Second Cleo Integral

On Feb 16, 2015 at 6:51 pm, famous Math Stack Exchange user, Cleo, posted the following result on the message board.

$$\int_0^\infty \arctan\left(\frac{2\pi}{x-\ln(x)+\ln\left(\frac{\pi}{2}\right)}\right)\frac{\mathrm{d}x}{x+1} = \pi\,\ln\left(\frac{1+\pi^2+\ln^2\left(\frac{\pi}{2}\right)-2\ln\left(\frac{\pi}{2}\right)}{1+\frac{\pi^2}{4}}\right)$$

In these pages we seek to rederrive this result.

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1 Setting Up The Integral

While we may attempt the integral in its current form, the nested natural logarithms make integration exceedingly difficult to approach directly (Note: complex arctangent involves natural logarithms). Therefore, it may be beneficial to find an equivalent integral that does not involve nested natural logarithms.

$$\boxed{\text{Let}}\qquad I(\xi) \triangleq \int_0^\infty \arctan\left(\frac{2\pi}{x-\ln(x)+\ln\left(\frac{\pi}{2}\right)}\right)\frac{\mathrm{d}x}{\xi x+1}$$

To find an alternative form to I we may turn to our arsenal of elementary integration techniques. A single application of integration by parts yeilds especially useful for eliminating nested natural logarithmic terms.

$$I(\xi) = \frac{1}{\xi}\int_0^\infty \left(\frac{2\pi(x-1)}{\left(x-\ln(x)+\ln\left(\frac{\pi}{2}\right)\right)^2+(2\pi)^2}\cdot\frac{\ln|\xi x+1|}{x}\right)\mathrm{d}x$$

Feynman's Trick may also be applied to eliminate the right-most logarithmic term, created by the integration by parts. In this step, It's important to note that we will remove the $1/\xi$ coefficient from the beginning of the integral to simplify this process. Later, after reintegrating with respect to ξ, we shall divide our solution by ξ.

$$\boxed{\text{Let}}\qquad \mathcal{I}(\xi) \triangleq \int_0^\infty \left(\frac{2\pi(x-1)}{\left(x-\ln(x)+\ln\left(\frac{\pi}{2}\right)\right)^2+(2\pi)^2}\cdot\frac{\ln|\xi x+1|}{x}\right)\mathrm{d}x$$

Regardless of this fact, we will continue to solve for $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$ as the branch points of the integrand are all at zero.

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\xi} = \int_0^\infty \left(\frac{2\pi(x-1)}{\left(x-\ln(x)+\ln\left(\frac{\pi}{2}\right)\right)^2+(2\pi)^2}\;\frac{\partial}{\partial\xi}\frac{\ln|\xi x+1|}{x}\right)\mathrm{d}x$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\xi} = \int_0^\infty \left(\frac{2\pi}{\left(x-\ln(x)+\ln\left(\frac{\pi}{2}\right)\right)^2+(2\pi)^2}\frac{x-1}{\xi x+1}\right)\mathrm{d}x$$

As we proceed with the integration, it is becoming increasingly apparent that we will use a keyhole contour to evaluate the integral. To determine the direction for the branch cut, the choice of the integrand, and to simplify the calculation of residues, we must first separate the integrand into a linear combination of functions using partial fraction decomposition. To prepare for this, consider the following substitution.

$$\boxed{\text{Let}} \qquad u\left(x\right) \triangleq x - \ln\left(x\right) + \ln\left(\frac{\pi}{2}\right), \quad Q\left(u\right) \triangleq \frac{2\pi}{u^2 + \left(2\pi\right)^2}$$

Immidiately following from these definitions,

$$Q(u) = \frac{2\pi}{\left(u + 2\pi i\right)\left(u - 2\pi i\right)} = \frac{A}{u + 2\pi i} + \frac{B}{u - 2\pi i}.$$

Applying the residue trick allows us to swiftly compute the partial fraction coefficients.

$$A = \operatorname*{Res}_{u=-2\pi i} Q\left(u\right) = \frac{i}{2}$$
$$B = \operatorname*{Res}_{u=2\pi i} Q\left(u\right) = -\frac{i}{2}$$

Finally, applying these findings to the integrand from earlier, we derive the following equivalence:

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\xi} = \frac{i}{2}\int_0^{\infty} \frac{x-1}{\xi x + 1}\left(\frac{1}{x - \ln\left(x\right) + \ln\left(\frac{\pi}{2}\right) + 2\pi i} - \frac{1}{x - \ln\left(x\right) + \ln\left(\frac{\pi}{2}\right) - 2\pi i}\right)\mathrm{d}x \tag{1}$$

Our goal in applying contour integration is to re-derive the integrand for $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$. Consider a keyhole contour deformed along $[0, \infty)$, where $\arg\left(x\right) \in \left[0, 2\pi\right]$. Integration of the integrand of $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$ along the keyhole contour will quickly devolve into a new, equally challenging integral. The complication occurs along the integral below the branch cut where $\arg\left(x\right) = 2\pi$. After adding the integral above and below the branch cut, the extra term created from the $\arg(x)$ term—created from the complex logarithm—forms an integral with $\pm 4\pi i$ rather than $\pm 2\pi i$. Therefore, one may reason to reform the integrand using a linear combination of the integrals above and below the branch cut.

Although reforming the integrand in this fashion may initially seem challenging, we may achieve it by removing the $+2\pi i$ term from the denominator of the integrand and flipping the sign on either term. This approach works because the $\arg(x)$ will cause the terms with coefficients of $\left(x - \ln\left(x\right) + \ln\left(\frac{\pi}{2}\right)\right)^{-1}$ to cancel out, leaving only the integrand of $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$.

We conclude this section by defining a new function as the ideal integrand for solving the original integral.

$$\boxed{\text{Let}} \qquad f\left(z, \xi\right) = \frac{i}{2}\frac{z-1}{\xi z + 1}\left(\frac{1}{z - \ln\left(z\right) + \ln\left(\frac{\pi}{2}\right)} + \frac{1}{z - \ln\left(z\right) + \ln\left(\frac{\pi}{2}\right) + 2\pi i}\right)$$

2 Contour Integration

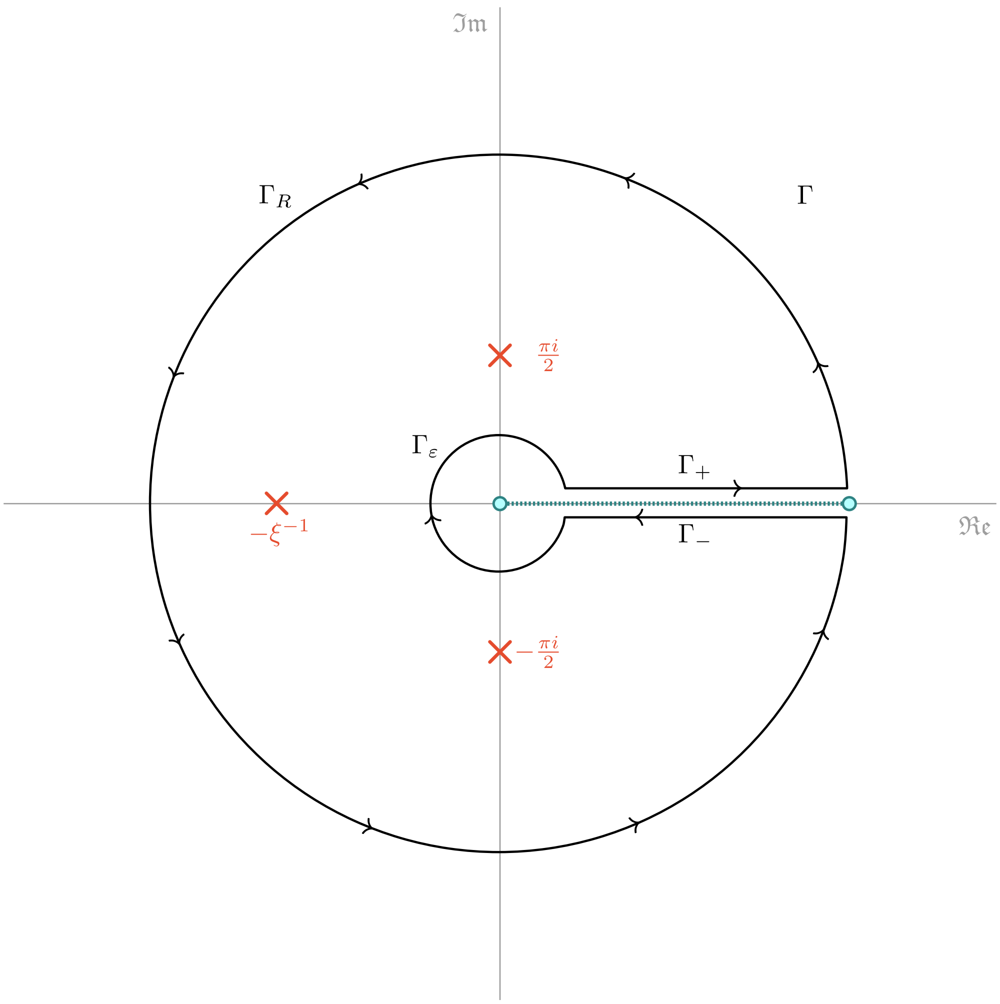
Before conducting any integration, we identify the poles of f.

$$\boxed{\text{Let}} \qquad \omega = \text{poles of } f$$

The first pole is trivially found via algebra ($\xi\omega_0 + 1 = 0 \implies \omega_0 = -\xi^{-1}$). The ladder two are particular values of the Lambert-W function.

$$\begin{aligned} z - \ln(z) &= -\ln\left(\frac{\pi}{2}\right) \\ z\mathrm{e}^{-z} &= \frac{\pi}{2} \\ \implies \omega_0 &= -W_0\left(-\frac{\pi}{2}\right) = -\frac{i\pi}{2} \\ \implies \omega_1 &= -W_{-1}\left(-\frac{\pi}{2}\right) = \frac{i\pi}{2} \end{aligned}$$

Now, consider the following contour, Γ, over the function f with a branch cut along $[0,\infty)$ and $\arg(z) \in [0, 2\pi]$.



2.1 Residues

Before parameterizing the contour, we must compute the sum of all residues—a necessary step for applying the Cauchy Residue Theorem. Computing the residues of f is surprisingly straightforward.

$$\begin{aligned}\operatorname*{Res}_{z=-1/\xi} f(z,\xi) &= -\lim_{z\to-1/\xi}\frac{i}{2}\left(\left(z+\frac{1}{\xi}\right)\left(\frac{z-1}{\xi z+1}\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\right)\right)\\ &= -\lim_{z\to-1/\xi}\frac{i}{2}\left(\frac{\xi z+1}{\xi}\frac{z-1}{\xi z+1}\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\right)\end{aligned}$$

$$\boxed{\operatorname*{Res}_{z=-1/\xi} f(z,\xi) = -\frac{\xi+1}{2\xi}\frac{2\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)}{\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)^2+(\xi\pi)^2}+\frac{\pi^2\xi}{\pi^2\xi^2+4}}$$

$$\begin{aligned}\operatorname*{Res}_{z=\pi i/2} f(z,\xi) &= \lim_{z\to\pi i/2}\left(\frac{i}{2}\frac{z-1}{\xi z+1}\left(z-\frac{\pi i}{2}\right)\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\right)\\ &= \frac{i}{2}\frac{\pi i-2}{\xi\pi i+2}\lim_{z\to\pi i/2}\frac{z-\pi i/2}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)}\\ &= \frac{i}{2}\frac{\pi i-2}{\xi\pi i+2}\lim_{z\to\pi i/2}\frac{\frac{\mathrm{d}}{\mathrm{d}z}(z-\pi i/2)}{\frac{\mathrm{d}}{\mathrm{d}z}\left(z-\ln(z)+\ln\left(\frac{\pi}{2}\right)\right)}\\ &= \frac{i}{2}\frac{\pi i-2}{\xi\pi i+2}\frac{1}{1+\frac{2i}{\pi}}\end{aligned}$$

$$\boxed{\operatorname*{Res}_{z=\pi i/2} f(z,\xi) = \frac{i\pi}{2\pi\xi-4i}}$$

$$\begin{aligned}\operatorname*{Res}_{z=-\pi i/2} f(z,\xi) &= \lim_{z\to-\pi i/2}\left(\frac{i}{2}\frac{z-1}{\xi z+1}\left(z+\frac{\pi i}{2}\right)\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)-2\pi i}\right)\right)\\ &= \frac{i}{2}\frac{2+i\pi}{i\pi\xi-2}\lim_{z\to-\pi i/2}\left(\left(z+\frac{\pi i}{2}\right)\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)}+\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)-2\pi i}\right)\right)\\ &= \frac{i}{2}\frac{2+i\pi}{i\pi\xi-2}\frac{1}{1-\frac{2i}{\pi}}\end{aligned}$$

$$\boxed{\operatorname*{Res}_{z=-\pi i/2} f(z,\xi) = \frac{i\pi}{2\pi\xi+4i}}$$

With each residue computed, we may find the sum of residues in preparation for utilizing the Residue Theorem.

Let

$$\sigma \triangleq \sum_{z_0\in\omega}\operatorname*{Res}_{z=z_0} f(z,\xi)$$

$$\sigma = -\frac{\xi+1}{2\xi}\frac{2\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)}{\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)^2+(\xi\pi)^2}+\frac{\pi^2\xi}{\pi^2\xi^2+4}$$

2.2 Parametrization

To avoid repeating recurrent mathematical syntax, we will use shorthand notation.

$$\oint_C = \oint_C f(z,\xi)\,\mathrm{d}z$$

We begin integrating by parameterizing the closed integral over Γ.

$$\oint_\Gamma = \int_{\Gamma_\varepsilon} + \int_{\Gamma_R} + \int_{\Gamma_+} + \int_{\Gamma_-}$$

Applying Cauchy's Residue Theorem to the left-hand side, we find the following equality.

$$2\pi i\sigma - \int_{\Gamma_R} = \int_{\Gamma_+} + \int_{\Gamma_-} + \int_{\Gamma_\varepsilon}$$

From the previous section, we discussed how we expect the contour integral over Γ_+ and Γ_- to reform the integrand for $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$. If we would like to focus on finding unknowns, we must evaluate the integral along Γ_R and Γ_ε .

2.2.1 Analysis of Γ_R

In many keyhole integrals, the large contour encompassing the entire complex plane tends to be zero. However, when conducting an ML proof on Γ_R, one may realize this is not the case with Γ_R.

To prepare for direct antidifferentiation, we begin by conducting a change of variables.

$$\boxed{\text{Let}} \qquad \begin{aligned} z &= R\mathrm{e}^{i\theta} \\ \mathrm{d}z &= Ri\mathrm{e}^{i\theta}\,\mathrm{d}\theta \end{aligned}$$

$$\int_{\Gamma_R} f(z,\xi)\,\mathrm{d}z = \frac{i}{2}\int_0^{2\pi} \frac{R\mathrm{e}^{i\theta}-1}{\xi R\mathrm{e}^{i\theta}+1}\left(\frac{Ri\mathrm{e}^{i\theta}}{R\mathrm{e}^{i\theta}-\ln\left(R\mathrm{e}^{i\theta}\right)+\ln\left(\frac{\pi}{2}\right)} + \frac{Ri\mathrm{e}^{i\theta}}{R\mathrm{e}^{i\theta}-\ln\left(R\mathrm{e}^{i\theta}\right)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\mathrm{d}\theta$$

Though the integral looks daunting, the behavior of the integrand asymptotes as $R\to\infty$—making integration far simpler.

$$\begin{aligned} \int_{\Gamma_R} f(z,\xi)\,\mathrm{d}z &= \lim_{R\to\infty}\frac{i}{2}\int_0^{2\pi} f^*(\theta,R,\xi)\,\mathrm{d}\theta \\ &= \frac{i}{2}\int_0^{2\pi}\lim_{R\to\infty} f^*(\theta,R,\xi)\,\mathrm{d}\theta \\ &= \frac{i}{2}\int_0^{2\pi}\frac{1}{\xi}(i+i)\,\mathrm{d}\theta \\ \int_{\Gamma_R} f(z,\xi)\,\mathrm{d}z &= -\frac{2\pi}{\xi} \end{aligned}$$

2.2.2 Analysis of Γ_ε

Though the integral long Γ_R was not zero, the same is not true for the integral along Γ_ε. In this section, we will show that

$$\int_{\Gamma_\varepsilon} f(z,\xi)\ \mathrm{d}z = 0.$$

Proof. The integral of f along the contour Γ_ε is zero.

By the estimation lemma

$$\begin{aligned}
\left|\int_{\Gamma_\varepsilon} f(z,\xi)\ \mathrm{d}z\right| &\le \sup_{z\in\Gamma_\varepsilon} |f(z)| \cdot \int_{\Gamma_\varepsilon} |\mathrm{d}z| \\
\left|\int_{\Gamma_\varepsilon} f(z,\xi)\ \mathrm{d}z\right| &\le 2\pi\varepsilon \sup_{z\in\Gamma_\varepsilon} |f(z)| \qquad (2)
\end{aligned}$$

We may analyze $\sup_{z\in\Gamma_\varepsilon} |f(z)|$ indevidually.

$$\begin{aligned}
\sup_{z\in\Gamma_\varepsilon} |f(z,\xi)| &\le \frac{1}{2} \sup_{z\in\Gamma_\varepsilon} \left|\frac{\varepsilon \mathrm{e}^{i\theta}-1}{\xi\varepsilon \mathrm{e}^{i\theta}+1}\right| \cdot \sup_{z\in\Gamma_\varepsilon} \left|\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)} + \frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right| \\
&\le \sup_{z\in\Gamma_\varepsilon} \left|\frac{\varepsilon \mathrm{e}^{i\theta}-1}{\xi\varepsilon \mathrm{e}^{i\theta}+1}\right| \cdot \sup_{z\in\Gamma_\varepsilon} \left|\frac{1}{\varepsilon \mathrm{e}^{i\theta} - \ln|\varepsilon \mathrm{e}^{i\theta}| - i\theta + \ln\left(\frac{\pi}{2}\right)}\right| \\
&\le \left|\frac{\varepsilon+1}{\xi\varepsilon+1}\right| \cdot \sup_{z\in\Gamma_\varepsilon} \frac{1}{\left|\varepsilon \mathrm{e}^{i\theta} - \ln|\varepsilon \mathrm{e}^{i\theta}| - i\theta + \ln\left(\frac{\pi}{2}\right)\right|} \\
\sup_{z\in\Gamma_\varepsilon} |f(z,\xi)| &\le \left|\frac{\varepsilon+1}{\xi\varepsilon+1}\right| \cdot \frac{1}{\ln|\varepsilon| - \varepsilon - 2\pi + \ln\left(\frac{\pi}{2}\right)} \qquad (3)
\end{aligned}$$

Combining (3) and (2) we may find the upper bound of the original integral.

$$\left|\int_{\Gamma_\varepsilon} f(z,\xi)\ \mathrm{d}z\right| \le \left|\frac{\varepsilon+1}{\xi\varepsilon+1}\right| \cdot \frac{2\pi\varepsilon}{\ln|\varepsilon| - \varepsilon - 2\pi + \ln\left(\frac{\pi}{2}\right)}$$

As $\varepsilon \to 0$,

$$\left|\int_{\Gamma_\varepsilon} f(z,\xi)\ \mathrm{d}z\right| \le 0$$

Therefore, $\displaystyle\int_{\Gamma_\varepsilon} f(z,\xi)\ \mathrm{d}z = 0$. □

2.3 Contour Integration Conclusion

Combining the parametrization of f over Γ with 2.2.1 and 2.2.2, the integrals on the right-hand side are simplified as $R \to \infty$ and $\varepsilon \to 0$.

$$2\pi i\sigma + \frac{2\pi}{\xi} = \int_{\Gamma_+} f(z,\xi)\ \mathrm{d}z + \int_{\Gamma_-} f(z,\xi)\ \mathrm{d}z$$

From here, we would like to expand Γ_+ and Γ_-.

$$\begin{aligned}
2\pi i\sigma + \frac{2\pi}{\xi} &= \int_0^\infty \frac{i}{2}\frac{z-1}{\xi z+1}\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)} + \frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\mathrm{d}z + \\
&\quad \int_\infty^0 \frac{i}{2}\frac{z-1}{\xi z+1}\left(\frac{1}{z-(\ln(z)+2\pi i)+\ln\left(\frac{\pi}{2}\right)} + \frac{1}{z-(\ln(z)+2\pi i)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\mathrm{d}z \\
&= \int_0^\infty \frac{i}{2}\frac{z-1}{\xi z+1}\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)} + \frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)+2\pi i}\right)\mathrm{d}z - \\
&\quad \int_0^\infty \frac{i}{2}\frac{z-1}{\xi z+1}\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)-2\pi i} + \frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)-2\pi i}\right)\mathrm{d}z
\end{aligned}$$

$$2\pi i\sigma + \frac{2\pi}{\xi} = \int_0^\infty \frac{i}{2}\frac{z-1}{\xi z+1}\left(\frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)-2\pi i} - \frac{1}{z-\ln(z)+\ln\left(\frac{\pi}{2}\right)-2\pi i}\right)\mathrm{d}z \tag{4}$$

Recall the form of $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$ from (1). The RHS of (4) has matched this exact form.

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\xi} = \frac{2\pi}{\xi} + 2\pi i\sigma$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\xi} = 2\pi\left(\frac{1}{\xi} + \frac{\xi+1}{2\xi}\frac{2\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)}{\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)^2+(\xi\pi)^2} - \frac{\pi^2\xi}{\pi^2\xi^2+4}\right) \tag{5}$$

3 Finalizing Integration

Although we now have a closed form for $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$, our final solution has yet to be found. To find the original integral, we must re-integrate $\mathrm{d}\mathcal{I}/\mathrm{d}\xi$ with respect to ξ.

$$\begin{aligned}
\mathcal{I}(\xi) &= 2\pi \int \left(\frac{1}{\xi} + \frac{\xi+1}{2\xi} \frac{2\left(\xi \ln(\xi) + \xi \ln\left(\frac{\pi}{2}\right) - 1\right)}{\left(\xi \ln(\xi) + \xi \ln\left(\frac{\pi}{2}\right) - 1\right)^2 + (\xi\pi)^2} - \frac{\pi^2 \xi}{\pi^2 \xi^2 + 4} \right) \mathrm{d}\xi \\
&= 2\pi \left(\ln|\xi| - \frac{1}{2} \ln\left|\pi^2\xi^2 + 4\right| + \int \frac{\xi+1}{2\xi} \frac{2\left(\xi \ln(\xi) + \xi \ln\left(\frac{\pi}{2}\right) - 1\right)}{\left(\xi \ln(\xi) + \xi \ln\left(\frac{\pi}{2}\right) - 1\right)^2 + (\xi\pi)^2} \mathrm{d}\xi \right) \\
\mathcal{I}(\xi) &= \pi \left(2\ln|\xi| - \ln\left|\pi^2\xi^2 + 4\right| + \int \frac{2(\xi+1)}{\xi^2} \frac{\ln(\xi) - \frac{1}{\xi} + \ln\left(\frac{\pi}{2}\right)}{\left(\ln(\xi) - \frac{1}{\xi} + \ln\left(\frac{\pi}{2}\right)\right)^2 + \pi^2} \mathrm{d}\xi \right)
\end{aligned}$$

Conducting a u-substitution:

$$\boxed{\text{Let}} \qquad \begin{aligned} u(\xi) &\triangleq \left(\ln(\xi) - \frac{1}{\xi} + \ln\left(\frac{\pi}{2}\right) \right)^2 \\ \mathrm{d}u &= \frac{2(\xi+1)}{\xi^2} \left(\ln(\xi) - \frac{1}{\xi} + \ln\left(\frac{\pi}{2}\right) \right) \mathrm{d}\xi \end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\xi) &= \pi \left(2\ln|\xi| - \ln\left|\pi^2\xi^2 + 4\right| + \int \frac{1}{u + \pi^2} \mathrm{d}u \right) \\
&= \pi \left(2\ln|\xi| - \ln\left|\pi^2\xi^2 + 4\right| + \ln\left| \left(\ln(\xi) - \frac{1}{\xi} + \ln\left(\frac{\pi}{2}\right) \right)^2 + \pi^2 \right| \right) + C \\
\mathcal{I}(\xi) &= \pi \ln \left(\frac{\left(\ln(\xi) - \frac{1}{\xi} + \ln\left(\frac{\pi}{2}\right) \right)^2 + \pi^2}{\pi^2 + \frac{4}{\xi^2}} \right) + C
\end{aligned} \tag{6}$$

Now, we must solve for the integration constant. Recall the definition of $\mathcal{I}$.

$$\mathcal{I}(\xi) \triangleq \int_0^\infty \left(\frac{2\pi(x-1)}{\left(x - \ln(x) + \ln\left(\frac{\pi}{2}\right)\right)^2 + (2\pi)^2} \cdot \frac{\ln|\xi x + 1|}{x} \right) \mathrm{d}x$$

When $\xi = 0$, the integrand becomes zero. As such, the $\mathcal{I}$ has a zero at $\xi = 0$. The implications of this fact allow us to express C in the following limit.

$$\begin{aligned}
C &= -\lim_{\xi \to 0^+} \pi \ln \left(\frac{\left(\ln(\xi) - 1/\xi + \ln\left(\frac{\pi}{2}\right)\right)^2 + \pi^2}{\pi^2 + 4\xi^{-2}} \right) \\
&= -\pi \lim_{\xi \to 0^+} \ln \left(\frac{\left(\xi \ln(\xi) + \xi \ln\left(\frac{\pi}{2}\right) - 1\right)^2 + \xi^2\pi^2}{\xi^2\pi^2 + 4} \right) \\
C &= \pi \lim_{\xi \to 0^+} \left(\ln\left(\xi^2\pi^2 + 4\right) - \ln\left(\left(\xi \ln(\xi) + \xi \ln\left(\frac{\pi}{2}\right) - 1 \right)^2 + \xi^2\pi^2 \right) \right)
\end{aligned}$$

Note that $\xi \ln(\xi)$ is continuous for all values greater than zero. This implies that we may bring the limit into the natural log terms.

$$\begin{aligned}
C &= \pi \lim_{\xi\to 0^+}\left(\ln\left(\xi^2\pi^2+4\right)-\ln\left(\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)^2+\xi^2\pi^2\right)\right)\\
&= \pi\left(\ln\left(\lim_{\xi\to 0^+}\left(\xi^2\pi^2\right)+4\right)-\ln\left(\lim_{\xi\to 0^+}\left(\left(\xi\ln(\xi)+\xi\ln\left(\frac{\pi}{2}\right)-1\right)^2+\xi^2\pi^2\right)\right)\right)\\
&= \pi\left(\ln(4)-\ln\left((-1)^2\right)\right)\\
&= \pi\left(\ln(4)-\ln(1)\right)\\
C &= \pi\ln(4)
\end{aligned}$$

Combining with (6), yeilds:

$$\mathcal{I}(\xi)=\pi\ln\left(\frac{\left(\ln(\xi)-\frac{1}{\xi}+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}{\pi^2+\frac{4}{\xi^2}}\right)+\pi\ln(4)$$

$$\mathcal{I}(\xi)=\pi\ln\left(\frac{\left(\ln(\xi)-\frac{1}{\xi}+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}{\frac{1}{\xi^2}+\frac{\pi^2}{4}}\right)$$

Using $I(\xi)=\mathcal{I}(\xi)/\xi$, we gain our final answer:

$$\boxed{I(\xi)=\frac{\pi}{\xi}\ln\left(\frac{\left(\ln(\xi)-\frac{1}{\xi}+\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}{\frac{1}{\xi^2}+\frac{\pi^2}{4}}\right)}$$

The original integral is identified as $I(1)$.

$$I(1)=\pi\ln\left(\frac{\left(1-\ln\left(\frac{\pi}{2}\right)\right)^2+\pi^2}{1+\frac{\pi^2}{4}}\right)$$

$$\boxed{\int_0^\infty \arctan\left(\frac{2\pi}{x-\ln(x)+\ln\left(\frac{\pi}{2}\right)}\right)\frac{\mathrm{d}x}{x+1}=\pi\ln\left(\frac{1+\pi^2+\ln^2\left(\frac{\pi}{2}\right)-2\ln\left(\frac{\pi}{2}\right)}{1+\frac{\pi^2}{4}}\right)}$$

While this solution does evaluate the integral, it may not be the most direct solution. One hour and five minutes after posting the question, Cleo responded with her solution.