Ex) **Let** us apply the recipe to a particle of mass m, with a force of 0 N applied.

- This particle is either ~~moving~~ still, or moving at a constant velocity:

$$F = ma = 0 = m\frac{d^2x}{dt^2}$$

1- Define the hamiltonian operator: convert observables to quantum observables

$x = x(0) + vt$ , $v$ is constant. $p = mv$

$E = K + U \longrightarrow E = \frac{p^2}{2m} + U$. What is potential energy in this case?

We defined $F = -\frac{dU}{dx}$ (The definition of potential energy) $\rightarrow \int F dx = U$

If the force is 0, then $0 = -\frac{dU}{dx}$ and U must be a constant.

Let us just define $U = 0$, for simplicity

Therefore $E = K + U = \frac{p^2}{2m} + 0 = \frac{p^2}{2m}$. We have already defined the total hamiltonian before:

$\hat{p}\psi = -i\hbar\frac{d}{dx}\psi$ from previous definition.

Since $E = \frac{p^2}{2m}$, $p = \pm\sqrt{2mE}$

$\hat{p}^2\psi = (-i\hbar\frac{d}{dx}\psi)(-i\hbar\frac{d}{dx}\psi)$

$\hat{p}^2\psi = -\hbar^2\frac{d^2}{dx^2}\psi$

and $\hat{K}\psi$ therefore is equal to $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi$

Since $E = K + U$, $\hat{E}\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + U\psi$

The total hamiltonian $\hat{H}$ is $\frac{\hbar^2}{2m}\cancel{\psi}\frac{d^2}{dx^2}{}_0 + U\psi$

2- Write Schrödinger's Equation

For this course, we will focus on the Time Independent Schrödinger's Equation:

$$\hat{H}\psi = E\psi$$
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + U\psi = E\psi$$
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = (\underbrace{E - U}_{U=0})\psi$$

$U = 0 \rightarrow$ We set potential energy to 0 at the start.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

3- Define boundary conditions

Normalize: $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$

4- Solve Schrödinger's Equation

$\hat{H}\psi = E\psi$, $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$

$\therefore \frac{d^2}{dx^2}\psi = -\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi$

Substitute $\frac{2mE}{\hbar^2} = k^2$ and $E = \frac{\hbar^2k^2}{2m}$

Since $\frac{d^2}{dx^2}\psi = -k^2\psi$

We can solve this using differential equations.

↳ We take the second derivative of a function to result in the same function times a constant. Therefore the solution must be either an exponential or sin(x) or cos(x).

$\psi = ce^{bx}$, where b and c are constants.

$\frac{d^2}{dx^2}\psi = -k^2\psi$, $\therefore \cancel{c}b^2\cancel{e^{bx}} = -k^2(\cancel{ce^{bx}})$. We see that $b^2 = -k^2$,

and $b = \pm ik$ : 2 solutions to wave fxn.

$$\left.\begin{aligned}\psi_+ &= c_+e^{ikx}\\ \psi_- &= c_-e^{ikx}\end{aligned}\right\}\text{ Oscillating exponential}$$

What can we deduce from our result, that

$$\Psi_+ = C_+ e^{ikx} \quad \text{and} \quad \Psi_- = C_- e^{ikx} ?$$

Put into Time Dependent Schrödinger Equation:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \underbrace{\hat{E}\Psi(x,t)}_{C_+ e^{ikx}} \rightarrow \frac{\partial \Psi(x,t)}{\partial t} = C_+ e^{ikx}\left(-i/\hbar\right)$$

$$\Psi_+(x,t) = C_+ e^{ikx} e^{-\frac{i}{\hbar}Et} \rightarrow \omega$$

Recall that $E/\hbar = \omega$ and $k = \sqrt{\frac{2mE}{\hbar^2}}$

What we are left with is

$$\Psi_+(x,t) = C_+ e^{ikx - i\omega t} = C_+ e^{-i(\omega t - kx)}$$

↳ wave propogating to the right

Similarly

$$\Psi_-(x,t) = C_- e^{-ikx - i\omega t} = C_- e^{-i(\omega t + kx)}$$

↳ wave propogating to the left

What happens if we use the momentum operator on this wave function?

$$\Psi_+ = C_+ e^{ikx}$$

$$\hat{p}\Psi_+ = \underbrace{-i\hbar ik}_{\hbar k} e^{ikx} C_+$$

$$p = \hbar k \qquad k = \sqrt{\frac{2mE}{\hbar^2}},$$

$$\therefore \quad p = \frac{\hbar\sqrt{2mE}}{\hbar} = \sqrt{2mE}$$

Which is exactly the momentum of a classical particle.

The difference between the wave and classical models is the probability:
- The probability everywhere in the range is constant, and differs from C.M. interpretation, which states the particle may only exist at a single point.

Are we able to normalize this function from $-\infty$ to $\infty$?

$$\int_{-\infty}^{\infty} |\Psi_+|^2 dx = \int_{-\infty}^{\infty} |C_+|^2 \underbrace{|e^{ikx}|^2}_{=1} dx = \int_{-\infty}^{\infty} |C_+|^2 dx = \infty$$

→ We cannot: the range must be restricted.

Restricting the range from $-L$ to $L$:

$$\int_{-L}^{L} |C_+|^2 dx = C_+^2 \cdot 2L.$$

Recall that to normalize a wave function we use $\tilde{\Psi} = \frac{\Psi}{\sqrt{\int |\Psi|^2 dx}}$

$$\tilde{\Psi}_+ = \frac{1}{\sqrt{2L}} e^{ikx} \rightarrow$$

Notice that as $L \to \infty$, the wave function behaves more and more linearly.

Since $\Psi_+ = \frac{1}{\sqrt{2L}} e^{ikx}$, $\Psi_- = \frac{1}{\sqrt{2L}} e^{-ikx}$. Any linear combination of these two are also solutions to schrödinger's equation, since the wave functions are normalized.

For example, $b_1\Psi_+ + b_2\Psi_-$ is also a solution.

$$\left(b_1 \frac{1}{\sqrt{2L}} e^{ikx} + b_2 \frac{1}{\sqrt{2L}} e^{-ikx}\right)$$

Notice also that we found the momentum for the wave function, represented as $\hbar k$. We can now compare this to wavelength:

$$p = \hbar k = \hbar\left(\frac{2\pi}{\lambda}\right) = \frac{h}{2\pi}\left(\frac{2\pi}{\lambda}\right) = \frac{h}{\lambda}.$$

De broglie's equation is $\lambda = \frac{h}{p}$: these statements are in agreement.