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CHARGE SCREENING AND SPACE DIMENSION*

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We consider two static "charges" interacting through the exchange of massless boson "photon" and study the screening which arises when the medium, into which the system is placed, gives a non-vanishing effective mass to the boson. We investigate how the functional form of screening depends on the space dimensions of the problem using a generalized Debye-Hückel formalism and calculate, in particular, the screened form of the linearly rising potential in a three dimensional space.

1. Introduction

The force between two given charges q and $-q$ becomes modified if the system is placed into a medium containing many such charges. In an ionized plasma of electric charges this results in the well-known Debye screening[1] which turns the $1/r$ potential energy into an exponentially damped form:

$$V_0(r) = -\frac{q^2}{r} \to V(r) = -\frac{q^2}{r} e^{-\mu r} . \tag{1}$$

where the screening mass μ (the inverse of the screening length Λ, $\mu = 1/\Lambda$) increases with the charge density of the medium. In atomic physics, such screening leads to an insulator-conductor transition (Mott transition).[2] It is expected that in QCD, a similar screening effect is responsible for deconfinement transition from hadronic matter to a quark-gluon plasma.[3] The confining part of the heavy quark potential in QCD has the form $V_0(r) \sim r$ in the absence of other color charges. The functional form of screening in this case is of great interest. Since confinement is related to one-dimensional electrodynamics, it was suggested[4] that the answer be

$$V_0(r) = \sigma r \to V(r) = \sigma r \left[\frac{1 - e^{-\mu r}}{\mu r}\right] . \tag{2}$$

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Here, σ denotes the string tension for the heavy quark-antiquark system. The form (2) has in fact been used to study the effect of screening on the binding of such systems in dense matter.[5] This form is the natural prediction for the screened, linearly rising potential in one space dimension in the Debye-Hückel model. We shall derive the corresponding screened form in three spatial dimensions.

On purely phenomenological grounds, the potential models have been reasonably successful in giving the mass spectra of the quarkoniums. The confining parts of the potentials have been variously parameterized as r, $r^{3/4}$, $r^{1/2}$, etc. see Refs. 6, 7 and 8 respectively. In this note, we want to see in general how the screening factor depends on the functional form of the potential; Eqs. 1 and 2 are two specific instances. Before proceeding with these calculations we shall show that varying the space dimension is essentially equivalent to changing the form of the potential.

2. Screened Potential

Screening in a medium implies that the interaction becomes "short ranged" due to the "exchanged photons" acquiring an effective mass μ. Thus, in the momentum space, the net effect is the change in the form of propagator from $1/k^2$ to $1/(k^2 + \mu^2)$. The corresponding potential energy $V(r)$ in the d-dimensional Euclidean configuration space is given by the Fourier transform of the propagator. For a function depending on $|\mathbf{k}|$ only, its Fourier transform can be written as a Hankel transform[9]

$$\phi(r) = \frac{1}{r^{d/2-1}} \int_0^\infty k^{d/2} \phi(k) J_{d/2-1}(kr) dk \,, \tag{3}$$

where $J_\nu(x)$ denotes a Bessel function. Thus, the screened potential energy corresponding to the propagator $1/(k^2 + \mu^2)$ is given by

$$V(r) \sim \frac{1}{r^{d/2-1}} \int_0^\infty k^{d/2} \frac{1}{k^2 + \mu^2} J_{d/2-1}(kr) dk \,. \tag{4}$$

Straightforward attempts to evaluate the above integral give rise to apparent divergences for $d \geq 5$ which can be avoided if $V(r)$ is interpreted as a distribution. Thus, we can first write

$$\frac{1}{k^2 + \mu^2} = \int_0^\infty dy \; e^{-y(k^2+y^2)} \,, \tag{5}$$

and then calculate the Fourier transform as[10]

$$V(r) \sim \frac{1}{r^{d/2-1}} \int_0^\infty dy \; e^{-y\mu^2} \int_0^\infty dk k^{d/2} e^{-yk^2} J_{d/2-1}(kr) \tag{6}$$

$$= \left(\frac{\mu}{r}\right)^{d/2-1} K_{d/2-1}(\mu r) \tag{7}$$

for any Re $d > 0$, where $K_\nu(x)$ is the modified Bessel function of the second kind.[11]

We shall now consider some particular cases.

2A. One space dimension.

Substituting $d = 1$ in Eq. 7, we get the screened potential energy to be of the form $(1/\mu)e^{-\mu|\mathbf{x}|}$. This expression then has to be corrected for the infinity which results in the limit of $\mu \to 0$ (no screening). Thus, one obtains the familiar "confining potential" with screening in one dimension, as already given by Eq. 2.

2B. Two space dimension.

The case $d = 2$ results in $K_0(\mu r)$ for $V(r)$, which when corrected for the logarithmic divergence as $\mu \to 0$, can be written as $V(r) \sim K_0(\mu r) + \ln \mu$ in two space dimensions.

2C. Three space dimensions.

For $d = 3$, the screened potential energy is of the form $\sqrt{\mu/r}K_{1/2}(\mu r)$, which is the familiar $e^{-\mu r}/r$ result of the Debye-Hückel mechanism.

3. Debye-Hückel Formalism

To get a qualitative picture for the change in propagator from $1/k^2$ to $1/(k^2 + \mu^2)$, we shall study the problem of screening in the classical Debye-Hückel formalism.

Let us start by considering an Abelian "charge" q placed at the origin in a space of d-dimensions such that the "field" at a point at distance r from the charge is given by

$$\mathbf{E}_0(r) = qr^{\alpha-1}\hat{\mathbf{r}}\,, \tag{8}$$

where α is a parameter which can take any value. The corresponding potential, defined by $\mathbf{E}_0 = -\nabla\phi_0$ is thus

$$\phi_0(r) = -\frac{qr^\alpha}{\alpha} \qquad \text{for } \alpha \neq 0\,, \tag{9}$$

and

$$\phi_0(r) = -q \ln r \qquad \text{for } \alpha = 0\,, \tag{10}$$

up to a constant of integration, which is conveniently chosen to be zero for $\alpha < 0$. The interaction energy of a pair of charges q and $-q$ at a distance r is given by

$$V_0(r) = -q\phi_0(r) \tag{11}$$

in the absence of any other charges. Now, by Gauss's theorem in d-dimensions, we can write

$$\int \nabla \cdot \left(\frac{\mathbf{E}_0}{r^{\alpha+d-2}} \right) dV = q \int \frac{\hat{\mathbf{r}} \cdot d\mathbf{s}}{r^{d-1}} = qD = qD \int \delta(\mathbf{r}) dV \,, \tag{12}$$

where $D = 2\pi^{d/2}/\Gamma(d/2)$ is the solid angle for a sphere in the d-dimensions. Therefore the bare field satisfies the relation

$$\nabla \cdot \left(\frac{\mathbf{E}_0}{r^{\alpha+d-2}} \right) = qD\delta(\mathbf{r}) \,. \tag{13}$$

Let us now consider an overall neutral plasma, in equilibrium (at temperature T), consisting of equal number densities (n_0) of particles of the same kind but of opposite "charges" q and $-q$ in the d-dimensional space. We place a test charge q at the origin. The presence of the test charge gives rise to an induced charge density. The resulting field $\mathbf{E}$ ($= -\nabla\Phi$) then satisfies the Debye-Hückel relation[12]

$$\nabla \cdot \left(\frac{\mathbf{E}}{r^{\alpha+d-2}} \right) = D[q\delta(\mathbf{r}) + \langle \rho(\mathbf{r}) \rangle] \,, \tag{14}$$

where $\langle \rho(\mathbf{r}) \rangle$ describes the variation in the charge density. We assume Boltzmann's distribution and thus write

$$\langle \rho(\mathbf{r}) \rangle = q\left[n_0 \exp\left(-\frac{q\Phi(r)}{kT} \right) - n_0 \right] - q\left[n_0 \exp\left(\frac{V\Phi(r)}{kT} \right) - n_0 \right] . \tag{15}$$

At high temperatures $\langle \rho(\mathbf{r}) \rangle$ can be approximated as

$$\langle \rho(\mathbf{r}) \rangle = \frac{-2q^2 n_0 \Phi(r)}{kT} \,. \tag{16}$$

This approximation is thus equivalent to a linear response theory. The equation satisfied by the effective potential $\Phi(r)$ in the Debye-Hückel approximation is therefore

$$-\frac{\nabla^2\Phi}{r^{\alpha+d-2}} + \frac{(\alpha + d - 2)}{r^{\alpha+d-1}} \nabla\Phi \cdot \hat{\mathbf{r}} = qD\delta(\mathbf{r}) - \frac{2q^2 n_0 D\Phi}{kT} \,, \tag{17}$$

or, taking into account radially symmetric cases only,

$$\frac{1}{r^{\alpha+d-2}} \frac{d^2\Phi}{dr^2} + \frac{(1-\alpha)}{r^{\alpha+d-1}} \frac{d\Phi}{dr} - A\Phi = -qD\delta(\mathbf{r}) \,, \tag{18}$$

where

$$A = \frac{2q^2 n_0 D}{kT} = \frac{4\pi^{d/2} q^2 n_0}{kT\Gamma\left(\frac{d}{2}\right)} . \tag{19}$$

We shall now look for the "screened" potential solutions of Eq. 18 which lead to $\mathbf{E} \sim q_0 r^{\alpha-1}\hat{\mathbf{r}}$ near $r = 0$ and which do not increase with r.

Case I. $(\alpha + d) > 0$. The solutions of the homogeneous part of Eq. 18 are given by the combinations of the cylindrical functions[11]

$$\Phi(r) \sim r^{\alpha/2} C_{\pm\alpha/(\alpha+d)}[i(\mu r)^{(\alpha+d)/2}] , \tag{20}$$

where

$$\mu = \left[\frac{16 q^2 n_0 \pi^{d/2}}{(\alpha + d)^2 kT\Gamma\left(\frac{d}{2}\right)}\right]^{1/(\alpha+d)} \tag{21}$$

Before considering the general case, let us first discuss the combination $\alpha + d = 2$ which is of particular significance as it represents the "generalized" Coulomb's law in d-spatial dimensions. For these values of α and d, the screened potential satisfying the boundary conditions can be written as

$$\Phi(r) = q \cdot \frac{2^{1-d/2}}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{r}{\mu}\right)^{1-d/2} K_{d/2-1}(\mu r) , \tag{22}$$

which is the form obtained earlier also.

The corresponding Fourier transform $\phi(k)$ is then given by

$$\phi(k) = q \cdot \frac{2^{1-d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{\mu^2 + k^2} \qquad \text{for } d > 0 . \tag{23}$$

It can also be seen directly from Eq. 18, which, for the case $\alpha + d = 2$, becomes

$$\frac{d^2\Phi}{dr^2} + \frac{(d-1)}{r}\frac{d\Phi}{dr} - \mu^2\Phi = -qD\delta(\mathbf{r}) , \tag{24}$$

and, as long as Φ is a function of r only, is equivalent to

$$\nabla^2\Phi(r) - \mu^2\Phi(r) = -qD\delta(\mathbf{r}) . \tag{25}$$

The Fourier transform $\phi(k)$ is then obviously given by Eq. 23. The case $\alpha + d = 2$ thus corresponds to "free field" between the sources, i.e., no direct interaction between the exchanged "photons". For $\alpha + d \neq 2$, this situation does not exist and the exchanged particles themselves represent some kind of "source" term.

We shall now consider the general situation.

Case IA. $-d < \alpha < 0$. The solution of Eq. 18, satisfying the boundary conditions is

$$\Phi(r) = \frac{-q \cdot 2^{(2\alpha+d)/(\alpha+d)}}{\alpha\Gamma\left(\dfrac{-\alpha}{\alpha + d}\right)} \left(\frac{r}{\mu}\right)^{\alpha/2} K_{-\alpha/(\alpha+d)}[(\mu r)^{(\alpha+d)/2}] \,. \tag{26}$$

For fixed μ, as $r \to \infty$, the solutions fall off exponentially,

$$\Phi(r) \sim \mu^{-\alpha}(\mu r)^{(\alpha-d)/4} e^{-[(\mu r)((\alpha+d)/2)]} \,. \tag{27}$$

Similarly, for fixed r, as $\mu \to 0$,

$$\Phi(r) \sim r^{\alpha} \,, \tag{28}$$

as desired.

Case IB. $\alpha \geq 0$. The solution $\Phi(r)$ can be written as

$$\Phi(r) = \frac{q 2^{(2\alpha+d)/(\alpha+d)}}{(\alpha + d)\Gamma\left(\dfrac{d}{\alpha + d}\right)} \left(\frac{r}{\mu}\right)^{\alpha/2} K_{\alpha/(\alpha+d)}[(\mu r)^{(\alpha+d)/2}] \,. \tag{29}$$

The solutions $\Phi(r)$ in Eqs. 26 and 29 still have an arbitrary constant which can be adjusted to satisfy the requirement that for any fixed value of r, the screened form $\phi_s(r)$ should tend to $\phi_0(r)$ is the limit of $n_0 \to 0$ (no screening). For $\alpha < 0$, with the potential $\phi_0(r)$ decreasing with r, this requirement is already satisfied, $\phi_s(r) = \Phi(r)$ and the constant can be chosen to be zero, for $\alpha \geq 0$, this is not the case. For $\alpha = 0$, in any dimensions d, the screened potential is given by

$$\phi_s(r) = \frac{2q}{d} K_0[(\mu r)^{d/2}] + q \ln \mu \tag{30}$$

and

$$V(r) = -\frac{2q^2}{d} K_0[(\mu r)^{d/2}] - q^2 \ln \mu \,, \tag{31}$$

which grows like $q^2 \ln r$ for large r. Similarly, for $\alpha > 0$,

$$\phi_s = \frac{q2^{(2\alpha+d)/(\alpha+d)}\left(\dfrac{r}{\mu}\right)^{\alpha/2}}{(\alpha+d)\Gamma\left(\dfrac{d}{\alpha+d}\right)} K_{\alpha/(\alpha+d)}[(\mu r)^{(\alpha+d)/2}] - \frac{q}{(\alpha+d)} \frac{\Gamma\left(\dfrac{\alpha}{\alpha+d}\right) 2^{2\alpha/(\alpha+d)}}{\Gamma\left(\dfrac{d}{\alpha+d}\right)\mu^{\alpha}} . \tag{32}$$

Thus, for $\alpha = 1$, $d = 1$, we recover the result of Eq. 2. As another example, the choice of $\alpha = 2$ and $d = 2$ results in

$$\phi_s = \frac{q}{2\mu^2} e^{-(\mu r)^2} - \frac{q}{2\mu^2} , \tag{33}$$

corresponding to the Fourier transform of $\Phi(r)$ as

$$\phi(k) = \frac{q}{4\mu^4} e^{-k^2/(4\mu^2)} . \tag{34}$$

Case II. $\alpha + d = 0$. The screening term now scales exactly as the first two terms. Solutions to the homogeneous Eq. 18 are linear combinations of $r^{(-d + \sqrt{d^2 + 4A}/2)}$ and $r^{(-d - \sqrt{d^2 + 4A}/2)}$. Thus, it is still not possible to get the desired proper behavior near $r = 0$.

Case III. $(\alpha + d) < 0$. The third term on the left hand side of Eq. 18, which can be literally called the "screening term" dominates the first two terms near $r = 0$ and it is impossible to get a desired solution.

4. Application to Quarkoniums

As mentioned earlier, the screened form of the charmonium potential used by Karsch, Mehr and Satz,[5] Eq. 2, corresponds to the linearly rising potential $\alpha = 1$ in one spatial dimension in the Debye-Hückel formalism. Only at zero temperature is the gluonic interaction supposed to 'condense' the gluonic flux to one-dimensional 'flux tubes'. At higher temperatures, this simple approximation should no longer be valid, particularly in the region of 'melting' of the quarkoniums. The phenomenological potential models of the quarkoniums treat them as bound states in $d = 3$ dimensions. Thus, for example, for the 'Cornell' potential, substituting $\alpha = 1$, $d = 3$ in Eq. 32, we get

$$V(r) = -\frac{q^2}{2^{3/4}\Gamma(\frac{3}{4})}\left(\frac{r}{\mu}\right)^{1/2} K_{1/4}[(\mu r)^2] + \frac{q^2\Gamma(\frac{1}{4})}{2^{3/2}\Gamma(\frac{3}{4})} \cdot \frac{1}{\mu} , \tag{35}$$

which, for large r values, has the form

$$V(r) \sim \frac{1}{\sqrt{\mu^3 r}} e^{-(\mu r)^2} . \tag{36}$$

Similar expressions can be obtained for other values of α (e.g., $\alpha = .5$ or $\alpha = .75$, etc) used in the literature. For the sake of tractability, our considerations have been limited so far to the potentials ϕ_0 of the type r^α. Actually, the potentials used in studying the bound states of heavy quarks-antiquarks are of the form $\phi_0(r) = Ar^\alpha + Br^{-\beta}$. It is straightforward to write the 'generalized' Debye-Hückel equation for these potentials. The resulting equations seem to be amenable to numerical methods only.

Discussion and Summary

In the first part, we show that the screened potential $V(r)$ is a manifestation, via the Fourier transform, of the basic screening mechanism imparting a nonzero mass to the exchanged 'photon' due to the interaction with a medium. The dependence on this process on the spatial dimensions is brought out explicitly. In the context of one particular model, the classical Debye-Hückel model, these result correspond to an unscreened potential $V_0(r) \sim r^{2-d}$ in d-spatial dimensions. In the second part, we have derived the screened form of the various confining potentials used in the literature for quarkoniums. In particular, for the linearly rising potential in three spatial dimensions, Eq. 36, the screened form of the potential comes out to be a Gaussian, instead of an exponential, as suggested by the perturbative QCD and lattice calculations.[13] The perturbative QCD and lattice calculations are not yet in agreement with each other and are beset with difficulties like gauge-dependence, lattice size dependence, etc. Most significantly, the results have been reported for the $T > T_c$ region and do not necessarily provide a clue to the effective screening in the $T \leq T_c$ region which is relevant for the study of the 'melting' of the quarkoniums. With suitable adaptation, the model calculations presented here can provide some guideline, in the Abelian, static limit, for the form of the dynamical screening.

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