Integral to calculate:

$$\int \frac{d^3p}{(2\pi)^3} \cdot 4\pi\tilde{\alpha}_s \frac{e^{i\vec{p}\vec{r}}}{p^2+Re(\Pi_R)} \qquad (1)$$

$Re(\Pi_R(\theta))$ is symmetric around $\frac{\pi}{2}$ so write:

$$\int \frac{d^3p}{(2\pi)^3} 4\pi\tilde{\alpha}_s \frac{e^{i\vec{p}\vec{r}}}{p^2+Re(\Pi_R)} = \frac{4\pi\tilde{\alpha}_s}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} p^2 dp \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi_R)}$$

$$= \frac{\tilde{\alpha}_s}{\pi}\int_0^{\infty} p^2 dp \left[\int_0^{\pi/2} \sin\theta d\theta \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi_R)} + \int_{\pi/2}^{\pi} \sin\theta d\theta \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi_R)}\right]$$

$$\theta' = \pi - \theta, \; d\theta' = -d\theta$$

$$= \frac{\tilde{\alpha}_s}{\pi}\int_0^{\infty} p^2 dp \left[\int_0^{\pi/2} \sin\theta d\theta \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi_R)} + \int_{\pi/2}^{0} \sin(\pi-\theta')(-d\theta') \frac{e^{ipr\cos(\pi-\theta')}}{p^2+Re(\Pi_R(\pi-\theta'))}\right]$$

$$= \frac{\tilde{\alpha}_s}{\pi}\int_0^{\infty} p^2 dp \left[\int_0^{\pi/2} \sin\theta d\theta \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi_R)} + \int_0^{\pi/2} \sin(\theta') d\theta' \frac{e^{-ipr\cos\theta'}}{p^2+Re(\Pi(\theta'))}\right]$$

$$= \frac{\tilde{\alpha}_s}{\pi}\left[\int_0^{\infty} p^2 dp \int_0^{\pi/2} \sin\theta d\theta \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi_R)} + \int_0^{-\infty} p^2 d(-p) \int_0^{\pi/2} \sin\theta d\theta \frac{e^{ipr\cos\theta}}{p^2+Re(\Pi)}\right]$$

$$= \frac{\tilde{\alpha}_s}{\pi}\int_0^{\pi/2} \sin\theta d\theta \int_{-\infty}^{\infty} dp \frac{p^2 e^{ipr\cos\theta}}{p^2+Re(\Pi_R)}$$

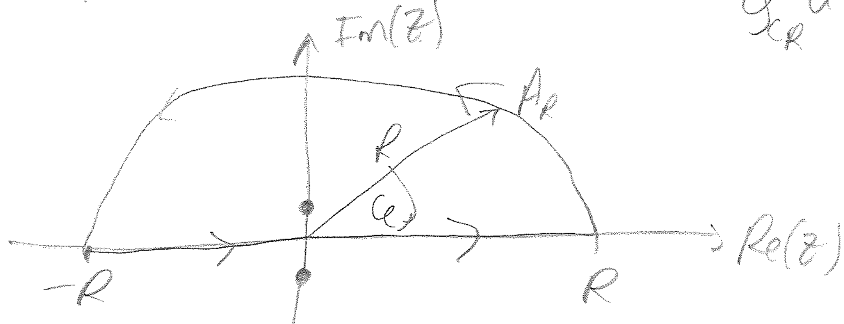
Now perform p-integral via contour integration:

$$\oint_{C_R} dz \frac{z^2 e^{izr\cos\theta}}{(z+i\sqrt{Re(\Pi_R)})(z-i\sqrt{Re(\Pi_R)})}$$

where path C_R is

$$\oint_{C_R} dz(\ldots) = \int_{-R}^{R} dz(\ldots) + \int_{A_R} dz(\ldots)$$

and take $R \to \infty$ limit.



Residue theorem:

$$\oint_{C_R} dz \frac{z^2 e^{izr\cos\theta}}{(z+i\sqrt{Re(\Pi_R)})(z-i\sqrt{Re(\Pi_R)})} = 2\pi i \sum Res$$

assume $Re(\Pi_R) > 0$

$$= 2\pi i \lim_{z \to i\sqrt{Re(\Pi_R)}} \frac{z^2 e^{izr\cos\theta}\,(z-i\sqrt{Re(\Pi_R)})}{(z+i\sqrt{Re(\Pi_R)})(z-i\sqrt{Re(\Pi_R)})}$$

$$= 2\pi i \frac{-Re(\Pi_R) e^{-\sqrt{Re(\Pi_R)}\, r\cos\theta}}{2i\sqrt{Re(\Pi_R)}}$$

$$= -\pi\sqrt{Re(\Pi_R)}\, e^{-\sqrt{Re(\Pi_R)}\, r\cos\theta}$$

Now consider integral over half circle. Let $z = Re^{i\varphi}$ then

$$\int_{A_R} dz \frac{z^2 e^{izr\cos\theta}}{p^2 + Re(\Pi_R)} = \lim_{R\to\infty} \int_0^\pi iRe^{i\varphi} d\varphi \frac{R^2 e^{2i\varphi} e^{iRe^{i\varphi} r\cos\theta}}{R^2 e^{2i\varphi} + Re(\Pi_R)}$$

$$= \lim_{R\to\infty} \int_0^\pi d\varphi\, i \frac{R^3 e^{3i\varphi} e^{iRe^{i\varphi} r\cos\theta}}{R^2 e^{2i\varphi}}$$

neglect $Re(\Pi_R)$ in $R\to\infty$ limit.

In full integral (1), this is contained within θ integral:

$$\int_0^{\pi/2} \sin\theta\, d\theta \cdot \lim_{R\to\infty} \int_0^\pi d\varphi\, i Re^{i\varphi} e^{iRe^{i\varphi} r\cos\theta}$$

$$= -i\int_1^0 dy \lim_{R\to\infty} \int_0^\pi d\varphi\, Re^{i\varphi} e^{iRe^{i\varphi} yr} = +i \lim_{R\to\infty} \int_0^\pi d\varphi\, Re^{i\varphi} \frac{1}{iRe^{i\varphi} r} e^{iRe^{i\varphi} ry}\Big|_0^1$$

$$= + \lim_{R\to\infty} \int_0^\pi d\varphi \left(e^{iRe^{i\varphi} r} - 1\right) \frac{1}{r} = -\frac{\pi}{r} + \Big(\underbrace{\lim_{R\to\infty} \int_0^\pi d\varphi\, e^{iRe^{i\varphi} r}}_{-2\pi \text{ according to Mathematica}}\Big) \cdot \frac{1}{r}$$

$$= -\frac{3\pi}{r}$$

↳ result should be $-\frac{\pi}{r}$, Mathematica says order of integration matters(?) in this case result matches Thakur et al.

Finally, full integral is then

$$\int \frac{d^3p}{(2\pi)^3} 4\pi\hat{\alpha}_s \frac{e^{i\vec{p}\vec{r}}}{p^2+Re(\Pi_R)} = \frac{\tilde{\alpha}_s}{\pi}\int_0^{\pi/2} \sin\theta d\theta \int_{-\infty}^{\infty} dp \frac{p^2 e^{ipr\cos\theta}}{p^2+Re(\Pi_R)}$$

$$= \frac{\hat{\alpha}_s}{\pi}\left[\int_0^{\pi/2} \sin\theta d\theta \oint_{C_R} dz \frac{z^2 e^{izr\cos\theta}}{z^2+Re(\Pi_R)} - \int_0^{\pi/2}\sin\theta d\theta \int_{A_R} dz \frac{z^2 e^{izr\cos\theta}}{z^2+Re(\Pi_R)}\right]$$

$$= \frac{\tilde{\alpha}_s}{\pi}\left[\int_0^{\pi/2} \sin\theta d\theta \cdot \left(-\pi\sqrt{Re(\Pi_R)}\, e^{-\sqrt{Re(\Pi_R)}\, r\cos\theta}\right) + \frac{\pi}{r}\right]$$

$$= \frac{\tilde{\alpha}_s}{r} - \tilde{\alpha}_s \int_0^{\pi/2} \sin\theta d\theta \sqrt{Re(\Pi_R(\theta))}\, e^{-\sqrt{Re(\Pi_R(\theta))}\, r\cos\theta}$$