Covariant calculations at finite temperature: The relativistic plasma

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It is shown that finite-temperature calculations in field theory are manifestly Lorentz covariant at all stages if the Minkowski-space form of the temperature-dependent propagators is used and if the four-velocity u_μ of the heat bath is taken into account. New tensor structures involving u_μ generally arise but are severely constrained by covariant current conservation. A complete high-temperature ($T \gg m$) expansion of the vacuum polarization tensor for non-Abelian gauge theories is computed to order g^2 and displays the separate dependence on frequency ω and wave number k that occurs at finite temperature. A covariant phenomenology of "electric" and "magnetic" properties is applied to the collective plasma effects, characterized by a plasma frequency $\omega_p{}^2 = (N_f+2N)g^2T^2/6$ for SU(N) with N_f fermions. The longitudinal normal modes of the "electric" field exist only for $\omega > \omega_p$; for $\omega < \omega_p$ all "electric" fields are screened. The transverse normal modes are plane waves along $\widetilde{E}\times\widetilde{B}$ for $\omega > \omega_p$; for $\omega < \omega_p$ both transverse "electric" and "magnetic" fields are shielded except for the static ($\omega = 0$) case.

I. INTRODUCTION

It is thought that the confinement mechanism of zero-temperature QCD does not persist at high temperature.[1] This justifies the use of quarks and gluons as constituents of the early universe at temperatures $T \gg 1$ GeV. By extension, any non-Abelian gauge theory is expected to be nonconfining at a sufficiently high temperature. One can therefore treat the various particles in grand unified theories by perturbative methods in computing, for example, the generation of a baryon excess at $T \sim 10^{14}$ GeV.

The absence of confinement does not, however, imply the absence of coherent behavior. The infinite range of the Abelian Coulomb potential produces a variety of collective effects in low-temperature gases ($T \ll m$). The simplest of these is the screening of an external charge introduced into a neutral gas containing an equal density n of electrons and protons. The induced polarization shields the external charge so that its electric field falls off exponentially with a characteristic Debye screening length $\lambda = (T/ne^2)^{1/2}$. The early universe is, of course, very hot ($T \gg m$) and also very dense since $n \sim T^3$ for both particles and antiparticles. In terms of the dimensionless ratio n/T^3, the density of the early universe is much larger than that of interstellar gases or the solar corona ($n/T^3 \sim 10^{-12}$) and roughly comparable to a tenuous laboratory plasma ($n/T^3 \sim 1$). A perturbative analysis is reasonable for such a plasma because the typical interaction energy is smaller than the thermal energy:

$$\left[\frac{e^2}{4\pi\lambda}\right]\frac{1}{T} = \frac{e^3}{4\pi}\left[\frac{n}{T^3}\right]^{1/2} \ll 1 \ .$$

Indeed, the perturbative calculations that have been performed for ultrarelativistic plasma, both Abelian[2] and non-Abelian,[3,4] do find that static electric fields are shielded, with a sceening length $\lambda \sim (eT)^{-1}$. This indicates the nonconfinement of electric charge at high temperature. (Static magnetic fields are, however, not shielded.) Of course, low-temperature plasma physics has advanced well beyond static effects and now encompasses a vast assortment of time-dependent phenomena.[5] The purpose of this paper is to investigate more fully the properties of high-temperature plasmas.

An ambiguity that immediately arises is whether the motion of a charge through the plasma is best analyzed in its own rest frame or in the rest frame of the plasma, and how one should deal with temperature in the different frames. The popular notion that finite-temperature corrections to quantum field theory destroy its Lorentz covariance is, of course, untrue. Planck and Einstein formulated thermodynamics in a relativistically covariant manner; readable discussions are given in the textbooks of Tolman[6] and Pauli.[7] Criticism of this formulation by Ott[8] in 1963 provoked enormous interest in the subject that led to considerable progress in properly formulating both equilibrium and nonequilibrium thermodynamics. In the modern approach due to Israel[9] a fluid in thermodynamic equilibrium is characterized by a four-velocity u_μ and a Lorentz-invariant parameter T, equal to the temperature of the fluid in its own rest frame. With this formulation it is possible to quantize finite-temperature field theory in any frame.

However, there is a second source of noncovariance in the conventional method: quantizing field configurations that are periodic in imaginary time leads to discrete energies but continuous momenta. The less familiar method of quantizing in real time eliminates this awkward distinction and, when combined with a covariant statistical mechanics, leads to fully covariant propagators, viz.,

$$S_F(p)=(\not p+m)\left[\frac{1}{p^2-m^2+i\eta}+\frac{2\pi i\delta(p^2-m^2)}{e^{|p\cdot u/T|}+1}\right].$$

In the fluid rest frame, $u_\mu=(1,0,0,0)$, such propagators are discussed by Fetter and Walecka[10] and by Dolan and Jackiw.[11]

Radiative corrections will generally introduce dependence on the four-vector u^α. A virtual particle with momentum K^α is no longer characterized by the single invariant $K_\alpha K^\alpha$ but by two invariants $\omega=K^\alpha u_\alpha$ and k such that $K_\alpha K^\alpha=\omega^2-k^2$. The vacuum polarization will depend on two covariant tensors, each satisfying current conservation. Accordingly the gauge boson propator will contain two Lorentz-invariant self-energy functions $\pi_L(k,\omega)$ and $\pi_T(k,\omega)$, which characterize the longitudinal and transverse modes. The existence of two such functions is equivalent to characterizing the plasma by both a dielectric permittivity ϵ and a magnetic permeability μ with $\epsilon\neq\mu^{-1}$. A simple separation of dielectric and diamagnetic phenomena is usually possible only in the rest frame of the medium; in a moving frame the field equations are a mess. To avoid this awkwardness it is convenient to introduce a pair of four-vectors $\widetilde{E}^\alpha$ and $\widetilde{B}^\alpha$, each orthogonal to u^α, defined by

$$F^{\alpha\beta}\equiv u^\alpha\widetilde{E}^\beta-u^\beta\widetilde{E}^\alpha+\epsilon^{\alpha\beta\mu\nu}\widetilde{B}_\mu u_\nu\ .$$

Despite the presence of ϵ and μ, the field equations for $\widetilde{E}$ and $\widetilde{B}$ in any frame have the same form as the familiar equations in the rest frame of the medium. Thus one can solve for $\widetilde{E}$ and $\widetilde{B}$ in any frame and, from the solution, compute E and B.

The formalism described above is constructed in Sec. II. In the Appendix the two self-energy functions π_L and π_T are computed to one-loop order for an SU(N) gauge theory with N_f fermions in the fundamental representation and a complete high-temperature expansion is obtained. The leading terms in this expansion determine the long-wavelength, coherent phenomena and are explored in Sec. III. These phenomena are characterized by a plasma frequency ${\omega_p}^2=(N_f+2N)g^2T^2/6$. At frequencies $\omega>\omega_p$ there are longitudinal normal modes of the "electric" field $\widetilde{E}_L$ [defined by $\epsilon_{\alpha\mu\nu\lambda}u^\mu\widetilde{\nabla}^\nu\widetilde{E}_L^\lambda=0$ where $\widetilde{\nabla}^\nu\equiv(\delta_\beta^\nu-u^\nu u_\beta)\partial^\beta$] coupled to a charge density wave. Also for $\omega>\omega_p$ there are transverse normal modes of the "electric" fields $\widetilde{E}_T$ and the "magnetic" induction $\widetilde{B}_T$ (defined by $\widetilde{\nabla}\cdot\widetilde{E}_T=0$, $\widetilde{\nabla}\cdot\widetilde{B}_T=0$) which together constitute plane-wave propagation. (The possibility of normal modes which are distinctly non-Abelian is not discussed since this is an unsolved problem even at $T=0$). For $\omega<\omega_p$ there are neither longitudinal nor transverse waves; all fields produced by external currents or charges fall off exponentially in space with a screening length that depends on frequency. The one exception is that the screening length for "magnetic" fields diverges as $\omega\to0$. Consequently $\widetilde{B}$ is screened for arbitrarily small ω but not for $\omega=0$.

The impatient reader may wish to turn directly to Sec. III, which is fairly self-contained. Section IV compares the results with those for a low-temperature ($T\ll m$) plasma.

II. LORENTZ-COVARIANT FORMALISM

A. Thermodynamics[9]

An arbitrary state of a system of matter and radiation is described by a conserved energy-momentum tensor $T^{\nu\mu}$, an entropy flux σ^μ, and a set of conserved currents J_A^μ ($A=1,2,\ldots$). The first law states that under an arbitrary displacement from equilibrium, the changes in the thermodynamic variables are related by

$$\beta_\nu dT^{\nu\mu}=d\sigma^\mu+\sum_A\alpha_A dJ_A^\mu\ , \tag{2.1}$$

where the α_A are Lorentz scalars and β_ν is a timelike Lorentz four-vector. There is no distinction made between heat flow and work performed. One can always express β_ν as

$$\beta_\nu=u_\nu/T\ , \tag{2.2}$$

where $u^\nu u_\nu=1$ and T is a Lorentz-invariant parameter related to temperature.

For a system in equilibrium the velocity u^μ

characterizes all the primary variables:

$$T^{\nu\mu}=(\rho+P)u^{\nu}u^{\mu}-P\eta^{\nu\mu}\ , \tag{2.3a}$$

$$\sigma^{\mu}=su^{\mu}\ , \tag{2.3b}$$

$$J_A^{\mu}=n_A u^{\mu}\ , \tag{2.3c}$$

where P, ρ, s, and n_A are the Lorentz-invariant pressure, energy density, entropy density, and particle density, respectively. When (2.2) and (2.3) are substituted into the first law (2.1) one obtains two scalar equations because u^{μ} and du^{μ} are orthogonal vectors. The terms proportional to u^{μ} give a more recognizable statement of the first law in terms of Lorentz invariants:

$$\frac{1}{T}d\rho=ds+\sum_A \alpha_A dn_A\ .$$

The terms proportional to du^{μ} yield the algebraic (not differential) relation

$$P=-\rho+Ts+\sum_A T\alpha_A n_A\ .$$

With the chemical potential $\mu_A=T\alpha_A$, this is exactly the standard Gibbs relation. For the equilibrium system the Lorentz-invariant pressure P may be expressed in terms of the primary tensors by[12]

$$P\beta^{\mu}=-\beta_{\nu}T^{\nu\mu}+\sigma^{\mu}+\sum_A \alpha_A J_A^{\mu}\ . \tag{2.4}$$

For later purposes it is useful to note that

$$d(P\beta^{\mu})=\sum_A J_A^{\mu}d\alpha_A-T^{\mu\nu}d\beta_{\nu} \tag{2.5}$$

because of the first law (2.1).

B. Statistical mechanics

Conventional nonrelativistic statistical mechanics is based on computing the thermodynamic potential $\Omega(T,V,\mu)$ from the grand partition function

$$\exp(-\beta\Omega)=\mathrm{Tr}\left[\exp\left[-\beta H+\sum_A \alpha_A N_A\right]\right]\ ,$$

where $\alpha_A=\mu_A/T$. In covariant mechanics[9] one computes a four-vector potential $\Phi^{\mu}(\alpha,\beta_{\nu})$. (This potential can later be identified as $\Omega u^{\mu}/TV$.) Quantization is performed on the spacelike surface normal to the fluid velocity u_{μ}. Letting dS_{μ} be the infinitesimal surface element, one may compute the grand partition density operator $\hat{Z}_G$ from the operators $\hat{J}_A^{\mu}$ and $\hat{T}^{\mu\nu}$:

$$\hat{Z}_G\equiv\exp\left[\int dS_{\mu}\left[\sum_A \alpha_A\hat{J}_A^{\mu}-\beta_{\nu}T^{\nu\mu}\right]\right]\ . \tag{2.6}$$

The potential $\Phi^{\mu}(\alpha,\beta_{\nu})$ is given by

$$\exp\left[-\int dS_{\mu}\Phi^{\mu}\right]=\mathrm{Tr}(\hat{Z}_G)\ .$$

From this potential the currents and the energy-momentum tensor may be immediately calculated by differentiation:

$$d\Phi^{\mu}=-\sum_A J_A^{\mu}d\alpha_A+T^{\mu\nu}d\beta_{\nu}\ .$$

Comparison of the right-hand side with (2.5) shows that the pressure is given by

$$P\beta^{\mu}=-\Phi^{\mu}\ . \tag{2.7}$$

Knowing $T^{\nu\mu}$, J_A^{μ}, and P automatically fixes the entropy flux σ^{μ} by (2.4).

C. Finite-temperature propagators

There are two ways of expressing free particle propagators at finite temperature. The imaginary-time method leads to Feynman rules in which energies are discrete but three-momenta are continuous. The real-time method leads to Feynman rules with energies and momenta that are continuous.[10,11] To preserve manifest Lorentz covariance the real-time formulation is much preferred. The fermion propagator is

$$S_F(x-y)\equiv i\,\mathrm{Tr}\{\hat{Z}_G T[\psi(x)\bar{\psi}(y)]\}/\mathrm{Tr}(\hat{Z}_G)\ , \tag{2.8}$$

where $\hat{Z}_G$ is the grand partition density operator (2.6). The operator ordering is with respect to the Lorentz-invariant time $u^{\alpha}x_{\alpha}$ and $u^{\alpha}y_{\alpha}$, which evolves orthogonally to the spacelike surface of quantization in (2.6).

From the momentum p^{α} of a virtual fermion two Lorentz invariants can be constructed: $(p\cdot u)$ and $\rho^2=(p\cdot u)^2-p^2>0$. If ρ is interpreted as the Lorentz-invariant three-momentum, the Lorentz-invariant energy variable is

$$E\equiv(m^2+\rho^2)^{1/2}>0$$

for a particle of mass m. The propagator (2.8) in momentum space is

$$S_F(p)=S_+(p')(e^{-p^0/T}+1)^{-1}+S_-(p')(e^{p^0/T}+1)^{-1}\ , \tag{2.9}$$

where $p'^{\alpha}=p^{\alpha}+\mu u^{\alpha}$, T is the Lorentz-invariant temperature, and $S_{\pm}$ are the retarded and advanced

Green's fuctions

$$S_{\pm}(p')=\frac{\not p'+m}{p'^2-m^2\pm i\eta\epsilon(u\cdot p')}\ .$$

The result (2.9) is closely analogous to the usual nonrelativistic form.[10] When the retarded and advanced Green's functions are substituted into (2.9) it simplifies to

$$S_F(p)=\frac{\not p'+m}{p'^2-m^2+i\eta}+(\not p'+m)\left[\frac{\theta(p'\cdot u)}{e^{(E+\mu)/T}+1}+\frac{\theta(-p'\cdot u)}{e^{(E-\mu)/T}+1}\right]2\pi i(p'^2-m^2)\ .$$

The exponential factor can be rewritten by using the fact that $|p\cdot u|=E\pm\mu$ when $p'^2=m^2$:

$$S_F(p)=\frac{\not p'+m}{p'^2-m^2+i\eta}+\frac{\not p'+m}{e^{|p\cdot u|/T}+1}2\pi i\delta(p'^2-m^2)\ . \tag{2.10}$$

This propagator is fully covariant; it coincides with that found by Dolan and Jackiw[11] in the fluid rest frame ($\vec{u}=0$) with no chemical potential ($\mu=0$). The separation of the T-dependent and the T-independent contributions is particularly useful and the exponential damping in the second term guarantees the known result that there are no additional one-loop divergences when $T>0$. It is very important that the support of the δ function includes $p_0'<0$. This will later be essential in discussing the imaginary part of the plasmon mass.

D. Vacuum polarization

Because the temperature-dependent propagator for the free fermion (2.10) satisfies $(\not p'-m)S_F(p)=1$ and similarly for the bosons, the usual proof that

$$K^\mu\pi_{\mu\nu}(K)=0 \tag{2.11}$$

is unchanged. The most general linear combination of the available symmetric tensors ($\eta_{\mu\nu}$, $K_\mu K_\nu$, $u_\mu u_\nu$, and $u_\mu K_\nu+u_\nu K_\mu$) that satisfies (2.11) contains only two unknown scalar functions. These functions can depend on two Lorentz scalars:

$$\begin{aligned}&\omega\equiv K^\alpha u_\alpha\ ,\\ &k\equiv[(K^\alpha u_\alpha)^2-K^2]^{1/2}\ .\end{aligned} \tag{2.12}$$

Since $K^2=\omega^2-k^2$ we may interpret ω and k as a Lorentz-invariant energy and three-momentum of the virtual boson. It is useful to define a tensor and a vector orthogonal to u_μ by

$$\begin{aligned}&\tilde\eta_{\mu\nu}\equiv\eta_{\mu\nu}-u_\mu n_\nu\ ,\\ &\tilde K_\mu\equiv K_\mu-\omega u_\mu\ .\end{aligned} \tag{2.13}$$

The vector is automatically spacelike:

$$\tilde K^\mu\tilde K_\mu=-k^2<0\ . \tag{2.14}$$

The general solution to (2.11) is

$$\pi_{\mu\nu}=\pi_T(k,\omega)P_{\mu\nu}+\pi_L(k,\omega)Q_{\mu\nu}\ , \tag{2.15}$$

where π_T and π_L are unknown functions and[13]

$$\begin{aligned}&P_{\mu\nu}\equiv\tilde\eta_{\mu\nu}+\frac{\tilde K_\mu\tilde K_\nu}{k^2}\ ,\\ &Q_{\mu\nu}\equiv\frac{-1}{K^2k^2}(k^2u_\mu+\omega\tilde K_\mu)(k^2u_\nu+\omega\tilde K_\nu)\ .\end{aligned} \tag{2.16}$$

One could choose a different pair of tensors as a basis for $\pi_{\mu\nu}$ but this choice is especially convenient because of the simple multiplication properties

$$\begin{aligned}&P^\mu{}_\nu P^\nu{}_\alpha=P^\mu{}_\alpha,\quad Q^\mu{}_\nu Q^\nu{}_\alpha=Q^\mu{}_\alpha\ ,\\ &P^\mu{}_\nu Q^\nu{}_\alpha=0,\quad Q^\mu{}_\nu P^\nu{}_\alpha=0\ .\end{aligned} \tag{2.17}$$

At $T=\mu=0$, the two scalar functions are equal, i.e., $\pi_T=\pi_L$, and the usual vacuum polarization tensor results from the identity

$$P_{\mu\nu}+Q_{\mu\nu}=\eta_{\mu\nu}-K_\mu K_\nu/K^2\ . \tag{2.18}$$

The full propagator of the gauge boson is obtained from the $T=0$, free-field propagator

$$D_{\mu\nu}=\left[-\eta_{\mu\nu}+\alpha\frac{K_\mu K_\nu}{K^2}\right]\frac{1}{K^2}\ ,$$

where α is a gauge parameter, by summing all the vacuum polarization insertions to obtain

$$\Delta_{\mu\nu}=-\frac{P_{\mu\nu}}{K^2-\pi_T}-\frac{Q_{\mu\nu}}{K^2-\pi_L}+(\alpha-1)\frac{K_\mu K_\nu}{K^4}\ . \tag{2.19}$$

This collapses to the known results[14] for the special case $\omega=0$, $\vec{u}=0$, and Feynman gauge ($\alpha=1$):

$$\Delta_{00}=\frac{1}{k^2+\pi_L}\ ,$$

$$\Delta_{ij}=\frac{-\delta_{ij}+K_iK_j/k^2}{k^2+\pi_T}\ ,$$

and $\Delta_{0j}=0$. The complete temperature-dependent propagator contains an additional imaginary part analogous to the mass-shell δ function in (2.10) but the remainder of the paper will only require (2.19).

To exhibit the interaction of two currents due to the exchange of a virtual gauge boson that is implies by (2.19), it is convenient to separate the current $J^\mu(K)$ into a charge density that moves with the fluid velocity u^μ and a spacelike current flow either longitudinal or transverse to $\tilde{K}^\mu$:

$$J^\mu(K)=u^\mu u^\alpha J_\alpha(K)+J_L^\mu(K)+J_T^\mu(K)\ . \qquad (2.20)$$

Current conservation places no constraint on the transverse current but does relate the longitudinal and timelike currents:

$$J_L^\mu(K)=\frac{\tilde{K}^\mu\omega}{k^2}u^\alpha J_\alpha(K)\ ,$$
$$J_T^\mu(K)=\left[\tilde{\eta}^\mu{}_\lambda+\frac{\tilde{K}^\mu\tilde{K}_\lambda}{k^2}\right]J^\lambda(K)\ . \qquad (2.21)$$

The current-current interaction due to (2.19) is

$$J^\mu(K)\Delta_{\mu\nu}J^\nu(K)=\frac{-J_T\cdot J_T}{K^2-\pi_T}-\left[\frac{K^2}{\omega^2}\right]\frac{J_L\cdot J_L}{K^2-\pi_L}\ . \qquad (2.22)$$

In this sense $P_{\mu\nu}$ is a transverse projection and $Q_{\mu\nu}$ is a mixture of longitudinal and timelike projections.

E. Electric and magnetic phenomena

It is useful to describe the electric and magnetic properties of the plasma covariantly. Such a description cannot be based on

$$E^m=F^{0m},\quad B^m=\tfrac{1}{2}\epsilon^{mab}F^{ab}$$

due to their awkward transform properties. Because the fluid velocity u_λ provides a physical distinction between timelike and spacelike directions, it is natural to instead use the four-vectors

$$\tilde{E}^\nu\equiv u_\lambda F^{\lambda\nu},\quad \tilde{B}^\nu=\tfrac{1}{2}\epsilon^{\alpha\beta\lambda\nu}F_{\alpha\beta}u_\lambda\ . \qquad (2.23)$$

Given $\tilde{E}^\nu$ and $\tilde{B}^\nu$ one can immediately construct $F^{\mu\nu}$ by

$$F^{\mu\nu}=u^\mu\tilde{E}^\nu-\tilde{E}^\mu u^\nu+\epsilon^{\mu\nu\alpha\beta}\tilde{B}_\alpha u_\beta\ .$$

Whenever $\vec{u}\neq 0$ the components of $\tilde{E}$ depend upon both $\vec{E}$ and $\vec{B}$ and likewise for $\tilde{B}$. Since $\tilde{E}^2$ and $\tilde{B}^2$ and Lorentz scalars can one evaluate them in the rest frame of the fluid ($\vec{u}=0$),

$$\tilde{E}^2=-\vec{E}^2_{\text{rest}},\quad \tilde{B}^2=-\vec{B}^2_{\text{rest}}\ .$$

There is a stronger result: we let Λ_μ^ν be a particular Lorentz boost from the fluid rest frame to the laboratory frame such that $u^\nu=\Lambda_\mu^\nu\delta^{\mu 0}$. With $F^{\mu\nu}$ related to $F^{\alpha\beta}_{\text{rest}}$ by the same boost, it immediately follows from (2.23) that

$$\tilde{E}^\nu=\Lambda_m^\nu E^m_{\text{rest}},\quad \tilde{B}^\nu=\Lambda_m^\nu B^m_{\text{rest}}\ .$$

The utility of $\tilde{E}$ and $\tilde{B}$ only emerges when interactions with the plasma are considered. Before including these interactions the action can be expressed in terms of the Fourier-transform fields $\tilde{E}(K)$ and $\tilde{B}(K)$ as

$$S_0=-\tfrac{1}{2}\int\frac{d^4K}{(2\pi)^4}(\tilde{E}^\alpha\tilde{E}_\alpha-\tilde{B}^\alpha\tilde{B}_\alpha)\ , \qquad (2.24)$$

with all group indices summed. From interacting with a finite-temperature, isotropic plasma the most general action that can arise that is both gauge invariant and Lorentz invariant is

$$S=-\tfrac{1}{2}\int\frac{d^4K}{(2\pi)^4}\left[\epsilon\tilde{E}^\alpha\tilde{E}_\alpha-\frac{1}{\mu}\tilde{B}^\alpha\tilde{B}_\alpha\right]\ , \qquad (2.25)$$

where ϵ and μ are unknown functions of momentum.[15] Because the "electric" and "magnetic" contributions to (2.25) are separately Lorentz invariant, ϵ and μ can only be functions of the Lorentz invariants k and ω defined in (2.12).

To obtain these functions, we write the finite-temperature, quantum corrections as

$$S_{\text{quantum}}=-\tfrac{1}{2}\int\frac{d^4K}{(2\pi)^4}A^\alpha(-K)\pi_{\alpha\beta}A^\beta(K)+\cdots\ ,$$

where the terms cubic and quartic in A are omitted. Because of gauge invariance the omitted terms must combine with the explicit quadratic term to form $\tilde{E}^2$ and $\tilde{B}^2$. Consequently we need only use the tensor structure (2.15) of $\pi_{\alpha\beta}$ to obtain

$$S_{\text{quantum}}=\tfrac{1}{2}\int\frac{d^4K}{(2\pi)^4}\left[\frac{\pi_L}{K^2}\tilde{E}^\alpha\tilde{E}_\alpha+\frac{K^2\pi_T-\omega^2\pi_L}{k^2K^2}\tilde{B}^\alpha\tilde{B}_\alpha\right]\ .$$

The electric permittivity and magnetic permeability functions are then

$$\epsilon(k,\omega)=1-\frac{\pi_L}{K^2}\,,$$
$$\frac{1}{\mu(k,\omega)}=1+\frac{K^2\pi_T-\omega^2\pi_L}{k^2K^2}\,. \tag{2.26}$$

For the familar case of $T=\mu=0$ vacuum polarization, $\pi_L=\pi_T$ so that $\epsilon=1/\mu$. This will not be the case at finite temperature.

F. External currents

To investigate the coherence behavior of the plasma we will consider its response to a weak external current J^{β}_{ext}. For a very weak source, linear response theory applies; the current-current interaction (2.22) implies that the vector potential induced in the plasma is

$$\langle A_\alpha(K)\rangle=\Delta_{\alpha\beta}(K)J^{\beta}(K)_{\text{ext}}\,. \tag{2.27}$$

As this point we specialize to the case of only one external current which points in a fixed direction in group space, so that $F^{\alpha\beta}$ is linear in A (i.e., the $[A,A]$ commutator vanishes). Using the projection tensors, (2.27) can be solved for the external current and the result projected either along u_ν or orthogonal to u_ν,

$$\epsilon K_\nu\langle\widetilde{E}^{\nu}(K)\rangle=iu_\nu J^{\nu}(K)_{\text{ext}}\,, \tag{2.28a}$$
$$\frac{1}{\mu}\epsilon^{\alpha\beta\gamma\nu}K_\alpha\langle\widetilde{B}_\beta(K)\rangle u_\gamma+\omega\epsilon\langle\widetilde{E}^{\nu}(K)\rangle=-i\widetilde{J}^{\nu}(K)_{\text{ext}}\,,$$

with ϵ and μ as defined in (2.26). Obviously these are covariant forms of the inhomogeneous pair of Maxwell's equations; the Yang-Mills theory is effectively Abelian for the case of a single weak source. The homogeneous pair of Maxwell's equations

$$K_\nu\langle\widetilde{B}^{\nu}(K)\rangle=0\,, \tag{2.28b}$$
$$\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\nu}K_\alpha\langle\widetilde{E}_\beta(K)\rangle u_\gamma-\omega\langle\widetilde{B}^{\nu}(K)\rangle=0$$

are automatically satisfied.

To extract the physical content of (2.28) it is useful to separate vectors longitudinal and transverse to the spacelike vector $\widetilde{K}^{\nu}$ by defining

$$\widetilde{E}^{\nu}_L(K)\equiv-\frac{\widetilde{K}^{\nu}\widetilde{K}^{\mu}}{k^2}\widetilde{E}_\mu(K)\,,$$
$$\widetilde{E}^{\nu}_T(K)\equiv\left[\widetilde{\eta}^{\nu\mu}+\frac{\widetilde{K}^{\nu}\widetilde{K}^{\mu}}{k^2}\right]\widetilde{E}_\mu(K) \tag{2.29}$$

and employing the definitions (2.21) for the current. The longitudinal and transverse parts of (2.28a) become[16]

$$\omega\epsilon\langle\widetilde{E}^{\nu}_L(K)\rangle=-iJ^{\nu}_L(K)_{\text{ext}}\,, \tag{2.30}$$
$$\left[-\frac{k^2}{\omega\mu}+\omega\epsilon\right]\langle\widetilde{E}^{\nu}_T(K)\rangle=-iJ^{\nu}_T(K)_{\text{ext}}\,. \tag{2.31}$$

The analysis in Sec. III will focus on these equations and, because of their familiar form, will parallel the usual rest-frame discussion.

III. COLLECTIVE PHENOMENA

A. Perturbative calculation

The general formalism of Sec. II will now be applied to a neutral plasma (i.e., having no chemical potential) of SU(N) gauge bosons with N_f fermions in the fundamental representation. For example, in color SU(3), N_f is the number of quark flavors. The lowest-order fermion contribution to the vacuum polarization is

$$\pi_{\mu\nu}(K)=ig^2N_f\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}[\gamma_\mu S_F(p+K)\gamma_\nu S_F(p)]+\cdots\,, \tag{3.1}$$

where S_F are the temperature-dependent propagators (2.10) and the g^2 contribution of gauge-boson loops is not displayed. Because of Lorentz covariance and current conservation, this tensor necessarily is of the form (2.15). The two Lorentz-invariant functions π_L and π_T are obtained by contraction:

$$\pi_L(k,\omega)=-\frac{K^2}{k^2}u^{\mu}u^{\nu}\pi_{\mu\nu}\,,$$
$$\pi_T(k,\omega)=-\tfrac{1}{2}\pi_L+\tfrac{1}{2}\eta^{\mu\nu}\pi_{\mu\nu}\,. \tag{3.2}$$

The computation of these integrals is performed in the Appendix for nonzero k and ω. Here only the leading term in the high-temperature expansion ($T\gg k$ and $T\gg\omega$) will be used[17]:

$$\mathrm{Re}\,\pi_L(k,\omega)=\frac{e^2T^2}{3}\left[1-\frac{\omega^2}{k^2}\right]\times\left[1-\frac{\omega}{2k}\ln\left|\frac{\omega+k}{\omega-k}\right|\right]\,,$$
$$\mathrm{Re}\,\pi_T(k,\omega)=\frac{e^2T^2}{6}\left[\frac{\omega^2}{k^2}+\left[1-\frac{\omega^2}{k^2}\right]\times\frac{\omega}{2k}\ln\left|\frac{\omega+k}{\omega-k}\right|\right]\,. \tag{3.3}$$

Because the contribution of the fermion and boson loops is of the same form, in this approximation, the gauge coupling g^2 occurs only in the combination

$$e^2 \equiv (N_f + 2N)g^2/2 \ . \tag{3.4}$$

Equation (3.3) agrees[2-4] with the known results at $\omega=0$, $k\neq 0$:

$$\mathrm{Re}\pi_L(k,0)=\frac{e^2T^2}{3}\ , \qquad \mathrm{Re}\pi_T(k,0)=0\ ; \tag{3.5a}$$

and at $k=0$, $\omega\neq 0$:

$$\mathrm{Re}\pi_L(0,\omega)=\frac{e^2T^2}{9}\ , \qquad \mathrm{Re}\pi_T(0,\omega)=\frac{e^2T^2}{9}\ . \tag{3.5b}$$

The real part of the electric permittivity and magnetic permeability functions defined in (2.26) can be computed from (3.3) and are plotted in Figs. 1 and 2. Figure 1 shows that at low frequencies (viz., $\omega/k<0.8$) the plasma is strongly dielectric; the induced charge density

$$u^\alpha J_\alpha^{\mathrm{ind}}(k,\omega)=u^\alpha J_\alpha^{\mathrm{ext}}(k,\omega)\left[-1+\frac{1}{\epsilon(k,\omega)}\right]$$

almost cancels the external charge because $\epsilon \gg 1$. Figure 2 shows that the plasma is diamagnetic[18] at low frequencies (viz., $\omega/k<0.9$); however at $\omega=0$ the diamagnetic and paramagnetic contributions exactly cancel and $\mu=1$.

Before discussing the further consequences of (3.3) it is necessary to examine the imaginary parts of π_L and π_T. The imaginary parts arise from integrals of the form

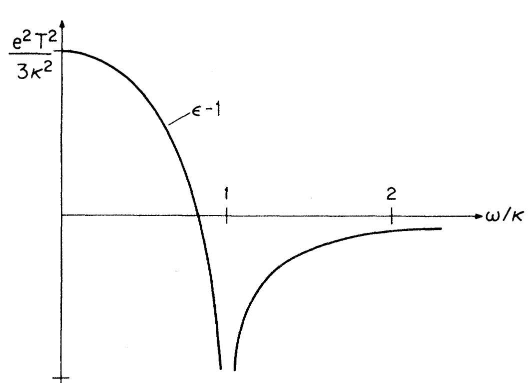

FIG. 1. The real part of the "electric" susceptibility, $\epsilon(k,\omega)-1$, as a function of ω for fixed k.

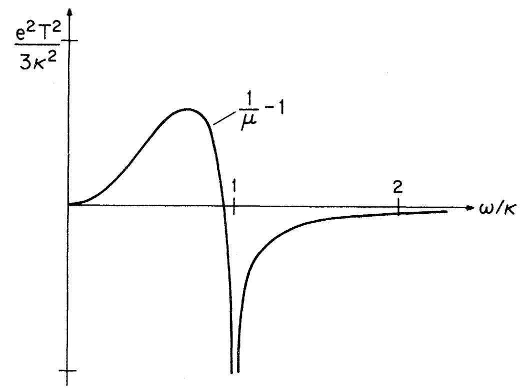

FIG. 2. The real part of the "magnetic" susceptibility, $1/\mu(k,\omega)-1$, as a function of ω for fixed k.

$$I(k,\omega)=\int \frac{d^4p}{(2\pi)^4}\delta((p+K)^2-m_1{}^2)\times\delta(p^2-m_2{}^2)f(p,K,T)\ , \tag{3.6}$$

where f contains all the finite-temperature factors from the propagators which exponentially suppress the contributions of large p. We are interested in whether $I(k,\omega)$ can be as large as the real parts (3.3) for $T\gg\omega$, $T\gg k$. To answer this it is essential to note that the δ functions in (3.6) allow both positive and negative energy contributions. [See the discussion following (2.10).] Consequently there are three regions of ω and k to be considered. (a) For $\omega^2-k^2>(m_1+m_2)^2$ the p integrations in (3.6) are limited by ordinary two-body phase space to be of order ω or k and can never be as large at T. Thus $I(k,\omega)$ is of order ω^2, k^2, or ωk and negligible compared to the real parts (3.3). (b) For $0<\omega^2-k^2<(m_1+m_2)^2$ $I(k,\omega)=0$ and there is no imaginary part at all. (c) For $\omega^2-k^2<0$ the p integrations in (3.6) are not limited by the δ functions to a finite range as may be easily verified. Instead it is the Boltzmann suppression factors in f that limit $p=O(T)$. In this region $I(k,\omega)\sim O(T^2)$; this serve damping precludes the existence of any reasonably stable normal modes.

B. Longitudinal properties

The behavior of the fields and currents that are longitudinal to the spacelike vector $\tilde{K}^\mu$ (i.e., with no three-curl) is determined by π_L. This function is necessarily zero at $K^2=0$ as is evident from the definition (3.2) or from the explicit form (3.3). Reference to (2.22), (2.26), and (2.30) shows that the correlation functions and the field equations depend on $\epsilon(k,\omega)=(K^2-\pi_L)/K^2$, which does not have the kinematic zero at $K^2=0$. This function has zeros at values of ω and k shown in Fig. 3, whose physical consequences are now examined.

1. Plasma oscillations

The analytic form of the dispersion relation for small k is

$$\omega^2=\frac{e^2T^2}{9}+\frac{3}{5}k^2+\cdots \quad (\omega\sim eT/3)\ . \tag{3.7a}$$

Because of (2.30), at any such frequency for which $\epsilon=0$, an oscillating longitudinal "electric" field can exist in the plasma with no external currents at all. This is a true normal mode of the plasma produced by a coherent density wave of the charged particles. At extremely high frequencies the dispersion relation goes over into

$$\omega^2=k^2+4k^2\exp\left[\frac{-6k^2}{e^2T^2}\right]+\cdots \quad (\omega\gg eT/3)\ . \tag{3.7b}$$

This mode exists entirely within the region $\omega^2>k^2$; consequently the damping is produced by pair production and contributes an imaginary part to π_L of order $e^2\omega^2$ rather than e^2T^2 as discussed following (3.6). The corresponding imaginary part on the right-hand side of (3.7) is of order e^4T^2 and the oscillation is therefore quite stable against decay.

2. Dynamical screening

The plasma oscillations do not exist at frequencies $\omega<eT/3$ because the wave number k in Fig. 3

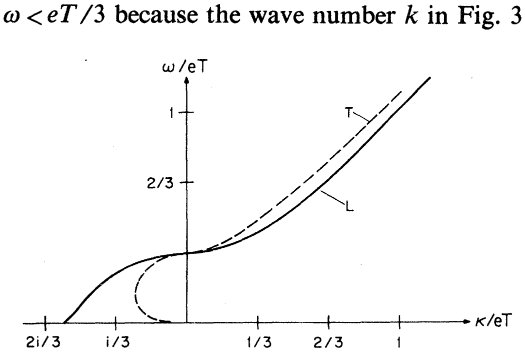

FIG. 3. Location of the poles in the gauge boson propagator as determined by $\omega^2=k^2+\mathrm{Re}[\pi_L(k,\omega)]$ and $\omega^2=k^2+\mathrm{Re}[\pi_T(k,\omega)]$ for $\omega^2>k^2$. In this region the imaginary parts of π_L and π_T are negligible. Note that for $\omega<eT/3$, both the longitudinal and transverse solutions have imaginary wave vector k.

is pure imaginary. For ω just below $eT/3$, (3.7a) determines k; at lower frequencies (3.7a) ceases to apply and is replaced at very small frequencies by

$$k=i\left[\frac{eT}{\sqrt{3}}-\frac{\pi\omega}{4}+\cdots\right] \quad (\omega\ll eT/3)\ . \tag{3.8}$$

Although there are no stable normal modes of the plasma in the frequency range $0<\omega<eT/3$, there is collective behavior. This is signalled by the screening of external charges introduced into the plasma. To analyze the "electric" field it is useful to use the Lorentz-invariant time $\tau=x^\alpha u_\alpha$ and an orthogonal spacelike coordinate $\tilde{x}^\alpha$:

$$x^\alpha\equiv u^\alpha\tau+\tilde{x}^\alpha\ .$$

The τ dependence of $\tilde{E}(\tilde{x},\tau)$ may be Fourier transformed to frequency dependence: $\tilde{E}(\tilde{x},\omega)$. The longitudinal "electric" field (2.30) produced by a longitudinal current flow, or equivalently by the charge density (2.21), is

$$\tilde{E}_L^\nu(\tilde{x},\omega)=-\frac{\partial}{\partial\tilde{x}_\nu}\int d^3\tilde{y}G_L(\tilde{x}-\tilde{y},\omega)u^\alpha J_\alpha(\tilde{y},\omega)\ ,$$

$$G_L(\tilde{x}-\tilde{y},\omega)\equiv\int\frac{d^3\tilde{K}}{(2\pi)^3}e^{-i\tilde{K}\cdot(\tilde{x}-\tilde{y})}\frac{1}{k^2\epsilon(k,\omega)}\ ,$$

where $\tilde{K}^2=-k^2$. For $\omega<eT/3$ this Green's function falls off exponentially in $\tilde{x}-\tilde{y}$. In particular, at very low frequencies

$$G_L(\tilde{x}-\tilde{y},\omega)=\frac{e^{-M_L|\tilde{x}-\tilde{y}|}}{4\pi|\tilde{x}-\tilde{y}|}\ ,$$

$$M_L\equiv\frac{eT}{\sqrt{3}}-\frac{\pi\omega}{4}+\cdots \quad (\omega\ll eT/3)$$

so that the longitudinal "electric" field has a frequency-dependent, or dynamic, screening length $M_L{}^{-1}$ that characterizes the exponential falloff.

C. Transverse properties

In (2.22), correlations of the currents transverse to the spacelike vector $\tilde{K}^\mu$ (i.e., with vanishing three-divergence) are determined by $K^2-\pi_T$. Moreover the field set up in the plasma due to the application of an external transverse current (2.31) depends on this same function because

$$\omega^2\epsilon-\frac{k^2}{\mu}=K^2-\pi_T \tag{3.9}$$

as follows from the definitions (2.26). The loci of the zeros of $K^2-\pi_T$ in the ω-k plane are shown in Fig. 3.

1. Plane-wave propagation

For frequencies above $eT/3$ the zeros of $K^2-\pi_T$ occur at real k and ω. For small k the analytic form of the dispersion relation is

$$\omega^2=\frac{e^2T^2}{9}+\tfrac{6}{5}k^2+\cdots \quad (\omega\sim eT/3) \qquad (3.10a)$$

and for very high frequencies it goes over into

$$\omega^2=k^2+\frac{e^2T^2}{6}+\cdots \quad (\omega\gg eT/3) . \qquad (3.10b)$$

There are qualitatively similar to the longitudinal dispersion relations (3.7) but the physical consequences are quite different. At these frequencies, because the left-hand side of (2.31) vanishes it is possible for a transverse "electric" field to exist even without an external current. The transverse "electric" field is accompanied by a (transverse) "magnetic" induction because of Faraday's law (2.28b). Thus $\widetilde{E}$, $\widetilde{B}$, and $\widetilde{K}$ are mutually perpendicular[19] and span the three-space orthogonal to the timelike fluid velocity u_μ. The plane wave propagates with a phase velocity greater than one since

$$\omega/k=(\epsilon\mu)^{-1/2}>1 ,$$

as can be seen from Fig. 3.

2. Dynamical screening of transverse currents

No plane wave can propagate at frequencies $\omega<eT/3$ because the wave number k in Fig. 3 becomes imaginary. In laboratory plasmas with boundaries, the nonpenetration of waves gives rise to the skin effect, but this has no application in the early universe. Here the phenomenon of more interest is the screening of the transverse part of the current. The dispersion relation in Fig. 3 may be approximated by (3.10a) for ω just below $eT/3$; as ω nears zero a useful approximation is

$$k=i\left[\frac{\pi e^2T^2\omega}{12}\right]^{1/2}+\cdots \quad (\omega\ll eT/3) . \qquad (3.11)$$

From (2.31) and (2.28b) the transverse "electric" field and "magnetic" induction produced by an external current is

$$\langle \widetilde{E}_T^\alpha(\widetilde{x},\omega)\rangle=i\omega\int d^3\widetilde{y}G_T(\widetilde{x}-\widetilde{y},\omega)J_T^\alpha(\widetilde{y},\omega) ,$$

$$\langle \widetilde{B}_T^\alpha(\widetilde{x},\omega)\rangle=\frac{i}{\omega}\epsilon^{\alpha\beta\gamma\lambda}u_\beta\partial_\gamma\langle \widetilde{E}_{T,\lambda}(\widetilde{x},\omega)\rangle ,$$

$$G_T(\widetilde{x}-\widetilde{y},\omega)\equiv\int\frac{d^3\widetilde{K}}{(2\pi)^3}\frac{e^{-i\widetilde{K}\cdot(\widetilde{x}-\widetilde{y})}}{\omega^2-k^2-\pi_T(k,\omega)} ,$$

where $\widetilde{K}^2=-k^2$ as usual. For $\omega<eT/3$ but $\omega\neq0$ both $\widetilde{E}$ and $\widetilde{B}$ are exponentially screened. For very small ω this is shown explicitly by using (3.11) to obtain

$$G_T(\widetilde{x}-\widetilde{y},\omega)=\frac{e^{-M_T|\widetilde{x}-\widetilde{y}|}}{6\pi|\widetilde{x}-\widetilde{y}|} ,$$

$$M_T=\left[\frac{\pi e^2T^2\omega}{12}\right]^{1/3} \quad (\omega\ll eT/3) . \qquad (3.12)$$

However, $\widetilde{B}_T$ is not screened for $\omega=0$ because of the vanishing of M_T. (There is no problem with the transverse "electric" field because $\widetilde{E}_T=0$ at $\omega=0$.)

IV. DISCUSSION

The results found here for an ultrarelativistic neutral plasma ($T\gg m$) of particles and antiparticles are summarized in Table I. It is interesting to compare these with the standard results for a simple nonrelativistic, neutral plasma ($T\ll m$) of

TABLE I. Summary of the coherent plasma effects with $\omega_p=eT/3$ determined from Fig. 3 and $e^2\equiv(N_f+2N)g^2/2$.

Frequency	Corresponding wave number	Longitudinal	Transverse
$\omega>\omega_p$	k=real	Plasma oscillations of $\widetilde{E}_L$ and $u^\alpha J_\alpha$	Plane waves in $\widetilde{E}_T$ and $\widetilde{B}_T$
$\omega<\omega_p$	k=imaginary	Dynamical screening of $\widetilde{E}_L$	Dynamical screening of $\widetilde{E}_T$ and $\widetilde{B}_T$ (except $\omega=0$)

mobile electrons and immobile positive ions, each with the same density n.[5] In both cases there is a plasma frequency [$\omega_p = eT/3$ vs $\omega_p = (ne^2/T)^{1/2}$].

1. *Normal modes* ($\omega > \omega_p$)

The transverse modes of the two plasmas are qualitatively similar; plane waves propagate in the direction of $\widetilde{E} \times \widetilde{B}$ at a phase velocity $\omega/k > 1$ in both cases. The longitudinal modes are not so similar: at high temperatures $\omega/k > 1$ always. At low temperatures the phase velocity can be less than one when k is large.

The modes with $\omega/k < 1$ are associated with two phenomena not found in Sec. III: (a) the modes with very low phase velocity $\omega/k < (T/m)^{1/2} \ll 1$ suffer severe Landau damping because they accelerate the charges moving slower and decelerate those moving faster than the wave; (b) The modes with intermediate phase velocities $(T/m)^{1/2} < \omega/k < 1$ are not damped. These are unique in that they can be excited by a charge particle moving through the plasma at constant velocity, thus producing a current

$$J^\mu(K) = 2\pi Q(u^\mu + \widetilde{V}^\mu)\delta(\omega + \widetilde{K}\cdot\widetilde{V}) ,$$

where $\widetilde{V}$ is the spacelike particle velocity ($\widetilde{V}^2 \equiv -V^2 < 0$) relative to the fluid motion ($\widetilde{V}\cdot u = 0$). The support of the δ function allows the moving charge to emit waves along $\widetilde{K}$ at an angle θ relative to $\widetilde{V}$ that satisfies

$$V\cos\theta = \omega/k .$$

This Čerenkov emission of the longitudinal plasma oscillations [$(T/m)^{1/2} < \omega/k < 1$] is produced only by particles moving faster than the speed of sound in the plasma [$(T/m)^{1/2} < V < 1$].

In the high-temperature plasma all modes have $\omega/k > 1$ as shown in Fig. 3; consequently charged particles (with $V < 1$) propagate rather freely and do not lose energy by Čerenkov emission. The same argument shows that virtual charged particles (with $V > 1$) do suffer energy loss by Čerenkov emission. This is confirmed by the fact that the self-energy of a charged particle has a much larger imaginary part for spacelike momenta than for timelike momenta in accordance with the discussion of (3.6).

2. *Dynamical Screening* ($\omega < \omega_p$)

The frequency-dependent screening of both longitudinal and transverse fields with $\omega < \omega_p$ occurs in both the high- and low-temperature plasmas. However, they differ in that for $T \gg m$ the screening mass of the transverse "magnetic" induction (3.12) vanishes as $\omega \to 0$.[20] This difference is likely due to the fact that the low-temperature plasma effectively has a chemical potential that is not present in the high-temperature plasma considered here.[21]

It is perhaps worth noting that the vanishing of the screening mass like $\omega^{1/3}$ in (3.12) implies that when the frequency dependence of $\widetilde{B}(\widetilde{x},\omega)$ is Fourier transformed, the resulting time-dependent "magnetic" induction will fall faster than at zero temperature by two additional powers of $\widetilde{x}$. Thus $\widetilde{B}(\widetilde{x},\tau) \sim |\widetilde{x}|^{n-2}/e^2T^2$ instead of $\widetilde{x}^{-n}$ for each multipole.

ACKNOWLEDGMENTS

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APPENDIX: COMPUTATION OF $\pi_{\mu\nu}$

The analysis of Sec. III is based on the perturbative calculation of the vacuum polarization tensor to order g^2 for an SU(N) gauge theory with N_f massless fermion fields in the fundamental representation.[4] Only the real part of $\pi_{\mu\nu}$ will be calculated. Because its tensor structure is known, it is only necessary to calculate four Lorentz-invariant functions:

$$\begin{aligned}\mathrm{Re}(\eta^{\mu\nu}\pi_{\mu\nu}) &\equiv g^2N_fG_f(k,\omega) + g^2NG_b(k,\omega) ,\\ \mathrm{Re}(u^\mu u^\nu\pi_{\mu\nu}) &\equiv g^2N_fH_f(k,\omega) + g^2NH_b(k,\omega) .\end{aligned} \tag{A1}$$

The $T=0$ vacuum polarization (dimensionally regularized) is

$$\pi_{\mu\nu}(T=0) = g^2\left[\frac{2N_f - 5N}{48\pi^2}\right]\ln\left[\frac{-K^2}{\sigma^2}\right] \times (K^2\eta_{\mu\nu} - K_\mu K_\nu) , \tag{A2}$$

where σ is the renormalization scale. The finite-T corrections to this will be denoted by a prime:

$$\pi_{\mu\nu} = \pi_{\mu\nu}(T=0) + \pi'_{\mu\nu} . \tag{A3}$$

These corrections can be isolated because the entire

T dependence of the covariant propagators occurs in the combinations

$$\Gamma(p)=2\pi\delta(p^2)n(p)\ ,\qquad n(p)=(e^{|p\cdot u/T|}\pm 1)^{-1} \tag{A4}$$

for massless fermions and bosons, respectively. [See (2.10).]

Step 1. Performing the fermion trace in (3.1) and the analogous sums for the two gauge-boson loops and the one ghost loop yields, in the Feynman gauge,[17]

$$\begin{aligned}\mathrm{Re}\{\pi'_{\mu\nu}\}=&-2g^2N_f\int\frac{d^4p}{(2\pi)^4}I_{\mu\nu}\Gamma_f(p)\frac{\mathscr{P}}{(K+p)^2}\\&-2g^2N\int\frac{d^4p}{(2\pi)^4}(I_{\mu\nu}+K^2\eta_{\mu\nu}-K_\mu K_\nu)\\&\times\Gamma_b(p)\frac{\mathscr{P}}{(K+p)^2}\ ,\end{aligned}$$

where $\mathscr{P}$ denotes the principal value and

$$\begin{aligned}I_{\mu\nu}=&\ 2(K+p)_\mu p_\nu+2(K+p)_\nu p_\mu\\&-2(p^2+p\cdot K)\eta_{\mu\nu}\ .\end{aligned}$$

Contracting $\pi_{\mu\nu}$ as in (A1) yields four Lorentz-invariant integrals. In each of these the first three integrations are straightforward if the p^0 axis is chosen parallel to u^μ and the p^3 axis chosen parallel to $\tilde{K}^\mu$ for the two angular integrations. In each case there remains a nontrivial integral over $|\vec{p}|\equiv p$ containing the functions

$$\begin{aligned}L(p)\equiv&\ln\left[\frac{p+\omega_+}{p+\omega_-}\right]-\ln\left[\frac{p-\omega_+}{p-\omega_-}\right]\ ,\\M(p)\equiv&\ (p+\omega_+)(p+\omega_-)\ln\left[\frac{p+\omega_+}{p+\omega_-}\right]\\&-(p-\omega_+)(p-\omega_-)\ln\left[\frac{p-\omega_+}{p-\omega_-}\right]\ ,\\\omega_\pm\equiv&\ \tfrac{1}{2}(\omega\pm k)\end{aligned} \tag{A5}$$

with ω and k as defined in (2.12). The logarithms are to be understood in the principal-value sense:

$$\ln(p+x)\equiv\tfrac{1}{2}\ln(p+x-i\epsilon)+\tfrac{1}{2}\ln(p+x+i\epsilon)\ . \tag{A6}$$

The integrals to be evaluated are

$$\begin{aligned}G'_f(k,\omega)=&\int_0^\infty\frac{dp}{2\pi^2}\left[4p+\frac{K^2}{2k}L(p)\right]n_f(p)\ ,\\G'_b(k,\omega)=&\int_0^\infty\frac{dp}{2\pi^2}\left[4p+\left[\frac{5}{4}\right]\frac{K^2}{k}L(p)\right]n_b(p)\ ,\\H'_f(k,\omega)=&\int_0^\infty\frac{dp}{2\pi^2}\left[2p\left[1-\frac{\omega}{k}\ln\left|\frac{\omega_+}{\omega_-}\right|\right]\right.\\&\left.+\frac{1}{k}M(p)\right]n_f(p)\ ,\\H'_b(k,\omega)=&\int_0^\infty\frac{dp}{2\pi^2}\left[2p\left[1-\frac{\omega}{k}\ln\left|\frac{\omega_+}{\omega_-}\right|\right]\right.\\&\left.+\frac{1}{k}M(p)-\frac{k}{4}L(p)\right]n_b(p)\ .\end{aligned} \tag{A7}$$

For the special limit $k=0$ the integrands become polynomials in p so that performing the p integration yields functions G' and H' that are analytic (in fact polynomials) in ω. However, it is not possible to obtain high-temperature expansions ($T\gg k$, $T\gg\omega$) by expanding the integrand for small k and ω because the resulting p integrations diverge.

Step 2. The high-temperature expansion of (A7) is obtained by first expanding the thermal distribution function as

$$n_i(p)=\sum_{m=1}^{\infty}\sigma_i(m)e^{-m\beta p}$$

with $\beta\equiv 1/T$, $\sigma_f(m)=(-1)^{m+1}$, and $\sigma_b(m)=1$. A typical integral to be evaluated is

$$I=\int_0^\infty dp\ln(p-A+i\epsilon)\epsilon^{-m\beta p}\ ,$$

where the $i\epsilon$ prescription (A6) is anticipated. To evaluate I, first integrate by parts

$$I=\frac{\ln(-A+i\epsilon)}{m\beta}+\frac{1}{m\beta}\int_0^\infty dp\frac{e^{-m\beta p}}{p-A+i\epsilon}\ .$$

Since the singularity of this integrand is in the lower-half p plane, one may distort the p contour to run along the positive imaginary axes from 0 to $i\infty$ and then in an arc from $i\infty$ to $+\infty$. The contribution of the arc vanishes exponentially. For the integral along the imaginary axis, put $p=ip'$ (p' real) to obtain

$$I=\frac{\ln(-A+i\epsilon)}{m\beta}+\frac{1}{m\beta}\int_0^\infty dp'\frac{e^{-im\beta p'}}{p'+iA}\ .$$

Computing the principal value in the sense of (A6) amounts to taking the real part of I:

$$\int_0^\infty dp \ln(p-A)e^{-\beta mp}=\frac{\ln|A|}{m\beta}+\frac{1}{m\beta}\int_0^\infty dx\frac{x\cos(\beta Ax/\pi)-m\pi\sin(\beta Ax/\pi)}{x^2+(m\pi)^2}, \tag{A8}$$

where the dimensionless integration variable x is related to p' by $p'=xA/m\pi$.

Equation (A8) can be applied to all the integrals in (A7). For example, that part of (A8) which is odd in A gives

$$\int_0^\infty dp \ln\left[\frac{p-A}{p+A}\right]e^{-\beta mp}=\frac{-2\pi}{\beta}\int_0^\infty dx\frac{\sin(\beta Ax/\pi)}{x^2+(m\pi)^2}$$

and the sum over m is elementary[22]:

$$\int_0^\infty dp \ln\left[\frac{p-A}{p+A}\right]n_i(p)=\frac{-2\pi}{\beta}\int_0^\infty dx\sin(\beta Ax/\pi)U_i(x),$$

$$U_f(x)\equiv\sum_{m=1}^\infty\frac{(-1)^{m+1}}{x^2+(m\pi)^2}=\frac{1}{2x^2}-\frac{1}{2x\sinh x}, \tag{A9}$$

$$U_b(x)\equiv\sum_{m=1}^\infty\frac{1}{x^2+(m\pi)^2}=\frac{1}{2x}-\frac{1}{2x^2}+\frac{e^{-x}}{2x\sinh x}.$$

This is immediately applicable to G_f' and G_b' in (A7). To compute H_f' and H_b', first differentiate (A8) with respect to β to construct the appropriate polynomial in p multiplying $\ln(p-A)$. The integration variable x on the right-hand side of (A8) is judiciously chosen so that after differentiating with respect to β, the summation over m will be elementary. The two new sums that enter are[22]

$$V_f(x)\equiv\sum_{m=1}^\infty\frac{(-1)^{m+1}}{(m\pi)^2[x^2+(m\pi)^2]}=\frac{1}{12x^2}-\frac{1}{2x^4}+\frac{1}{2x^3\sinh x}, \tag{A10}$$

$$V_b(x)\equiv\sum_{m=1}^\infty\frac{1}{(m\pi)^2[x^2+(m\pi)^2]}=\frac{1}{6x^2}-\frac{1}{2x^3}+\frac{1}{2x^4}-\frac{e^{-x}}{2x^3\sinh x}.$$

Note that the functions $U_i(x)$ and $V_i(x)$ are positive definite and finite for all real x. In applying (A8) in this manner, all the ω and k dependence is contained in

$$S(x)=\sin(\beta\omega_+x/\pi)-\sin(\beta\omega_-x/\pi),$$

$$C(x)=\cos(\beta\omega_+x/\pi)+\cos(\beta\omega_-x/\pi).$$

The integrals in (A7) may then be written

$$G_f'(k,\omega)=\frac{T^2}{6}+\frac{T}{2\pi}\left[\frac{K^2}{k}\right]\int_0^\infty dxS(x)U_f(x),\quad G_b'(k,\omega)=\frac{T^2}{3}+\frac{5T}{4\pi}\left[\frac{K^2}{k}\right]\int_0^\infty dxS(x)U_b(x),$$

$$H_f'(k,\omega)=\frac{2\pi T^3}{k}\int_0^\infty dxS(x)V_f(x)-T^2\int_0^\infty dxxC(x)V_f(x), \tag{A11}$$

$$H_b'(k,\omega)=\frac{2\pi T^3}{k}\int_0^\infty dxS(x)V_b(x)-T^2\int_0^\infty dxxC(x)V_b(x)-\frac{Tk}{4\pi}\int_0^\infty dxS(x)U_b(x).$$

Step 3. A complete high-temperature expansion of (A11) is now possible. For example, G_f' requires

$$\int_0^\infty dx\sin(Bx)\left[\frac{1}{2x^2}-\frac{1}{2x\sinh x}\right], \tag{A12}$$

where $B=\omega_\pm/\pi T$. Although this integrand is finite at $x=0$, the two pieces of the integrand are not. Introducing a regulator x^λ with $\lambda>0$ allows each intergration to be performed separately using[22]

$$\int_0^\infty dxx^{\lambda-2}\sin(Bx)=-\epsilon(B)|B|^{1-\lambda}\Gamma(\lambda-1)\cos\left[\frac{\lambda\pi}{2}\right],$$

$$\int_0^\infty dxx^{\lambda-1}\frac{\sin(Bx)}{\sinh x}=2\sum_{n=0}^\infty\frac{(-1)^nB^{2n+1}}{(2n+1)!}\Gamma(2n+\lambda+1)\zeta(2n+\lambda+1)\left[1-\frac{1}{2^{2n+\lambda+1}}\right]. \tag{A13a}$$

Both these integrals have a pole in λ at $\lambda=0$ but the pole cancels in (A12) and a finite answer results. Thus G_f' may be evaluated. To compute G_b' one needs[22]

$$\int_0^\infty dx x^{\lambda-1}\frac{\sin(Bx)e^{-x}}{\sinh x}=\sum_{n=0}^{\infty}\frac{(-1)^n B^{2n+1}}{(2n+1)!}\frac{\Gamma(2n+\lambda+1)\zeta(2n+\lambda+1)}{2^{2n+\lambda}}\ . \tag{A13b}$$

The necessary formulas for H_f' and H_b' are obtained by differentiating (A13) with respect to B.

A useful check on the results, in addition to the cancellation of singularities in λ, is that the $T=0$ terms $\ln(\sigma^2/K^2)$ in (A2) always combine with $\ln(K^2/\pi^2T^2)$ to produce $\ln(\sigma^2/\pi^2T^2)$. The final results are

$$G_f(k,\omega)=\frac{T^2}{6}-\frac{K^2}{4\pi^2}\left[\gamma-1+\frac{\omega}{2k}\ln\left|\frac{\omega_+}{\omega_-}\right|+\ln(\sigma/\pi T)\right]+O(T^{-2})\ ,$$

$$G_b(k,\omega)=\frac{T^2}{3}+\frac{5K^2}{8\pi^2}\left[\frac{T}{\omega}\theta(-K^2)+\gamma-1+\frac{\omega}{2k}\ln\left|\frac{\omega_+}{\omega_-}\right|+\ln(\sigma/4\pi T)\right]+O(T^{-2})\ ,$$

$$H_f(k,\omega)=\frac{T^2}{6}\left[1-\frac{\omega}{2k}\ln\left|\frac{\omega_+}{\omega_-}\right|\right]+\frac{\omega^2}{24\pi^2}\left[1-\frac{\omega}{k}\ln\left|\frac{\omega_+}{\omega_-}\right|\right]+\frac{k^2}{12\pi^2}[\gamma-\tfrac{4}{3}+\ln(\sigma/\pi T)]+O(T^{-2})\ ,$$

$$H_b(k,\omega)=\frac{T^2}{3}\left[1-\frac{\omega}{2k}\ln\left|\frac{\omega_+}{\omega_-}\right|\right]-\frac{T}{4}\left[k-\frac{\omega^2}{2k}\right]\theta(-K^2)-\frac{\omega^2}{24\pi^2}\left[1-\frac{\omega}{2k}\ln\left|\frac{\omega_+}{\omega_-}\right|\right]-\frac{\omega k}{8\pi^2}\ln\left|\frac{\omega_+}{\omega_-}\right|$$
$$+\frac{5k^2}{24\pi^2}[\gamma-\tfrac{17}{15}-\ln(\sigma/4\pi T)]+O(T^{-2})\ .$$

Note that even the T^2 contribution to H_f and H_b required keeping the integrands $L(p)$ and $M(p)$ in (A7). With these results, the longitudinal and transverse self-energies in (3.2) are obtained by

$$\mathrm{Re}\{\pi_L\}=-\frac{K^2}{k^2}g^2(N_fH_f+NH_b),\quad \mathrm{Re}\{\pi_T\}=-\tfrac{1}{2}\,\mathrm{Re}\{\pi_L\}+\frac{g^2}{2}(N_fG_f+NG_b)\ .$$

The terms of order T^2 are displayed in (3.3).

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[5]For a readable account of modern plasma physics see S. Ichimaru, *Basic Principles of Plasma Physics* (Benjamin, Reading, Mass. 1973).

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[11]L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974), particularly Appendix A.

[12]For off-equilibrium states the first law (2.1) still holds but the Gibbs relation acquires an additional term which describes heat flux and viscosity.

[13]In the fluid rest frame ($\vec{u}=0$) these tensors reduce to those used in Ref. 3.

[14]See, for example, A. D. Linde, Ref. 2.

[15]One can also define the "electric" induction $\tilde{D}=\epsilon\tilde{E}$ and "magnetic" field $\tilde{H}=(1/\mu)\tilde{B}$, but these will not be used. The covariant relations are due to Minkowski. See Ref. 6, p. 10, Ref. 7, p. 99, or L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), p. 243.

[16]Alternatively one can relate $\tilde{E}$ to the induced current by $\tilde{J}_L^{\mathrm{ind}}=\sigma_L\tilde{E}_L$ and $\tilde{J}_T^{\mathrm{ind}}=\sigma_T E_T$. The longitudinal and transverse conductivities are $\sigma_L=i\omega(1-\epsilon)=i\omega\pi_L/K^2$ and $\sigma_T=i(k^2/\mu-\omega^2\epsilon+K^2)/\omega=i\pi_T/\omega$.

[17]The leading T^2 contribution is gauge invariant. See Y. Ueda, Phys. Rev. D 23, 1383 (1981).

[18]In contrast, the nonrelativistic plasma is paramagnetic; the intrinsic spin magnetic moment there dominates the induced orbital magnetic moment by a factor of 3.

[19]In an arbitrary frame, the conventional description of these waves in terms of the three-vectors $\vec{E}$, $\vec{B}$, and $\vec{K}$ is more complicated: $\vec{E}$ and $\vec{B}$ are perpendicular to each other but not to $\vec{K}$.

[20]Order-g^4 corrections do give a finite screening length for the static magnetic induction as discussed in Ref. 4. See also M. Dine and W. Fischler, Phys. Lett. 105B, 207 (1981); T. W. Appelquist and R. T. Pisarski, Phys. Rev. D 23, 2291 (1981); A. Billoire, G. Lazarides, and Q. Shafi, Phys. Lett. 103B, 450 (1981); T. A. DeGrand and D. Toussaint, Phys. Rev. D 25, 526 (1982).

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