$\psi_j(\omega) = \begin{cases} 0 & , j \leq \bar{n} \\ \mathbb{1}_A(\omega)\, \phi_j(\omega) & , j > \bar{n} \end{cases} \qquad \text{with } A = \{\tilde{G}_{\bar{n}}(\phi) < 0\} \in \mathcal{F}_{\bar{n}}.$$

it is free that is self financing since is from 1 to d so $\exists!\, \psi_0$ s.t. ...

So ψ is predictable (until $\bar{n}$ is constant to 0, then ϕ_j is predictable so $\mathcal{F}_{j-1}$-measurable and $\mathbb{1}_A$ $\mathcal{F}_{\bar{n}}$-measurable with $\bar{n} < j \Rightarrow \mathcal{F}_j$-measurable too). Furthermore:

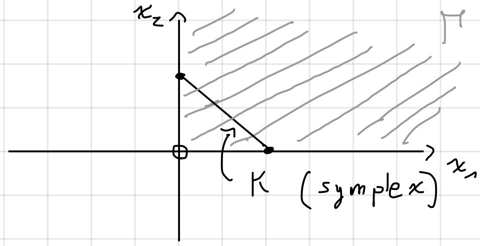
$$\tilde{G}_j(\psi) = \begin{cases} 0 & , j \leq n \\ \sum_{k=1}^{j} \psi_k \cdot \Delta \tilde{S}_k = \sum_{k=\bar{n}+1}^{j} \mathbb{1}_A\, \phi_k \cdot \Delta \tilde{S}_k = \mathbb{1}_A \sum_{k=\bar{n}+1}^{j} \phi_k \cdot \Delta \tilde{S}_k = \mathbb{1}_A \big(\overbrace{\tilde{G}_j(\phi)}^{\geq 0} - \overbrace{\tilde{G}_{\bar{n}}(\phi)}^{<0 \text{ on } A} \big) & , j > \bar{n} \end{cases}$$

>0 on A

$\Rightarrow \tilde{G}_j(\phi) \geq 0 \quad \forall j \in \{0, \dots, N\}$ and $\tilde{G}_N(\phi) > 0$ on A and since A is not negligible $\Rightarrow$ the market is viable ($\tilde{G}_n = \tilde{V}_n$ is always true). $\square$

(4) $\mathbb{R}^{\Omega} := \{ \Omega \to \mathbb{R} \text{ r.v.} \}$, $K := \{ X \in \mathbb{R}^{\Omega} \quad X(\omega) \geq 0 \quad \forall \omega, \ \sum_{\omega} X(\omega) = 1 \}$.

In 2-dimension if $X(\omega_1) = x_1$ and $X(\omega_2) = x_2$:



Now, K is convex and compact:
$\hookrightarrow$ obvious

$Z = tX + (1-t)Y \quad t \in [0,1] \ ; X, Y \in K \quad \Rightarrow \sum_{\omega} Z(\omega) = t \overbrace{\sum_{\omega} X(\omega)}^{1} + (1-t) \overbrace{\sum_{\omega} Y(\omega)}^{1} = 1$

And $V := \{$ random variables $\tilde{G}_N(\phi)$: ϕ predictable $\}$ is a linear subspace of $\mathbb{R}^{\Omega}$:

$$\alpha \tilde{G}_N(\phi) + \beta \tilde{G}_N(\bar{\phi}) = \alpha \sum_{j=1}^{N} \phi_j \cdot \Delta \tilde{S}_j + \beta \sum_{j=1}^{N} \bar{\phi}_j \cdot \Delta \tilde{S}_j = \sum_{j=1}^{N} \underbrace{(\alpha \phi_j + \beta \bar{\phi}_j)}_{\text{predictable}} \cdot \Delta \tilde{S}_j .$$

(5) Linear version of Hahn Banach: $K \subseteq \mathbb{R}^n$ convex and compact, V a subspace of $\mathbb{R}^n$ $\Rightarrow$ if $V \cap K = \emptyset$, there exists a linear functional ξ defined on $\mathbb{R}^n$ satisfying: (1) $\forall x \in K$ $\xi(x) > 0$ (2) $\forall x \in V$, $\xi(x) = 0$. (In the case it is the dual as in our case $\xi(x) = \xi \cdot x$ ↖ scalar product).

So since $V \cap K = \emptyset$ for (4), $\exists \lambda$ s.t. $\lambda \cdot X = \sum_{\omega} X(\omega) \lambda(\omega) > 0 \ \forall X \in K$ and

$\lambda \cdot \tilde{G}_N(\phi) = \sum_{\omega} \lambda(\omega) \tilde{G}_N(\phi)(\omega) = 0 \quad \forall \tilde{G}_N(\phi) \in V$. In particular, since $X = \mathbb{1}_{\{\omega\}} \in K$

$\Rightarrow \lambda(\omega) > 0 \quad \forall \omega \in \Omega$. So define:

$$\mathbb{Q}(\{\omega\}) = \frac{\lambda(\omega)}{\sum_{\omega \in \Omega} \lambda(\omega)} > 0 ,$$

- $\mathbb{P} \sim \mathbb{Q}$ $(\lambda(\omega) > 0 \ \forall \omega)$
- $\sum_{\omega} \mathbb{Q}(\omega) = 1$ (probability)

(7) Martingale probability: $\sum_{\omega} \lambda(\omega) \tilde{G}_N(\phi)(\omega) = 0 \Rightarrow \sum_{\omega} \frac{\lambda(\omega)}{\sum_{\omega} \lambda(\omega)} \tilde{G}_N(\phi)(\omega) = 0$

$\Rightarrow \mathbb{E}^{\mathbb{Q}}[\tilde{G}_N(\phi)] = 0 \Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{N} \phi_j \cdot \Delta \tilde{S}_j\right] \stackrel{def}{=} \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{N} \sum_{k=1}^{d} \phi_j^k \Delta \tilde{S}_j^k\right]$, now I choose

$\phi = (0, \dots, 0, \phi^k, 0, \dots, 0) \Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{N} \phi_j^k \Delta \tilde{S}_j^k\right] = 0 \quad \forall \phi_j^k$ predictable $\underset{\substack{\uparrow \\ \text{characterization of martingale}}}{\Rightarrow}$ $\tilde{S}^k$ martingale

$\forall k = 1, \dots, d.$ $\square$

Remark: If $\mathbb{Q}$ is a martingale probability: $\tilde{V}_n$ is a $\mathbb{Q}$-martingale and so:

$$\boxed{\tilde{V}_n = \mathbb{E}^{\mathbb{Q}}[\tilde{V}_N \mid \mathcal{F}_n]}$$, in case $V_N = X$ payoff we have $\tilde{V}_n = \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_n]$.

This value is not unique because $\mathbb{Q}$ it isn't.

Contingent claim: is a nonnegative $\mathcal{F}_N$-measurable random variable.

↳ **Example**: • European call: $H = (S_N - K)^+$ • European put: $H = (K - S_N)^+$ • European digital: $H = \mathbb{1}_{\{S_N > K\}}$

Attainable claim: The contingent claim defined by H is attainable if there exists an admissible strategy worth H at time N.

Complete market: The market is complete if every contingent claim is attainable.

Theorem (Fundamental theorem of Asset Pricing 2): A viable market is complete if and only if there exists a unique probability measure $\mathbb{Q} \sim \mathbb{P}$, under which $\tilde{S}^j_n$ are martingales $\forall j$.

Proof $\boxed{\Rightarrow}$ Suppose $\mathbb{Q}_1, \mathbb{Q}_2$ two martingale probability. So: $\mathbb{E}^{\mathbb{Q}_i}[\tilde{V}_N] = \mathbb{E}^{\mathbb{Q}_i}[V_0(\phi)] \overset{\mathcal{F}_0 = \{\emptyset, \Omega\}}{=} V_0(\phi)$

Now consider the contingent claim $H = \mathbb{1}_A S^0_N$, $A \in \mathcal{F}_N$. $\exists \phi$ admissable s.t. $V_N = \mathbb{1}_A S^0_N$

$\Rightarrow \tilde{V}_N = \mathbb{1}_A \Rightarrow \mathbb{E}^{\mathbb{Q}_1}[\mathbb{1}_A] = V_0(\phi) = \mathbb{E}^{\mathbb{Q}_2}[\mathbb{1}_A] \Rightarrow \mathbb{Q}_1 = \mathbb{Q}_2 \quad (\mathcal{F}_N = \mathcal{F})$.

$\boxed{\Leftarrow}$ We prove the counternominal. Suppose a viable and incomplete market. So $\exists H \geq 0$ s.t. H is not attainable. $\tilde{V} := \{ \text{r.v. } Z : \exists \phi \text{ admissable } \underset{\text{constant } (\mathcal{F}_0\text{-meas.})}{V_0} + \sum_{i=1}^{N} \phi_n \cdot \Delta \tilde{S}_n = Z \}$. Step by step:

1. $\tilde{V} \subsetneq \{\text{r.v. on the measurable space}\}$ since $\tilde{H} \notin \tilde{V}$

2. $\mathbb{Q}_1$ is an equivalent martingale, we define $\langle Z, Y \rangle \xmapsto{\text{scalar product}} \mathbb{E}^{\mathbb{Q}_1}[Z Y]$. Since $\tilde{H} \notin \tilde{V}$ if $\pi: \{\text{r.v.}\} \longmapsto \tilde{V} \oplus \tilde{V}^{\perp}$ we have $\pi(\tilde{H}) \neq 0$, set $\pi(\tilde{H}) =: X$. Now:

$$\mathbb{Q}_2(\{\omega\}) = \left(1 + \frac{X(\omega)}{2\|X\|_\infty}\right) \mathbb{Q}_1(\{\omega\}) > 0 \quad (\Rightarrow \text{equivalent to } \mathbb{P})$$

↖ $|X(\omega)| \leq \|X\|_\infty \Rightarrow \frac{|X(\omega)|}{\|X\|_\infty} \leq 1 \Rightarrow$

$\Rightarrow \left| \frac{X(\omega)}{2\|X\|_\infty} \right| \leq \frac{1}{2} \Rightarrow 1 + \frac{X(\omega)}{2\|X\|_\infty} > 0$

$$\sum_\omega \mathbb{Q}_2(\omega) = \sum_\omega \mathbb{Q}_1(\omega) + \frac{\sum_\omega X(\omega)\mathbb{Q}_1(\omega)}{2\|X\|_\infty} =$$

$$= 1 + \frac{\mathbb{E}^{\mathbb{Q}_1}[X]}{2\|X\|_\infty} = 1$$

↖ but $\phi = (0, \dots, 0)$ admiss. $\Rightarrow U_0 > 0$ constant $\in \tilde{V} \Rightarrow \mathbb{E}[U_0 X] = 0$ since X orthogonal $\tilde{V}$ $\Rightarrow U_0 \mathbb{E}[X] = 0 \Rightarrow \mathbb{E}[X] = 0$

(3) $\mathbb{Q}_2$ is a martingale probability:

$$\mathbb{E}^{\mathbb{Q}_2}\Big[\underbrace{\sum_{n=1}^{N} \phi_n \cdot \Delta\tilde{S}_n}_{Z}\Big] = \sum_{\omega} Z(\omega)\, \mathbb{Q}_2(\omega) = \underbrace{\sum_{\omega} Z(\omega)\,\mathbb{Q}_1(\omega)}_{\mathbb{E}^{\mathbb{Q}_1}[Z] = 0} + \underbrace{\sum_{\omega} \frac{Z(\omega)\, X(\omega)}{2\|X\|_\infty}\, \mathbb{Q}_1(\omega)}_{=0}$$

$\mathbb{E}^{\mathbb{Q}_1}[Z] = 0$ ← Z mart.

$= 0$ since $\mathbb{E}^{\mathbb{Q}_1}[ZX] = 0$ $(Z \perp X)$

$= 0$ and so we conclude as in FTAP 1.

$\square$

<u>Remark</u>: Given H claim, if the market is complete and viable there exists ϕ_H self-financing with $V_N(\phi_H) = H$. And so:

$$\boxed{\tilde{V}_n(\phi_H) \underset{\text{FTAP 1}}{=} \mathbb{E}^{\mathbb{Q}}[\tilde{V}_N(\phi_H)] \underset{\text{FTAP 2}}{=} \mathbb{E}^{\mathbb{Q}}[\tilde{H} \mid \mathcal{F}_n]}$$

(and is unique!)

Lecture 15/03/2024 (4)

<u>Assumption:</u> market viable and complete.

<u>American option</u> : can be exercised at any time until maturity. At any time n the buyer decide whether or not he wants to exercise and at the same time the seller has to hedge the exposure.

↳ <u>Payoff at time n of american option</u> : $Z_n := (S_n - K)^+$ (call)

$Z_n := (K - S_n)^+$ (put)

↳ <u>Value at time n of american option</u> : we indicate the value with U_n. Now:

- at time $n = N$: $\boxed{U_N = Z_N}$
- at time $n = N-1$: we have two options: we exercise and we will have a payoff of Z_{N-1} ; or we wait until maturity and so the value at time $N-1$ is

$$V_{n-1} \overset{\text{FTAP 1}}{=} S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{V}_N \mid \mathcal{F}_{N-1}] \overset{\text{FTAP 2}}{=} S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{U}_N \mid \mathcal{F}_{N-1}] \overset{\tilde{U}_N = \tilde{Z}_N}{=} S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{Z}_N \mid \mathcal{F}_{N-1}]$$

(V_{n-1} is the value of the portfolio and coincides with the value of the claims at time n because at maturity $V_N = U_N$ so they have to be equal previously too). So:

$$\boxed{U_{N-1} = \max\left(Z_{N-1}, \; S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{Z}_N \mid \mathcal{F}_{N-1}]\right)}$$

- at time $n < N-1$: same argument as before but we don't have the equality $\tilde{U}_n \neq \tilde{Z}_n$, so:

$$\boxed{U_{n-1} = \max\left(Z_{n-1}, \; S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{U}_n \mid \mathcal{F}_{n-1}]\right)} \quad \text{(backward induction)}$$

$(\tilde{U}_n)$ is called Snell envelope of $(\tilde{Z}_n)$.

<u>Proposition</u>: The sequence $(\tilde{U}_n)$ is a supermartingale under Q. Also, it is the smallest Q-supermartingale that dominates the sequence $(\tilde{Z}_n)$, i.e. $\tilde{U}_n \geq \tilde{Z}_n$, $\forall n = 0, \dots, N$.

<u>Proof</u> :

- Supermartingale: $U_{n-1} = \max\left(Z_{n-1}, S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{U}_n \mid \mathcal{F}_{n-1}]\right) \geq S_{n-1}^{0} \mathbb{E}^{Q}[\tilde{U}_n \mid \mathcal{F}_{n-1}] \Rightarrow$
  $\tilde{U}_{n-1} \geq \mathbb{E}^{Q}[\tilde{U}_n \mid \mathcal{F}_{n-1}]$;
- Domination: directly from definition;
- Smallest: Assume $\tilde{T}_n$ is another process that satisfies the hypothesis. We see $\tilde{T}_n \overset{?}{\geq} \tilde{U}_n$ $\forall n$

i.e. U_n is the smallest

using backward induction:

- $n = N$: $\tilde{U}_N = \tilde{Z}_N$ and $\tilde{T}_n$ dominates $\tilde{Z}_n$ $\Rightarrow \tilde{T} \geq \tilde{Z}_N$;
- $n \to n-1$: $\tilde{T}_{n-1} \overset{\text{supermart.}}{\geq} \mathbb{E}^{\mathbb{Q}}[\tilde{T}_n \mid \mathcal{F}_{n-1}] \overset{\text{induction hp.}}{\geq} \mathbb{E}^{\mathbb{Q}}[\tilde{U}_n \mid \mathcal{F}_{n-1}]$ and $\tilde{T}_{n-1} \geq \tilde{Z}_{n-1}$

$$\Rightarrow \tilde{T}_{n-1} \geq \max\left(\mathbb{E}^{\mathbb{Q}}[\tilde{U}_n \mid \mathcal{F}_{n-1}], \tilde{Z}_{n-1}\right) = \tilde{U}_{n-1}.$$

$\square$

<u>Theorem</u> (Doob - Meyer Decomposition): $\tilde{U}_n$ supermartingale. It has a unique decomposition:

$$\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$$

where $\tilde{M}_n$ is a martingale and $\tilde{A}_n$ is a non decreasing, predictable process, vanishing at zero.

<u>Stopping time</u>: A random variable $\tau: \Omega \longrightarrow \{0, \dots, N\}$ is a stopping time if:

$$\forall n \in \{0, 1, \dots, N\}: \{\tau \leq n\} \in \mathcal{F}_n$$

<u>Prop.</u>: In discrete time τ is a stopping time $\Leftrightarrow \{\tau = n\} \in \mathcal{F}_n \ \forall n$.

<u>Proof</u>: $\boxed{\Rightarrow}$ $\{\tau = n\} = \{\tau \leq n\} \cup \{\tau \leq n-1\}^c \in \mathcal{F}_n$

$\boxed{\Leftarrow}$ $\{\tau \leq n\} = \bigcup_{i=1}^{n} \{\tau = i\} \in \mathcal{F}_n$

$\square$

<u>Stopped process at a stopping time</u>: (X_n) stochastic process adapted to $(\mathcal{F}_n)$ and τ a stopping time w.r.t. $(\mathcal{F}_n)$. We define: $X_n^{\tau}(\omega) = X_{n \wedge \tau(\omega)}(\omega)$

In particular, on the set $\{\tau = j\}$:

$$X_n^{\tau} = \begin{cases} X_j, & n \geq j \\ X_n, & n < j \end{cases}$$

<u>Proposition</u>: Let (X_n) be an adapted process with respect to $\mathcal{F}_n$ and τ a stopping time. The stopped sequence (X_n^{τ}) is adapted and if (X_n) is a $\mathcal{F}_n$-martingale (resp. supermart.) $\Rightarrow (X_n^{\tau})$ is also a martingale (resp. supermartingale).

<u>Proof.</u>:

- Adapted: $n \geq 1$

$$X_{\tau \wedge n} \overset{(*)}{=} X_0 + \sum_{j=1}^{\tau \wedge n} (X_j - X_{j-1}) = X_0 + \sum_{j=1}^{n} \overbrace{\mathbb{1}_{\{j \leq \tau\}}}^{\{j \leq \tau\}^c = \{j > \tau\} = \{\tau \leq j-1\} \in \mathcal{F}_{j-1} \Rightarrow \text{predictable}} \underbrace{(X_j - X_{j-1})}_{\text{adapted}}$$

$\Rightarrow X_{\tau \wedge n}$ is adapted

- Martingale. $(*)$ is a martingale transform $\Rightarrow$ so a martingale.

• Supermartingale: analogous Lemma used for martingale:

Lemma: $Y_n = \begin{cases} \phi_0 X_0 & , n=0 \\ \phi_0 X_0 + \sum_{j=1}^{n} \phi_j (X_j - X_{j-1}) & , n \geq 1 \end{cases}$ with X_n supermart., (ϕ_n)-non negative

and predictable $\Rightarrow (Y_n)$ supermartingale

Proof. $\mathbb{E}[Y_{n+1} - Y_n \mid \mathcal{F}_n] = \mathbb{E}[\phi_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n] = \underbrace{\phi_{n+1}}_{\geq 0}\big(\underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - X_n}_{\leq 0}\big) \leq 0.$

□

So the thesis follows immediately by the Lemma.

□

Prop: The random variable defined by: $\nu = \inf\{n \geq 0 : \tilde{U}_n = \tilde{Z}_n\}$

is a stopping time. Furthermore, the stopped sequence $(\tilde{U}_n^{\nu})$ is a martingale (in general it is only a supermartingale). Denote by $\mathcal{T}_{n,N}$ the set of all stopping time taking values in $\{n, n+1, \dots, N\}$.

Proof: • Stopping time: it is well defined since $\tilde{U}_N = \tilde{Z}_N \Rightarrow \tau \in \{0, \dots, N\}$; for $n=0$

$\{\nu = 0\} = \{\tilde{U}_0 = \tilde{Z}_0\} \in \mathcal{F}_0$; for $k \geq 1$ $\{\nu = k\} = \{\tilde{Z}_0 < U_0\} \cap \dots \cap \{\tilde{Z}_{k-1} < \tilde{U}_{k-1}\} \cap \{\tilde{U}_k = \tilde{Z}_k\} \in \mathcal{F}_k$.

• Martingale: $\tilde{U}_n^{\nu} = \tilde{U}_{n \wedge \nu} = U_0 + \sum_{j=1}^{n} \mathbb{1}_{\{j \leq \nu\}} \Delta \tilde{U}_j$ adapted as in the previous Proposition.

Now, $\tilde{U}_{n+1}^{\nu} - \tilde{U}_n^{\nu} \overset{(*)}{=} \mathbb{1}_{\{\nu \geq n+1\}}(\tilde{U}_{n+1} - \tilde{U}_n)$ and on $\mathbb{1}_{\{\nu \geq n+1\}}$ by definition of ν

$\tilde{U}_n > \tilde{Z}_n$ and so $\tilde{U}_n = \mathbb{E}^{Q}[\tilde{U}_{n+1} \mid \mathcal{F}_n]$ $\left(\tilde{U}_n = \max\{\tilde{Z}_n, \mathbb{E}^{Q}[\tilde{U}_{n+1} \mid \mathcal{F}_n]\}\right)$ So we

can rewrite (*): $\tilde{U}_{n+1}^{\nu} - \tilde{U}_n^{\nu} = \mathbb{1}_{\{\nu \geq n+1\}}\left(\tilde{U}_{n+1} - \mathbb{E}^{Q}[\tilde{U}_{n+1} \mid \mathcal{F}_n]\right) \xRightarrow{\mathbb{E}[\cdot \mid \mathcal{F}_n]}$

$\mathbb{E}[\tilde{U}_{n+1}^{\nu} - \tilde{U}_n^{\nu} \mid \mathcal{F}_n] = \mathbb{E}\Big[\mathbb{1}_{\{\nu \geq n+1\}}\left(\tilde{U}_{n+1} - \mathbb{E}^{Q}[\tilde{U}_{n+1} \mid \mathcal{F}_n]\right) \Big| \mathcal{F}_n\Big] =$

$\{\nu \geq n+1\}^c = \{\nu < n+1\} = \{\nu \leq n\} \subset \mathcal{F}_n$

$= \mathbb{1}_{\{\nu \geq n+1\}}\left(\mathbb{E}^{Q}[\tilde{U}_{n+1} \mid \mathcal{F}_n] - \mathbb{E}^{Q}[\tilde{U}_{n+1} \mid \mathcal{F}_n]\right) = 0.$

□

Corollary: The stopping time ν satisfies: $\mathbb{E}^{Q}[\tilde{Z}_{\nu}] \overset{1}{=} \tilde{U}_0 \overset{2}{=} \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}^{Q}(\tilde{Z}_{\tau})$

Proof.

1. $\tilde{U}_n^{\nu}$ is a martingale $\Rightarrow \tilde{U}_0 = \tilde{U}_0^{\nu} = \mathbb{E}^{Q}[\tilde{U}_N^{\nu}] = \mathbb{E}[\tilde{U}_{\nu}] = \mathbb{E}[Z_{\nu}]$.
$\quad \nu \in \{0, \dots, N\}$

2. We know $\mathbb{E}^{Q}[\tilde{Z}_{\nu}] = \tilde{U}_0$ by 1 and $\nu \in \mathcal{T}_{0,N} \Rightarrow \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}^{Q}[Z_{\tau}] \geq U_0$.

But, $\underset{\tau \in \mathcal{T}_{0,N}}{\tilde{U}_0 = \tilde{U}_0^{\tau}} \geq \mathbb{E}[\tilde{U}_N^{\tau}] = \mathbb{E}[\tilde{U}_{\tau}] \geq \mathbb{E}[\tilde{Z}_{\tau}] \Rightarrow$ this is an $\boxed{=}$.
$\quad \tilde{U}_n^{\tau}$ supermart.

□

<u>Optimal stopping time</u>: A stopping time τ^* is called optimal for the sequence $(\tilde{Z}_n)_{0\le n\le N}$ if: $\mathbb{E}^{\mathbb{Q}}[\tilde{Z}_{\tau^*}] = \sup_{\tau\in \mathcal{T}_{0,N}} \mathbb{E}^{\mathbb{Q}}[\tilde{Z}_\tau]$.

<u>Corollary</u>: ν is an optimal stopping time.

<u>Proposition</u> (The smallest optimal stopping time): The stopping time τ^* is optimal if and only if

$$\begin{cases} \tilde{U}_{\tau^*} = \tilde{Z}_{\tau^*} \\ (U_{\tau^*\wedge n})_n \text{ is a } \mathcal{F}_n\text{-martingale.}\end{cases}$$

<u>dim.</u> $\boxed{\Leftarrow}$ $\tilde{U}_0 \overset{\text{constant}}{=} \mathbb{E}[\tilde{U}_0] = \mathbb{E}[\tilde{U}_{\tau^*\wedge 0}] \overset{\tilde{U}_{\tau\wedge n}\text{ mart.}}{=} \mathbb{E}[\tilde{U}_{\tau^*\wedge N}] = \mathbb{E}[\tilde{U}_{\tau^*}] = \mathbb{E}[\tilde{Z}_{\tau^*}] \Rightarrow$ from

2 of the previous Corollary we have the thesis.

$\boxed{\Rightarrow}$ If τ^* is optimal from the previous Corollary: $\tilde{U}_0 = \mathbb{E}[\tilde{Z}_{\tau^*}] \overset{\tilde{U}\text{ dominate }\tilde{Z}}{\le} \mathbb{E}[\tilde{U}_{\tau^*}] =$

$= \mathbb{E}[\tilde{U}_{\tau^*\wedge N}] \overset{\text{supermart.}}{\le} \mathbb{E}[\tilde{U}_{\tau^*\wedge 0}] = \mathbb{E}[\tilde{U}_0] = \tilde{U}_0 \Rightarrow$ (sandwich) $\mathbb{E}[\tilde{Z}_{\tau^*}] = \mathbb{E}[\tilde{U}_{\tau^*}]$

$\Rightarrow \mathbb{E}[\tilde{U}_{\tau^*} - \tilde{Z}_{\tau^*}] = 0$ but $\tilde{U}_{\tau^*} \ge \tilde{Z}_{\tau^*} \Rightarrow \tilde{U}_{\tau^*} = \tilde{Z}_{\tau^*}$ a.s.

Let's see that $\tilde{U}_{n\wedge\tau^*}$ is a martingale. In particular we'll prove that $\mathbb{E}[\tilde{U}_{\tau\wedge N} \mid \mathcal{F}_n] = \tilde{U}_{\tau\wedge n}$ $\forall n$ (it implies that is a martingale because $\mathbb{E}[\tilde{U}_{\tau\wedge n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[\tilde{U}_{\tau\wedge N} \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] =$

$= \mathbb{E}[\tilde{U}_{\tau\wedge N} \mid \mathcal{F}_n] = \tilde{U}_{\tau\wedge n}$ $\forall n$). So:

$\tilde{U}_0 = \mathbb{E}[\tilde{U}_0] = \mathbb{E}[\tilde{U}_{\tau^*\wedge 0}] \underset{\text{supermar.}}{\ge} \mathbb{E}[\tilde{U}_{\tau\wedge n}] \ge \mathbb{E}[\tilde{U}_{\tau^*\wedge N}] = \mathbb{E}[\tilde{U}_{\tau^*}] \overset{\text{optimal + Corollary + } U_{\tau^*}=Z_{\tau^*} \text{ just proved}}{=} U_0^* \Rightarrow$

$\mathbb{E}[\tilde{U}_{\tau^*\wedge n}] = \mathbb{E}[\tilde{U}_{\tau^*}] = \mathbb{E}[\mathbb{E}[\tilde{U}_{\tau^*} \mid \mathcal{F}_n]] \Rightarrow \mathbb{E}[\tilde{U}_{\tau^*\wedge n} - \mathbb{E}[U_{\tau^*} \mid \mathcal{F}_n]] = 0$

but $\mathbb{E}[U_{\tau^*} \mid \mathcal{F}_n] = \mathbb{E}[U_{\tau^*\wedge N} \mid \mathcal{F}_n] \underset{\text{Supermar.}}{\le} \tilde{U}_{\tau^*\wedge n} \Rightarrow \tilde{U}_{\tau^*\wedge n} - \mathbb{E}[U_{\tau^*} \mid \mathcal{F}_n] \ge 0$

$\Rightarrow \tilde{U}_{\tau^*\wedge n} = \mathbb{E}[U_{\tau^*} \mid \mathcal{F}_n] = \mathbb{E}[U_{\tau^*\wedge N} \mid \mathcal{F}_n]$. □

<u>Remark</u>: ν is the smallest optimal stopping time because if τ^* is another stopping time

$\Rightarrow \tilde{U}_{\tau^*} = \tilde{Z}_{\tau^*}$ but $\nu = \inf\{n : \tilde{U}_n = \tilde{Z}_n\} \Rightarrow \nu \le \tau^*$.

<u>Proposition</u> (The largest stopping time): The largest optimal stopping time for $(\tilde{Z}_n)$ is given by

$$\nu_{max} = \begin{cases} N & \text{, if } \tilde{A}_N = 0 \\ \inf\{n : \tilde{A}_{n+1} \ne 0\} & \text{, if } \tilde{A}_N \ne 0 \end{cases}$$

$\to$ ($\tilde{A}_n \ge 0$ and is not decreasing so if $\tilde{A}_N = 0 \Rightarrow \tilde{A}_n = 0$ $\forall n$)

where $\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$ is the Doob decomp.

Proof ν_{max} is a stopping time because $\hat{A}_n$ is predictable. Now we want to prove that is optimal using the previous proposition. So:

- $U_{\nu_{max}\wedge n}$ is a martingale: $\forall n \le \nu_{max}$ $\hat{A}_n = 0 \Rightarrow \tilde{U}_{\nu_{max}\wedge n} = \tilde{M}_{\nu_{max}\wedge n} - \hat{A}_{\nu_{max}\wedge n} =$

$= \tilde{M}_{\nu_{max}\wedge n}$ but $\tilde{M}_n$ is a martingale and so also the stopped process $\tilde{M}_{\nu_{max}\wedge n}$.

- $\tilde{U}_{\nu_{max}} = \tilde{Z}_{\nu_{max}}$: $\tilde{U}_{\nu_{max}} = \sum_{j=0}^{N-1} 1_{\{\nu_{max}=j\}} \tilde{U}_j + 1_{\{\nu_{max}=N\}} \tilde{U}_N$. Now $\tilde{U}_N = \tilde{Z}_N$;

on $\{\nu_{max}=j\}$ we have $\tilde{U}_j \overset{def.}{=} \underbrace{\max\{\tilde{Z}_j, E[\tilde{U}_{j+1}|\mathcal{F}_j]\} = \tilde{M}_j}$, but on

on $\{\nu_{max}=j\}$ $\tilde{U}_j = \tilde{M}_j$ $(\hat{A}_j=0)$

$\{\nu_{max}=j\}$ $\tilde{U}_{j+1} = \tilde{M}_{j+1} - \underset{>0}{\tilde{A}_{j+1}} \Rightarrow E[\tilde{U}_{j+1}|\mathcal{F}_j] \underset{\tilde{M}\text{ mart.},\ \tilde{A}\text{ predictable}}{=} \tilde{M}_j - \tilde{A}_{j+1} < \tilde{M}_j \Rightarrow$

since $\max\{\tilde{Z}_j, E[\tilde{U}_{j+1}|\mathcal{F}_j]\} = \tilde{M}_j \Rightarrow \max\{\tilde{Z}_j, E[\tilde{U}_{j+1}|\mathcal{F}_j]\} = \tilde{Z}_j \Rightarrow$

$\tilde{U}_j = \tilde{Z}_j$ on $\{\nu_{max}=j\}$. So $\tilde{U}_{\nu_{max}} = \sum_{j=0}^{N-1} 1_{\{\nu_{max}=j\}}\tilde{Z}_j + 1_{\{\nu_{max}=N\}}\tilde{Z}_N = \tilde{Z}_{\nu_{max}}$.

Now we prove that is the largest:

take τ^* optimal $\Rightarrow$ $\tilde{U}_{\tau^*\wedge n}$ is a martingale $\Rightarrow E[\tilde{U}_{\tau^*\wedge n}] \overset{A}{=} E[\tilde{U}_{\tau^*\wedge 0}] = E[\tilde{U}_0] = \tilde{U}_0$ $(*_1)$

but $E[U_{\tau^*\wedge n}] \overset{Doob}{=} E[\tilde{M}_{\tau^*\wedge n}] - E[\hat{A}_{\tau^*\wedge n}] \overset{mart.}{=} E[\tilde{M}_0] - E[\hat{A}_{\tau^*\wedge n}] =$ $\tilde{M}_0 = \tilde{U}_0$ $(\hat{A}_0 = 0)$

$= \tilde{U}_0 - E[\hat{A}_{\tau^*\wedge n}]$ $(*_2)$. So $(*_1)$ and $(*_2)$ gives: $E[\hat{A}_{\tau^*\wedge n}] = 0$ $\forall n$

$\Rightarrow 0 = E[\hat{A}_{\tau^*\wedge N}] = E[\tilde{A}_{\tau^*}]$ but $\tilde{A} \ge 0 \Rightarrow \tilde{A}_{\tau^*} \ge 0 \Rightarrow \hat{A}_{\tau^*} = 0$ a.s. So,

by definition $\nu_{max} \ge \tau^*$.

□

Application to american option: Recall that:

- Call: $Z_n = (S_n - K)^+$
- Put: $Z_n = (K - S_n)^+$
- value process
$$\begin{cases} U_N = Z_N \\ U_n = \max\left\{Z_n, S_n^0 E^Q\left[\frac{U_{n+1}}{S_{n+1}^0} \,\middle|\, \mathcal{F}_n\right]\right\} & \forall n \le N-1 \end{cases}$$

→ What is the value at time 0?

From a previous prop. we have $\tilde{U}_0 = \sup_{\tau\in\mathcal{T}_{0,N}} E^Q[\tilde{Z}_\tau] \left(= \sup_{\tau\in\mathcal{T}_{0,N}} E^Q\left[\frac{Z_\tau}{S_\tau^0}\right]\right)$.

In case we have an optimal time $\tilde{U}_0 = E^Q[\tilde{Z}_{\tau^*}]$ by def.

→ How can we manage a short position for the American call?

We have to hedge the short position. So take the Doob decomposition $\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$.

Since the market is complete we can replicate $S_N^0 \tilde{M}_N$ by a strategy ϕ, so

$\exists \phi$ self-financing s.t. $V_N(\phi) = S_N^0 \tilde{M}_N \Rightarrow \tilde{V}_N = \tilde{M}_N \Rightarrow \mathbb{E}^{\mathbb{Q}}[\tilde{V}_N(\phi) \mid \mathcal{F}_n] =$

$= \mathbb{E}^{\mathbb{Q}}[\tilde{M}_N \mid \mathcal{F}_n] = \tilde{M}_n$ but we also have that $\tilde{V}_n$ is a $\mathbb{Q}$-martingale $\Rightarrow$

$\mathbb{E}^{\mathbb{Q}}[\tilde{V}_N \mid \mathcal{F}_n] = \tilde{V}_n \Rightarrow \tilde{M}_n = \tilde{V}_n \Rightarrow \tilde{U}_n = \tilde{V}_n - \tilde{A}_n \Rightarrow U_n = V_n - A_n$

$\Rightarrow V_n \geq U_n \geq Z_n$ so having V_n you can pay the payoff Z_n if the option buyer exercises.

→ <u>Optimal date to exercise</u>?

The date has to be chosen among the stopping times. Now:

- if $\tau < \nu$: it means we exercise when $U_n > Z_n$ but in this case we exercise an option that has value $U_n >$ the payoff Z_n, is not optimal. ($\Rightarrow \tau$ has to be s.t. $U_\tau = Z_\tau$)
- if $\tau > \nu_{max}$: in general $U_n = M_n - A_n$ but by definition $\tilde{U}_{\nu_{max}} = \tilde{M}_{\nu_{max}}$ and $\tilde{U}_{\nu_{max+1}} = \tilde{M}_{\nu_{max+1}} - \tilde{A}_{\nu_{max+1}}$ with $\tilde{A}_{\nu_{max+1}} > 0$.

$$\tilde{U}_{\nu_{max}} = \tilde{M}_{\nu_{max}} = \mathbb{E}[\tilde{M}_{\nu_{max+1}} \mid \mathcal{F}_{\nu_{max}}] = \mathbb{E}[\tilde{U}_{\nu_{max+1}} \mid \mathcal{F}_{\nu_{max}}] + \tilde{A}_{\nu_{max+1}} > \mathbb{E}[\tilde{U}_{\nu_{max+1}} \mid \mathcal{F}_{\nu_{max}}]$$

So we expect that the value of $\tilde{U}_n$ will decrease, and so it is better to exercise at ν_{max}.

Easier: by def. of optimal we are maximizing the expeted value of the payoff so τ has to be optimal $\Rightarrow \nu \leq \tau \leq \nu_{max}$

Continuous Time Models

A little recap of stochastic analysis and assumptions:

__Itō process__: an Itō process is a stochastic process X_t that can be written as:

$$dX_t = a(\omega,t)\,dt + b(\omega,t)\,dB_t \quad \text{or} \quad X_t = X_0 + \int_0^t a(\omega,s)\,ds + \int_0^t b(\omega,s)\,dB_s$$

for $t \in [0,T]$, with $a(\cdot,\cdot)$ measurable with respect to $\mathcal{F} \times \mathcal{B}([0,T])$ and adapted.

Obviously, the integrals have to make sense and so:

$$\mathbb{P}\left(\int_0^T |a(\omega,s)|\,ds < +\infty\right) = 1 \qquad \left(a \in L^1(ds)\right)$$

$$\mathbb{P}\left(\int_0^T |b(\omega,s)|^2 ds < +\infty\right) = 1 \qquad \left(b(\omega,s) \in L^2_{loc}([0,T])\right)$$

So X_t is a semimartingale (finite variation + local martingale). We can also define the integral with respect to X_t:

$$\int_0^t f(\omega,t)\,dX_t = \int_0^t f(\omega,s)\,a(\omega,t)\,dt + \int_0^t f(\omega,s)\,b(\omega,t)\,dB_t$$

with integrability hypothesis: $f(\omega,s)\,a(\omega,t) \in L^1(ds)$ and $f(\omega,s)\,b(\omega,t) \in L^2_{loc}([0,T])$.

__Filtration__: $(\Omega, \mathcal{F}, \mathbb{P})$ probability space. $\overline{\mathcal{F}}_t = \sigma(X_s, s \le t)$, we want that the filtration satisfies the usual conditions: complete (if $A' \subset A \in \mathcal{F}$ s.t. $\mathbb{P}(A) = 0 \Rightarrow \forall t \; A' \in \mathcal{F}_t$), right-continuous ($\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$). So we set $\mathcal{F}_t := \sigma(\overline{\mathcal{F}}_t \cup \mathcal{N})$ ($\mathcal{N}$ σ-algebra generated by $\mathbb{P}$-null sets of $\mathcal{F}$) and in the case of BM it turns out that $\mathcal{F}_t$ satisfies the usual condition.

Reasons to choose usual conditions:

1. If $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ and $X = Y$ $\mathbb{P}$ a.s. and Y is $\mathcal{F}_t$-measurable $\Rightarrow X$ is also $\mathcal{F}_t$-measurable

2. If $\mathcal{F}_t$ is right-continuous $\Rightarrow$ every martingale has a cadlag version.

Black-Scholes model

↳ Contains two assets:

- __Risk-free asset__: $S^0 = (S^0_t)_{0 \le t \le T}$. It deterministic and it follows the ODE:

$\boxed{dS_t^0 = r S_t^0 \, dt}$, with $S_0^0 = 1$ and $r \geq 0$. Hence, $\boxed{S_t^0 = e^{rt}}$

- **Stock**: a risky asset with price process denoted by $S = (S_t)_{0 \leq t \leq T}$. Its dynamic follows the SDE:

$$\boxed{dS_t = \mu S_t \, dt + \sigma S_t \, dB_t}$$

with $\sigma > 0$ the volatility (annualised standard deviation of the log of the price change factors) and μ costants.

In the following exercise sheet we prove that:

$$\boxed{S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right)}$$ and this solution is unique up to indistinguishability

Properties of the risky asset:

1. Continuous pathes

obvious. there is $B_t - B_s$ that's $\mathcal{F}_s$-indipendent

2. Indipendence of the relative increments: if $s \leq t$, $\frac{S_t}{S_s}$ and $\frac{S_t - S_s}{S_s}$ are indipendent of $\mathcal{F}_s$

3. Stationarity of the relative increments: if $u \leq t$ the law of $\frac{S_t - S_u}{S_u}$ shares the distribution with $\frac{S_{t-u} - S_0}{S_0}$.

$= \frac{S_{t-u}}{S_0} - 1$ $\rightarrow N(0, t-u)$

$\frac{S_t - S_u}{S_u} = \frac{S_t}{S_u} - 1 \rightarrow B_t - B_u \sim N(0, t-u)$

4. $\forall t$, $\log S_t \sim N\left(\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$

Proof $d(\log S_t) = \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} d\langle S \rangle_t = \mu \, dt + \sigma \, dB_t - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma \, dB_t$. So:

$$\log S_t = \log S_0 + \int_0^t \mu - \frac{1}{2}\sigma^2 \, ds + \int_0^t \sigma \, dB_s = \overbrace{\log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t}^{\text{deterministic}} + \sigma \overbrace{B_t}^{\sim N(0,t)} \Rightarrow$$

$$\log S_t \sim N\left(\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right). \quad \square$$

5. If $X \sim N(\mu, \sigma^2) \Rightarrow$ the characteristic function $\varphi_X(u) = \mathbb{E}[e^{iuX}] = \exp\left(iu\mu - \frac{1}{2}\sigma^2 u^2\right)$ $\forall u \in \mathbb{R}$. In the case of normal distribution it can be extended to $\mathbb{C}$, and so:

$$\mathbb{E}[S_t] = \mathbb{E}\left[S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t\right)\right] = S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t\right)\mathbb{E}\left[e^{i(-i)\sigma \overbrace{B_t}^{\sim N(0,t)}}\right] =$$

$$= S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t\right)\exp\left(-\frac{1}{2}(-i)^2\sigma^2 t\right) = S_0 \exp(\mu t) \quad (\text{expected yield})$$

<u>Strategy</u> : is a process $\phi = (\phi_t)_{0\le t\le T} = (H_t^0, H_t)_{0\le t\le T}$ adapted to $(\mathcal{F}_t)_{0\le t\le T}$.

H_t^0 = quantity of risk-less asset and H_t = quantity of risky asset.

<u>Value of portfolio</u> : $\boxed{V_t(\phi) = H_t^0 S_t^0 + H_t S_t}$.

<u>Self-financing</u>: has to satisfy 2 conditions:

1 : $\int_0^T |H_t^0| dt + \int_0^T H_t^2 \, dt < \infty$ a.s. (usual integrability conditions)

2 : $V_t(\phi) \underset{\text{def.}}{=} H_t^0 S_t^0 + H_t S_t \underset{\text{self-fin. cond}}{=} H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_s^0 \, dS_s^0 + \int_0^t H_s \, dS_s$

or in differential form $\boxed{dV_t = H_t^0 \, dS_t^0 + H_t \, dS_t}$ a.s. $\forall t \in [0,T]$.

<u>Discounted values</u> : $\tilde{S}_t := \dfrac{S_t}{e^{rt}}$ $\quad \tilde{V}_t = \dfrac{V_t}{e^{rt}}$

<u>Proposition</u> : Let ϕ be a strategy satisfying the integrability condition. Then, ϕ is a self-financing strategy $\iff$ $\boxed{\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_s \, d\tilde{S}_s \quad \text{a.s.}}$

<u>Proof</u>

$\boxed{\Longrightarrow}$ $d\tilde{V}_t(\phi) = d(V_t(\phi)\cdot e^{-rt}) = V_t(\phi)\, d(e^{-rt}) + e^{-rt} dV_t + \underbrace{\langle e^{-rt}, V_t(\phi)\rangle}_{=0} =$

$$= V_t(\phi)\,(-re^{-rt})\,dt + e^{-rt} dV_t \underset{\text{self-financing}}{=}$$

$$= \cancel{-re^{-rt} H_t^0 \underbrace{S_t^0}_{=e^{rt}} dt} - re^{-rt} H_t S_t \, dt + \cancel{e^{-rt} H_t^0 \underbrace{dS_t^0}_{=re^{rt}dt}} + e^{-rt} H_t dS_t =$$

$$= H_t \, d\tilde{S}_t$$

$$\hat{d\tilde{S}_t} = d(S_t e^{-rt}) = e^{-rt} dS_t + S_t \, d(e^{-rt}) + 0$$

$\boxed{\Longleftarrow}$ $V_t(\phi) = e^{rt}\tilde{V}_t(\phi) \Rightarrow dV_t = e^{rt} d\tilde{V}_t(\phi) + \tilde{V}_t(\phi)\, re^{rt} dt + 0 \overset{\text{hyp.}}{=}$

$$= e^{rt} H_t \, d\tilde{S}_t + H_t^0 re^{rt} dt + H_t S_t e^{-rt} re^{rt} dt =$$

$d\tilde{S}_t = d(S_t \cdot e^{-rt}) =$

$= e^{-rt} dS_t - S_t re^{-rt} dt$

$$= \cancel{e^{rt}} H_t \cancel{e^{-rt}} dS_t - \cancel{e^{rt}} \cancel{H_t S_t r e^{-rt}} dt + H_t^0 re^{rt} dt + \cancel{H_t S_t} r\,dt$$

$$= H_t \, dS_t + H_t^0 \, d(e^{rt}) = H_t \, dS_t + H_t^0 \, dS_t^0.$$

$\square$

Absolutely continuity : A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is absolutely continuous with respect to $\mathbb{P}$ if $\forall A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$, in which case we write $\mathbb{Q} << \mathbb{P}$.

Equivalent probabilities : two probabilities $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if $\mathbb{P} << \mathbb{Q}$ and $\mathbb{Q} << \mathbb{P}$.

Theorem (Radon - Nikodym) : a probability measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if and only if $\exists Z$ r.v. s.t. $\mathbb{P}(Z \geq 0) = 1$ on $(\Omega, \mathcal{F})$ and $\forall A \in \mathcal{F}$: $\mathbb{Q}(A) = \int_A Z \, d\mathbb{P}$.

In this case Z is the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ and denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

Remark : If $\mathbb{Q} << \mathbb{P}$ with density Z : $\mathbb{P} \sim \mathbb{Q} \Leftrightarrow \mathbb{P}(Z > 0) = 1$.

Theorem (Girsanov) : Let $\theta = (\theta_t)_{t \in [0,T]}$ be an adapted process satisfying $\int_0^T \theta_s^2 \, ds < \infty$ a.s. and such that the process $(L_t)_{t \in [0,T]}$ defined by: $L_t := \exp\left(- \int_0^t \theta_s \, dB_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right)$ is a martingale. Then, under the probability $\mathbb{P}^{(L)}$ with density L_T with respect to $\mathbb{P}$, the process $(W_t)_{t \in [0,T]}$ defined by $W_t = B_t + \int_0^t \theta_s \, ds$, is an $(\mathcal{F}_t)$-standard Brownian motion. In this case, we have
$$\left. \frac{d\mathbb{P}^{(L)}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \mathbb{E}\left[L_T \mid \mathcal{F}_t \right] = L_t$$

Remark : a sufficient condition for $(L_t)_{t \in [0,T]}$ to be a martingale is $\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^T \theta_t^2 \, dt \right) \right] < +\infty$ (Novikov's condition).

Theorem (Martingale Representation) : Let $M = (M_t)_{0 \leq t \leq T}$ be a square-integrable martingale, with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by standard Brownian motion $(W_t)_{0 \leq t \leq T}$ (the augmented one (complete + right-continuous)). There exists an adapted process $(K_t)_{0 \leq t \leq T}$ such that $\mathbb{E}\left[\int_0^T K_u^2 \, du \right] < +\infty$ and $\forall t \in [0,T]$:

$$M_t = M_0 + \int_0^t K_u \, dW_u \quad \text{a.s.}$$

Remark : If U is a r.v. $\mathcal{F}_T$-measurable, square integrable, it can be written as:

$$U = \mathbb{E}[U] + \int_0^T H_s \, dB_s \quad \text{a.s.}$$

where (H_t) is an adapted process such that $\mathbb{E}\left[\int_0^T H_s^2 \, ds \right] < +\infty$.

Proof We consider $M_t = \left(\mathbb{E}\left[U \mid \mathcal{F}_t\right]\right)_t$. It is a martingale: $\mathbb{E}\left[\mathbb{E}\left[U \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] =$

$= \mathbb{E}\left[U \mid \mathcal{F}_s\right] = M_s$. It is square integrable: $\sup_{t \geq 0} \mathbb{E}\left[\mathbb{E}\left[U \mid \mathcal{F}_t\right]^2\right] \underset{\text{Jensen}}{\leq} \sup_{t \geq 0} \mathbb{E}\left[\mathbb{E}\left[U^2 \mid \mathcal{F}_t\right]\right] =$

$= \sup_{t \geq 0} \mathbb{E}\left[U^2\right] = \mathbb{E}\left[U^2\right] < +\infty$. So we can apply the Martingale Representation theorem, and we obtain that exists (H_t) adapted s.t. $\mathbb{E}\left[\int_0^T H_s^2 ds\right] < +\infty$ and:

$$M_t = M_0 + \int_0^t H_s \, dW_s$$

so, for $t = T$ $M_T = \mathbb{E}\left[U \mid \mathcal{F}_T\right] = U$ and $M_0 = \mathbb{E}\left[U \mid \mathcal{F}_0\right] = \mathbb{E}[U]$.

□

<u>$\tilde{S}_t$ is a martingale under a certain probability</u>:

- Write the SDE for $\tilde{S}_t$: $\boxed{d\tilde{S}_t = \tilde{S}_t\left((\mu - r)\,dt + \sigma\, dB_t\right)}$. So usual solution for geometric SDE $\boxed{\tilde{S}_t = S_0 \exp\left(-\frac{1}{2}\sigma^2 t + \sigma\left(\frac{\mu - r}{\sigma} t + B_t\right)\right)}$

<u>Proof.</u>

$$d\tilde{S}_t = d\left(S_t \cdot e^{-rt}\right) = S_t \, d(e^{-rt}) + e^{-rt} dS_t + 0 = -r S_t e^{-rt} dt + e^{-rt} S_t\left(\mu\, dt + \sigma\, dB_t\right) =$$

$$= \tilde{S}_t\left((\mu - r)\,dt + \sigma\, dB_t\right).$$

□

the volatility is not changed! The volatility is the same in real world and in this model

- We set $\boxed{W_t := B_t + \frac{\mu - r}{\sigma} t}$, obtaining $\boxed{d\tilde{S}_t = \tilde{S}_t \sigma\, dW_t}$. Now, we claim exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $(W_t)_{t \in [0,T]}$ is a standard BM.

<u>Proof</u>

We want to use Girsanov: $W_t = B_t + \frac{\mu - r}{\sigma} t = B_t + \int_0^t \frac{\mu - r}{\sigma} ds$, so we want to use Girsanov with $\theta_t = \frac{\mu - r}{\sigma}$. Now: $\int_0^T \theta_s^2 ds = \left(\frac{\mu - r}{\sigma}\right)^2 T < +\infty$. We have to show that:

$$e^{-\int_0^t \frac{\mu - r}{\sigma} dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma}\right)^2 ds}$$

is a martingale. We can use Novikov's condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] = \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 dt\right)\right] = \mathbb{E}\left[\exp\left(\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right)\right] = \exp\left(\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right) < \infty$$

□

- Since $(W_t)_t$ is a standard BM we can immediately obtain that the solution for $\tilde{S}_t$ is:

$$\boxed{\tilde{S}_t = S_0 \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)}$$ (solution for a brownian equation)

And also $\tilde{S}_t$ is a $\mathbb{Q}$-martingale.

<u>Proof</u>

Adapted: obviously Integrable: the exponential moment of a normal distribution always exists.

Martingale property: $\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}[S_0 \exp(\sigma W_t - \frac{\sigma^2}{2} t) \mid \mathcal{F}_s] =$

$= \mathbb{E}^{\mathbb{Q}}[S_0 \exp(\sigma W_t + \sigma W_s - \sigma W_s - \frac{1}{2}\sigma^2 t) \mid \mathcal{F}_s] = S_0 \exp(\sigma W_s - \frac{1}{2}\sigma^2 t)\, \mathbb{E}^{\mathbb{Q}}[e^{\sigma(W_t - W_s)} \mid \mathcal{F}_s] =$

$\underset{[W_t - W_s\ \mathcal{F}_s\text{-ind.}\ (W_t \text{ is a } (\mathbb{Q}, \mathcal{F}_t)\text{-BM})}{=} S_0 \exp(\sigma W_s - \frac{1}{2}\sigma^2 t)\, \mathbb{E}^{\mathbb{Q}}[e^{\sigma(W_t - W_s)}] = S_0 \exp(\sigma W_s - \frac{1}{2}\sigma^2 t)\, \mathbb{E}^{\mathbb{Q}}[e^{i(-i\sigma) N(0, t-s)}] =$

$= S_0 \exp(\sigma W_s - \frac{1}{2}\sigma^2 t) \exp(-\frac{1}{2}(-i\sigma)^2 (t-s)) = S_0 \exp(\sigma W_s - \frac{1}{2}\sigma^2 s) = \tilde{S}_s.$ □

Pricing

Admissible strategy: A strategy $\phi = (H_t^0, H_t)_{0 \le t \le T}$ is admissible if it is self-financing and if the discounted value $\tilde{V}_t(\phi) = H_t^0 + H_t \tilde{S}_t$ of the corresponding portfolio is, for all t, non negative, and if $\mathbb{E}^{\mathbb{Q}}[\int_0^T H_t^2 (\sigma \tilde{S}_t)^2 dt] < +\infty$

Theorem (Pricing contingent claims): In the Black-Scholes model, any option defined by a non-negative $\mathcal{F}_T$-measurable (for example American option are NOT $\mathcal{F}_T$-meas.) random variable h, which is square-integrable under the probability measure $\mathbb{Q}$, is replicable and the value at time t of any replicating portfolio is given by:

$$\boxed{V_t = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}[h \mid \mathcal{F}_t]}$$

Hence, the option value at time t can be naturally defined by expression $e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[h \mid \mathcal{F}_t]$.

Proof

- First assume that an admissible strategy exists $(\phi = (H^0, H)) \Rightarrow V_t = H_t^0 S_t^0 + H_t S_t$ and $V_T(\phi) = h$.

$\tilde{V}_t(\phi) \underset{\text{self-fin.}}{=} V_0 + \int_0^t H_u \, d\tilde{S}_u = V_0 + \int_0^t H_u \sigma \tilde{S}_u \, dW_u$. So, for Itô integral definition and assumption of admissible strategy, $\tilde{V}_t(\phi)$ is a square-integrable $\mathbb{Q}$-martingale. Hence:

$$\tilde{V}_t(\phi) \underset{\text{martingale}}{=} \mathbb{E}[\tilde{V}_T(\phi) \mid \mathcal{F}_t] \underset{\text{replicates } h}{=} \mathbb{E}[\tfrac{h}{e^{rT}} \mid \mathcal{F}_t] \Rightarrow V_t(\phi) = e^{-r(T-t)}\mathbb{E}[h \mid \mathcal{F}_t].$$

- It remains to show that h is indeed replicable. We prove that:

$V_t(\phi) = H_t^0 S_t^0 + H_t S_t = e^{rt}\, \mathbb{E}^{\mathbb{Q}}[e^{-rT} h \mid \mathcal{F}_t]$, in particular for $t = T$ we would have $V_T(\phi) = h$. We set $M_t := \mathbb{E}^{\mathbb{Q}}[e^{-rT} h \mid \mathcal{F}_t]$. M_t is a martingale (straight by def.) and is square integrable: $\mathbb{E}^{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}}[e^{-rT} h \mid \mathcal{F}_t]^2] \underset{\text{Jensen}}{\le} \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[e^{-2rT} h^2 \mid \mathcal{F}_t]] = \mathbb{E}^{\mathbb{Q}}[e^{-2rT} h^2] = e^{-2rT}\mathbb{E}[h^2] < \infty$

(Note the augmented filtration of B_t is the same of W_t). Then by martingale representation

$\exists$ an adapted process $(K_t)_t$ such that $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T K_s^2\, ds\right] < +\infty$ and $\forall t \in [0,T]$: $M_t = M_0 + \int_0^t K_s\, dW_s$.

So, we define $H_t := \frac{K_t}{\sigma \tilde{S}_t}$ $\quad H_t^0 := M_t - H_t \tilde{S}_t$. It is self-financing:

$$\tilde{V}_t(\phi) = M_0 + \int_0^t K_s\, dW_s = M_0 + \int_0^t H_s\, \sigma \tilde{S}_s\, dW_s = M_0 + \int_0^t H_s\, d\tilde{S}_s = V_0 + \int_0^t H_s\, d\tilde{S}_s.$$

$M_0 = H_0^0 + H_0 \tilde{S}_0 = V_0$

And: $\tilde{V}_t^{\phi} = H_t^0 + H_t \tilde{S}_t = M_t \Rightarrow V_t = e^{rt}\, \mathbb{E}\left[e^{-rT} h \mid \mathcal{F}_t\right]$.

Note also that $\tilde{V}_t(\phi)$ is non-negative ($\tilde{V}_t = M_t$ and h non-negative). Furthermore we have the integrability condition: $\mathbb{E}\left[\int_0^T H_t^2 (\sigma \tilde{S}_t)^2\, dt\right] = \mathbb{E}\left[\int_0^T K_t^2\, dt\right] < +\infty$.

$\square$

Case of the European call option (and put option)

We can compute the value of the call option in the Black-Scholes model. (S_T $\mathcal{F}_T$-meas. and normal distr. $\Rightarrow$ integrable)

$$V_t = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \mid \mathcal{F}_t\right] \overset{\frac{S_t}{e^{rt}} = S_0 \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)}{=}$$

$$= e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\Big[\big(\underbrace{S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}}_{S_t\ \mathcal{F}_t\text{-meas.}}\ \underbrace{e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}}_{\mathcal{F}_t\text{-ind.}} - K\big)_+ \mid \mathcal{F}_t\Big] \overset{\text{Freezing Lemma}}{=}$$

$$\Rightarrow V_t = F(t, S_t) \text{ where } F(t,x) = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\left[\left(x\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_s)} - K\right)_+\right]$$

If $\sigma(W_T - W_t) + (r - \frac{1}{2}\sigma^2)(T-t) =: Z + r(T-t)$ we have $Z \sim N\left(-\frac{1}{2}\sigma^2(T-t),\ \sigma^2(T-t)\right)$.

Denote $\tilde{\tau} = T - t$.

$$F(t,x) = e^{-r\tilde{\tau}} \cdot \int_{\mathbb{R}} \big(\underbrace{x e^{z + r\tilde{\tau}} - K}_{*}\big)_+ \cdot \overbrace{\frac{e^{-(z + \frac{1}{2}\sigma^2\tilde{\tau})^2/2\sigma^2\tilde{\tau}}}{\sqrt{2\pi\sigma^2\tilde{\tau}}}}^{\text{density of } Z}\, dz =$$

$\to * \geq 0 \Leftrightarrow x e^{z + r\tilde{\tau}} \geq K$

$\Leftrightarrow z \geq \log\left(\frac{K}{x}\right) - r\tilde{\tau}$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tilde{\tau}}} \int_{\log(\frac{K}{x}) - r\tilde{\tau}}^{+\infty} x e^{z} e^{-(z + \frac{1}{2}\sigma^2\tilde{\tau})^2/2\sigma^2\tilde{\tau}} - \frac{K e^{-r\tilde{\tau}}}{\sqrt{2\pi\sigma^2\tilde{\tau}}} \int_{\log(\frac{K}{x}) - r\tilde{\tau}}^{+\infty} e^{-(z + \frac{1}{2}\sigma^2\tilde{\tau})^2/2\sigma^2\tilde{\tau}}\, dz =$$

$v = -\frac{z + \frac{1}{2}\sigma^2\tilde{\tau}}{\sigma\sqrt{\tilde{\tau}}}$

so

$z = -\sigma\sqrt{\tilde{\tau}}\, v - \frac{1}{2}\sigma^2\tilde{\tau}$

and $\frac{dz}{dv} = -\sigma\sqrt{\tilde{\tau}}$

$$= \frac{\sigma\sqrt{\tilde{\tau}}}{\sqrt{2\pi\sigma^2\tilde{\tau}}} \int_{-\infty}^{\boxed{\frac{\log(\frac{x}{K}) + r\tilde{\tau} - \frac{1}{2}\sigma^2\tilde{\tau}}{\sigma\sqrt{\tilde{\tau}}}}\ d_-} x\, e^{-\sigma\sqrt{\tilde{\tau}}\, v - \frac{1}{2}\sigma^2\tilde{\tau} - \frac{1}{2}v^2} -$$

$$- \frac{K e^{-r\tilde{\tau}} \sigma\sqrt{\tilde{\tau}}}{\sqrt{2\pi\sigma^2\tilde{\tau}}} \int_{-\infty}^{d_-} e^{-\frac{1}{2}v^2}\, dv$$

- $z \to +\infty \quad v \to -\infty$
- $z = \log\left(\frac{K}{x}\right) - r\tilde{\tau}$

$\Rightarrow v = \frac{\log(\frac{x}{K}) + r\tilde{\tau} - \frac{1}{2}\sigma^2\tilde{\tau}}{\sigma\sqrt{\tilde{\tau}}}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_- + \sigma\sqrt{\tau}} x\, e^{-\frac{1}{2}\omega^2} d\omega \; - \underbrace{\frac{e^{r\tau} K}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-\frac{1}{2}v^2} dv}_{\text{non lo cambio}} =$$

$\omega = v + \sigma\sqrt{\tau}$

$\frac{dv}{d\omega} = 1 \qquad v = d_- \Rightarrow \omega = d_- + \sigma\sqrt{\tau}$

$-\frac{1}{2}(v+\sigma\sqrt{\tau}) = -\frac{1}{2}v^2 - v\sigma\sqrt{\tau} - \frac{1}{2}\sigma^2\tau$

$$= x\, N(d_+) - K e^{-r(T-t)} N(d_-) \quad \text{with} \quad N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2} dx$$

$d_+ := d_- + \sigma\sqrt{\tau} =$

$= \dfrac{\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$

So
$$\boxed{C_t = S_t\, N(d_+) - K e^{-r(T-t)} N(d_-) \quad \text{with} \quad d_\pm = \frac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}}$$
$$\boxed{\text{and} \quad N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2} dx \quad (\text{cumulative distribution function of an } N(0,1))}$$

For the European put:

$$e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[S_T - K \mid \mathcal{F}_t] = e^{rt}\, \mathbb{E}[\tilde{S}_T - K e^{-rT} \mid \mathcal{F}_t]$$

$\tilde{S}_t$ mart. $= \underbrace{e^{rt}\tilde{S}_t}_{S_t} - K e^{-r(T-t)} \quad (*_1)$

$= e^{rt}\left(\mathbb{E}^{\mathbb{Q}}[(\tilde{S}_T - K e^{-rT})_+ \mid \mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}}[(K e^{-rT} - \tilde{S}_T)_+ \mid \mathcal{F}_t]\right) \quad (*_2)$

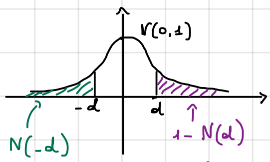
$$\Rightarrow \underbrace{e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(K - S_T)_+ \mid \mathcal{F}_t]}_{\text{Put price (Pricing theorem)}} = \underbrace{e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(S_T - K)_+ \mid \mathcal{F}_t]}_{\text{Call price}} + K e^{-r(T-t)} - S_t =$$

use $(*_1)$ and $(*_2)$ where $\tilde{S}_T = S_T \cdot e^{-rT}$

$$= S_t\, N(d_+) - K e^{-r(T-t)} N(d_-) + K e^{-r(T-t)} - S_t =$$

$$= -S_t(1 - N(d_+)) + K e^{-r(T-t)}(1 - N(d_-))$$

$$\boxed{P_t = -S_t(1 - N(d_+)) + K e^{-r(T-t)}(1 - N(d_-)) \quad \text{or}}$$
$$\boxed{P_t = -S_t\, N(-d_+) + K e^{-r(T-t)} N(-d_-) \quad \text{since :}}$$



for simmetry $N(-d) = 1 - N(d)$

Hedging (Explicit form)

Consider a European claim. So:

$$V_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\overset{h}{\overset{\shortparallel}{f(S_T)}} \mid \mathcal{F}_t] =$$

$$= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[f\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_t} \right) \mid \mathcal{F}_t \right] =$$

$$= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\Big[f\Big(\underbrace{S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}}_{S_t \text{ (measurable)}} \underbrace{e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}}_{\text{indipendent}} \Big) \mid \mathcal{F}_t \Big]$$

so $V_t \overset{(*)}{=} F(t, S_t)$ with:

$$F(t,x) = e^{-r(T-t)} \mathbb{E}\Big[f\Big(e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma \underbrace{(W_T - W_t)}_{\mathcal{N}(0,\,T-t)}} x \Big) \Big] = e^{-r(T-t)} \int_{\mathbb{R}} f\left(e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, z} \right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

dunque moltiplico e divido per $\sqrt{T-t}$ così da avere una $\mathcal{N}(0,1)$

Note that by integrations the payoff function by the above density we obtain a "regularisatiering effect".

Furthermore, recall that:

$$S_t = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)t + \sigma W_t \right) \qquad \leftarrow (W_t)\ \mathbb{Q}\text{-st. BM.}$$

$$dS_t = r S_t\, dt + \sigma S_t\, dW_t \quad , \text{ by Itô's formula}$$

$$dF(t,S_t) = \left(r S_t \frac{\partial F}{\partial x}(t,S_t) + \frac{\partial F}{\partial t}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t) \right) dt + \sigma S_t \frac{\partial F}{\partial x}(t,S_t)\, dW_t =$$

$$\overset{(*)}{\shortparallel}$$

$$= dV_t(\phi) \overset{\text{self-financing}}{=} H_t^0\, d(e^{rt}) + H_t\, dS_t = H_t^0\, r e^{rt} dt + H_t r S_t dt + H_t \sigma S_t dW_t =$$

$$= (H_t^0 e^{rt} + H_t S_t) r\, dt + H_t \sigma S_t\, dW_t$$

Thanks to a deep theorem of uniquess of decomposition of "special" semimartingales we have that:

$$\sigma S_t \frac{\partial F}{\partial x}(t,S_t) = H_t \sigma S_t \quad \Rightarrow \quad \boxed{H_t = \frac{\partial F}{\partial x}(t,S_t) \ \text{ - called } \Delta\text{-hedging}}$$

$$\uparrow F(t,S_t) = V_t$$

Hence
$$V_t(\phi) = F(t,S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\overset{h}{\overbrace{f(S_T)}} \mid \mathcal{F}_t] =$$

$$= e^{rt}\left(e^{-rt} F(t,S_t) - H_t \tilde{S}_t \right) + H_t S_t =$$

$$= S_t^0 \Big(\underbrace{e^{-rt}\left(F(t,S_t) - \frac{\partial F}{\partial x}(t,S_t) S_t \right)}_{\substack{\shortparallel \\ H_t^0}} \Big) + \frac{\partial F}{\partial x}(t,S_t) S_t$$

$$H_t^0 = \left(e^{-rT} \mathbb{E}^{\mathbb{Q}}[h \mid \mathcal{F}_t] - H_t \tilde{S}_t \right)$$

$$\boxed{H_t^0 = e^{-rt}\left(F(t,S_t) - \frac{\partial F}{\partial x}(t,S_t) S_t \right)}$$

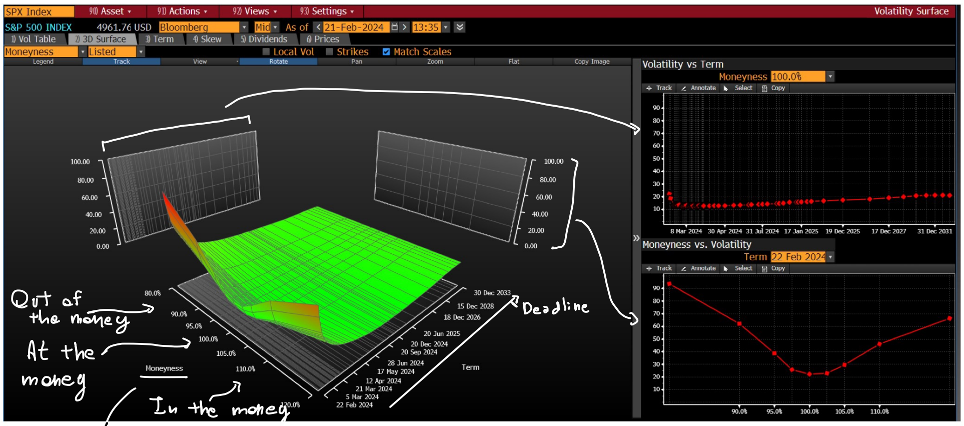
$$\uparrow F(t,S_t) = V_t$$

Comments on the Black - Scholes model

3/05/24 (9)

↳ <u>Hypothesis for BS</u> :

- log-normal diffusion (restrictive, e.g. continuous, we are in a discrete world)
- No bid-ask spread
- No taxes or transition costs
- Borrowing and lending equals
- Underlying instruments are unlimited divisible
- Own transactions have no influence on the price (very liquid markets)
- Complete informations, which is reflected in the traded prices
- In BS, σ is constant but the data tell us is not:



Terminologie

<u>Moneyness</u> : an option is called to be :

- <u>at the money</u>: if the underlying is (around) the strike and so $\frac{S_t}{K}\,\% = 100\%$
- <u>in the money</u>: if the intrinsic value (value if the option would exercised immediatly, like it was an american option) and so $\frac{S_t}{K}\,\% > 100\%$
- <u>out of the money</u>: if the intrinsic value is negative $\left(\frac{S_t}{K}\,\% < 100\%\right)$

<u>Time value</u> : the price and the intrinsic value (ex. for a call is $C_t - (S_t - K)$)

↳ <u>Obvious attempt to solve the problems</u> → <u>$\sigma \rightsquigarrow \sigma(t)$ and after $\sigma \rightsquigarrow \sigma(t, S_t)$</u>

<u>Exponential integral Brownian model</u>:

$$S_t = S_0 \exp\left(\int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s\right)$$

<u>Exponential time - changed Brownian model</u> :

$$S_t = S_0 \exp\left(\mu\, T(t) + B_{T(t)}\right)$$

time change in BM

↳ Pragmatic attempts to tackle the problem : In particular for European call and put options, we have formulae depending on $S_t, K, r, T-t, \sigma$ so it makes sense to define some objects that stress out the effect obtained by variating the parameters :

↓

Greeks :

generally Greeks are computed at time $=0$

① Delta : partial derivative of the price formula with respect to underlying $\left(\text{ex. } \Delta = \frac{\partial C_t}{\partial S_t} \right)$

② Gamma : second partial " " " $\left(\text{ex. } \Gamma = \frac{\partial^2 C_t}{\partial S_t^2} \right)$

(if is high $\Rightarrow \Delta$ is sensitive $\Rightarrow$ you have to trade frequently)

③ Theta : $(-1) \times$ partial derivative of the pricing formula with respect to time to maturity

$\left(\text{ex. } \Theta = \frac{\partial C_t}{\partial T-t} \right)$

④ Rho : partial derivative of the pricing formula with respect to the interest rate

$\left(\text{ex. } \rho = \frac{\partial C_t}{\partial r} \right)$

⑤ Vega : partial derivative of the pricing formula with respect to volatility $\left(\text{ex. } \vartheta = \frac{\partial V}{\partial \sigma} \right)$

↑ very important

↳ Implicit volatility : we know :

$$C_0 = S_0 \, \underbrace{N\left(\frac{\log\left(\overbrace{\frac{S_0 e^{rT}}{K}}^{F} \right) + \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right)}_{\underbrace{= S_0 N\left(\frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \right)}} - K e^{-rT} \underbrace{N\left(\frac{\log\left(\overbrace{\frac{S_0 e^{rT}}{K}}^{F} \right) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right)}_{\underbrace{= S_0 N\left(\frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \right)}}$$

$\sigma \to 0$:

$F > K$: $\frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy = S_0$

$F = K$: $\frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}y^2} = \frac{S_0}{2}$

$F < K$: $\frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} " = 0$

$\sigma \to 0$:

$F > K$: Ke^{-rT}

$F = K$: $\frac{Ke^{-rT}}{2}$

$F < K$: 0

So for $\sigma \to 0$:

- if $F > K$: $C_0 = S_0 - Ke^{-rT}$ (reasonable we are "in the money" since $S_0 - Ke^{-rT} > 0$ infact by hyp. $S_0 e^{rT} - K > 0$) (without volatility)

- if $F = K$: $C_0 = \frac{1}{2}\left(S_0 - Ke^{-rT} \right) = \frac{1}{2} e^{-rT} \left(F - K \right) = 0$ (reasonable, we are "at the money")

(without volatility)

• if $F < K$: $C_0 = 0$ (reasonable we are "out of the money" without volatility).

Now :

$$C_0 = \underbrace{S_0 \, N\left(\frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \right)}_{\downarrow \sigma \to +\infty} - \underbrace{K e^{-rT} N\left(\frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \right)}_{\downarrow \sigma \to +\infty}$$

$$\frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2/2} \, dy = S_0 \qquad \frac{K e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} e^{-y^2/2} \, dy = 0$$

So for $\sigma \to +\infty$ $C_0 = S_0$.

Recall that $\max\left(S_0 - K e^{-rT}, 0\right) \le C_0 \le S_0$ so we have obtained the lower static arbitrage bound for $\sigma \to 0$ and the upper static arbitrage bound for $\sigma \to +\infty$.

Vega > 0 $\Longrightarrow$ as long as the market price of the european call is between the statistic arbitrage bounds, $\sigma \longmapsto C_0(S_0, K, t, r, \sigma)$ is strictly increasing $\Rightarrow$ admits inverse and so we define σ as the output of the inverse function; this is called <u>implicit volatility</u>.

$\downarrow$

This idea leads to "<u>the inverse problem</u>" : risk neutral model with parameters $\theta \in \Theta$ (not necessarly only σ). We want to find θ s.t. $(\tilde{S}_t)$ is a martingale ; so we choose some prices C_i with T_i time maturities and strikes K_i and we want:

$$\forall i \in I \quad C_i = e^{-rT_i} \, \mathbb{E}_{\mathbb{Q}^\theta}\left[(S_{t_i} - K_i)_+ \right]$$

Problem : no solution / infinite solutions ; numerically instable ; we can use the RSS:

$$\sum_{i=1}^{N} \underset{\substack{\nearrow \\ \text{weight}}}{\omega_i} \left| e^{-rT_i} \, \mathbb{E}_{\mathbb{Q}^\theta}\left[(S_{t_i} - K_i)_+ - C_i \right] - C_i \right|^2 + F(\mathbb{Q}^\theta, \mathbb{P}_0, \alpha)$$

↳ <u>Historic volatilities</u> : volatility computed by time series in real word

↳ <u>Historic vs implicit</u> : thanks to Girsanov we know that the volatility under the $\mathbb{P}$-probab. real world and $\mathbb{Q}$ are the same $\Longrightarrow$ we should have $\hat{\sigma} \approx \sigma_{IMP}$ but we observe $\sigma_{IMP} \gg \hat{\sigma}$.

We use implicit because we want a model where we price close to the market price.

Hp.: • $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$ filtered probability space where $\mathbb{T} = [0,T]$, $[0,+\infty)$.

• $(\mathcal{F}_t)_t$ is assumed to satisfy the usual conditions: right continuous $\left(\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s\right)$ and complete $(\mathcal{F}_0 \supseteq \{A \in \mathcal{F} \text{ s.t. } P(A) = \emptyset\})$. Information at time t.

<u>Cadlag process</u>: $(X_t)_t$ stochastic process (can be vectorial, i.e. $\mathrm{Im}\, X_t \subseteq \mathbb{R}^n$) is cadlag if the paths are right continuous and the lef limits exist finite.

<u>Stopping time</u>: $\tau: \Omega \longrightarrow [0,+\infty)$ is a stopping time if $\{\tau \le t\} \in \mathcal{F}_t$ $\forall t \ge 0$. The stopping time σ-algebra $\mathcal{F}_\tau$ is defined as: $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \ \forall t \ge 0\}$.

<u>Stopped process</u>: $(X_t)_t$ and τ stopping time. $(X_t^\tau)_t := X_{\tau \wedge t}$ $\forall t \ge 0$.

<u>Martingale</u>: $(M_t)_{t\ge0}$ is a $(\mathcal{F}_t)$-martingale if M_t is integrable $\forall t \ge 0$, $\mathcal{F}_t$-adapted and $E[M_t | \mathcal{F}_s] = M_s$ a.s. $\forall s \le t$.

<u>Local martingale</u>: $(M_t)_t$ is a local martingale if $\exists (\tau_K)_{K=1}^{+\infty}$ sequence of stopping time s.t. $\tau_K \nearrow +\infty$ a.s. and $(M_t^{\tau_K})_t$ is a martingale $\forall K$.

<u>Bounded variation process</u>: $(A_t)_{t\ge0}$ is a bounded variation process if:

$$\sup_{\Delta} \sum_{i=1}^{n} |A_{t_i} - A_{t_{i-1}}| < +\infty \text{ a.s. where } \Delta = \{0 = t_0 < t_1 < \dots < t_n = t\} \text{ is}$$

a partition of $[0,t]$, $\forall t$.

<u>Vectorial process</u>: the same definitions where each component has the property of being martingale/local martingale/bounded variation.

<u>Stochastic intervals with stopping time</u>: $\tau_1 \le \tau_2$ stopping times; $[\![\tau_1, \tau_2]\!] = \{(t,\omega) : t \in \mathbb{T}, \omega \in \Omega$ and $\tau_1(\omega) \le t \le \tau_2(\omega)\}$. The sigma algebra generated by $[\![0, \tau]\!]$ with τ stopping time is called <u>predictable σ-algebra</u> and is denoted by $\mathcal{P}$.

<u>Predictable process</u>: $(X_t)_t$ is predictable if $X: \mathbb{T} \times \Omega \longrightarrow \mathbb{R}\ (\mathbb{R}^n)$, $(t,\omega) \longmapsto X_t(\omega)$ is $\mathcal{P}$-measurable.

<u>Remark</u>: $\mathcal{P}$ is also generated by the left-continuous adapted processes.

<u>Semimartingale</u>: a n-dimensional cadlag process X is called semimartingale if it admits

a decomposition: $X_t = X_0 + A_t + M_t$, $t \geq 0$, where $(A_t)_{t\geq 0}$ is a n-dimensional adapted process of bounded variation, $(M_t)_{t\geq 0}$ is a n-dimensional local martingale and $A_0 = M_0 = 0$.

Special semimartingale : X semimartingale is special if a decomposition as above is with A predictable. If such decomposition $\exists \Rightarrow$ is ! . In particular this is the case for continuous semimartingales.

Integration for semimartingale : see in his the theoretical appendix.

σ-martingale : $X = (X_t^1, \dots, X_t^n)_{t\geq 0}$ semimartingale is a σ-martingale if there exists an $\mathbb{R}^n$-valued local martingale M and a process $H = (H_t^1, \dots, H_t^n)_{t\geq 0}$ such that for each i, H^i is a M^i-integrable ($H^i \cdot M^i$ well defined) and $X^i = X_0^i + H^i \cdot M^i$.

Remark : σ-martingale $\underset{H^i = 1 \ \forall i}{\overset{\supsetneq}{}}$ local martingale $\underset{\{\tau_n = n\}}{\overset{\supsetneq}{}}$ martingale

The market: there traded asset whose prices are represented by a semimartingale

$S = (S_t^0, \dots, S_t^n)_{t\geq 0}$ where $(S_t^0)_{t\geq 0}$ is the money market account

S is adapted with respect to the filtered space

Self-financing strategy : a self financing strategy π is a pair $\pi = (x, H)$ where $x \in \mathbb{R}$ and $H = (H_t^1, \dots, H_t^n)_t$ is a predictable process and $\tilde{S}$ integrable (i.e. $\exists$ vector stochastic integral $H \cdot \tilde{S}$ where $\tilde{S}$ stands for the discounted price processes of the risky asset:

$$\tilde{S}_t = \left(\frac{S_{t1}}{B_t}, \dots, \frac{S_{tn}}{B_t} \right)_{t\geq 0} \qquad B_t := S_t^0 > 0^{\text{hp.}})$$

Discounted capital : given $\pi = (x, H)$ strategy, the discounted capital is:

$$V_t^\pi = x + \underbrace{(H \cdot \tilde{S})_t}_{\text{is the discounted return of } \pi}, \quad t \geq 0 \qquad (1)$$

Remark: It also holds:

$$V_t^\pi = \langle H_t, \tilde{S}_t \rangle + \underbrace{H_t^0}_{\text{amount invested in } H_t^0} \qquad (2)$$

Cumulative discounted cost : $C_t := V_t^\pi - (H \cdot \tilde{S})_t$, π is not necessarily self-financing.

If π is self financing $C_t = x \ \forall x$, i.e. a self-financing is completely described by

x and H, determining H_t^0 $\forall t$ (we can determine H_t^0 from (1) and (2)).

<u>Arbitrage strategy</u>: a strategy $\pi = (x, H)$ realises arbitrage if:

1. $x = 0$ ("from nothing")
2. $\exists a \geq 0$ s.t. $\mathbb{P}(V_t^\pi \geq -a \quad \forall t \geq 0) = 1$ (admissability)
3. $\lim_{t \to \infty} V_t^\pi =: V_\infty^\pi$ $\exists$ $\mathbb{P}$-a.s.
4. $V_\infty^\pi \geq 0$ $\mathbb{P}$-a.s. ("without risk")
5. $\mathbb{P}(V_\infty^\pi > 0) > 0$ ("have gain")

A model satisfies the No-arbitrage condition (NA) if such a strategy does not exist.

<u>Free lunch with vanishing risk strategies</u>: A sequence of strategies $\pi^K = (x^K, H^K)$ $K \geq 0$, realises free lunch with vanishing risk if for all $K \geq 0$:

1. $x^K = 0$ $\forall K$;
2. $\exists a_K > 0$ s.t. $\mathbb{P}(V_t^{\pi_K} \geq -a_K \quad \forall t \geq 0) = 1$;
3. $\lim_{t \to +\infty} V_t^{\pi_K} =: V_\infty^{\pi_K}$ $\exists$ $\mathbb{P}$-a.s. $\forall K$;
4. $V_\infty^{\pi_K} \geq -\frac{1}{K}$ $\mathbb{P}$-a.s.
5. $\exists \delta_1, \delta_2 > 0$ s.t. $\mathbb{P}(V_\infty^{\pi_K} > \delta_1) > \delta_2$ (δ_1, δ_2 indipendent of K) $\forall K$.

A model satisfies No free lunch with vanishing risk (NFLVR) if such a sequence $\nexists$.
$\downarrow$ stronger than NA (taking the constant sequence)

<u>Theorem (Fundamental theorems of asset pricing 1)</u>: The following assertions are equivalent for an $\mathbb{R}^n$-valued semimart. model $\tilde{S}$ of a financial market:

1. $\exists \mathbb{Q} \sim \mathbb{P}$ such that $\tilde{S}$ is a σ-martingale under $\mathbb{Q}$;
2. $\tilde{S}$ satisfies NFLVR

<u>Remark</u>: if the components of $\tilde{S}$ are nonnegative, NFLVR is equivalent to obtain $\tilde{S}$ local martingale under $\mathbb{Q}$.

$\downarrow$

<u>Consequence for risk neutral valuation</u>: consider a finite time horizon $T > 0$ and a claim C.

$dS_t = \mu S_t \, dt + \sigma(t) S_t \, dB_t$ (*) with $\sigma(t) = e^{Y_t}$ s.t. $dY_t = (\alpha - \beta Y_t)dt + \gamma \, d\tilde{B}_t$

and $d\langle B, \tilde{B}\rangle_t = \rho \, dt$.

$f(Y) = e^Y$

$d\sigma(t) = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} d\langle Y\rangle_t =$

we assume >0

$d\langle S, \sigma\rangle_t = \sigma(t)^2 S_t \gamma \, d\langle B, \tilde{B}\rangle_t = \sigma(t)^2 S_t \gamma \rho \, dt$

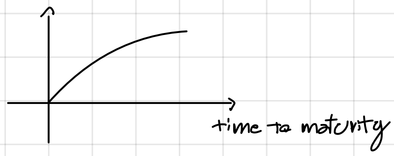
$\Rightarrow$ If I want $d\langle S,\sigma\rangle_t < 0$ we have to set $\rho < 0$ (we want a negative covariation between prices and volatility).

Other application of OU-processes:

$$dr_t = (\alpha - \beta r_t)dt + \sigma \, dW_t \quad \text{with } W_t \text{ - sBM}$$

$$S_t^0 = B_t = \exp\left(\int_0^t r_s \, ds\right)$$

Vasiček - model yield



Or, more generally for K, θ, σ bounded Borel functions:

$$dr_t = K_t(\theta_t - r_t)dt + \sigma_t \, dW_t$$

Hull-White model (K positive)

<u>Valuation with stochastich r_t</u>

Let X be a T claim.

$$B_t = S_t^0 = \exp\left(\int_0^t r_s \, ds\right) \quad \text{with} \quad \mathbb{E}_Q\left[\frac{|X|}{S_T^0}\right] < +\infty$$

The most important claim example is $X = 1$ - zerobond

Price $P(t,T) = S_t^0 \, \mathbb{E}_Q\left[\frac{1}{S_T^0} \middle| \mathcal{F}_t\right] = \mathbb{E}_Q\left[\exp\left(-\int_t^T r_s \, ds\right) \middle| \mathcal{F}_t\right]$

For X more complex $P_t = S_t^0 \, \mathbb{E}_Q\left[\frac{X}{S_T^0} \middle| \mathcal{F}_t\right]$ ← two random variable $\Rightarrow$ complex. It makes no sense to suppose X or S_T^0 indipendent, we have to do something else:

$$\frac{dQ^T}{dQ} = \frac{\overbrace{P(T,T)}^{=1}}{P(0,T) S_T^0}$$, assumed to be Q integrable

$\left(\frac{1}{P(0,T)} \frac{P(t,T)}{S_t^0}\right)_t$ is a true martingale w.r.t. Q (assumption)

For $t \leq T$:

$$Z_t = \frac{dQ^T}{dQ}\Big|_{\mathcal{F}_t} = \underbrace{\mathbb{E}_Q\left[\frac{dQ^T}{dQ} \middle| \mathcal{F}_t\right]}$$ ← $\left(\mathbb{E}_Q\left[\frac{dQ^T}{dQ} \middle| \mathcal{F}_t\right]\right)_t$ is a martingale w.r.t. Q by tower property

$$\overset{\text{assumption (true mart.)}}{=} \frac{P(t,T)}{P(0,T)S_t^0} \qquad t \le T$$

Thus, $P_t = S_t^0 \, \mathbb{E}_Q\left[\frac{X}{S_T^0} \,|\, \mathcal{F}_t\right] \underset{\uparrow \text{change of measure}}{=} S_t^0 \, \mathbb{E}_{Q^T}\left[\frac{X}{S_T^0}\frac{1}{Z_T} Z_t \,|\, \mathcal{F}_t\right] =$

$$= \cancel{S_t^0} \, \mathbb{E}_Q\left[\frac{X}{S_T^0} \frac{P(t,T)}{\cancel{P(0,T)}\cancel{S_t^0}} \underbrace{\frac{\cancel{P(0,T)}\cancel{S_T^0}}{P(T,T)}}_{=1} \,|\, \mathcal{F}_t\right] \underset{\uparrow P(t,T) \text{ measurable}}{=}$$

$$= P(t,T) \, \mathbb{E}_{Q^T}\left[X \,|\, \mathcal{F}_t\right] \longleftrightarrow \text{Remark: is } = P(t,T) \, \mathbb{E}_{Q^T}\left[\frac{X}{\underbrace{P(T,T)}_{=1}} \,|\, \mathcal{F}_t\right] \Rightarrow \text{is}$$

two separate random variable   a change of numeraire!

Remark: with the measure also the distribution of r.v. processes are changing.

Q^T is called "former measure"

Extra remarks on Levy Processes

⓪ Consider a cadlag processes, $L_0 = 0$

① Indipendent increments : $X \in PII$ (processes indip. increments) if $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$ $\forall$ $0 \le s < t < +\infty$

⚠ $X \in PII \not\Rightarrow X$ semimartingale but $X = D + S$
(D: deterministic process (maybe of infinite variation); S: semimartingale)

② Stationary increments: the distribution of $X_t - X_s$ depends only on $t-s$ and therefore

$$\mathcal{L}aw\left(X_{t+h} - X_{s+h}\right) = \mathcal{L}aw\left(X_t - X_s\right) \quad \forall h > 0$$

③ $\forall \varepsilon > 0 \quad \lim_{s \to t} P\left(|L_s - L_t| > \varepsilon\right) = 0$ (i.e. continuous in probability)

Note (L_t) - Lévy, $S_t = S_0 e^{L_t}$ and assume $\mathbb{E}_Q\left[|f(S_t)|\right] < +\infty$, then

$$e^{-r(T-t)} \mathbb{E}_Q\left[f(S_0 e^{L_T}) \,|\, \mathcal{F}_t\right] = e^{-r(T-t)} \mathbb{E}_Q\left[f(\underbrace{S_0 e^{L_t}}_{S_t} \underbrace{e^{L_T - L_t}}_{\text{indipendent}}) \,|\, \mathcal{F}_t\right] = \varphi(S_t)$$

where $\varphi(s) = e^{-r(T-t)} \mathbb{E}_Q\left[f(s\, e^{L_T - L_t})\right] \quad \forall s > 0$

Hence, a framework with "good valuation properties".

Decompositions: $(X_t)_{t \ge 0}$ semimartingale with starting point at zero.

$$\underbrace{X_t - \underbrace{\sum_{s \le t} \Delta X_s \mathbb{1}_{\{|\Delta X_s| > 1\}}}_{\text{subtract the big jump}}}_{} \quad \text{process with bounded jumps}$$

is not obvious ← special semimartingale ⇒ unique decomposition into a local martingale + predictable process of bounded variation

$$\Rightarrow X_t = \underbrace{A_t + M_t}_{\text{depend on the truncation}} + \underbrace{\sum_{s\le t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}}}_{\text{depends on the truncation}}$$

(truncation function here $h(x) = x \mathbb{1}_{\{|x|\le 1\}} \Rightarrow (x - h(x)) = x \mathbb{1}_{\{|x|>1\}}$)

$M_t = M_t^c + M_t^d$ ~ M_t^d purely discontinuous i.e. $M^d N$ local martingale $\forall N$ continuous local martingales

M_t^c - continuous

Remark: it turns out that the continuous martingale component M^c of a semimartingale does not depend on the function h.

Lévy-case: · $A_t = bt$ and $N_c^t = \sqrt{c}\, W_t$, (W_t)-sBM

· μ - random measure of jumps of the Lévy process $L = (L_t)_{t\ge 0}$

$\mu(\omega; (0,t]\times A)$ counts how many jumps of size within A occure for path ω from 0 to t.

· $\nu((0,t]\times A) = t \underbrace{F(A)}_{\text{Levy-process}} = \mathbb{E}[\mu(\cdot, (0,t]\times A)]$

$F(A) = \mathbb{E}[\# L_t \in [0,1] \cdot \Delta L_t \neq 0,\ \Delta L_t \in A\}]$ counts the expected number per unite time of jumps where size belongs to A. Does not depend on h.

M^d purely discontinuous (local) martingale:

$$M_t^d = \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x|\le 1\}} (\underbrace{\mu(ds, dx)}_{\mu} - \underbrace{ds\, F(dx)}_{\nu})$$

Then: $$L_t = bt + \sqrt{c}\, W_t + \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x|\le 1\}} (\mu(ds, dx) - ds\, F(dx)) + \underbrace{\int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x|>1\}} \mu(ds, x)}_{\text{big jumps}}$$

The four term on the decomposition are indipendent

$F(dx)$ do not depend on the truncation

· $M_t^d = \lim_{\varepsilon \downarrow 0} \left(\sum_{s\le t} \Delta L_s \mathbb{1}_{\{\varepsilon \le |\Delta L_s| \le 1\}} - t \int x \mathbb{1}_{\{\varepsilon \le |x| \le 1\}} F(dx) \right)$ (convergence is almost surely and uniform in t on $[0,T]$)

· $\int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x|>1\}} \mu(ds, dx) = \sum_{s\le t} \Delta L_s \mathbb{1}_{\{|\Delta L_s|>1\}}$

· Ha finito il capitolo 6 31 Maggio

· Per l'esame:

- manderò una mail con la time-table e la stanza
- 20-25 minuti, orale
- si fa su un foglio (se uno preferisce alla lavagna)
- local question?

Overview:

Starting point: · Underlying assets

· Stochastic processes

Definitions: · Call, Put; European vs American

· Forward (future contract)

· Barrier options

· Contingent claims ($\mathcal{F}_T/\mathcal{F}_N$ measurable, nonnegative, suitably integrable)

discrete time ↑ always inte

continuous: ↗ squared-int.

Main tasks discussed in this course:

- pricing
- hedging

Focus will be on the theoretical basis

Key result: Translation of the economical property of "absence of arbitrage opportunity" ⟺ existence of an equivalent martingale measure:

① Discrete time, finite probability space:

- d-risky assets
- 1-riskless bond
- Strategies: predictable processes (sequences, "$\mathcal{F}_{n-1}$" measurable)
- Value process $V(n) = \sum_{i=0}^{d} \phi_n^i S_n^i = \phi_n \cdot S_n$ or (for the arbitrage theory of finite spaces) equivalently: $\tilde{V}_n(\phi) = \frac{1}{S_n^0} \phi_n \cdot S_n = \phi_n \cdot \tilde{S}_n$
- Self-financing: Equivalent definitions (only initial costs, no investment and no consumption before maturity)
- Admissable strategies: value of the portfolio must remain nonnegative and strategy self-financing
- Arbitrage: admissable strategy with $V_0 = 0$ and $P(V_N > 0) > 0$.

- FTAP 1: Viable (free of arbitrage) $\Leftrightarrow \exists \mathbb{Q}$ equivalent to $\mathbb{P}$ st. $\tilde{S}$ is a $\mathbb{Q}$-martingale, i.e. $\tilde{S}_n$'s are $\mathbb{Q}$-martingales
  - Tools: . Results from discrete-time martingales (all proved)
    - Convex separation theorem (Hahn-Banach linearly version) (not proved)

<u>Proof</u>: $\Leftarrow$: ← important one
play with martingale property and the admissibility prop.

$\Rightarrow$: Construction of an equivalen measure: Hahn Banach

- Complete markets: every contingent claim is attenaible $\rightsquigarrow \exists$ an admissible strategy worth H at N

- FTAP 2: $\boxed{\text{A viable market}}$ ← it does not make sense to speak about! if there is no $\exists$. Complete $\Leftrightarrow \exists!$ equivalent martingale measure

Proofs: $\Rightarrow$ relatively easy

$\Leftarrow$ relatively difficult (viable and incomplete $\Rightarrow$ we can construct another different martingale measure)

Note: For pricing is relevant $\mathbb{Q}$ not $\mathbb{P}$

- Pricing and hedging contingent claims in complete markets: no arbitrage arguments can be applied very directly (as B&S)

- H-claim:

$$V_n = S_n^0 \; E^{\mathbb{Q}}\left[\frac{H}{S_N^0} \,|\, \mathcal{F}_n\right]$$ with V_n value of the replicating strategy in complete markets

- Based on the FTAP I this so-called "risk neutral valuation" provides an arbitrage-free rule for all eq. martingale measures in an arbitrage free and incomplete market. (tower property)

- When choosing the specific martingale measure, practitioners often rely on calibration on market data

- Arbitrage free - Incomplete markets: all martingales measures deliver the same value for <u>attenaible</u> claims.

- Slides s: analogous results in the "most general setting" in the classical understanding. Focus will be on Slides 1 and not "Slides s"

- We have had many exercises related to discrete time models: before some easy general results: Forward pricing, European put-call parity, American options; CRR-model (binomial model) - complete model; trinomial-model (as an example of an incomplete model).

- American options:
  - short seller point of view $\rightsquigarrow$ backward induction

$$V_{n-1} = \max\left(Z_{n-1},\ S^0_{n-1}\, \mathbb{E}_Q\left[\frac{V_n}{S^0_n} \,\Big|\, \mathcal{F}_n\right]\right) \quad ⚠ \neq Z_n$$

Z_{n-1}: to pay if exercise; $S^0_{n-1}\mathbb{E}_Q[\dots]$: value to be replicated if the long position does not exercise

  - $\tilde{U}_n$ is the smallest Q-supermartingale that dominates $\tilde{Z}_n$
  - How does that fit into the theory of pricing by excluding arbitrage. Answer: by optional stopping.
  - $\nu = \inf\{n \geq 0 : \tilde{U}_n = \tilde{Z}_n\}$ stopping time where $(\tilde{U}_{n\wedge\nu})$-martingale
  - $U_0 = \mathbb{E}_Q[\tilde{Z}_\nu] = \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}_Q[\tilde{Z}_\tau]$ hence ν is optimal

    It is also the smallest optimal stopping time
  - Doob decomp $\Rightarrow \tilde{U}_n = \tilde{M}_n - \tilde{A}_n$

    $\tilde{M}_n$ mart; $\tilde{A}_n$ non decreasing, predictable, $A_0 = 0$
  - Largest optimal stopping time: $\nu_{max} = \begin{cases} N, & \text{if } \tilde{A}_N = 0 \\ \inf\{n : \tilde{A}_{n+1} \neq 0\}, & \text{if } \tilde{A}_N \neq 0 \end{cases}$

    furthermore: $\left.\begin{array}{l} \tilde{U}^\nu_n \text{ -martingale} \\ \tilde{U}_\nu = \tilde{Z}_\nu \end{array}\right\}$ is a charac. of the optimal stopping time
  - Hence, if the long position exercises in an optimal way, everything suits fine in to theory of absence of arbitrage
  - More applied exercises: pricing on a binomial tree by backward induction; early ex. call with $r > 0$ and no dividends (makes no sense); early ex. of puts: why this could make sense; Doob decomp.

- Black-Scholes:
  - Model: $dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$ SDE (B_t) BM under P

    $dS^0_t = r S^0_t\, dt$ deterministic saving accounts
  - For given S_0, $S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$ S_t is the unique

strong (solution) of SDE

- Strategies in this setting are adapted
- Self financing: again analogue definitions, the meaning remains, e.g.

$$dV_t(\phi) = H_t^0\, dS_t^0 + H_t\, dS_t \qquad \left(\int_0^T |H_t^0|\, dt < \infty \text{ a.s. } \int_0^T H_t^2\, dt < +\infty \right)$$

- In this framework: admissable strategies are adapted, self-financing with $\tilde{V}_t(\phi)$ being non negative for all t and such that $\mathbb{E}\left[\int_0^T H_t^2 (\tilde{S}_t)^2 dt\right] < +\infty$

$\phi = (H_t^0, H_t)$

- If h is $\mathcal{F}_T$-measurable, nonnegative and square integrable under $\mathbb{Q}$

$$\Rightarrow \boxed{V_t = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}}[h \mid \mathcal{F}_t]} \quad (1).$$

Proof: we show the market is complete: girsanov tells us that $\exists \mathbb{Q}$ being an equivalent messure, then we have the Martingale representation theorem

$\Rightarrow$ exists self financing strategy (replicating h), so with no arbitrage $\Rightarrow$ (1) follows.

- Hence, we can price claims: example for European calls, barrier options (S7, Ex. 4), sometimes there are possible abbreviation based known results: European put via European Put-Call parity, on S8 Ex 3.

- Hedging strategy can be concretely derived: $H_t = \frac{\partial f}{\partial x}(t, S_t)$

$H_t^0 = e^{-rT}\, \mathbb{E}_{\mathbb{Q}}[h \mid \mathcal{F}_t] - H_t \tilde{S}_t$

- Hence, closed formulas have some advantages deriving the hedge. They are also useful for calculation slides 4 (S.8.)

- Sheet 7: reflection principle: important for the application, e.g. barrier options and model extension (not part of the exam)

Model extension: • explanation how to construct concret models ($\sigma \leadsto \sigma_t$ Merton models, $\sigma \leadsto \sigma(t, S_t)$ Dupire it is also deterministic) (complete market models)

- There are also more complex integral trasformations in particular stochastic volatility

$$S_t = S_0 \exp\left(\int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s \right) \quad \text{or time change} \quad S_t = S_0 \exp\left(\mu T_t + B_{T_t} \right)$$

time change

<u>Levy process</u>: · definition

· Why practically in valuation: same trick as in B&S with expected value

· Why are independent time changes often analytically particularly useful.

· Ex 1,2 Sheet 9

Theoretical Appendix

metric space

- Indicate a stochastic process with $\xi(t,\omega)$. For each fixed ω the function $\xi(\cdot,\omega): T \longrightarrow S$ is a realisation of the stochastic process (a path). So $\xi(\cdot,\omega) \in S^T$. From now $S = \mathbb{R}, \mathbb{R}^d$.

In order to generate a σ-algebra on S^T we may use evaluations maps:

$$\pi_t : S^T \longrightarrow S \qquad x \longmapsto \pi_t(x) = x(t), \ t \in T$$

and set $\mathcal{E} := \sigma(\pi_t, t \in T)$ - cylinderial σ-algebra is the minimal σ-algebra which makes all evaluation maps π_t measurable. So:

$$\begin{tikzcd}
\Omega \arrow[r, "\xi(\cdot{,}\omega)"] \arrow[rr, bend left, ":= \xi_t"] & S^T \arrow[r, "\pi_t"] & S \\
\omega \arrow[r, mapsto] & \xi(\cdot,\omega) \arrow[r, mapsto] & \xi(t,\omega)
\end{tikzcd}$$

Some technical problems can be settled by requiring that all values ξ_t can be explored by using only a countable dense set of time moments (<u>separable process</u>)

<u>Fact</u>: Every stochastic process has an equivalent separable version.

<u>Prop.</u>: Let ξ and η be two stochastic processes. Then $\xi \overset{\mathcal{D}}{=} \eta \iff$ they're modification of each other.

push-forward

<u>Gaussian process</u>: ξ is a gaussian process if $(\xi_{t_1}, \dots, \xi_{t_n})$ have a gaussian distribution with mean $(m(t_1), \dots, m(t_n))$ $\left(\begin{array}{l} m: T \longrightarrow \mathbb{R} \\ \quad t \longmapsto \mathbb{E}[\xi_t] \end{array} \right)$ and covariance matrix

$$\left(K(t_j, t_k) \right)_{j,k=1}^{n} \qquad \left(\begin{array}{l} K: T \times T \longrightarrow \mathbb{R} \\ (t_1, t_2) \longmapsto Cov(X_{t_1}, X_{t_2}) \end{array} \right).$$

<u>Prop.</u>: $(\xi_{t_1}, \dots, \xi_{t_n})$ is jointly gaussian $\iff \sum_{i=1}^{n} \lambda_i \xi_{t_i}$ is a normally distributed r.v. $\forall t_1, \dots, t_n \in T$.

- <u>Matching Coefficients</u>: we want to find the solution of SDE.

Fore example:

(1) $dX_t = \mu X_t dt + \sigma X_t dB_t$, $X_0 = x_0$. We suppose that the solution is of the form: $X_t = f(t, B_t)$.

So by Itô:

$$dX_t = \frac{\partial f}{\partial x}(t,B_t)\, dB_t + \frac{\partial f}{\partial t}(t,B_t)\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)\, dt =$$

$$= \left(\frac{\partial f}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)\right) dt + \frac{\partial f}{\partial x}(t,B_t)\, dB_t$$

So, matching coefficients:

$$\begin{cases} \mu f(t,B_t) = \frac{\partial f}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t) \\ \sigma f(t,B_t) = \frac{\partial f}{\partial x}(t,B_t) \end{cases}$$

$\Rightarrow$ $f(t,B_t) \overset{\text{a solution}}{=} \exp(\sigma B_t + g(t))$ $\Rightarrow$ putting in first equation we have:

$$\mu \cancel{\exp(\sigma B_t + g(t))} = \cancel{\exp(\sigma B_t + g(t))} \frac{\partial g}{\partial t}(t) + \frac{1}{2}\sigma^2 \cancel{\exp(\sigma B_t + g(t))}$$

$$\Rightarrow \mu - \frac{1}{2}\sigma^2 = \frac{\partial g}{\partial t}(t) \Rightarrow g(t) = \left(\mu - \frac{1}{2}\sigma^2\right)t + c \Rightarrow f(t,B_t) = \overset{\text{initial condition}}{x_0} \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right)$$

(2) Ornstein-Uhlenbeck: $dX_t = -cX_t dt + \sigma dB_t$. The solution is not of the form $f(t,B_t)$, we have to consider a larger set, so we consider the Gaussian processes:

$$X_t = a(t)\left(x_0 + \int_0^t b(s)\, dB_s\right)$$ (product of a Gaussian process and deterministic function is Gaussian proc)

So, by Itô product:

$$dX_t = \frac{\partial a}{\partial t}(t) \underbrace{\left(x_0 + \int_0^t b(s)\, dB_s\right)}_{X_t / a(t)} + a(t)\, b(t)\, dB_t + 0$$

Assume $a(0) = 1$ $a(t) > 0$ $\forall t$.

So:

$$\begin{cases} \frac{\partial a(t)}{\partial t} \cdot \frac{1}{a(t)} = -c \Rightarrow a(t) = \exp(-ct) \quad (a(0) = 1) \\ a(t)\, b(t) = \sigma \Rightarrow b(t) = \sigma \exp(ct) \end{cases}$$

So $$X_t = \exp(-ct)\left(x_0 + \int_0^t \sigma \exp(cs)\, dB_s\right)$$