String part of potential

From Laine et al.:

$$\varepsilon^{-1}(\vec{p}, v) = \lim_{w\to 0} p^2 D_{11}^L(w, \vec{p}, v)$$

where

$$\mathrm{Re}[D_{11}^L] = \frac{1}{2}(D_R^L + D_A^L) = \frac{1}{2}(D_R^L + D_R^{L*}) = \mathrm{Re}(D_R^L) = \mathrm{Re}\left(\frac{1}{p^2+\Pi_R}\right)$$

$$\mathrm{Im}(D_{11}^L) = \frac{1}{2}D_S^L = \begin{cases} \parallel: & \dfrac{-\pi i T m_D^2 (1-v^2)^{3/2}(2+v^2\sin^2\theta)}{p(1-v^2\sin^2\theta)^{5/2}(p^2+\Pi_R)(p^2+\Pi_R^*)\cdot 2} \\ \perp: & \dfrac{-\pi i T m_D^2 (1-v^2)^{3/2}(2+v^2-v^2\cos^2(\phi-\beta)\sin^2\theta)}{p(1-v^2\cos^2(\phi-\beta)\sin^2\theta)^{5/2}(p^2+\Pi_R)(p^2+\Pi_R^*)\cdot 2} \end{cases}$$

Insert into modified Gauss law ansatz:

$$p^2 V_c(\vec{p}) = \frac{4\pi\alpha_s^2}{\varepsilon(\vec{p}, v)}$$

$$\Rightarrow p^2 V_c(\vec{p}) = 4\pi\alpha_s^2\left[\frac{p^2}{p^2+\mathrm{Re}(\Pi_R)} - i\pi T m_D^2 \Phi(\theta,\phi)\frac{p^2}{(p^2+\Pi_R)(p^2+\Pi_R^*)p}\right]$$

$$\Rightarrow \underbrace{p^2 V_c(\vec{p}) + M_D^2 V_c(\vec{p}) = 4\pi\alpha_s^2\left[\frac{p^2+m_D^2}{p^2+\mathrm{Re}(\Pi_R)}\right.}_{} \underbrace{\left. - i\pi T m_D^2 \Phi(\theta,\phi)\frac{p^2+m_D^2}{(p^2+\Pi_R)(p^2+\Pi_R^*)p}\right]}_{}$$

Fourier transforms to real space differential equation from Gauss law, with modified RHS (usually const.) that contains everything we are ignorant of above finite v in this approach (ASSUMPTION)

Assume that Fourier transform of this also gives $\mathrm{Im} V_c$ when solving differential equation.

↳ Check for $v \to 0$:

$$p^2 V_c(\vec{p}) + m_D^2 V_c(\vec{p}) = 4\pi\alpha_s^2\left[1 - i\pi T m_D^2 \frac{1}{(p^2+m_D^2)p}\right]$$

✓ Recovers our expression

Now we proceed in similar manner to previously; Fourier transform RHS and assume this modification holds for string potential Gauss law differential equation.

We need:

$$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{r}} \Phi(\theta,\phi) \frac{p^2+m_D^2}{p(p^2+\Pi_R)(p^2+\Pi_R^*)}$$

The term

$$m_D^2 \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{r}} \Phi(\theta,\phi) \frac{1}{p(p^2+\Pi_R)(p^2+\Pi_R^*)}$$

has already been calculated $(Im(V_c))$. Result:

$$+\frac{2\pi}{(2\pi)^3} m_D^2 \int_0^{\pi/2} d\theta \sin\theta \, \Phi(\theta,\phi) \frac{i}{Im(\Pi_R)} \Big[\sinh(r\cos\theta\sqrt{\Pi_R})\, Si(r\cos\theta\sqrt{\Pi_R})$$

$$- \cosh(r\cos\theta\sqrt{\Pi_R})\, Ci(r\cos\theta\sqrt{\Pi_R}) - \sinh(r\cos\theta\sqrt{\Pi_R^*})\, Si(r\cos\theta\sqrt{\Pi_R^*})$$

$$+ \cosh(r\cos\theta\sqrt{\Pi_R^*})\, Ci(r\cos\theta\sqrt{\Pi_R^*}) \Big]$$

where

$$Si(z) = \int_0^z \frac{\sinh(t)}{t} dt, \quad Ci(z) = \gamma_E + \log(z) + \int_0^z \frac{\cosh(t)-1}{t} dt$$

Now need to calculate:

$$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{r}} \Phi(\theta,\phi) \frac{p}{(p^2+\Pi_R)(p^2+\Pi_R^*)}$$

$$= \frac{1}{(2\pi)^2} \int_0^\pi \sin\theta \, d\theta \int_0^\infty p^2 dp \, e^{i\vec{p}\vec{r}} \Phi(\theta,\phi) \frac{p}{(p^2+\Pi_R)(p^2+\Pi_R^*)}$$

$$= \frac{1}{(2\pi)^2} \int_0^\pi d\theta \int_0^\infty dp \, \Phi(\theta,\phi) \sin\theta \frac{p^3 e^{ipr\cos\theta}}{(p^2+\Pi_R)(p^2+\Pi_R^*)}$$

Use same trick as before, splitting the polar integral:

$$\frac{1}{(2\pi)^2}\int_0^{\pi/2} d\theta \sin\theta\, \Phi(\theta,\phi)\left[\int_0^\infty dp\, \frac{p^3 e^{ipr\cos\theta}}{(p^2+\Pi_R)(p^2+\Pi_R^*)} - \int_{-\infty}^0 dp\, \frac{p^3 e^{ipr\cos\theta}}{(p^2+\Pi_R)(p^2+\Pi_R^*)}\right]$$

Give momentum integrals to Mathematica:

$$\frac{1}{(2\pi)^2}\int_0^{\pi/2} d\theta \sin\theta\, \Phi(\theta,\phi)\,\frac{1}{2i\,\mathrm{Im}(\Pi_R)}\left[\sqrt{\pi}\left(\sqrt{\Pi_R}\, G^{21}_{13}\left(\tfrac14 r^2\cos^2\theta\sqrt{\Pi_R}\,\middle|\,{-1 \atop -1,0,\frac12}\right)\right.\right.$$

$$\left.\left. - \sqrt{\Pi_R^*}\, G^{21}_{13}\left(\tfrac14 r^2\cos^2\theta\sqrt{\Pi_R^*}\,\middle|\,{-1 \atop -1,0,\frac12}\right)\right)\right]$$

So in total we have (drop overall i for imaginary part)

$$-i4\pi^2\hat{\alpha}_s T M_D^2 \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{r}}\, \Phi(\theta,\phi)\, \frac{p^2+M_D^2}{(p^2+\Pi_R)(p^2+\Pi_R^*)\,p}$$

$$= -\hat{\alpha}_s T M_D^2 \int_0^{\pi/2} d\theta \sin\theta\, \Phi(\theta,\phi)\, \frac{i}{\mathrm{Im}(\Pi_R)}\Big[M_D^2\big[\sinh(r\cos\theta\sqrt{\Pi_R})\,\mathrm{Si}(r\cos\theta\sqrt{\Pi_R})$$

$$-\cosh(r\cos\theta\sqrt{\Pi_R})\,\mathrm{Ci}(r\cos\theta\sqrt{\Pi_R}) - \sinh(r\cos\theta\sqrt{\Pi_R^*})\,\mathrm{Si}(r\cos\theta\sqrt{\Pi_R^*})$$

$$+\cosh(r\cos\theta\sqrt{\Pi_R^*})\,\mathrm{Ci}(r\cos\theta\sqrt{\Pi_R^*})\big] - \frac{\sqrt{\pi}}{2}\Big[\sqrt{\Pi_R}\, G^{21}_{13}\left(\tfrac14 r^2\cos^2\theta\sqrt{\Pi_R}\,\middle|\,{-1 \atop -1,0,\frac12}\right)$$

$$-\sqrt{\Pi_R^*}\, G^{21}_{13}\left(\tfrac14 r^2\cos^2\theta\sqrt{\Pi_R^*}\,\middle|\,{-1 \atop -1,0,\frac12}\right)\Big]\Big]$$

which we call $-\hat{\alpha}_s T M_D^2 g(M_D, r, v)$

~~string potential then satisfies~~

$$\boxed{\cancel{-\frac{1}{r^2}\frac{d^2 u_s(r)}{dr^2} + \mu^4 u_s(r) = \sigma\left(f\left(4\pi\frac{p^2+M_D^2}{p^2+\mathrm{Re}(\Pi_R)}\right) - \hat{\alpha} T M_D^2 g(M_D, r, v)\right)}}$$

~~and we can proceed to solve as before.~~