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ON A SYSTEMATIC DERIVATION OF THE QUARK-ANTIQUARK POTENTIAL

A. BARCHIELLI, E. MONTALDI and G.M. PROSPERI

Dipartimento di Fisica dell'Università di Milano and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy

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A systematic method of deriving the quark-antiquark potential is developed. The whole procedure is organized as a $1/c$ expansion. Up to order $1/c^2$, the Eichten-Feinberg spin-dependent potential is recovered and new velocity-dependent terms are obtained. An essential step of the derivation is the identification of a Pauli-type two-particle propagator, which satisfies a suitable Schrödinger equation. The short- and large-distance behaviour of the potentials is studied.

1. Introduction and new results

In the last years considerable attention has been paid in the literature to the problem of describing the interaction between quarks by a nonrelativistic potential [1]. The goal is to reproduce the spectrum of mesons and baryons by solving an appropriate Schrödinger equation. Purely phenomenological and semitheoretical potentials have been used. In principle such a potential should be derivable from QCD, which is believed to be the correct field theory for strong interactions.

The various methods traditionally used for deriving a potential between two charged particles from QED rest more or less directly on the particularly simple properties of the Coulomb gauge or on perturbative arguments. Both fail in QCD, due to the conceptual and formal complications of the Coulomb gauge in non-abelian gauge theories and to the essential inapplicability of perturbative methods to a theory which is believed to produce confinement.

In this context important progress has been made by Wilson [2], who essentially derived a potential between a static quark and a static antiquark in terms of a functional integral involving the so called Wilson loop. The Wilson potential can be written as

$$V(r) = \lim_{\tau \to +\infty} \frac{i}{\tau} \log \left\langle \tfrac{1}{3} \mathrm{Tr}\, \mathrm{P}_{\Gamma_\mathrm{W}} \exp \left[\frac{ig}{c} \oint_{\Gamma_\mathrm{W}} \mathrm{d}x_\mu A^\mu(x) \right] \right\rangle , \tag{1.1}$$

where Γ_W is a rectangular loop with a temporal side of length $c\tau$ and a spatial side of length r. Here as usual $A^{\mu}(x) \equiv \sum_a A^{\mu}_a(x) T^a$, Tr denotes the trace over the gauge matrices and $\mathrm{P}_{\Gamma_\mathrm{W}}$ the path-ordering operator for the gauge matrices. Finally, we have set

$$\langle f[A] \rangle := \int \mathscr{D}[A] f[A] \exp(iS[A]) \Big/ \int \mathscr{D}[A] \exp(iS[A]), \tag{1.2}$$

$S[A]$ being the gauge-field action (while we take $\hbar = 1$, we find convenient to keep $c \neq 1$, as we shall explain).

The first nonstatic spin-dependent corrections V_{sd} to the potential (1.1) have been considered by Eichten and Feinberg [3], (see also refs. [4–6]) who obtained

$$\begin{aligned} V_{\mathrm{sd}} = & \left(\frac{1}{m_1^2} \boldsymbol{L}_1 \cdot \boldsymbol{S}_1 + \frac{1}{m_2^2} \boldsymbol{L}_2 \cdot \boldsymbol{S}_2 \right) \frac{1}{2c^2 r} \frac{\mathrm{d}}{\mathrm{d}r} \left(V(r) + 2V_1(r) \right) \\ & + \left(\boldsymbol{L}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{L}_2 \cdot \boldsymbol{S}_1 \right) \frac{1}{m_1 m_2 c^2 r} \frac{\mathrm{d}}{\mathrm{d}r} V_2(r) \\ & + \frac{1}{m_1 m_2 c^2 r^2} \left(\frac{1}{r^2} \boldsymbol{r} \cdot \boldsymbol{S}_1 \boldsymbol{r} \cdot \boldsymbol{S}_2 - \tfrac{1}{3} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \right) V_3(r) + \frac{1}{3 m_1 m_2 c^2} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 V_4(r), \end{aligned} \tag{1.3}$$

where V is given by eq. (1.1), while for V_1, V_2, V_3, V_4 we have

$$\frac{\boldsymbol{r}}{r} \frac{\mathrm{d}V_1(r)}{\mathrm{d}r} = - \lim_{\tau \to +\infty} \tfrac{1}{2} i g^2 \int_{-\tau/2}^{\tau/2} \mathrm{d}t\, ct \langle\langle \boldsymbol{B}(t, \boldsymbol{x}_1) \times \boldsymbol{E}(0, \boldsymbol{x}_1) \rangle\rangle_{\Gamma_\mathrm{W}}, \tag{1.4a}$$

$$\frac{\boldsymbol{r}}{r} \frac{\mathrm{d}V_2(r)}{\mathrm{d}r} = \lim_{\tau \to +\infty} \frac{i g^2}{2\tau} \int_{-\tau/2}^{\tau/2} \mathrm{d}t\, \mathrm{d}t'\, ct' \langle\langle \boldsymbol{B}(t, \boldsymbol{x}_1) \times \boldsymbol{E}(t', \boldsymbol{x}_2) \rangle\rangle_{\Gamma_\mathrm{W}}, \tag{1.4b}$$

$$\begin{aligned} & \frac{1}{r^2} \left(\frac{r^h r^k}{r^2} - \tfrac{1}{3} \delta^{hk} \right) V_3(r) \\ & \quad = \lim_{\tau \to +\infty} \tfrac{1}{2} i g^2 \int_{-\tau/2}^{\tau/2} \mathrm{d}t \langle\langle B^h(t, \boldsymbol{x}_1) B^k(0, \boldsymbol{x}_2) + B^k(t, \boldsymbol{x}_1) B^h(0, \boldsymbol{x}_2) \\ & \qquad - \tfrac{2}{3} \delta^{hk} \boldsymbol{B}(t, \boldsymbol{x}_1) \cdot \boldsymbol{B}(0, \boldsymbol{x}_2) \rangle\rangle_{\Gamma_\mathrm{W}}, \end{aligned} \tag{1.4c}$$

$$\begin{aligned} V_4(r) = & \, i g^2 \lim_{\tau \to +\infty} \int_{-\tau/2}^{\tau/2} \mathrm{d}t \langle\langle \boldsymbol{B}(t, \boldsymbol{x}_1) \cdot \boldsymbol{B}(0, \boldsymbol{x}_2) \rangle\rangle_{\Gamma_\mathrm{W}} \\ = & \, 2\Delta_2 V_2(r) - \lim_{\tau \to +\infty} \frac{g^3}{\tau} \\ & \times \int_{-\tau/2}^{\tau/2} \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3 \Big[\langle\langle \boldsymbol{B}(t_1, \boldsymbol{x}_2) \times \boldsymbol{E}(t_2, \boldsymbol{x}_2) \cdot \boldsymbol{E}(t_3, \boldsymbol{x}_2) \rangle\rangle_{\Gamma_\mathrm{W}} \\ & - \langle\langle \boldsymbol{B}(t_1, \boldsymbol{x}_1) \times \boldsymbol{E}(t_2, \boldsymbol{x}_2) \rangle\rangle_{\Gamma_\mathrm{W}} \cdot \langle\langle \boldsymbol{E}(t_3, \boldsymbol{x}_2) \rangle\rangle_{\Gamma_\mathrm{W}} \Big]. \end{aligned} \tag{1.4d}$$

Here the index 1 refers to the particle, the index 2 to the antiparticle, $\boldsymbol{r} := \boldsymbol{x}_1 - \boldsymbol{x}_2$, $\boldsymbol{L}_1 := \boldsymbol{r} \times \boldsymbol{p}_1$, $\boldsymbol{L}_2 := -\boldsymbol{r} \times \boldsymbol{p}_2$, and we have set

$$\langle\langle f[A]\rangle\rangle_\Gamma := \left\{\left\langle \operatorname{Tr} \mathrm{P}_\Gamma \exp\left[\frac{ig}{c}\oint_\Gamma \mathrm{d}x^\mu A_\mu(x)\right]\right\rangle\right\}^{-1}$$

$$\times \left\langle \operatorname{Tr} \mathrm{P}_\Gamma \exp\left[\frac{ig}{c}\oint_\Gamma \mathrm{d}x^\mu A_\mu(x)\right] f[A]\right\rangle. \tag{1.5}$$

Note that we have given the result in a slightly more symmetric form than the original one and that we have used a different notation. Among the potentials V, V_1, V_2 the following relation has been found in ref. [5]:

$$V(r) + V_1(r) - V_2(r) = \text{const}. \tag{1.6}$$

The proof is given in the appendix. Finally, recall that the "electric" and "magnetic" fields are given by

$$\boldsymbol{E}(x) = -\operatorname{grad} A^0(x) - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(x) - \frac{ig}{c}\left[A^0(x), \boldsymbol{A}(x)\right], \tag{1.7a}$$

$$\boldsymbol{B}(x) = \operatorname{curl} \boldsymbol{A}(x) - \frac{ig}{c}\boldsymbol{A}(x) \times \boldsymbol{A}(x). \tag{1.7b}$$

Although extremely interesting in its results, the original Eichten-Feinberg derivation makes a large use of intuitive arguments, and it is not immediately applicable to the evaluation of the neglected terms. For such reasons we want to carry out this analysis in a more systematic way, organizing the whole procedure in the form of a $1/c$ expansion, starting consistently from QCD basic principles.

Although we are mainly interested in the methodological aspects of the derivation, our net result, at the order $1/c^2$, is an additional spin-independent, but velocity-dependent term V_{vd} of electrodynamic type besides the usual kinetic energy correction $-\sum_{j=1}^2 p_j^4/8m_j^3c^2$. This additional potential is given by

$$V_{\mathrm{vd}} = \frac{1}{8c^2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\Delta_2\left(V(r) + V_a(r)\right)$$

$$+ \frac{1}{2m_1m_2c^2}\left\{ p_1^h p_2^k, \delta^{hk}V_b(r) + \left(\tfrac{1}{3}\delta^{hk} - \frac{r^h r^k}{r^2}\right)V_c(r)\right\}$$

$$+ \frac{1}{2c^2}\left\{\sum_j \frac{p_j^h p_j^k}{m_j^2}, \delta^{hk}V_d(r) + \left(\tfrac{1}{3}\delta^{hk} - \frac{r^h r^k}{r^2}\right)V_e(r)\right\}, \tag{1.8}$$

where $\{A, B\}$ denotes the anticommutator between A and B and

$$\Delta_2 V_a(r) = ig^2 \lim_{T\to+\infty} \int_{-T/2}^{T/2} \mathrm{d}t \Big[\langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\cdot\boldsymbol{E}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \cdot \langle\langle \boldsymbol{E}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle \boldsymbol{B}(t, \boldsymbol{x}_1)\cdot\boldsymbol{B}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} + \langle\langle \boldsymbol{B}(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \cdot \langle\langle \boldsymbol{B}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \Big], \tag{1.9a}$$

$$V_b(r) = -\tfrac{1}{6} ig^2 \lim_{T\to+\infty} \int_{-T/2}^{T/2} \mathrm{d}t\, c^2 t^2 \Big[\langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\cdot\boldsymbol{E}(0, \boldsymbol{x}_2)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \cdot \langle\langle \boldsymbol{E}(0, \boldsymbol{x}_2)\rangle\rangle_{\Gamma_{\mathrm{W}}} \Big], \tag{1.9b}$$

$$\begin{aligned}\left(\frac{r^h r^k}{r^2} - \tfrac{1}{3}\delta^{hk}\right) V_c(r) &= \tfrac{1}{4} ig^2 \lim_{T\to+\infty} \int_{-T/2}^{T/2} \mathrm{d}t\, c^2 t^2 \\ &\times \Big\{ \Big[\langle\langle E^h(t, \boldsymbol{x}_1) E^k(0, \boldsymbol{x}_2)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle E^h(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \langle\langle E^k(0, \boldsymbol{x}_2)\rangle\rangle_{\Gamma_{\mathrm{W}}} \\ &- \tfrac{1}{3}\delta^{hk} \Big(\langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\cdot\boldsymbol{E}(0, \boldsymbol{x}_2)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \cdot \langle\langle \boldsymbol{E}(0, \boldsymbol{x}_2)\rangle\rangle_{\Gamma_{\mathrm{W}}} \Big) \Big] + (h \rightleftarrows k) \Big\},\end{aligned} \tag{1.9c}$$

$$V_d(r) = \tfrac{1}{12} ig^2 \lim_{T\to+\infty} \int_{-T/2}^{T/2} \mathrm{d}t\, c^2 t^2 \Big[\langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\cdot\boldsymbol{E}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \cdot \langle\langle \boldsymbol{E}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \Big], \tag{1.9d}$$

$$\begin{aligned}\left(\tfrac{1}{3}\delta^{hk} - \frac{r^h r^k}{r^2}\right) V_e(r) &= \tfrac{1}{8} ig^2 \lim_{T\to+\infty} \int_{-T/2}^{T/2} \mathrm{d}t\, c^2 t^2 \\ &\times \Big\{ \Big[\langle\langle E^h(t, \boldsymbol{x}_1) E^k(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle E^h(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \langle\langle E^k(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \\ &- \tfrac{1}{3}\delta^{hk} \Big(\langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\cdot\boldsymbol{E}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} - \langle\langle \boldsymbol{E}(t, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \cdot \langle\langle \boldsymbol{E}(0, \boldsymbol{x}_1)\rangle\rangle_{\Gamma_{\mathrm{W}}} \Big) \Big] + (h \rightleftarrows k) \Big\}.\end{aligned} \tag{1.9e}$$

In the case of QED the functional integrals are gaussian and can be explicitly computed. Then eq. (1.1) (without the trace and the normalization factor $\frac{1}{3}$) gives the Coulomb potential, while eq. (1.3) gives the ordinary spin-dependent terms, which are considered for instance in the theory of positronium or in atomic physics. The other potentials give rise to the well known velocity-dependent terms, discussed, for instance, by Bethe and Salpeter [7]. Similar results can be obtained for a nonabelian gauge theory in the weak-coupling limit, when the direct gluon-gluon interaction is neglected; they are usually believed to give the correct short-range behaviour of the potentials even in QCD (via asymptotic freedom). The short-range behaviour turns out to be (see sect. 6)

$$V(r)=V_2(r)=-\tfrac{1}{3}V_3(r)=-\tfrac{3}{2}V_b(r)=2V_c(r)=-\frac{\alpha}{r}, \tag{1.10a}$$

$$V_4(r)=2\Delta_2 V_2(r), \tag{1.10b}$$

$$V_1(r)=V_a(r)=V_d(r)=V_e(r)=0, \tag{1.10c}$$

where $\alpha=e^2/4\pi$ for QED and $\alpha=\frac{4}{3}g^2/4\pi=\frac{4}{3}\alpha_s$ for QCD.

For QCD, in the strong coupling limit, numerical calculations in the lattice approximation suggest a linear behaviour of $V(r)$ for large r, $V(r)\sim ar$, and provide a value for the string constant a, once the parameter Λ controlling the running coupling constant has been fixed on the basis of the high energy phenomenology [8, 9].

Apart from numerical calculations, by exploiting an idea of Gromes [4], it is possible to make a guess about the long-range behaviour of the various potentials. As shown in the last section, for large r we have

$$V(r)=-V_1(r)=\tfrac{9}{4}V_b(r)=\tfrac{3}{2}V_c(r)=\tfrac{9}{4}V_d(r)=\tfrac{3}{2}V_e(r)=ar, \tag{1.11a}$$

$$V_2(r)=V_3(r)=V_4(r)=V_a(r)=0. \tag{1.11b}$$

By putting together eqs. (1.10) and (1.11), the following simple phenomenological potentials can be proposed

$$V(r)=-\frac{\alpha}{r}+ar, \qquad V_a(r)=0, \tag{1.12a}$$

$$V_b(r)=\frac{2}{3}\frac{\alpha}{r}+\tfrac{4}{9}ar, \qquad V_c(r)=-2\frac{\alpha}{r}+\tfrac{2}{3}ar, \tag{1.12b}$$

$$V_d(r)=\tfrac{4}{9}ar, \qquad V_e(r)=\tfrac{2}{3}ar, \tag{1.12c}$$

$$V_1(r)=-ar, \qquad V_2(r)=-\frac{\alpha}{r}, \qquad V_3(r)=3\frac{\alpha}{r}. \tag{1.12d}$$

The potential $V_4(r)$ is given by eq. (1.10b). Obviously, interpolating constants could be added to $V(r)$ (cf. ref. [10]) and to all the other potentials, but we omit them because there is no way to determine them in the present context.

Note that, as already stressed in ref. [3] for the spin-dependent potential, new terms arise in QCD, which have no counterpart in QED (cf. eqs. (1.10) and (1.11)). About the spin-independent terms, we expect they play some role in the case of bound states involving light quarks (in this respect cf. ref. [11]).

Apart from the new terms we have obtained, one of the more significant aspects of our work is perhaps to have shown explicitly that the ordinary Dirac-type $q\bar{q}$-propagator of QCD can be reexpressed in terms of a Pauli-type two-particle propagator, satisfying a Schrödinger-like equation. An important intermediate step in this proof is the use for the two-particle propagator of an expression in terms of a path-integral in phase space. Somewhat similar ideas, but in a less systematic framework, have been used also in ref. [6]. To give a global view of our approach, we illustrate in the second section the main lines of the derivation and we leave the technical details to the subsequent sections.

2. Quark-antiquark potential: main lines of the derivation

Consider a gauge field theory (abelian or nonabelian) corresponding to a colour group G; the gauge field interacts with some "matter fields" ψ_j (the index j denotes different "flavours"). We shall use the notation $[\psi_j(x)]^{\alpha a}$ for denoting the Dirac component α and the gauge component a of the field ψ_j. For the charge-conjugate fields we have (sums over repeated indices are understood)

$$\left[\psi_j^{\mathrm{C}}(x)\right]^{\alpha a}=\left[\bar{\psi}_j(x)\right]^{\beta a}C_{\beta\alpha}, \tag{2.1a}$$

$$\left[\bar{\psi}_j^{\mathrm{C}}(x)\right]^{\alpha a}=-[C^{-1}]_{\alpha\beta}\left[\psi_j(x)\right]^{\beta a}, \tag{2.1b}$$

where C is the usual charge-conjugation matrix. By using a tilde for transpose, we have

$$C^{-1}\gamma_\mu C=-\tilde{\gamma}_\mu, \qquad C^{-1}=C^{\dagger}. \tag{2.2}$$

We take as starting point the following modified Wightman function

$$W(x_1,x_2;y_1,y_2;\tau|\alpha,\beta;\gamma,\delta)$$

$$:=\frac{1}{N}\langle 0|[\psi_2^{\mathrm{C}}(x_2)]^{\gamma a}\left[P\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]^{ab}[\psi_1(x_1)]^{\alpha b}[\bar{\psi}_1(y_1)]^{\beta c}\left[P\begin{pmatrix}y_2\\y_1\end{pmatrix}\right]^{cd}[\bar{\psi}_2^{\mathrm{C}}(y_2)]^{\delta d}|0\rangle, \tag{2.3}$$

where N is the dimension of the fermion representation of the gauge group G and

$$x_1^0 = x_2^0 = \tfrac{1}{2}c\tau\,, \qquad y_1^0 = y_2^0 = -\tfrac{1}{2}c\tau\,, \tag{2.4}$$

$$P\binom{x}{y} := \mathrm{P}\exp\left[\frac{ig}{c}\int_y^x \mathrm{d}z_\mu A^\mu(z)\right]. \tag{2.5}$$

The strings between quarks have been introduced in order to have a gauge-invariant function. The paths in $P\binom{x_1}{x_2}$ and $P\binom{y_2}{y_1}$ are the space-like straight lines joining x_2, x_1 and y_1, y_2; P is the path-ordering prescription. Quark and antiquark are in a singlet state.

The τ-Fourier spectrum of the function $W(\dots)$ gives the energy spectrum of the particle-antiparticle system; moreover, suitable limits on $W(\dots)$ give also the corresponding S-matrix, if this quantity exists (in practice, when there is no confinement). By a $1/c$ expansion and some other manipulations on this W-function, we shall succeed in identifying a Pauli-type two-particle propagator, for which a Schrödinger-like equation holds.

By using eqs. (2.1) and the time-ordering prescription, we can write

$$\begin{aligned} &W(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}_1, \boldsymbol{y}_2; \tau|\alpha, \beta; \gamma, \delta) \\ &= \frac{1}{N}\langle 0|\mathrm{T}\left[P\binom{x_1}{x_2}\right]^{ab}[\psi_1(x_1)]^{\alpha b}[\bar{\psi}_1(y_1)]^{\beta c}\left[P\binom{y_2}{y_1}\right]^{cd} \\ &\quad\times[\psi_2(y_2)]^{\eta d}[\bar{\psi}_2(x_2)]^{\varepsilon a}|0\rangle C_{\varepsilon\gamma}[C^{-1}]_{\delta\eta}\,. \end{aligned} \tag{2.6}$$

If we use the functional integral formalism, after integrating explicitly over the Grassmann variables, we obtain

$$\begin{aligned} &W(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}_1, \boldsymbol{y}_2; \tau|\alpha, \beta; \gamma, \delta) \\ &= C_{\varepsilon\gamma}[C^{-1}]_{\delta\eta}\frac{1}{N}\mathrm{Tr}\left\langle P\binom{x_1}{x_2}[S_1^{\mathrm{F}}(x_1, y_1|A)]_{\alpha\beta}P\binom{y_2}{y_1}[S_2^{\mathrm{F}}(y_2, x_2|A)]_{\eta\varepsilon}\right\rangle, \end{aligned} \tag{2.7}$$

where Tr denotes the trace over the gauge indices. The notation (1.2) has been used with $S[A]$ in principle replaced by the effective action

$$S_{\mathrm{eff}}[A] = S[A] + \Omega[A]\,, \tag{2.8a}$$

with

$$\Omega[A] := -i\log\frac{\int\mathscr{D}[\psi, \bar{\psi}]\exp(iS[\psi|A])}{\int\mathscr{D}[\psi, \bar{\psi}]\exp(iS[\psi|0])}\,. \tag{2.8b}$$

Here $S[\psi|A]$ and $S_j^{\mathrm{F}}(x, y|A)$ denote the matter field action and the one-particle Feynman propagator in an external gauge field. The term $\Omega[A]$ takes into account the fermion-loop contributions.

Recall that the one-particle propagator satisfies the equation

$$\left\{\gamma^\mu\left[i\partial_\mu - \frac{g}{c}A_\mu(x)\right] - m_j c + i\varepsilon\right\} S_j^{\mathrm{F}}(x, y|A) = -\delta_4(x-y). \tag{2.9}$$

Then, we set

$$S_j^{\mathrm{F}}(x, y|A) = \vartheta(x_0 - y_0) S_j^+(x, y|A) - \vartheta(y_0 - x_0) S_j^-(x, y|A), \tag{2.10}$$

where $S_j^\pm$ satisfy the homogeneous Dirac equation

$$\left\{\gamma^\mu\left[i\partial_\mu - \frac{g}{c}A_\mu(x)\right] - m_j c\right\} S_j^\pm(x, y|A) = 0, \tag{2.11}$$

and the Cauchy condition

$$\left[S_j^+(x, y|A) + S_j^-(x, y|A)\right]_{x_0 = y_0} = i\gamma_0 \delta_3(\boldsymbol{x} - \boldsymbol{y}). \tag{2.12}$$

The quantities $S_j^\pm$ reduce asymptotically to the usual positive and negative frequency parts of S_j^{F}. Moreover, eq. (2.9) is equivalent to

$$S_j^{\mathrm{F}}(x, y|A)\left\{\gamma^\mu\left[-i\frac{\overleftarrow{\partial}}{\partial y^\mu} - \frac{g}{c}A_\mu(y)\right] - m_j c + i\varepsilon\right\} = -\delta_4(x-y) \tag{2.13}$$

and, by eqs. (2.2) and (2.9), we have also

$$\left\{\tilde{\gamma}^\mu\left[-i\frac{\partial}{\partial y^\mu} + \frac{g}{c}A_\mu(y)\right] - m_j c + i\varepsilon\right\} C^{-1} S_j^{\mathrm{F}}(y, x|A) C = -\delta_4(x-y). \tag{2.14}$$

By comparing eq. (2.13) with the transposed of eq. (2.14) and taking into account eq. (2.10), we obtain

$$\left[C^{-1} S_j^\mp(y, x|A) C\right]^\sim = -S_j^\pm(x, y| -\tilde{A}). \tag{2.15}$$

By using the equations above, the function $W(\dots)$ can be rewritten as

$$W(\boldsymbol{x}_1, \boldsymbol{x}_2; y_1, y_2; \tau|\alpha, \beta; \gamma, \delta)$$

$$= -\left\langle \frac{1}{N}\mathrm{Tr}\, P\binom{x_1}{x_2}\left[S_1^+(x_1, y_1|A)\right]_{\alpha\beta} P\binom{y_2}{y_1}\left[\tilde{S}_2^+(x_2, y_2| -\tilde{A})\right]_{\delta\gamma}\right\rangle. \tag{2.16}$$

Up to now we have considered two different flavours, but we are interested also in the bound states of a quark with its own antiquark. In this case an additional term appears in eq. (2.16) (the annihilation term); however, in the present paper we do not consider it, because we are interested only in obtaining the interaction potential.

As a first step, by some analogue of the Foldy-Wouthuysen transformation, we define a quantity $K_j(x, y|A)$, $x_0 > y_0$, by

$$\begin{aligned}
&\int \mathrm{d}_3 x_1\, \mathrm{d}_3 x_2\, \mathrm{d}_3 y_1\, \mathrm{d}_3 y_2 \exp(-i\boldsymbol{p}_1\cdot\boldsymbol{x}_1 - i\boldsymbol{p}_2\cdot\boldsymbol{x}_2 + i\boldsymbol{p}_1'\cdot\boldsymbol{y}_1 + i\boldsymbol{p}_2'\cdot\boldsymbol{y}_2)\\
&\quad\times\left[\bar{u}^{(1)}_{\sigma_1}(p_1)\gamma_0\right]^\alpha\left[\bar{u}^{(2)}_{\sigma_2}(p_2)\gamma_0\right]^\gamma W(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}_1, \boldsymbol{y}_2; \tau|\alpha,\beta;\gamma,\delta)\\
&\quad\times\left[\gamma_0 u^{(2)}_{\sigma_2'}(p_2')\right]^\delta\left[\gamma_0 u^{(1)}_{\sigma_1'}(p_1')\right]^\beta\\
&\quad=\int \mathrm{d}_3 x_1\, \mathrm{d}_3 x_2\, \mathrm{d}_3 y_1\, \mathrm{d}_3 y_2\\
&\qquad\times\exp\left[-i\boldsymbol{p}_1\cdot\boldsymbol{x}_1 - i\boldsymbol{p}_2\cdot\boldsymbol{x}_2 + i\boldsymbol{p}_1'\cdot\boldsymbol{y}_1 + i\boldsymbol{p}_2'\cdot\boldsymbol{y}_2 - i(m_1+m_2)c^2\tau\right]\\
&\qquad\times\left\langle \frac{1}{N}\operatorname{Tr} P\binom{x_1}{x_2}\left[v^\dagger_{\sigma_1}K_1(x_1, y_1|A)v_{\sigma_1'}\right]P\binom{y_2}{y_1}\left[v^\dagger_{\sigma_2'}\tilde{K}_2(x_2, y_2|-\tilde{A})v_{\sigma_2}\right]\right\rangle,
\end{aligned}\tag{2.17}$$

where v_σ is an ordinary two-dimensional Pauli spinor and $u^{(j)}_\sigma(p)$ the related plane-wave Dirac four-dimensional spinor

$$u^{(j)}_\sigma(p) = \frac{m_j c + \gamma^\mu p_\mu}{\sqrt{2p_0(m_j c + p_0)}}\begin{pmatrix} v_\sigma \\ 0\end{pmatrix}, \qquad p_0 = \sqrt{m_j^2c^2 + \boldsymbol{p}^2}\,, \tag{2.18}$$

$$\bar{u}^{(j)}_\sigma(p) = u^{(j)}_\sigma(p)^\dagger\gamma_0\,. \tag{2.19}$$

Note that S_j^+ and K_j are both square matrices of the same dimensionality in the gauge indices, while S_j^+ is a 4×4 matrix in the Dirac indices and K_j a 2×2 matrix in the spin indices.

In sect. 3 we shall show that, up to the order $1/c^2$, K_j satisfies the following Schrödinger-like equation

$$\begin{aligned}
i\partial_{t_x}K_j(x, y|A) = \Bigg(&\frac{1}{2m_j}\left(-i\vec{\partial} - \frac{g}{c}\boldsymbol{A}(x)\right)^2 + gA^0(x) - \frac{g}{m_j c}\boldsymbol{S}\cdot\boldsymbol{B}(x)\\
&-\frac{1}{8m_j^3c^2}\left(-i\vec{\partial} - \frac{g}{c}\boldsymbol{A}(x)\right)^4 - \frac{g}{8m_j^2c^2}\left[\partial_h - \frac{ig}{c}A^h(x), E^h(x)\right]\\
&+\frac{g}{4m_j^2c^2}S^l\varepsilon^{lhk}\left\{-i\partial_h - \frac{g}{c}A^h(x), E^h(x)\right\}\Bigg)K_j(x, y|A)
\end{aligned}\tag{2.20}$$

and the Cauchy condition

$$K_j(x, y|A)|_{t_x=t_y} = \delta_3(\boldsymbol{x}-\boldsymbol{y}), \tag{2.21}$$

where $x_0 = ct_x$, $y_0 = ct_y$, ε^{lhk} is the Ricci symbol and the summation over repeated indices is understood.

By standard techniques the solution of eq. (2.20), with the initial condition (2.21), can be expressed as a path integral in phase space

$$\begin{aligned} K_j(x, y|A) &= \int_{z(t_y)=y}^{z(t_x)=x} \mathscr{D}[z, p] \mathrm{T} \exp\Bigg(i \int_{t_y}^{t_x} \mathrm{d}t \Bigg\{ p(t) \cdot \dot{z}(t) \\ &\quad - \Bigg[\frac{1}{2m_j} \Big(p(t) - \frac{g}{c} A(t, z(t)) \Big)^2 + g A^0(t, z(t)) \\ &\quad - \frac{g}{m_j c} S_j \cdot B(t, z(t)) - \frac{1}{8 m_j^3 c^2} \Big(p(t) - \frac{g}{c} A(t, z(t)) \Big)^4 \\ &\quad - \frac{g}{8 m_j^2 c^2} \Big[\partial_h - \frac{ig}{c} A^h(t, z(t)), E^h(t, z(t)) \Big] \\ &\quad + \frac{g}{4 m_j^2 c^2} S_j^l \varepsilon^{lhk} \Big\{ p^h(t) - \frac{g}{c} A^h(t, z(t)), E^k(t, z(t)) \Big\} \Bigg] \Bigg\} \Bigg), \end{aligned} \tag{2.22}$$

where the time-ordering prescription T acts both on spin and gauge matrices.

By using eq. (2.17), we can identify the "large" components of the two-particle propagator (2.16) by

$$\begin{aligned} &\langle \sigma_1, \sigma_2 | K(x_1, x_2; y_1, y_2; \tau) | \sigma_1', \sigma_2' \rangle \\ &:= \Big\langle \frac{1}{N} \mathrm{Tr}\, P\binom{x_1}{x_2} [\langle \sigma_1 | K_1(x_1, y_1|A) | \sigma_1' \rangle] P\binom{y_2}{y_1} [\langle \sigma_2' | \tilde{K}_2(x_2, y_2|A) | \sigma_2 \rangle] \Big\rangle, \end{aligned} \tag{2.23}$$

where $x_i^0 = \frac{1}{2}c\tau$, $y_i^0 = -\frac{1}{2}c\tau$, $\tau > 0$, and the notation of Dirac kets and bras has been used.

In sect. 4 we shall show that, by using eq. (2.22), the quantity (2.23) can be rewritten as

$$
\begin{aligned}
&K(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}_1, \boldsymbol{y}_2; \tau) \\
&\quad = \int_{z_1(-\tau/2)=y_1}^{z_1(\tau/2)=x_1} \mathscr{D}[z_1, p_1] \int_{z_2(-\tau/2)=y_2}^{z_2(\tau/2)=x_2} \mathscr{D}[z_2, p_2] \\
&\quad\quad \times \exp\left\{ i\int_{-\tau/2}^{\tau/2} \mathrm{d}t \sum_j \left[\boldsymbol{p}_j(t)\cdot\dot{\boldsymbol{z}}_j(t) - \frac{\boldsymbol{p}_j(t)^2}{2m_j} + \frac{\boldsymbol{p}_j(t)^4}{8m_j^3c^2} \right] \right\} \\
&\quad\quad \times \left\langle \frac{1}{N}\mathrm{Tr}\,\mathrm{T_S P}_\Gamma \exp\left\{ \frac{ig}{c}\oint_\Gamma \mathrm{d}x_\mu A^\mu(x) + \sum_j \frac{ig}{m_jc^2} \right.\right. \\
&\qquad\qquad\qquad \times \int_{\Gamma_j} \mathrm{d}x^\mu \left(S_j^l \hat{F}_{l\mu}(x) - \frac{1}{2m_jc} S_j^l \varepsilon^{lkr} p_j^k(t) F_{\mu r}(x) \right. \\
&\qquad\qquad\qquad\qquad \left.\left.\left. - \frac{1}{8m_jc}\left[\partial^\nu + \frac{ig}{c}A^\nu(x), F_{\nu\mu}(x) \right] \right) \right\} \right\rangle, \qquad (2.24)
\end{aligned}
$$

where

$$F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \frac{ig}{c}\left[A_\mu(x), A_\nu(x)\right], \qquad (2.25a)$$

$$\hat{F}^{\mu\nu}(x) := \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}(x). \qquad (2.25b)$$

Here $\varepsilon^{\mu\nu\rho\sigma}$ is the four-dimensional Ricci symbol and ε^{lkr} the three-dimensional one. Γ_1 is the path going from $(\frac{1}{2}\tau, \boldsymbol{x}_1)$ to $(-\frac{1}{2}\tau, \boldsymbol{y}_1)$ along the trajectory $(t, \boldsymbol{z}_1(t))$; Γ_2 goes from $(-\frac{1}{2}\tau, \boldsymbol{y}_2)$ to $(\frac{1}{2}\tau, \boldsymbol{x}_2)$ along the trajectory $(t, \boldsymbol{z}_2(t))$; Γ is the closed path given by the union of Γ_1 and Γ_2 and closed by two straight lines joining $(-\frac{1}{2}\tau, \boldsymbol{y}_1)$ with $(-\frac{1}{2}\tau, \boldsymbol{y}_2)$ and $(\frac{1}{2}\tau, \boldsymbol{x}_2)$ with $(\frac{1}{2}\tau, \boldsymbol{x}_1)$. Moreover, $\mathrm{T_S}$ is the time-ordering prescription acting only on spin matrices and P_Γ the path-ordering prescription acting only on the gauge matrices; Tr denotes the trace on the gauge matrices. Note that the expression (2.24) is manifestly gauge invariant.

Now, if we are able to reexpress the angular bracket term in eq. (2.24) as

$$\left\langle \frac{1}{N}\mathrm{Tr}\ldots \right\rangle = \mathrm{T_S}\exp\left[-i\int_{-\tau/2}^{\tau/2} \mathrm{d}t\, U(\boldsymbol{z}_1(t), \boldsymbol{z}_2(t), \boldsymbol{p}_1(t), \boldsymbol{p}_2(t), \boldsymbol{S}_1, \boldsymbol{S}_2) \right], \qquad (2.26)$$

then we may conclude that the propagator K obeys a two-particle Schrödinger equation of the form

$$i\partial_\tau K(x_1, x_2; y_1, y_2; \tau) = \left[\sum_j \left(\frac{p_j^2}{2m_j} - \frac{p_j^4}{8m_j^3c^2}\right) + U\right] K(x_1, x_2; y_1, y_2; \tau) \quad (2.27)$$

and that U plays the role of a two-particle potential. Note that in eq. (2.24), by using the controgradient representation for the spin of particle 2, it would be possible to obtain a path-ordering prescription also for spins. But this alternative form would not be useful, because we must compare eq. (2.24) with the solution of a two-particle Schrödinger equation. In the path-integral expression of this solution both spins are time ordered (cf. eq. (2.26)).

We stress that the construction of a quantity U satisfying eq. (2.26) is not a trivial matter, because we are equating a complicate functional $\langle(1/N)\mathrm{Tr}\ldots\rangle$ of $z_j(t)$ and $p_j(t)$ to another functional of an apparently very specific form (the right-hand side of eq. (2.26)). We shall discuss this point in sect. 5.

We succeed in solving this problem, up to order $1/c^2$, first obtaining a potential of the form

$$U \simeq U_0 + \frac{1}{c^2} \sum_{i,j=1}^{2} \dot{z}_i^h X_{ij}^{hk} \dot{z}_j^k + \frac{1}{c^2} \sum_{j=1}^{2} Y_j \cdot \ddot{z}_j , \quad (2.28)$$

where U_0, X_{ij}^{hk}, Y_j depend only on z_1, z_2, p_1, p_2 (and S_1, S_2), but not their derivatives. Then the second derivatives $\ddot{z}_j$ are eliminated by partial integration and the first derivatives $\dot{z}_j$ may be replaced by p_j/m_j up to terms of higher order in $1/c$. In this way the potential (1.3)–(1.5), (1.8)–(1.9e) is obtained, provided that τ is large.

The reason why, in order to have a Schrödinger equation (2.27), a large time interval τ must be considered, may be understood in the following way. We have two particles interacting via an exchange of gauge particles. Once the gauge field is eliminated (this elimination is indicated by the angular brackets in eq. (2.24)), we obtain a noninstantaneous interaction between the two particles. In order to approximate this noninstantaneous interaction by an instantaneous one, we have to wait a sufficiently long time, so that all interactions have taken place.

Note that, if one wishes to go beyond the $1/c^2$ approximation, one gets in eq. (2.28) terms containing derivatives of p_j, and higher derivatives of z_j. However, since these derivatives cannot be reexpressed in a simple way in terms of z_j and p_j, a potential in the ordinary sense is not obtained. This difficulty is related in some way to the noninteraction theorem of the classical hamiltonian relativistic theory [12,13]. We have not explored whether this difficulty could be overcome by a suitable redefinition of the configurational variables as in the classical case*.

* Canonical configuration variables in classical hamiltonian theories in the presence of an interaction are complicated functions of position and momentum [13].

3. Schrödinger equation for the one-particle propagator

In this section we show that the one-particle propagator K_j, defined by eq. (2.17), satisfies – up to terms of order $1/c^2$ – the Schrödinger equation (2.20).

It is convenient to introduce the gauge covariant derivatives

$$D_\mu := i\frac{\partial}{\partial x^\mu} - \frac{g}{c}A_\mu(x), \tag{3.1}$$

for which the following identities hold

$$[D_0, D_h] = -\frac{ig}{c}E^h(x), \tag{3.2a}$$

$$\tfrac{1}{2}\varepsilon^{lhk}[D_h, D_k] = \frac{ig}{c}B^l(x), \tag{3.2b}$$

$$\varepsilon^{lhk}\left[D_h, E^k(x)\right] = -\left[D_0, B^l(x)\right], \tag{3.2c}$$

$$\left(\sigma^h D_h\right)^2 = \boldsymbol{D}^2 - \frac{g}{c}\sigma^h B^h(x), \tag{3.2d}$$

where the σ^k are the Pauli matrices.

With the choice

$$\gamma^0 = \beta, \qquad \gamma^k = \beta\alpha^k, \tag{3.3a}$$

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \tag{3.3b}$$

the Dirac equation (2.11) becomes

$$cD_0 S_j^+(x, y|A) = \left(m_j c^2\beta - c\alpha^k D_k\right)S_j^+(x, y|A). \tag{3.4}$$

By writing

$$\begin{pmatrix} U_{11}(x, y|A) & U_{12}(x, y|A) \\ U_{21}(x, y|A) & U_{22}(x, y|A) \end{pmatrix} := e^{im_j c(x_0 - y_0)} S_j^+(x, y|A), \tag{3.5}$$

where the U_{ij} are 2×2 matrices in the Dirac indices, we have from eq. (3.4)

$$cD_0 U_{1s}(x, y|A) = -c\sigma^k D_k U_{2s}(x, y|A), \tag{3.6a}$$

$$\left(cD_0 + 2m_j c^2\right)U_{2s}(x, y|A) = -c\sigma^k D_k U_{1s}(x, y|A). \tag{3.6b}$$

First, eq. (3.6b) gives

$$cU_{2s}(x, y|A) = -\frac{1}{2m_j}\left(1 - \frac{cD_0}{2m_jc^2}\right)\sigma^k D_k U_{1s}(x, y|A) \tag{3.7}$$

(here, and in the sequel, the expressions are correct up to order $1/c^2$). Then, by inserting this result into eq. (3.6a) and using the identities (3.2), we get

$$cD_0U_{1s}(x, y|A) = \left[\frac{\boldsymbol{D}^2}{2m_j} - \frac{\boldsymbol{D}^4}{8m_j^3c^2} - \frac{g}{2m_jc}\boldsymbol{\sigma}\cdot\boldsymbol{B}(x) - \frac{g}{8m_j^2c^2}\sigma^l\varepsilon^{lhk}\right.$$
$$\left.\times\{D_h, E^k(x)\} + \frac{ig}{4m_j^2c^2}D_hE^h(x)\right]U_{1s}(x, y|A). \tag{3.8}$$

With $s=1$, this is the equation for the component U_{11}. The other components can be expressed in terms of U_{11}; precisely (see eq. (3.7))

$$U_{21}(x, y|A) = -\frac{1}{2m_jc}\sigma^kD_kU_{11}(x, y|A) \tag{3.9}$$

and

$$U_{12}(x, y|A) = \frac{1}{2m_jc}U_{11}(x, y|A)\sigma^k\hat{D}_k, \tag{3.10}$$

$$U_{22}(x, y|A) = -\frac{1}{4m_j^2c^2}\sigma^kD_kU_{11}(x, y|A)\sigma^h\hat{D}_h, \tag{3.11}$$

where we have introduced the covariant derivatives from the right:

$$\hat{D}_\mu := -i\frac{\overleftarrow{\partial}}{\partial y^\mu} - \frac{g}{c}A_\mu(y). \tag{3.12}$$

Equations (3.10) and (3.11) follow from

$$cS_j^+(x, y|A)\hat{D}_0 = S_j^+(x, y|A)\left(m_jc^2\beta + c\alpha^k\hat{D}_k\right), \tag{3.13}$$

which, in turn, is a consequence of eqs. (2.10)–(2.13) ($C=\gamma^2\gamma^0$). Moreover, by eqs.

(2.15) and (3.5), the Cauchy condition (2.12) takes the form

$$U_{11}(x, y|A)|_{x_0=y_0} = i\left(1 - \frac{\boldsymbol{D}^2}{4m_j^2c^2}\right)\delta_3(\boldsymbol{x}-\boldsymbol{y}). \tag{3.14}$$

Let us now consider eq. (2.17). The Dirac spinors can be written as

$$u_\sigma^{(j)}(p) \simeq \begin{pmatrix} \left(1 - \dfrac{\boldsymbol{p}^2}{8m_j^2c^2}\right)v_\sigma \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m_jc}v_\sigma \end{pmatrix}; \tag{3.15}$$

due to the exponential factors, we are allowed to replace $\boldsymbol{p}_j$ by $-i\partial/\partial\boldsymbol{x}_j$ and $\boldsymbol{p}_j'$ by $i\partial/\partial\boldsymbol{y}_j$. These derivatives act on the product of the one-particle propagators and the gauge strings; the contribution of the gauge strings is to change the ordinary derivatives into covariant derivatives (from the left and from the right, respectively) acting only on the one-particle propagators. Bearing this in mind, and inserting eqs. (3.5), (3.9)–(3.11), (3.15) into eq. (2.17), we readily obtain

$$K_j(x, y|A) = -i\left(1 + \frac{\boldsymbol{D}^2}{8m_j^2c^2}\right)U_{11}(x, y|A)\left(1 + \frac{\hat{\boldsymbol{D}}^2}{8m_j^2c^2}\right). \tag{3.16}$$

Finally, by recalling eqs. (3.8) and (3.14), it is a simple matter to derive eqs. (2.20) and (2.21) for $K_j(x, y|A)$.

4. Path integral formulation in phase space

As is well known, a Schrödinger propagator can be expressed by means of a path integral in phase space. In particular, the solution of eq. (2.20), with the initial condition (2.21), is given by eq. (2.22). Then, by performing the translation $\boldsymbol{p} \to \boldsymbol{p} + g\boldsymbol{A}/c$, we obtain an equation containing the expression $\mathrm{d}t(gA^0 - g\dot{\boldsymbol{z}}\cdot\boldsymbol{A}/c) \equiv g\,\mathrm{d}x^\mu A_\mu/c$, which is useful for obtaining the Wilson loop. More precisely, since the A^h are matrices, the step $\int \mathrm{d}_3\boldsymbol{p} f(\boldsymbol{p} - g\boldsymbol{A}/c) = \int \mathrm{d}_3\boldsymbol{p} f(\boldsymbol{p})$ can be justified by expanding $f(\boldsymbol{p} - g\boldsymbol{A}/c)$ in powers of g; apart from the zeroth order term, all the other terms involve derivatives of $f(\boldsymbol{p})$ and do not contribute to the integral.

In the following it will be useful to have an expression for K_j in which the tensor field $F^{\mu\nu}$ and its dual $\hat{F}^{\mu\nu}$ appear. To this end we make the further translation $\boldsymbol{p}(t) \to \boldsymbol{p}(t) - g\boldsymbol{E}(t, \boldsymbol{z}(t)) \times \boldsymbol{S}/m_j c^2$. Apart from higher order terms, we obtain

$$K_j(x, y|A) = \int_{z(t_y)=y}^{z(t_x)=x} \mathscr{D}[z, p] \mathrm{T} \exp\left\{ i \int_{t_y}^{t_x} \mathrm{d}t \left[\boldsymbol{p}(t) \cdot \dot{\boldsymbol{z}}(t) - \left(\frac{\boldsymbol{p}(t)^2}{2m_j} - \frac{\boldsymbol{p}(t)^4}{8m_j^3 c^2} + gA^0(t, \boldsymbol{z}(t)) - \frac{g}{c} \dot{\boldsymbol{z}}(t) \cdot \boldsymbol{A}(t, \boldsymbol{z}(t)) - \frac{g}{m_j c} \boldsymbol{S} \cdot \boldsymbol{B}(t, \boldsymbol{z}(t)) - \frac{g}{8m_j^2 c^2} \left[\partial_h - \frac{ig}{c} A^h(t, \boldsymbol{z}(t)), E^h(t, \boldsymbol{z}(t)) \right] - \frac{g}{2m_j^2 c^2} \boldsymbol{S} \cdot \boldsymbol{p}(t) \times \boldsymbol{E}(t, \boldsymbol{z}(t)) + \frac{g}{m_j c^2} \boldsymbol{S} \cdot \dot{\boldsymbol{z}}(t) \times \boldsymbol{E}(t, \boldsymbol{z}(t)) \right) \right] \right\}. \quad (4.1)$$

Then, it is easy to show that, by eqs. (1.7) and (2.25), the following identities hold

$$\mathrm{d}x_\mu A^\mu = c\,\mathrm{d}t \left(A^0 - \frac{1}{c} \dot{\boldsymbol{z}} \cdot \boldsymbol{A} \right), \quad (4.2a)$$

$$\mathrm{d}x_\mu \hat{F}^{l\mu} = c\,\mathrm{d}t \left[B^l - \frac{1}{c} (\dot{\boldsymbol{z}} \times \boldsymbol{E})^l \right], \quad (4.2b)$$

$$\mathrm{d}x^\mu F^{l\mu} = c\,\mathrm{d}t \left[E^l + \frac{1}{c} (\dot{\boldsymbol{z}} \times \boldsymbol{B})^l \right], \quad (4.2c)$$

$$\mathrm{d}x^\mu \left[\partial^\nu + \frac{ig}{c} A^\nu, F_{\nu\mu} \right] = c\,\mathrm{d}t \left\{ \left[\partial_k - \frac{ig}{c} A^k, E^k + \frac{1}{c} (\dot{\boldsymbol{z}} \times \boldsymbol{B})^k \right] + \frac{1}{c^2} \dot{z}^k \left[\partial_t + igA^0, E^k \right] \right\}. \quad (4.2d)$$

Apart from higher order terms, we can write

$$K_j(x, y|A) = \int_{z(t_y)=y}^{z(t_x)=x} \mathscr{D}[z, p]$$

$$\times \mathrm{T} \exp\left\{ i\int_{t_y}^{t_x} \mathrm{d}t \left[p(t)\cdot \dot{z}(t) - \frac{p(t)^2}{2m_j} + \frac{p(t)^4}{8m_j^3c^2} - \frac{ig}{c} \right.\right.$$

$$\times \int_y^x \mathrm{d}x^\mu \left(A_\mu(x) + \frac{1}{m_j c} S^l \hat{F}_{l\mu}(x) - \frac{1}{2m_j^2c^2} S^l \varepsilon^{lkr} p^k(t) \right.$$

$$\left.\left.\left. \times F_{\mu r}(x) - \frac{1}{8m_j^2c^2} \left[\partial^\nu + \frac{ig}{c} A^\nu(x), F_{\nu\mu}(x) \right] \right) \right] \right\}. \tag{4.3}$$

Equation (4.3) can be used as it stands to express K_1 in eq. (2.23). Coming to K_2, we have to replace A by $-\tilde{A}$ in eq. (4.3), and then we can write

$$\langle \sigma' | \tilde{K}_2(x, y| -\tilde{A}) | \sigma \rangle$$

$$= \langle \sigma | \int_{z(t_y)=y}^{z(t_x)=x} \mathscr{D}[z, p] \mathrm{T_S T_A^*}$$

$$\times \exp\left\{ i\int_{t_y}^{t_x} \mathrm{d}t \left[p(t)\cdot \dot{z}(t) - \frac{p(t)^2}{2m_2} + \frac{p(t)^4}{8m_2^3c^2} \right] \right.$$

$$+ \frac{ig}{c} \int_y^x \mathrm{d}x^\mu \left(A_\mu(x) + \frac{1}{m_2 c} S^l \hat{F}_{l\mu}(x) - \frac{1}{2m_2^2c^2} S^l \varepsilon^{lkr} p^k(t) F_{\mu r}(x) \right.$$

$$\left.\left. - \frac{1}{8m_2^2c^2} \left[\partial^\nu + \frac{ig}{c} A^\nu(x), F_{\nu\mu}(x) \right] \right) \right\} | \sigma' \rangle, \tag{4.4}$$

where $\mathrm{T_S}$ denotes the chronological ordering for the spin matrices, and T_A^* the antichronological one for the gauge matrices.

By inserting eqs. (4.3), (4.4) and (2.5) into eq. (2.23), eq. (2.24) follows immediately.

5. Schrödinger equation for the two-particle propagator

As already observed, the derivation of the Schrödinger equation (2.27) for the two-particle propagator rests, in an essential way, on the possibility of the identification (2.26). Therefore, we first discuss this problem from a general point of view.

For a sufficiently regular functional $F[y]$ of $y(t) \equiv (y_1(t), \ldots, y_n(t))$ we may assume that the following expansion holds:

$$F[y] = \int_{-\tau/2}^{\tau/2} \mathrm{d}t \left[f(y(t)) + \sum_i \dot{y}_i(t) b_i(y(t)) \right.$$
$$\left. + \sum_{ij} \dot{y}_i(t) \bar{c}_{ij}(y(t)) \dot{y}_j(t) + \sum_i \ddot{y}_i(t) d_i(y(t)) + \cdots \right], \tag{5.1}$$

with $\bar{c}_{ij} = \bar{c}_{ji}$. In our case, by dimensional arguments, eq. (5.1) amounts to an expansion in powers of $1/c$.

The identification of the coefficients is straightforward. By putting

$$\bar{F}(y, v, a) := F\left[y_i + v_i t + \tfrac{1}{2} a_i t^2 \right], \tag{5.2}$$

we obtain immediately

$$f(y) = \frac{1}{\tau} \bar{F}|_{v=a=0}, \tag{5.3}$$

$$b_j(y) = \frac{1}{\tau} \frac{\partial \bar{F}}{\partial v_j}\bigg|_{v=a=0}, \tag{5.4}$$

$$\bar{c}_{ij}(y) = \frac{1}{2} \left(\frac{1}{\tau} \frac{\partial^2 \bar{F}}{\partial v_i \partial v_j} - \frac{\tau}{12} \frac{\partial^2 \bar{F}}{\partial y_i \partial y_j} \right)\bigg|_{v=a=0}, \tag{5.5}$$

$$d_j(y) = \left(\frac{1}{\tau} \frac{\partial \bar{F}}{\partial a_j} - \frac{\tau}{24} \frac{\partial \bar{F}}{\partial y_i} \right)\bigg|_{v=a=0}. \tag{5.6}$$

After a partial integration of the term with $\ddot{y}_i$, eq. (5.1) becomes

$$F[y] = \int_{-\tau/2}^{\tau/2} \mathrm{d}t \left[f(y(t)) + \sum_i \dot{y}_i(t) b_i(y(t)) + \sum_{ij} \dot{y}_i(t) c_{ij}(y(t)) \dot{y}_j(t) + \cdots \right]$$
$$+ \sum_i \left[\dot{y}_i\left(\frac{\tau}{2}\right) d_i\left(y\left(\frac{\tau}{2}\right)\right) - \dot{y}_i\left(-\frac{\tau}{2}\right) d_i\left(y\left(-\frac{\tau}{2}\right)\right) \right], \tag{5.7}$$

where

$$c_{ij}(y) = \frac{1}{2\tau}\left(\frac{\partial^2\bar{F}}{\partial v_i\,\partial v_j} - \frac{\partial^2\bar{F}}{\partial y_i\,\partial a_j} - \frac{\partial^2\bar{F}}{\partial y_j\,\partial a_i}\right)\Bigg|_{v=a=0}. \tag{5.8}$$

Let us now go back to the concrete case of eq. (2.26). The problem is the following identification (up to order $1/c^2$):

$$\begin{aligned}
&\Bigg\langle \frac{1}{N}\mathrm{Tr}\,\mathrm{T_S}\mathrm{P}_\Gamma \exp\Bigg\{\frac{ig}{c}\oint_\Gamma \mathrm{d}x_\mu\, A^\mu(x) + \sum_j \frac{ig}{m_j c^2}\int_{\Gamma_j}\mathrm{d}x^\mu \\
&\qquad\times\Bigg(S_j^l \hat{F}_{l\mu}(x) - \frac{1}{2m_j c} S_j^l \varepsilon^{lkr} p_j^k(t) F_{\mu r}(x) \\
&\qquad\qquad - \frac{1}{8m_j c}\left[\partial^\nu + \frac{ig}{c}A^\nu(x),\, F_{\nu\mu}(x)\right]\Bigg)\Bigg\}\Bigg\rangle \\
&\simeq \mathrm{T_S}\exp\left\{-i\int_{-\tau/2}^{\tau/2}\mathrm{d}t\left[U(t) + \sum_j \boldsymbol{Y}_j(t)\cdot \boldsymbol{S}_j + S_1^h T^{hk}(t) S_2^k\right]\right\},
\end{aligned} \tag{5.9}$$

where $U(t), Y_j^h(t), T^{hk}(t)$ are functions of $z_1(t), z_2(t), \boldsymbol{p}_1(t), \boldsymbol{p}_2(t)$ (and of their derivatives). In order to make this identification, it is convenient to expand the logarithm of both sides. The l.h.s. of eq. (5.9) gives

$$\begin{aligned}
&\log\left\langle \frac{1}{N}\mathrm{Tr}\,\mathrm{T_S}\mathrm{P}_\Gamma\exp\left\{\frac{ig}{c}\oint_\Gamma \mathrm{d}x_\mu\, A^\mu(x) + \sum_j \frac{ig}{m_j c}\cdots\right\}\right\rangle \\
&\simeq \log\left\langle \frac{1}{N}\mathrm{Tr}\,\mathrm{P}_\Gamma\exp\left[\frac{ig}{c}\oint_\Gamma \mathrm{d}x_\mu\, A^\mu(x)\right]\right\rangle \\
&\quad + \sum_j \frac{ig}{m_j c^2}\int_{\Gamma_j}\mathrm{d}x^\mu\Bigg\{S_j^l\langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_\Gamma - \frac{1}{2m_j c}S_j^l\varepsilon^{lkr}p_j^k(t)\langle\langle F_{\mu r}(x)\rangle\rangle_\Gamma \\
&\qquad\qquad - \frac{1}{8m_j c}\left\langle\left\langle\left[\partial^\nu + \frac{ig}{c}A^\nu(x),\, F_{\nu\mu}(x)\right]\right\rangle\right\rangle_\Gamma\Bigg\} \\
&\quad - \sum_{ij}\frac{g^2}{m_i m_j c^4}\int_{\Gamma_j}\mathrm{d}x^\mu\int_{\Gamma_i}\mathrm{d}x'^\sigma\left[\vartheta(t-t')S_j^l S_i^k + \vartheta(t'-t)S_i^k S_j^l\right] \\
&\quad\times\left(\langle\langle \hat{F}_{l\mu}(x)\hat{F}_{k\sigma}(x')\rangle\rangle_\Gamma - \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_\Gamma\langle\langle \hat{F}_{k\sigma}(x')\rangle\rangle_\Gamma\right),
\end{aligned} \tag{5.10}$$

where the double angular bracket is defined by eq. (1.5). From eqs. (5.9) and (5.10), it is apparent that the expansion of $Y_j^h\,(T^{hk})$ starts with the term of order $1/c\,(1/c^2)$. Bearing this in mind, it follows that

$$\log \mathrm{T_S}\exp\left\{-i\int_{-\tau/2}^{\tau/2}\mathrm{d}t\left[U(t)+\sum_j \boldsymbol{Y}_j(t)\cdot\boldsymbol{S}_j+S_1^h T^{hk}(t)S_2^k\right]\right\}$$
$$\simeq -i\int_{-\tau/2}^{\tau/2}\mathrm{d}t\left[U(t)+\sum_j \boldsymbol{Y}_j(t)\cdot\boldsymbol{S}_j+S_1^h T^{hk}(t)S_2^k\right]$$
$$-\tfrac{1}{2}i\int_{-\tau/2}^{\tau/2}\mathrm{d}t\int_{-\tau/2}^{t}\mathrm{d}t'\sum_j \boldsymbol{S}_j\cdot\boldsymbol{Y}_j(t)\times\boldsymbol{Y}_j(t'). \tag{5.11}$$

Then, by comparing eqs. (5.10) and (5.11) and using the properties of the spin matrices, we have

$$\int_{-\tau/2}^{\tau/2}\mathrm{d}t\,U(t)\simeq i\log\left\langle\frac{1}{N}\mathrm{Tr}\,\mathrm{P}_\Gamma\exp\left[\frac{ig}{c}\oint_\Gamma \mathrm{d}x_\mu\,A^\mu(x)\right]\right\rangle$$
$$+\sum_j\frac{g}{8m_j^2c^3}\int_{\Gamma_j}\mathrm{d}x^\mu\left\langle\!\!\left\langle\left[\partial^\nu+\frac{ig}{c}A^\nu(x),F_{\nu\mu}(x)\right]\right\rangle\!\!\right\rangle_\Gamma$$
$$-\sum_j\frac{ig^2}{8m_j^2c^4}\int_{\Gamma_j}\mathrm{d}x_\mu\int_{\Gamma_j}\mathrm{d}x'_\sigma\Big(\big\langle\big\langle\hat{F}^{k\mu}(x)\hat{F}^{k\sigma}(x)\big\rangle\big\rangle_\Gamma$$
$$-\big\langle\big\langle\hat{F}^{k\mu}(x)\big\rangle\big\rangle_\Gamma\big\langle\big\langle\hat{F}^{k\sigma}(x')\big\rangle\big\rangle_\Gamma\Big), \tag{5.12}$$

$$\int_{-\tau/2}^{\tau/2}\mathrm{d}t\,Y_j^l(t)+\tfrac{1}{2}\int_{-\tau/2}^{\tau/2}\mathrm{d}t\int_{-\tau/2}^{t}\mathrm{d}t'\big(\boldsymbol{Y}_j(t)\times\boldsymbol{Y}_j(t')\big)^l$$
$$\simeq -\frac{g}{m_jc^2}\int_{\Gamma_j}\mathrm{d}x^\mu\left(\big\langle\big\langle\hat{F}_{l\mu}(x)\big\rangle\big\rangle_\Gamma-\frac{1}{2m_jc}\varepsilon^{lkr}p_j^k(t)\big\langle\big\langle F_{\mu r}(x)\big\rangle\big\rangle_\Gamma\right)$$
$$+\frac{g^2}{2m_j^2c^4}\int_{\Gamma_j}\mathrm{d}x_\mu\int_{\Gamma_j}\mathrm{d}x'_\sigma\,\vartheta(t-t')\varepsilon^{rkl}$$
$$\times\Big(\big\langle\big\langle\hat{F}^{r\mu}(x)\hat{F}^{k\sigma}(x')\big\rangle\big\rangle_\Gamma-\big\langle\big\langle F^{r\mu}(x)\big\rangle\big\rangle_\Gamma\big\langle\big\langle\hat{F}^{k\sigma}(x')\big\rangle\big\rangle_\Gamma\Big), \tag{5.13}$$

$$\int_{-\tau/2}^{\tau/2}\mathrm{d}t\,T^{hk}(t)\simeq-\frac{ig^2}{m_1m_2c^4}\int_{\Gamma_1}\mathrm{d}x_\mu\int_{\Gamma_2}\mathrm{d}x'_\sigma$$
$$\times\Big(\big\langle\big\langle\hat{F}^{h\mu}(x)\hat{F}^{k\sigma}(x')\big\rangle\big\rangle_\Gamma-\big\langle\big\langle\hat{F}^{h\mu}(x)\big\rangle\big\rangle_\Gamma\big\langle\big\langle\hat{F}^{k\sigma}(x')\big\rangle\big\rangle_\Gamma\Big). \tag{5.14}$$

By using the trick explained at the beginning of this section (eqs. (5.1)–(5.8)), as well as the symmetry properties of the measure over the gauge fields, eqs. (5.12)–(5.14) give the following results (for technical details, see appendix)

$$U(t) \simeq V(r(t)) + \sum_j \frac{1}{8m_j^2c^2}\Delta_2[V(r(t)) + V_a(r(t))]$$
$$+\frac{\dot{z}_1^h(t)\dot{z}_2^k(t)}{c^2}\left[\delta^{hk}V_b(r(t)) + \left(\tfrac{1}{3}\delta^{hk} - \frac{r^hr^k}{r^2}\right)V_c(r(t))\right]$$
$$+\sum_j \frac{\dot{z}_j^h(t)\dot{z}_j^k(t)}{c^2}\left[\delta^{hk}V_d(r(t)) + \left(\tfrac{1}{3}\delta^{hk} - \frac{r^hr^k}{r^2}\right)V_e(r(t))\right], \quad (5.15)$$

$$\boldsymbol{Y}_j(t) = -\frac{\boldsymbol{L}_j(t)}{2m_j^2c^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}V(r(t)) + \frac{\boldsymbol{L}_j'(t)}{m_j^2c^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(V(r(t)) + V_1(r(t)))$$
$$+\sum_i (1-\delta_{ij})\frac{\boldsymbol{L}_i'(t)}{m_1m_2c^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}V_2(r(t)), \quad (5.16)$$

$$T^{hk}(t) = \frac{1}{3m_1m_2c^2}\delta^{hk}V_4(r(t)) + \frac{1}{m_1m_2c^2r^2}\left(\frac{r^hr^k}{r^2} - \tfrac{1}{3}\delta^{lk}\right)V_3(r(t)), \quad (5.17)$$

where $\boldsymbol{r}(t) = \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)$, $\boldsymbol{L}_j(t) = \varepsilon_j\boldsymbol{r}(t)\times\boldsymbol{p}_j(t)$, $\boldsymbol{L}_j'(t) = \varepsilon_m m_j\boldsymbol{r}(t)\times\dot{\boldsymbol{z}}_j(t)$ ($\varepsilon_1 = 1$, $\varepsilon_2 = -1$), $V(r)$ is given by eq. (1.1) (with the normalization factor $\frac{1}{3}$ replaced in general by $1/N$), V_a–V_e by eqs. (1.9), V_1–V_4 by eqs. (1.4).

We now insert eqs. (5.15)–(5.17) and (5.9) into eq. (2.24). By suitable translations on $\boldsymbol{p}_j$ we can manage to replace $\dot{\boldsymbol{z}}_j$ by $\boldsymbol{p}_j/m_j$ (apart from higher order terms). Having this in mind, eq. (2.24) becomes

$$K(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y}_1, \boldsymbol{y}_2; \tau)$$
$$= \int_{z_1(-\tau/2)=y_1}^{z_1(\tau/2)=x_1}\mathscr{D}[\boldsymbol{z}_1, \boldsymbol{p}_1]\int_{z_2(-\tau/2)=y_2}^{z_2(\tau/2)=x_2}\mathscr{D}[\boldsymbol{z}_2, \boldsymbol{p}_2]$$
$$\times \mathrm{T_S}\exp\left\{i\int_{-\tau/2}^{\tau/2}\mathrm{d}t\left[\sum_j \boldsymbol{p}_j(t)\cdot\dot{\boldsymbol{z}}_j(t) - \left(\sum_j \frac{\boldsymbol{p}_j(t)^2}{2m_j} - \sum_j \frac{\boldsymbol{p}_j(t)^4}{8m_j^3c^2}\right.\right.\right.$$
$$\left.\left.\left. + V(r(t)) + V_{\mathrm{vd}}(t) + V_{\mathrm{sd}}(t)\right)\right]\right\}. \quad (5.18)$$

Here, V_{sd} and V_{vd} are given by eqs. (1.3) and (1.8) respectively, obviously provided that, in eq. (1.8), the quantities $\boldsymbol{p}_j$ are now understood as c-numbers.

Equation (5.18) is just the functional integral expression for the propagator of a two-particle Schrödinger equation. The identification of the hamiltonian poses a delicate problem of operator ordering, because we are regarding the path integral as the limit of a suitable discretized expression, and – as is well known – any discretization prescription gives rise to a different operator symmetrization (see, for instance, ref. [14]). In going from eq. (2.20) to eq. (2.22) we have tacitly used the discretization prescription which corresponds to the anticommutator between functions of position and momentum. It is natural to assume that the same prescription holds for eq. (5.18), although this point is by no means obvious, because the manipulations leading to eq. (5.18) could have changed the discretization. A check of the correctness of our procedure lies in the fact that, in the QED case, the choice of the anticommutator gives exactly the electron-electron potential given in ref. [7].

In conclusion, the propagator (5.18) satisfies a Schrödinger equation with hamiltonian

$$H=\sum_j\left(\frac{\boldsymbol{p}_j^2}{2m_j}-\frac{\boldsymbol{p}_j^4}{8m_j^3c^2}\right)+V(r)+V_{\mathrm{vd}}+V_{\mathrm{sd}}\,. \tag{5.19}$$

This concludes the justification of the expressions for the potentials anticipated in sect. 1.

6. Short- and long-range behaviour of the potentials

In QED, if one neglects the photon-photon effective interaction due to the fermion loops, the functional integrals defining the potentials become gaussian and exactly computable. This amounts to take eq. (1.2) as definition of the angular brackets, without the correction indicated in eqs. (2.8). The same happens in QCD, provided that all the gluon-gluon interactions are neglected; this approximation is equivalent to a perturbative evaluation of the potentials, and should give their correct behaviour only for small distances.

By taking SU(3) as the gauge group, and choosing the normalization $\mathrm{Tr}\{T_aT_b\}=\frac{1}{2}\delta_{ab}$ for the gauge matrices, we have in this approximation

$$\frac{1}{N}\mathrm{Tr}\langle A^\mu(x)A^\nu(y)\rangle=-ic\frac{4}{3}\frac{g^{\mu\nu}}{4\pi}\int\mathrm{d}_4k\,\frac{\exp[-ik(x-y)]}{k^2+i0}\,. \tag{6.1}$$

Here, we adopt the Feynman gauge (remember that all the expressions for the potentials are gauge invariant). In the QED case the same result holds, provided that one replaces $\frac{4}{3}$ by 1. Then, by using the expressions for the potentials given in sect. 1, it is a simple matter to obtain eqs. (1.10). We remark that in these computations some divergences occur. More precisely, all the potentials, but V, V_a and V_d, are finite. The expressions of V, V_a, V_d contain infinite constants of self-energy type, which contribute to mass renormalization (cf. ref. [6], pp. 163–164).

In order to estimate the long-range behaviour of the potentials the following alternative expressions are more useful (the proofs are given in the appendix):

$$V(r) = \lim_{\tau \to +\infty} \bar{V}|_{v=a=0}, \tag{6.2}$$

$$\Delta_2 V_a(r) = \lim_{\tau \to +\infty} \frac{ig^2}{c^2\tau} \int_{\Gamma_W^{(j)}} dx^\mu \times dx'_\sigma \Big[\langle\langle \hat{F}_{\mu\nu}(x)\hat{F}^{\sigma\nu}(x')\rangle\rangle_{\Gamma_W} - \langle\langle \hat{F}_{\mu\nu}(x)\rangle\rangle_{\Gamma_W}\langle\langle \hat{F}^{\sigma\nu}(x')\rangle\rangle_{\Gamma_W} - \langle\langle F_{\mu\nu}(x)F^{\sigma\nu}(x')\rangle\rangle_{\Gamma_W} + \langle\langle F_{\mu\nu}(x)\rangle\rangle_{\Gamma_W}\langle\langle F^{\sigma\nu}(x')\rangle\rangle_{\Gamma_W}\Big], \tag{6.3}$$

$$\delta^{hk}V_b(r) + \left(\tfrac{1}{3}\delta^{hk} - \frac{r^h r^k}{r^2}\right)V_c(r) = \lim_{\tau \to +\infty} 2c^2 \left[\left(\frac{\partial^2}{\partial v_1^h \partial v_2^k} - \frac{\partial^2}{\partial x_1^h \partial a_2^k} - \frac{\partial^2}{\partial x_2^h \partial a_1^k}\right) + (h \rightleftarrows k)\right]\bar{V}|_{v=a=0}, \tag{6.4}$$

$$\delta^{hk}V_d(r) + \left(\tfrac{1}{3}\delta^{hk} - \frac{r^h r^k}{r^2}\right)V_e(r) = \lim_{\tau \to +\infty} 2c^2 \left(\frac{\partial^2}{\partial v_1^h \partial v_1^k} - \frac{\partial^2}{\partial x_1^h \partial a_1^k} - \frac{\partial^2}{\partial x_1^k \partial a_1^h}\right)\bar{V}|_{v=a=0}, \tag{6.5}$$

$$\frac{d}{dr}(V(r) + V_1(r)) = \lim_{\tau \to +\infty} \frac{g}{4\tau}\frac{r^h}{r}\varepsilon^{hlk}\left[\frac{\partial}{\partial v_1^k}\int_{\bar{\Gamma}_1} dx^\mu \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_{\bar{\Gamma}} - \frac{\partial}{\partial v_1^l}\int_{\bar{\Gamma}_1} dx^\mu \langle\langle \hat{F}_{k\mu}(x)\rangle\rangle_{\bar{\Gamma}}\right]\Bigg|_{v=a=0}, \tag{6.6}$$

$$\frac{d}{dr}V_2(r) = \lim_{\tau \to +\infty} \frac{g}{4\tau}\frac{r^h}{r}\varepsilon^{hlk} \times \left[\frac{\partial}{\partial v_2^l}\int_{\bar{\Gamma}_1} dx^\mu \langle\langle \hat{F}_{k\mu}(x)\rangle\rangle_{\bar{\Gamma}} - \frac{\partial}{\partial v_2^k}\int_{\bar{\Gamma}_1} dx^\mu \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_{\bar{\Gamma}}\right]\Bigg|_{v=a=0}, \tag{6.7}$$

$$\frac{1}{r^2}\left(\frac{r^h r^k}{r^2} - \tfrac{1}{3}\delta^{hk}\right)V_3(r)$$

$$= - \lim_{\tau \to +\infty} \frac{ig^2}{2c^2\tau}\int_{\Gamma_W^{(1)}} dx_\mu \int_{\Gamma_W^{(2)}} dx'_\sigma$$

$$\times\left[\left\langle\left\langle \hat{F}^{h\mu}(x)\hat{F}^{k\sigma}(x') + \hat{F}^{k\mu}(x)\hat{F}^{h\sigma}(x')\right\rangle\right\rangle_{\Gamma_W} - \tfrac{2}{3}\delta^{hk}\left\langle\left\langle \hat{F}^{l\mu}(x)\hat{F}^{l\sigma}(x')\right\rangle\right\rangle_{\Gamma_W}\right], \tag{6.8}$$

$$V_4(r) = \lim_{\tau \to +\infty} \frac{ig^2}{c^2\tau}\int_{\Gamma_W^{(1)}} dx^\mu \int_{\Gamma_W^{(2)}} dx'_\sigma \left\langle\left\langle \hat{F}_{h\mu}(x)\hat{F}^{h\sigma}(x')\right\rangle\right\rangle_{\Gamma_W}, \tag{6.9}$$

where

$$\bar{V} = \frac{i}{\tau}\log\left\langle \frac{1}{N}\mathrm{Tr}\,\mathrm{P}_{\bar{\Gamma}}\exp\left[\frac{ig}{c}\oint_{\bar{\Gamma}} dx^\mu A_\mu(x)\right]\right\rangle \tag{6.10}$$

and $\bar{\Gamma}$ is the usual closed loop Γ in which the trajectories of the two particles are given by $\boldsymbol{x}_j + \boldsymbol{v}_j t + \frac{1}{2}\boldsymbol{a}_j t^2$; $\bar{\Gamma}_j\,(\Gamma_W^{(j)})$ is the piece of the path $\bar{\Gamma}(\Gamma_W)$ along the trajectory of the particle j.

Now, by exploiting some ideas of Gromes [4,5], we are able to estimate the behaviour for large r of the potentials. First, for large r and τ, we can evaluate $\bar{V}$ (eq. (6.10)) by the area law [4]

$$\bar{V} = \frac{i}{\tau}\log\exp\left(-\frac{i}{c}a\mathscr{A}\right) = \frac{a}{\tau c}\mathscr{A}, \tag{6.11}$$

where $\mathscr{A}$ is the area of the minimal surface enclosed by the loop $\bar{\Gamma}$. As Gromes remarks, eq. (6.11) is the essential message from lattice gauge theory. As eqs. (6.4) and (6.5) involve only second order derivatives with respect to $\boldsymbol{v}_j$ and first order derivatives with respect to $\boldsymbol{a}_j$ at $\boldsymbol{v}_j = \boldsymbol{0}$, $\boldsymbol{a}_j = \boldsymbol{0}$, we need $\mathscr{A}$ solely up to first order in $\boldsymbol{a}_j$ and to second order in $\boldsymbol{v}_j$. Up to this order, the minimal surface can be identified (cf. ref. [4]) with the surface spanned by the straight lines joining $(ct, \boldsymbol{z}_1(t))$ to $(ct, \boldsymbol{z}_2(t))$, with

$$\boldsymbol{z}_j(t) \equiv \boldsymbol{x}_j + \boldsymbol{v}_j t + \tfrac{1}{2}\boldsymbol{a}_j t^2. \tag{6.12}$$

The generic point of this surface is

$$x^\mu(\sigma, t) \equiv (ct, \sigma\boldsymbol{z}_1(t) + (1-\sigma)\boldsymbol{z}_2(t)), \tag{6.13a}$$

$$0 \leqslant \sigma \leqslant 1, \qquad -\tfrac{1}{2}\tau \leqslant t \leqslant \tfrac{1}{2}\tau, \tag{6.13b}$$

and the infinitesimal surface element is given by

$$\mathrm{d}_2\mathscr{A} = c\,\mathrm{d}t\,\mathrm{d}\sigma\left[\sum_{\mu=0}^{3}\frac{\partial x^\mu}{\partial ct}\frac{\partial x^\mu}{\partial ct}\sum_{\nu=0}^{3}\frac{\partial x^\nu}{\partial \sigma}\frac{\partial x^\nu}{\partial \sigma} - \left(\sum_{\mu=0}^{3}\frac{\partial x^\mu}{\partial ct}\frac{\partial x^\mu}{\partial \sigma}\right)^2\right]^{1/2}$$

$$= c\,\mathrm{d}t\,\mathrm{d}\sigma\left\{|z_1(t)-z_2(t)|^2\left(1+\left|\frac{\sigma}{c}\dot{z}_1(t)+\frac{1-\sigma}{c}\dot{z}_2(t)\right|^2\right)\right.$$

$$\left.-\left[(z_1(t)-z_2(t))\cdot\left(\frac{\sigma}{c}\dot{z}_1(t)+\frac{1-\sigma}{c}\dot{z}_2(t)\right)\right]^2\right\}^{1/2}. \qquad (6.14)$$

At first order in $\boldsymbol{a}_j$ and second order in $\boldsymbol{v}_j$, we get

$$\mathscr{A} = c\tau\left\{r+\frac{\tau^2}{24r}\left[v^2+\boldsymbol{r}\cdot\boldsymbol{a}-\frac{(\boldsymbol{r}\cdot\boldsymbol{v})^2}{r^2}\right]\right.$$

$$\left.+\frac{r}{2c^2}\left[\frac{1}{3}v^2+\boldsymbol{v}_1\cdot\boldsymbol{v}_2-\frac{(\boldsymbol{r}\cdot\boldsymbol{v})^2}{3r^2}-\frac{\boldsymbol{r}\cdot\boldsymbol{v}_1\boldsymbol{r}\cdot\boldsymbol{v}_2}{r^2}\right]\right\}, \qquad (6.15a)$$

$$\boldsymbol{r}\equiv\boldsymbol{x}_1-\boldsymbol{x}_2, \qquad \boldsymbol{v}\equiv\boldsymbol{v}_1-\boldsymbol{v}_2, \qquad \boldsymbol{a}\equiv\boldsymbol{a}_1-\boldsymbol{a}_2. \qquad (6.15b)$$

From eqs. (6.2), (6.4), (6.5), (6.11), (6.15) we have immediately

$$V(r)\sim ar, \qquad (6.16a)$$

$$\delta^{hk}V_b(r)+\left(\tfrac{1}{3}\delta^{hk}-\frac{r^hr^k}{r^2}\right)V_c(r)\sim\tfrac{2}{3}ar\left(\delta^{hk}-\frac{r^hr^k}{r^2}\right), \qquad (6.16b)$$

$$\delta^{hk}V_d+\left(\tfrac{1}{3}\delta^{hk}-\frac{r^hr^k}{r^2}\right)V_e(r)\sim\tfrac{2}{3}ar\left(\delta^{hk}-\frac{r^hr^k}{r^2}\right), \qquad (6.16c)$$

from which V_b, V_c, V_d, V_e can be obtained [cf. eq. (1.11a)].

Next, we assume that the longitudinal colour electric field alone is responsible for the long-range part of the potentials, while magnetic field correlations are short-range [3]. In order to correctly apply this idea, relativistic covariance properties are important [5]. Eqs. (6.6)–(6.9) are written in a form making these properties quite evident. We do not repeat in detail the arguments of ref. [5]. Essentially, from eqs. (6.7)–(6.9) one sees that the potentials V_2, V_3, V_4 involve magnetic field (cf. also eqs. (1.4b)–(1.4d)) and, therefore, they vanish for large r. Then, from eqs. (1.6) and (6.16a) one has $V_1(r)\sim -ar$.

Finally, we have to discuss the behaviour of $V_a(r)$ (eq. (6.3)). We have seen that, for small r, $\Delta_2 V_a(r)$ reduces to a constant which contributes to mass renormalization. Then, as in the second of refs. [3], we consider the instanton contributions to V_a. For the pseudoparticle solutions of field equations satisfying the duality or antiduality condition $F^{\mu\nu} = \pm\hat{F}^{\mu\nu}$, the two terms in $\Delta_2 V_a$ are of opposite sign and cancel exactly. Therefore, these pseudoparticle solutions give a vanishing contribution to $\Delta_2 V_a$. It is then natural to assume $V_a = 0$ for any r, or at least that $\Delta_2 V_a$ is less important than $\Delta_2 V$. This ends the proof of eqs. (1.11).

Appendix

In this appendix we indicate how to obtain eqs. (5.15)–(5.17) from eqs. (5.12)–(5.14).

The equations (5.1)–(5.8), applied to eq. (5.12), give immediately

$$\int_{-\tau/2}^{\tau/2} \mathrm{d}t\, U(t) \simeq \frac{1}{c^2}\sum_j \left[\dot{z}_j\left(\frac{\tau}{2}\right)\cdot \boldsymbol{d}_j\left(z\left(\frac{\tau}{2}\right)\right) - \dot{z}_j\left(-\frac{\tau}{2}\right)\cdot \boldsymbol{d}_j\left(z\left(-\frac{\tau}{2}\right)\right)\right]$$
$$+\int_{-\tau/2}^{\tau/2}\mathrm{d}t\left[f(z(t)) + \frac{1}{c}\sum_j \dot{z}_j(t)\cdot \boldsymbol{b}_j(z(t)) + \frac{1}{c^2}\sum_{ij}\dot{z}_i^h(t)\, c_{ij}^{hk}(z(t))\dot{z}_j^k(t)\right], \tag{A.1}$$

where

$$f(z) = \overline{V}|_{v=a=0} + \sum_j \frac{g}{8m_j^2c^3\tau}\int_{\Gamma_W^{(j)}}\mathrm{d}x^\mu \left\langle\left\langle\left[\partial^\nu + \frac{ig}{c}A^\nu(x), F_{\nu\mu}(x)\right]\right\rangle\right\rangle_{\Gamma_W}$$
$$+\sum_j \frac{ig^2}{8m_j^2c^4\tau}\int_{\Gamma_W^{(j)}}\mathrm{d}x^\mu\int_{\Gamma_W^{(j)}}\mathrm{d}x'_\sigma\left[\left\langle\left\langle \hat{F}_{\nu\mu}(x)\hat{F}^{\nu\sigma}(x')\right\rangle\right\rangle_{\Gamma_W} - \left\langle\left\langle \hat{F}_{\nu\mu}(x)\right\rangle\right\rangle_{\Gamma_W}\left\langle\left\langle \hat{F}^{\nu\sigma}(x')\right\rangle\right\rangle_{\Gamma_W}\right], \tag{A.2}$$

$$b_j^h(z) = c\frac{\partial}{\partial v_j^h}\overline{V}|_{v=a=0}, \tag{A.3}$$

$$c_{ij}^{hk}(z) = \frac{c^2}{2}\left(\frac{\partial^2}{\partial v_i^h\,\partial v_j^k} - \frac{\partial^2}{\partial z_i^h\,\partial a_j^k} - \frac{\partial^2}{\partial z_j^k\,\partial a_i^h}\right)\overline{V}|_{v=a=0}, \tag{A.4}$$

$$d_j^h(z) = c^2\left(\frac{\partial}{\partial a_j^h} - \frac{\tau^2}{24}\frac{\partial}{\partial z_j^h}\right)\overline{V}|_{v=a=0}. \tag{A.5}$$

Here $\bar{V}$ is given by eq. (6.10) and $\Gamma_W^{(j)}$ is one of the two time-like straight lines in the Wilson loop Γ_W.

The first term in eq. (A.2) is, for large τ, the static potential (1.1) (with the normalization factor $\frac{1}{3}$ replaced by $1/N$); by symmetry arguments it depends only on $r=|z_1-z_2|$. In the following, it is useful to have its first and second derivatives. We have

$$\begin{aligned}\left.\frac{\partial \bar{V}}{\partial z_1^h}\right|_{v=a=0} &= -\frac{g}{c\tau}\left\langle\left\langle \int_{\Gamma_W^{(1)}} \mathrm{d}x_0\, \partial_h A^0(x) + \frac{1}{c}A^h\left(\frac{\tau}{2}, z_1\right) - \frac{1}{c}A^h\left(-\frac{\tau}{2}, z_1\right)\right\rangle\right\rangle_{\Gamma_W} \\ &= -\frac{g}{c\tau}\int_{\Gamma_W^{(1)}} \mathrm{d}x_0 \left\langle\left\langle \partial_h A^0(x) + \partial_0 A^h(x) + \frac{ig}{c}\left[A^0(x), A^h(x)\right]\right\rangle\right\rangle_{\Gamma_W} \\ &= \frac{g}{c\tau}\int_{\Gamma_W^{(1)}} \mathrm{d}x_0 \langle\langle E^h(x)\rangle\rangle_{\Gamma_W} = \frac{g}{c\tau}\int_{\Gamma_W^{(1)}} \mathrm{d}x^\mu \langle\langle F_{\mu h}(x)\rangle\rangle_{\Gamma_W}.\end{aligned}$$

In the first step, the first term is due to the variation in the time-like path along (t, z_1) and the other two terms are due to the variation of the extremal points in the two space-like straight lines. The second step is obvious in an abelian gauge theory; in the case of a non-abelian one, one can freely commute fields refering to different points, because of the presence of the path-ordering operator P_{Γ_W}. However, fields at the same point do not commute and this fact must be taken into account when partial integrations are carried out. The last term in the second step is precisely due to the lack of commutativity between A^h and a contribution at the same point coming from the Wilson loop (cf. ref. [5], p. 403). In this appendix many computations of this kind are needed. As a rule, one can make calculations in the abelian case; by rewriting the final result in a gauge invariant form, an expression valid also in the nonabelian case is obtained. In this way we obtain also

$$\left.\frac{\partial \bar{V}}{\partial z_j^h}\right|_{v=a=0} = \frac{g}{c\tau}\int_{\Gamma_W^{(j)}} \mathrm{d}x^\mu \langle\langle F_{\mu h}(x)\rangle\rangle_{\Gamma_W}, \tag{A.6}$$

$$\begin{aligned}\Delta_2 \bar{V}|_{v=a=0} &\equiv \left.\frac{\partial^2 \bar{V}}{\partial z_j^h\, \partial z_j^h}\right|_{v=a=0} \\ &= \frac{g}{c\tau}\int_{\Gamma_W^{(j)}} \mathrm{d}x^\mu \left\langle\left\langle \left[\partial^\nu + \frac{ig}{c}A^\nu(x), F_{\nu\mu}(x)\right]\right\rangle\right\rangle_{\Gamma_W} \\ &\quad + \frac{ig^2}{\tau c^2}\int_{\Gamma_W^{(j)}} \mathrm{d}x^\mu\, \mathrm{d}x'_\sigma \Big[\langle\langle F_{\mu\nu}(x) F^{\sigma\nu}(x')\rangle\rangle_{\Gamma_W} \\ &\qquad - \langle\langle F_{\mu\nu}(x)\rangle\rangle_{\Gamma_W} \langle\langle F^{\sigma\nu}(x')\rangle\rangle_{\Gamma_W}\Big].\end{aligned} \tag{A.7}$$

By introducing the potential (6.3) and taking the limit for large τ, eq. (A.2) becomes

$$f(r) = V(r) + \sum_j \frac{1}{8m_j^2c^2}\Delta_2[V(r) + V_a(r)]. \tag{A.8}$$

Equation (6.3) is trivially equivalent to eq. (1.9a). In writing the expression above, we have taken into account translational and rotational invariance properties.

Next, consider eq. (A.3). By computations symilar to the above ones, one gets

$$b_j^h = \frac{g}{\tau}\int_{\Gamma_W^{(j)}} \mathrm{d}t\, ct\big\langle\big\langle E^h(t, z_j)\big\rangle\big\rangle_{\Gamma_W}. \tag{A.9}$$

By considering time reversal ($\boldsymbol{x} \to \boldsymbol{x}$, $t \to -t$, $A^0 \to \tilde{A}^0$, $\boldsymbol{A} \to -\tilde{\boldsymbol{A}}$), one can show that the quantity (A.9) vanishes.

Consider now eq. (A.4). First, we obtain

$$c_{ij}^{hk}(z) = \frac{ig^2}{4\tau}\int_{\Gamma_W^{(i)}} \mathrm{d}t \int_{\Gamma_W^{(j)}} \mathrm{d}t'\, c^2(t-t')^2$$
$$\times\Big[\big\langle\big\langle E^h(t, z_i)E^k(t', z_j)\big\rangle\big\rangle_{\Gamma_W} - \big\langle\big\langle E^h(t, z_i)\big\rangle\big\rangle_{\Gamma_W}\big\langle\big\langle E^k(t', z_j)\big\rangle\big\rangle_{\Gamma_W}\Big]. \tag{A.10}$$

Then, by time translation invariance, we can write

$$\lim_{\tau\to+\infty} c_{ij}^{hk}(z) = \lim_{\tau\to+\infty}\big[\delta_{ij} - (1-\delta_{ij})\big]\tfrac{1}{4}ig^2\int_{-\tau/2}^{\tau/2} \mathrm{d}t\, c^2t^2$$
$$\times\Big[\big\langle\big\langle E^h(t, z_i)E^k(0, z_j)\big\rangle\big\rangle_{\Gamma_W} - \big\langle\big\langle E^h(t, z_i)\big\rangle\big\rangle_{\Gamma_W}\big\langle\big\langle E^k(0, z_j)\big\rangle\big\rangle_{\Gamma_W}\Big] \tag{A.11}$$

This tensor can be easily decomposed in the sum of a symmetric tensor and a pseudovector; but the contribution of this latter vanishes, because the only vector at our disposal is $\boldsymbol{r}$, which has the opposite parity. Thus, c_{ij}^{hk} is a symmetric tensor, and therefore it must involve δ^{hk} and $r^h r^k/r^2$. Having this in mind, one sees that the potentials (1.9b)–(1.9e) can be introduced, and eq. (5.15) is established. Going back to eq. (A.4), we see that equivalent expressions for the potentials V_b, V_c, V_d, V_e are given by eqs. (6.4) and (6.5).

Finally, the terms containing d_j^h, given by eq. (A.5), are simply a surface contribution and can be eliminated by a redefinition of the propagator K.

Let us now apply eqs. (5.1)–(5.8) to eq. (5.13). At lowest order, we obtain

$$Y_j = \frac{g}{m_j c\tau} \int_{\Gamma_W^{(j)}} \mathrm{d}t \langle\langle \boldsymbol{B}(x) \rangle\rangle_{\Gamma_W}, \tag{A.12}$$

which vanishes by parity. This shows that the term $Y_j(t) \times Y_j(t')$ in eq. (5.13) can be neglected. Then, we have

$$\begin{aligned}
Y_j^l = &- \frac{g}{m_j c^2 \tau} \sum_i \dot{z}_i^k \frac{\partial}{\partial v_i^k} \int_{\bar{\Gamma}_j} \mathrm{d}x^\mu \langle\langle \hat{F}_{l\mu}(x) \rangle\rangle_{\bar{\Gamma}} \bigg|_{v=a=0} \\
&+ \frac{g}{2m_j^2 c^3 \tau} \varepsilon^{lkr} p_j^k \int_{\Gamma_W^{(j)}} \mathrm{d}x^\mu \langle\langle F_{\mu r}(x) \rangle\rangle_{\Gamma_W} \\
&+ \frac{g^2}{2m_j^2 c^4 \tau} \int_{\Gamma_W^{(j)}} \mathrm{d}x_\mu \mathrm{d}x'_\sigma \vartheta(t-t') \varepsilon^{rkl} \\
&\times \left[\langle\langle \hat{F}^{r\mu}(x) \hat{F}^{k\sigma}(x') \rangle\rangle_{\Gamma_W} - \langle\langle \hat{F}^{r\mu}(x) \rangle\rangle_{\Gamma_W} \langle\langle \hat{F}^{k\sigma}(x') \rangle\rangle_{\Gamma_W} \right].
\end{aligned} \tag{A.13}$$

The third term can be rewritten as

$$\begin{aligned}
\frac{g^2}{2m_j^2 c^2 \tau} \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{t} \mathrm{d}t' \Big[&\langle\langle \boldsymbol{B}(t, z_j) \times \boldsymbol{B}(t', z_j) \rangle\rangle_{\Gamma_W} \\
&- \langle\langle \boldsymbol{B}(t, z_j) \rangle\rangle_{\Gamma_W} \times \langle\langle \boldsymbol{B}(t', z_j) \rangle\rangle_{\Gamma_W} \Big]^l,
\end{aligned}$$

which vanishes by parity, because $\boldsymbol{B} \times \boldsymbol{B}$ is a pseudovector. By eq. (A.6), the second term can be rewritten as

$$\frac{1}{2m_j^2 c^2} \varepsilon^{lkr} p_j^k \frac{\partial \bar{V}}{\partial z_j^r} \bigg|_{v=a=0} \overset{\tau \to +\infty}{\simeq} - \frac{1}{2m_j^2 c^2} L_j^l \frac{1}{r} \frac{\mathrm{d}V}{\mathrm{d}r}.$$

Consider now the first term in eq. (A.17). The expression

$$\frac{\partial}{\partial v_i^k} \int_{\bar{\Gamma}_j} \mathrm{d}x^\mu \langle\langle \hat{F}_{l\mu}(x) \rangle\rangle_{\bar{\Gamma}} \bigg|_{v=a=0} + \frac{\partial}{\partial v_i^l} \int_{\bar{\Gamma}_j} \mathrm{d}x^\mu \langle\langle \hat{F}_{k\mu}(x) \rangle\rangle_{\bar{\Gamma}} \bigg|_{v=a=0}$$

vanishes by parity. Then, by symmetry arguments we must have

$$\frac{\partial}{\partial v_i^k}\int_{\Gamma_j} dx^\mu \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_{\Gamma}\bigg|_{v=a=0} = \frac{1}{2}\left[\frac{\partial}{\partial v_i^k}\int_{\Gamma_j} dx^\mu \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle\bigg|_{v=a=0}\right.$$

$$\left.\bigg| - \frac{\partial}{\partial v_i^l}\int_{\Gamma_j} dx^\mu \langle\langle \hat{F}_{k\mu}(x)\rangle\rangle_{\Gamma}\bigg|_{v=a=0}\right] = r^s \varepsilon^{slk} h_{ij}(r),$$

where h_{ij} is some function of r, such that $h_{11}(r) = h_{22}(r)$, $h_{12} = -h_{21}(r)$. By introducing the potentials V_1 and V_2 by eqs. (6.6) and (6.7), eq. (5.16) is obtained.

In order to obtain the expressions (1.4a) and (1.4b) for the potentials V_1 and V_2, note that the first term in eq. (A.17) gives

$$-\frac{g}{m_j c^2 \tau}\sum_i \dot{z}_i^k \frac{\partial}{\partial v_i^k}\int_{\Gamma_j} dx^\mu \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_{\Gamma}\bigg|_{v=a=0}$$

$$= \frac{g}{m_j c^2 \tau}\dot{z}_j^k \int_{\Gamma_W^{(j)}} dt \left\{ ct \left\langle\left\langle \left[\partial_k - \frac{ig}{c}A^k(t,z_j), B^l(t,z_j)\right]\right\rangle\right\rangle_{\Gamma_W}\right.$$

$$\left. + \varepsilon^{lks}\langle\langle E^s(t,z_j)\rangle\rangle_{\Gamma_W}\right\} - \frac{ig^2}{m_j c^2 \tau}\sum_i \dot{z}_i^k \int_{\Gamma_W^{(j)}} dt \int_{\Gamma_W^{(i)}} dt'\, ct'$$

$$\times\left[\langle\langle B^l(t,z_j) E^k(t',z_i)\rangle\rangle_{\Gamma_W} - \langle\langle B^l(t,z_j)\rangle\rangle_{\Gamma_W}\langle\langle E^k(t',z_i)\rangle\rangle_{\Gamma_W}\right].$$

By noting that

$$0 = \frac{\partial}{\partial z_j^k}\langle\langle \boldsymbol{B}(t,z_j)\rangle\rangle_{\Gamma_W}$$

$$= \left\langle\left\langle \left[\partial_k - \frac{ig}{c}A^k(t,z_j), \boldsymbol{B}(t,z_j)\right]\right\rangle\right\rangle_{\Gamma_W} - ig\int_{\Gamma_W^{(j)}} dt' \langle\langle \boldsymbol{B}(t,z_j) E^k(t',z_j)\rangle\rangle_{\Gamma_W}, \tag{A.14}$$

we have

$$
\begin{aligned}
&-\frac{g}{m_j c^2 \tau}\sum_i \dot{z}_i^k \frac{\partial}{\partial v_i^k}\int_{\bar{\Gamma}_j} \mathrm{d}x^\mu \left\langle\left\langle \hat{F}_{l\mu}(x)\right\rangle\right\rangle_{\bar{\Gamma}}\Bigg|_{v=a=0} \\
&\quad= -\frac{g}{m_j c^2 \tau}\dot{z}_j^k \varepsilon^{lks}\int_{\Gamma_{\mathrm{W}}^{(j)}} \mathrm{d}t \left\langle\left\langle E^s(t, z_j)\right\rangle\right\rangle_{\Gamma_{\mathrm{W}}} \\
&\qquad+\frac{ig^2}{m_j c^2 \tau}\dot{z}_j^k \int_{-\tau/2}^{\tau/2} \mathrm{d}t\,\mathrm{d}t'\,c(t-t')\left\langle\left\langle B^l(t, z_j) E^k(t', z_j)\right\rangle\right\rangle_{\Gamma_{\mathrm{W}}} \\
&\qquad+\frac{ig^2}{m_j c^2 \tau}\sum_i (1-\delta_{ij})\dot{z}_i^k \int_{-\tau/2}^{\tau/2} \mathrm{d}t\,\mathrm{d}t'\,ct'\left\langle\left\langle B^l(t, z_j) E^k(t', z_i)\right\rangle\right\rangle_{\Gamma_{\mathrm{W}}}.
\end{aligned}
$$

The first term is (cf. eq. (A.6))

$$
\frac{L_j'^{\,l}}{m_j^2 c^2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}V(r).
$$

About the other two terms, by decomposing $\langle\langle B^k(t, z_j)E^h(t', z_i)\rangle\rangle$ in the sum of a symmetric pseudotensor (which vanishes by parity) and a vector, the expressions (1.4a) and (1.4b) for the potentials V_1 and V_2 are obtained.

Let us prove now eq. (1.6). By eqs. (6.6) and (6.7) we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\left(V(r) + V_1(r) - V_2(r)\right) \\
&\quad= \lim_{\tau \to +\infty} \frac{g}{2\tau}\frac{r^h}{r}\varepsilon^{hlk} \\
&\qquad\times\left[\frac{\partial}{\partial v_1^k}\int_{\bar{\Gamma}_1} \mathrm{d}x^\mu \left\langle\left\langle \hat{F}_{l\mu}(x)\right\rangle\right\rangle_{\bar{\Gamma}} + \frac{\partial}{\partial v_2^k}\int_{\bar{\Gamma}_1} \mathrm{d}x^\mu \left\langle\left\langle \hat{F}_{l\mu}(x)\right\rangle\right\rangle_{\bar{\Gamma}}\right]\Bigg|_{v=a=0}.
\end{aligned}
\tag{A.15}
$$

By setting $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)$, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\left(V(r) + V_1(r) - V_2(r)\right) \\
&\quad= \lim_{\tau \to +\infty} \frac{g}{2\tau}\frac{r^h}{r}\varepsilon^{hlk}\frac{\partial}{\partial w^k}\int_{\bar{\Gamma}_1} \mathrm{d}x^\mu \left\langle\left\langle \hat{F}_{l\mu}(x)\right\rangle\right\rangle_{\bar{\Gamma}}\Bigg|_{v=a=0}.
\end{aligned}
\tag{A.16}
$$

But, by Lorentz covariance, the quantity $\int_{\bar{\Gamma}_1} dx^\mu \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle_{\bar{\Gamma}}$ depends only on $v_1 - v_2$, and this proves eq. (1.6).

Finally, by eqs. (5.1)–(5.8), eq. (5.14) gives

$$T^{hk} = -\frac{ig^2}{m_1 m_2 c^4 \tau} \int_{\Gamma_W^{(1)}} dx_\mu$$

$$\times \int_{\Gamma_W^{(2)}} dx'_\sigma \left[\langle\langle \hat{F}^{h\mu}(x) \hat{F}^{k\sigma}(x') \rangle\rangle_{\Gamma_W} - \langle\langle \hat{F}^{h\mu}(x) \rangle\rangle_{\Gamma_W} \langle\langle \hat{F}^{k\sigma}(x') \rangle\rangle_{\Gamma_W} \right]$$

$$= -\frac{ig^2}{2m_1 m_2 c^4 \tau} \int_{\Gamma_W^{(1)}} dx_\mu \int_{\Gamma_W^{(2)}} dx'_\sigma \langle\langle \hat{F}^{h\mu}(x) \hat{F}^{k\sigma}(x') + \hat{F}^{k\mu}(x) \hat{F}^{h\sigma}(x') \rangle\rangle_{\Gamma_W}, \tag{A.17}$$

where the usual symmetry properties have been taken into account. We now decompose the tensor T^{hk} in a part proportional to $(\frac{1}{3}\delta^{hk} - r^h r^k / r^2)$ plus a part proportional to δ^{hk}. By introducing the potentials (6.8) and (6.9) eq. (5.17) is obtained. By using $F^{ko} = B^k$ and time-translation invariance, eq. (1.4c) and the first step of eq. (1.4d) are immediately obtained. Finally, by direct evaluation of $\Delta_2 V_2(r)$, the second step of eq. (1.4d) is obtained.

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