4. Let $Y_i = I(\text{reject } H_i)$, $i = 1, \cdots, 4$.

$$p_{iy} = P(Y_i = y)$$

Since the hypotheses are independent, we have

$$Q(y_1, y_2, y_3, y_4) \overset{\text{def}}{=} P(Y_i = y_i,\ i = 1, \cdots, 4)$$

$$= P(Y_1 = y_1) P(Y_2 = y_2) P(Y_3 = y_3) P(Y_4 = y_4)$$

$$= p_{1y_1} p_{2y_2} p_{3y_3} p_{4y_4}$$

Each Y_i has 2 possible values (0 or 1), so the total number of possible configurations of (Y_1, Y_2, Y_3, Y_4) is 16.

W.l.o.g. assume that H_1 and H_2 are the true nulls and H_3 and H_4 are the false nulls.

We are using Bonferroni with $m = 4$ hypotheses at level $\alpha = 0.10$, so each hypothesis is tested at level $\alpha' = \alpha/m = 0.10/4 = 0.025$.

Let Z_i denote the test statistic. With one-sided testing at the 0.025 level, the rejection rule is

$$Z_i \geq \Phi^{-1}(0.975) = 1.96$$

For the true nulls H_1 and H_2, we have

$$p_{10} = p_{20} = 0.975$$

$$p_{11} = p_{21} = 0.025$$

For the false nulls H_3 and H_4, we have

$$P(\text{reject}) = P(N(1.5, 1) \geq 1.96)$$

$$= P(N(0,1) \geq 0.46)$$

$$= 1 - P(N(0,1) \leq 0.46)$$

$$= 1 - 0.6772$$

$$= 0.3228.$$

As in class, define

R = total number of rejections
V = number of wrong rejections

We then have

$$FDP = \begin{cases} V/R & R > 0 \\ 0 & R = 0 \end{cases}$$

and $FDR = E[FDP]$

We can summarize the possible results in the following table:

(Y_1, Y_2, Y_3, Y_4)	R	V	FDP	Q
0000	0	0	0	0.4360
0001	1	0	0	0.2078
0010	1	0	0	0.2078
0011	2	0	0	0.0991
0100	1	1	1	0.0112
0101	2	1	1/2	0.0053
0110	2	1	1/2	0.0053
0111	3	1	1/3	0.0025
1000	1	1	1	0.0112
1001	2	1	1/2	0.0053
1010	2	1	1/2	0.0053
1011	3	1	1/3	0.0025
1100	2	2	1	0.0003
1101	3	2	2/3	0.0001
1110	3	2	2/3	0.0001
1111	4	2	1/2	0.0001

The distribution of FDP is therefore as follows:

Value	Prob
0	0.9507
1/3	0.0050
1/2	0.0213
2/3	0.0002
1	0.0227

$$\text{FDR} = (1/3)(0.0050) + (1/2)(0.0213) + (2/3)(0.0002) + (1)/(0.0227) = 0.0352$$

A shorter way is also possible, as follows:

We have

$$V = Y_1 + Y_2 \sim Bin(2, 0.025)$$

and if we define $S = Y_3 + Y_4$ we have

$S \sim Bin(2, 0.3228)$.

In addition, we have

$$FDP = \begin{cases} V/(V+S) & V > 0 \\ 0 & V = 0 \end{cases}$$

$P(FDP = 0) = P(V = 0) = (0.975)^2 = 0.9506$

The remaining cases can be tabulated as follows:

V	S	FDP	Probability
1	0	1	0.0224
1	1	1/2	0.0213
1	2	1/3	0.0051
2	0	1	0.0003
2	1	2/3	0.0003
2	2	1/2	0.0001

FDP	Probability
1/3	0.0051
1/2	0.0214
2/3	0.0003
1	0.0227

same as before up to ± 0.0001