

Now to find defining differential equation for V_s , we need

$$\int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{r}} \frac{p^2 + m_0^2}{p^2 + \text{Re}(\tau_R)}$$

First write

$$\frac{p^2 + m_0^2}{p^2 + \text{Re}(\tau_R)} = 1 + \frac{m_0^2 - \text{Re}(\tau_R)}{p^2 + \text{Re}(\tau_R)}$$

then

$$\int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{r}} \left(1 + \frac{m_0^2 - \text{Re}(\tau_R)}{p^2 + \text{Re}(\tau_R)} \right) = \delta(\vec{r}) + \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{r}} \frac{m_0^2 - \text{Re}(\tau_R)}{p^2 + \text{Re}(\tau_R)}$$

Consider second term:

$$\begin{aligned} & \frac{2\pi}{(2\pi)^3} \int_0^\pi d\theta \sin\theta \int_0^\infty dp \cdot p^2 \frac{m_0^2 - \text{Re}(\tau_R)}{p^2 + \text{Re}(\tau_R)} e^{ipr \cos\theta} \\ &= \frac{1}{(2\pi)^2} \int_0^{\pi/2} d\theta \sin\theta \int_{-\infty}^\infty dp \cdot p^2 \frac{m_0^2 - \text{Re}(\tau_R)}{p^2 + \text{Re}(\tau_R)} e^{ipr \cos\theta} \end{aligned}$$

same trick as before

Perform momentum integration via contour integration:

$$\oint_{C_R} dz \, z^2 \frac{m_0^2 - \text{Re}(\tau_R)}{z^2 + \text{Re}(\tau_R)} e^{izr \cos\theta}, \quad C_R: \text{Diagram of a contour in the complex plane. The contour consists of a large semi-circle in the upper half-plane (labeled with an arrow pointing counter-clockwise) and a horizontal line segment on the real axis from -R to R (labeled with an arrow pointing right). The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'. The semi-circle is labeled 'C_R'.$$

$$\begin{aligned} \oint_{C_R} dz (\dots) &= \lim_{R \rightarrow \infty} \left[\int_{-R}^R dz (\dots) + \underbrace{\int_{\text{arc}} dz (z \rightarrow Re^{i\varphi})}_{(A)} \right] \\ &= 2\pi i \sum \text{Res} \end{aligned}$$

For (A) let $z = Re^{i\varphi} \Rightarrow dz = iRe^{i\varphi} d\varphi$ and we have

$$\lim_{R \rightarrow \infty} \int_0^\pi d\varphi \cdot iRe^{i\varphi} (R^2 e^{2i\varphi}) \frac{m_0^2 - \text{Re}(\tau_R)}{R^2 e^{2i\varphi} + \text{Re}(\tau_R)} e^{-iRe^{i\varphi} r \cos\theta}$$

$$\sim \lim_{R \rightarrow \infty} i \int_0^\pi d\theta \operatorname{Re} e^{i\theta} (M_0^2 - \operatorname{Re}(\tau_R)) e^{i \operatorname{Re}(\tau_R) \cos \theta}$$

$$= \lim_{R \rightarrow \infty} \frac{2(\operatorname{Re}(\tau_R) - M_0^2) \sin(R \cos \theta)}{R \cos \theta}$$

according to Mathematica

Now, recall that $\operatorname{Re}(\tau_R(\theta))$ is symmetric around $\frac{\pi}{2}$, and $(\sin(\cos \theta))$ is anti-symmetric, and $\cos(\theta)$ is symmetric.
 ↳ Not important.

↳ From looks of above expression evaluated numerically (also with $d\theta$ integral) in Mathematica, does not go to zero or some easily perceptible combination of r, γ, M_0 .

Return to

$$\frac{1}{(2\pi)^2} \int_0^{\pi/2} d\theta \sin \theta \int_{-\infty}^{\infty} dp \, p^2 \frac{M_0^2 - \operatorname{Re}(\tau_R)}{p^2 + \operatorname{Re}(\tau_R)} e^{i p \cos \theta}$$

$$\sqrt{\operatorname{Re}(\tau_R)} (\operatorname{Re}(\tau_R) - M_0^2) \left[\pi \cosh(\sqrt{\operatorname{Re}(\tau_R)} r \cos \theta) - i \operatorname{Ci}(-i \sqrt{\operatorname{Re}(\tau_R)} r \cos \theta) \cdot \sinh(\sqrt{\operatorname{Re}(\tau_R)} r \cos \theta) + i \operatorname{Ci}(i \sqrt{\operatorname{Re}(\tau_R)} r \cos \theta) \sinh(\sqrt{\operatorname{Re}(\tau_R)} r \cos \theta) \right]$$

↳ according to Mathematica when damped with e^{-ap} .

$$= g_{\operatorname{Re}}(r, M_0, \gamma) \cdot \frac{1}{(2\pi)^2}$$