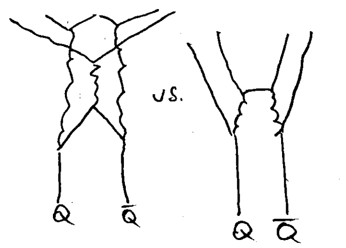
Lattice effective field theory methods for in-medium heavy quarkonium

In the course of this lecture we will both explore the physics of and some of the theoretical tools to study a particular class of hadrons under extreme conditions. These mesons, christened heavy-quarkonium are composed of a heavy quark and antiquark. When referring to extreme conditions we mean temperatures as high as $T \sim \mathcal{O}(100\,\mathrm{MeV})$ similar to the conditions shortly after the Big Bang or found in relativistic heavy-ion collisions.

Higgs contribution
$m_{u/d} \sim 4\,\mathrm{MeV}$ $m_s \sim 95\,\mathrm{MeV}$
$m_c \sim 1.27\,\mathrm{GeV}$ $m_b \sim 4.6\,\mathrm{GeV}$
$m_t \sim 173\,\mathrm{GeV}$

Let us recollect some details on heavy quarkonium in vacuum
$b\bar{b}$ Bottomonium $c\bar{c}$ Charmonium

Heavy $Q\bar{Q}$ is a unique system, as it represents exceptionally stable bound states of strongly interacting particles with life times of $\tau^{-1} \sim \Gamma \sim \mathrm{keV}$. The reason is that decays of color neutral states are suppressed due to the OZI rule below the heavy-light threshold (D or B mesons). Preferred decay into di-leptons e^+e^-, $\mu^+\mu^-$



$\Rightarrow$ Long lifetimes means easier experimental access @ e.g.
BELLE (KEKB), BABAR (SLAC), CLEO (FLAB), BESIII, LHCb (LHC)
via electron beam collisions e.g. e^+e^-. High precision data available on masses, widths and decay channels available. Successful description via quark model
[$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ $S=0$ $PC=-+$ $S=1$ $PC=--$ S-wave $L=0$ $\begin{smallmatrix} S & 0^{-+} \\ T & 1^{--} \end{smallmatrix}$ $L=1$ $\begin{smallmatrix} 1^{+-} \\ \{0,1,2\}^{++} \end{smallmatrix}$]
Current topic of interest exotic quarkonia with quantum numbers NOT available in the quark model: $\underset{1^{++}}{X(3872)}$ $Y(4260)$ $Z_c(3900)$ (proposals: meson molecule, tetraquarks)
Nomenclature similar to hydrogen $c\bar{c}$ $J/\psi \equiv {}^3S_1\,(n=1)$ $m_{J/\psi} = 3.096\,\mathrm{GeV}$ $\psi' \equiv {}^3S_1\,(n=2)$ $\eta_c \equiv {}^1S_0\,(n=1)$ $\chi_{c1} \equiv {}^3P_1\,(n=1)$

$\Rightarrow$ Hierarchy of scales justifies non-relativistic treatment $M_Q \gg Mv \sim \frac{1}{r} \sim p \gg Mv^2 \sim E_b$
E_b $\Upsilon(b\bar{b})$ 1.1 GeV ($m_\Upsilon = 9.4603\,\mathrm{GeV}$) $J/\psi(c\bar{c})$ 600 MeV $\chi_b(b\bar{b})$ 600 MeV $\Leftrightarrow$ $v^2 \approx 0.3\ (c\bar{c}),\ 0.1\ (b\bar{b})$
Very successful modeling of bound states below threshold via Cornell-potential
$$V(r) = -\frac{\alpha}{r} + \sigma r \equiv \underset{\text{gluon exchange}}{\overset{\text{perturbative}}{1}} + \text{non-perturbative confinement related to } \Lambda_{QCD}$$
$\boxed{\text{Fig1.}}$

In addition lattice QCD simulations recovered the quarkonium spectrum with high accuracy and were able to even predict the masses of e.g. $\eta_b(2S)$ $\boxed{\text{Fig2.}}$

While at $T=0$ we have a very good understanding of quarkonium bound states knowledge @ high temperatures is much less robust. $\boxed{\text{Fig3}}$

Our goal is TWOFOLD: On the one hand we want to understand how tightly bound $Q\bar{Q}$ states react to a heatbath with $T \lesssim E_b$, how do states dissociate

On the other hand we want to use that knowledge to investigate nuclear matter under extreme conditions, eg. produced in a relativistic heavy-ion collision.

Classic idea by Matsui & Satz: If in a heavy-ion collision deconfined matter (QGP) is produced it screens the color interaction between the $Q\bar{Q}$, which dissociates. Suppression of yields as sign of Quark-Gluon plasma formation

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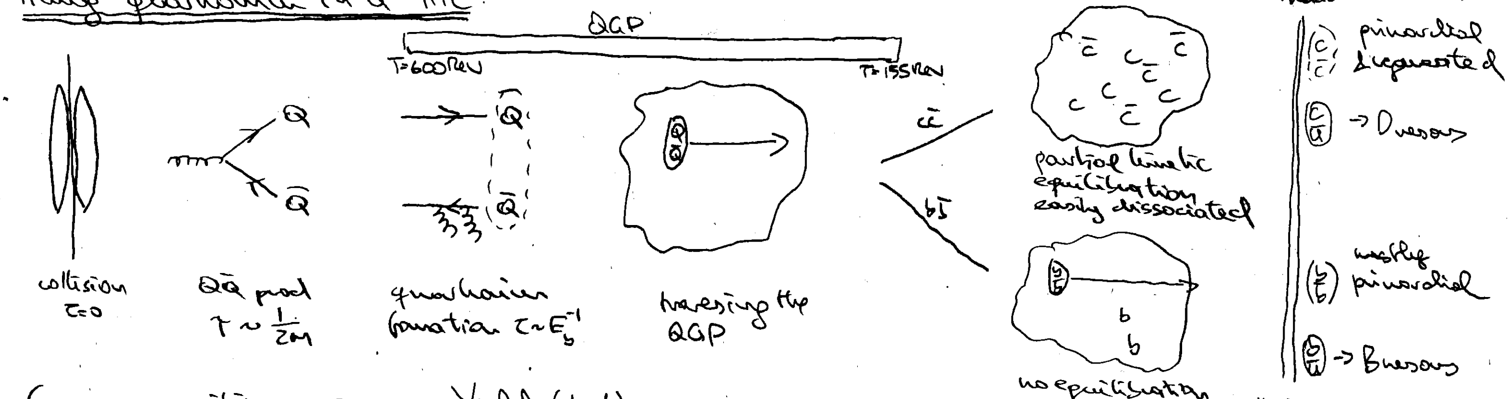
Heavy quarkonium in a HIC:



Common quantity $R_{AA} = \dfrac{\text{Yield}(A+A)}{\text{Yield}(p+p)\, N_{coll}}$

Currently established observations in HIC:

1. Quarkonia in HIC suppressed compared to pp $R_{AA} < 1$. Suppression hierarchically ordered with vacuum binding energy. Fig 4
2. Different mechanisms for $b\bar{b}$ and $c\bar{c}$: with increasing beam energy $\sqrt{s_{NN}}$ $R_{AA}(b\bar{b}) \downarrow$ (primordial suppression) $R_{AA}(c\bar{c}) \uparrow$ increased regeneration. Fig 5&6
3. J/ψ flows with the bulk matter, partial kinetic equilibration ($v_2 > 0$) Fig 7

Current question of interest: Many different models with different underlying physics mechanisms can reproduce R_{AA} of ground state. What are more discriminating observables eg. $J/\psi / \psi'$ ratio.

Concrete questions to theory:

1. How do masses and decay widths of $Q\bar{Q}$ change @ $T>0$
2. what is the real-time evolution of a $Q\bar{Q}$ in a HIC?

⇒ Since $T_{HIC} \lesssim 3T_c$ need nonperturbative methods to compute quarkonium correlation functions (Lattice QCD + effective field theory for heavy quarks)

⇒ Need numerical methods to extract spectral information (m, Γ) from correlators. (Bayesian inference)

⇒ Need to set up a potential description for in-medium $Q\bar{Q}$ similar to the T=0 case to implement real time evolution (EFT + Lattice + Bayesian inference)

You have already listened to dedicated lectures on lattice QCD, where the determination of meson properties is a central topic. So why should we use additional effective field theory techniques? ⇒ Separation of scales (III)

On the lattice: $\Lambda_{UV} \sim a^{-1} \gg M_{Q\bar{Q}} \gg E_{bind} > T \sim m_\pi \gg \Lambda_{IR} \sim L^{-1}$

$T = 150\,\mathrm{MeV}$ $\quad m_{J/\psi}\, a < \frac{1}{2} \to a < 0{,}03\,\mathrm{fm} \quad m_\Upsilon \cdot a < \frac{1}{2} \to a < 0{,}01\,\mathrm{fm}$

$0{,}197\,\mathrm{fm\,GeV} = 1 \quad T^{-1} = 1.3\,\mathrm{fm}$ $\quad$ Sane box size $3\,\mathrm{fm}$ $\quad N_x = 100 \ldots 300$ (current realistic $48^3 \times 12 \ldots 24$ full QCD, quenched $92^3 \times 96$)

Let us have a look at non-interacting standard Wilson fermions: (Euclidean time)

$$S_{lat} = \sum_x \bar{\psi}(x) \left\{ m_0 + \frac{\Delta^+_\mu + \Delta^-_\mu}{2}\gamma^\mu - a\frac{\Delta^-\Delta^+}{2} \right\} \psi(x)$$

$a\Delta^+_\mu \psi = U_\mu \psi(x + a\mu) - \psi(x)$

$\tilde{p}_\mu = \frac{1}{a}\sin(p_\mu a) \quad p_\mu \in \frac{\pi}{N a}\nu \quad \nu \in \left(-\frac{N}{2}, \ldots, \frac{N}{2}\right]$

$\hat{p}^2 = \sum_\mu (\hat{p}_\mu)^2 \quad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{a p_\mu}{2}\right)$

at first $T=0$ infinite volume limit $L \to \infty$

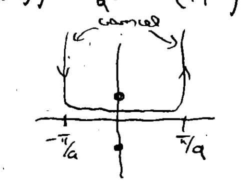
$$= \int_{-\pi/a}^{\pi/a} d^4p\, \tilde{\bar{\psi}}(p) \left\{ m_0 + i\tilde{p}_\mu\gamma_\mu + \frac{a}{2}\hat{p}^2 \right\} \tilde{\psi}(p)$$

⇒ Propagator reads $G_W(p) = \langle \bar{\psi}(-p)\psi(p) \rangle = \dfrac{(m_0 + \frac{a}{2}\hat{p}^2) - i\tilde{p}_\mu\gamma_\mu}{(m_0 + \frac{a}{2}\hat{p}^2)^2 + \tilde{p}^2}$ $\quad$ c.f. $\dfrac{m - i\gamma p}{m^2 + p^2}$

To read off the mass of the quark and dispersion relation, need to compute time-slice correlator (Rothe, Münster)

$$C(\tau, \vec{p}=0) = \frac{1}{V}\sum_x \langle \psi_{x,t}\bar{\psi}_{0,0} \rangle = \int_{-\pi/a}^{\pi/a} \frac{dk_4}{(2\pi)} e^{i\tau k_4} G(\vec{0}, k_4) \qquad \frac{\#}{\left(m_0 + \frac{2}{a}\sin^2\left(\frac{a p_4}{2}\right)\right)^2 + \frac{1}{a^2}\sin^2(p_4 a)}$$

Anticipate that the poles lie in the complex plane $E = -ip_4$



$$\left[m_0 - \frac{2}{a}\sinh^2\left(\frac{aE}{2}\right)\right]^2 - \sinh(Ea) = 0 \quad \Rightarrow \quad E_\pm = \pm a^{-1}\log(1 + a m_0)$$

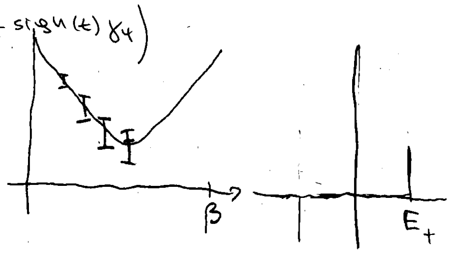
$$\approx m_0 - \underbrace{a\frac{m_0}{2}}_{O(a)\text{ enhanced by } m_0} + \ldots$$

$$C(\tau, \vec{p}=0) = e^{-aE_1|\tau|} + \delta_{\tau 0}\frac{1}{2(1 + a m_0)}$$

For finite Euclidean time extent and anti periodic boundary conditions: $\displaystyle\sum_{m=-\infty}^{\infty} e^{\frac{i m k_4}{N_\tau}} = \frac{1}{N_\tau}\sum_{\nu_4=0}^{N_\tau - 1} 2\pi\,\delta\left(k_4 - \frac{2\pi}{T}\nu_4\right)$

$$= \sum_{m=-\infty}^{\infty} e^{-aE|\tau + mN_\tau|} = \frac{e^{-aE\tau} + e^{-aE(N_\tau - \tau)}}{1 + e^{-aEN_\tau}} \qquad C(\tau) = e^{-aE|\tau|}\left(\frac{1}{\sinh(aE)} + \mathrm{sign}(\tau)\gamma_4\right)$$

$$\Rightarrow \quad C(\tau, \vec{p}=0) = \int_{-\infty}^{\infty} d\omega\, \frac{e^{-\tau\omega}}{1 + e^{-\beta\omega}} \big(\underbrace{\delta(\omega - E)}_{\text{backward}} + \underbrace{\delta(\omega + E)}_{\text{forward}}\big)$$



In general for relativistic correlators $\displaystyle G(\tau) = \int d\omega\, \frac{e^{-\tau\omega}}{1 \mp e^{-\beta\omega}}\,\rho(\omega)$

⇒ Practical problem: - We loose $N_\tau/2$ datapoints, since redundant information.
Makes extracting information difficult if realistic $N_\tau = 12 \ldots 24$
- Exponential decay with m_0 large requires extremely high statistics.

Instead of seeing the separation of scales as a drawback, we can embrace it and use it to our advantage => Effective Field Theory.

We give up the requirement that the theory is valid at all scales, as long as it produces accurate quantitative results at the relevant scales of the problem. Corrections need to be known.

General setup: Physics at a low energy scale described by effective d.o.f treated explicitly while physics @ higher scales enters via ① contact interactions ② low-energy constants (Wilson prefactors in L)

~~Konstruct~~ a Lagrangian in low energy d.o.f. that reproduces physical observables.

For heavy quarks: $\frac{\Lambda_{QCD}}{M} \ll 1$ no spontaneous $Q\bar{Q}$ pair production
$\frac{T}{M} \ll 1$ no thermal $Q\bar{Q}$ pair production } Non-relativistic quarks.

How to construct the most general Lagrangian for non-rel. quarks compatible with the symmetries of QCD? (No general formulas available)

Clever Trick: Foldy-Tani-Wouthuysen transformation

$$\bar{Q}(x)\,(D_\mu \gamma^\mu + m)\,Q \qquad Q \to Q' = e^{S'}e^{S}Q \qquad \bar{Q} \to \bar{Q}' = \bar{Q}e^{S}e^{S'}$$
$$S = \frac{1}{2M} D_i \gamma_i = -S^+ \qquad S' = \frac{1}{4M^2}\gamma_0\gamma_k F_{k0} \qquad Q = \begin{pmatrix}\psi \\ \chi\end{pmatrix}$$

From the appearing terms construct Lagrangian $L = \sum_n \frac{c_n(\alpha_s(M),\mu)}{M^n}\, O_n(\mu, mv, mv^2)$

$$L^{NRQCD} = \underbrace{\psi^+(iD_0 + H_\psi)\psi}_{L_\psi} + \underbrace{\chi^+(-iD_0 + H_\chi)\chi}_{L_\chi = L_\psi^*} + \text{contact terms} \underbrace{+ \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{q}(\dots)q}_{\text{light d.o.f @ scale } \mu}$$

$$iD^0 = i\partial^0 - gA^0 \qquad i\vec{D} = i\vec{\partial} + g\vec{A} \qquad E^i = F^{i0} \qquad B^i = -\varepsilon_{ijk}F^{jk}/2$$

$$L_\psi = \psi^+\left(iD_0 + \frac{c_1}{2m_1}D_i^2 + \underbrace{\frac{c_4}{8M^3}D^4}_{*} + \frac{c_F}{2M}\vec{\sigma}\cdot\vec{B}\,g + \frac{c_D}{8M}(\vec{D}\cdot g\vec{E} - g\vec{E}\cdot\vec{D}) + \dots\right)\psi$$

* relativistic correction $\sqrt{p^2+M^2} \approx M + \frac{p^2}{2M} - \frac{p^4}{8M^3}$ $\quad c_k = c_4 = 1$ $\quad$ Constraints on c_i's as remnant of underlying Lorentz invariance.

In general c's can receive non-trivial contributions from UV physics @ the scale $\mu \sim M$, can even be complex : Need matching

Compute correlation functions both in QCD and NRQCD with the same physics content and set equal at some scale. That gives the c_i's. Carry out QCD computations either perturbatively or using lattice QCD. (In the following neglected)

$$j^\mu_k(x) = b_1(\psi^+\sigma_k\chi) + O\!\left(\tfrac{1}{M^2}\right)$$

Note: Explicit mass term 2M from FTW removed from L_ψ. All energies are shifted and absolute energy scale needs to be calibrated.

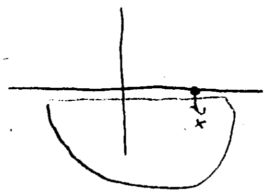
Let us have a look at the non-interacting theory ($A^\mu = 0$) (V)

$$\int D\psi D\psi^\dagger\, \psi^\dagger \psi\, e^{i S^{NRQCD}} = \int D\psi D\psi^\dagger\, \psi^\dagger\psi\, e^{i\psi^\dagger \underbrace{\left(+iD_0 + \frac{p^2}{2M}\right)}_{K_\psi^{-1}}\psi}$$

$$= K \det(K^{-1}) \approx K \qquad K_\psi^{-1} K_\psi = \text{"}\mathbb{1}\text{"} \Rightarrow \left(i\partial_0 + \frac{p^2}{2M}\right) K_\psi(x,x') = \delta(x-x')$$

Solve for the propagator explicitly: $\left(i\partial_0 + \frac{p^2}{2M}\right)K_\psi = \delta \qquad \underbrace{\left(-i\partial_0 + \frac{p^2}{2M}\right)K_\chi = \delta}_{\text{exercise}}$

$$K(t,\vec{x}) = \frac{1}{(2\pi)^4}\int d\omega \int d^3p\, e^{-i\omega t} e^{i\vec{p}\vec{x}} \frac{-1}{\omega - \frac{p^2}{2M} + i\varepsilon}$$



$$= \frac{i\Theta(t)}{(2\pi)^3}\int d^3p\, e^{-i\frac{p^2}{2M}t} e^{i\vec{p}\vec{x}}$$

$$= \frac{i\Theta(t)}{(2\pi)^2}\int_0^\infty dp \int_{-1}^{1} dy\, e^{-i\frac{p^2}{2M}t} e^{ipry} p^2 \qquad -\frac{it}{2M}\left(p^2 - pr\frac{2M}{t} + \frac{M^2}{t^2}r^2 - \frac{M^2}{t^2}r^2\right)$$

$$= \frac{i\Theta(t)}{(2\pi)^2}\int_0^\infty dp\, \frac{p}{ir}\left(e^{ipr} - e^{-ipr}\right) e^{+ip^2\frac{t}{2M}} \qquad = -\frac{it}{2M}\left[\left(p - \frac{Mr}{t}\right)^2 - \frac{M^2}{t^2}r^2\right]$$

$$= \frac{\Theta(t)}{(2\pi)^2}\int_{-\infty}^{\infty} dp\, \frac{p}{ir} e^{ipr} e^{+ip^2\frac{t}{2M}} = \frac{\Theta(t)}{(2\pi)^2} e^{i\frac{M}{2t}r^2}\int_{-\infty}^{\infty} dp'\, \frac{p' + \frac{M}{t}r}{r} e^{-i\frac{t}{2M}p'^2}$$

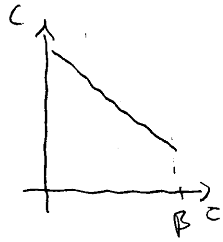
$$= \frac{\Theta(t)}{(2\pi)^2}\frac{\sqrt{\pi}}{i2}\left(\frac{2M}{t}\right)^{3/2} e^{i\frac{M}{t}r^2} \qquad \frac{1}{2}\left(\frac{2M}{t}\right)\left(\frac{2M}{t}\right)^{1/2}\sqrt{\pi}$$

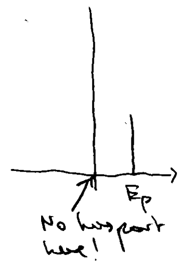
Quick check for $A \neq 0$ but $M \to \infty$ $\quad (i\partial_0 - gA^0)K = \delta \Rightarrow K = \Theta(t)\, e^{ig\int_t^{t'} A^0(s)\, ds}\, \delta^3(x)$

$\Rightarrow$ There is only a single forward propagating mode.

Time-slice correlator $\quad C(\tau) = K(\tau, \vec{p}) = e^{-\frac{p^2}{2M}\tau}\,\Theta(\tau)$

(single heavy quark) $\quad \rho(\omega) = \Theta(\omega)\,\delta\left(\omega - \frac{p^2}{2M}\right)$





Now put NRQCD on the lattice: $\quad t \to i\tau \quad D_t \to iD_\tau$

$a\Delta^+\psi(x) = U_{x,\mu}\psi(x + a\hat{\mu}) - \psi(x) \qquad \Delta^\pm = \frac{1}{2}(\Delta^+ + \Delta^-) \qquad \Delta^2 = \sum_i \Delta_i^- \Delta_i^+$

discretized Lagrangian becomes series in powers of $\frac{\Lambda}{Ma}$

$$S^{NRQCD}_{lat,\psi} = \psi^\dagger K^{-1}\psi = \sum_{x,\tau}\psi^\dagger\left(\Delta_4^+ - \sum_{j=1}^{3}\frac{\Delta_j^+\Delta_j^-}{2M} + \frac{\vec{\sigma}\vec{B}}{2M} + \dots\right)\psi_{x,\tau} \qquad B \equiv$$  clover leaf approximation.

The free propagator from $\int_{-\pi/a}^{\pi/a} d^4p\, \psi^\dagger(p)\left[i\tilde{p}_4 a - \frac{\tilde{p}_4^2 a^2}{2} + \frac{1}{2M}\sum_j(\hat{p}_j)^2\right]\psi(p)$

since $e^{i\tilde{p}_4 a} - 1 = i\tilde{p}_4 a - \frac{a^2\tilde{p}_4^2}{2}$

This expression only vanishes once in the Brillouin zone $\to$ no doublers.

How to compute heavy quark propagation on the lattice? (VI)

$$K^{-1}K = \mathbb{1} \qquad \left(\Delta_4^+ - \sum_{j=1}^{3} \frac{\Delta_j^+ \Delta_j^-}{2M}\right) K_{x,\tau} = \delta_{x,\tau}$$

This is a discretized stochastic diffusion equation (U_μ fluctuate)

① Initial conditions trick: $K_{x,0} = \eta$ with $\langle\eta\rangle = 0$ $\langle\eta\eta'\rangle = \delta_{xx'}$
use different noise for each configuration.

② $$K_{x,\tau+1} = \underbrace{U^+_{x,\tau}\,(1 - aH)}_{(\neq e^{-aH_{\text{eff}}})} K_{x,\tau}$$ for better stability usually define Lepage parameter n.

$$= \left(1 - \frac{a}{2n} H\right)^n U^+_{x,\tau} \left(1 - \frac{a}{2n} H\right)^n K_{x,\tau}$$

=> Much simpler than relativistic theory, no inversion of Dirac operator.

Now we are ready to compute heavy quarkonium current-current correlators:

Channel	Relativistic	NRQCD
$^3S_1$	$\bar{Q}\gamma^\mu Q$	$\chi^+ \sigma_i \psi$
$^3P_1$	$\bar{Q}\gamma_5\gamma^\mu Q$	$\chi^+(\overleftarrow{\Delta}_i \sigma_j - \overrightarrow{\Delta}_j \sigma_i)\psi$

$$D(\tau,\vec{x}) = \int D(\psi,\psi^+) \int D[\chi,\chi^+] \int DU \; (\chi^+\sigma^i\psi)(\chi^+\sigma_i\psi)^+ \, e^{-S[U] - S^{NRQCD}[\psi,\psi^+,\chi,\chi^+,U]}$$

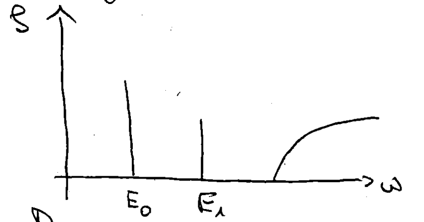
Integrating out the fermions gives $D(\tau,\vec{x}) = \int DU \, \mathrm{Tr}\left[K^+_\chi(\tau,x)\,\sigma_i\, K_\psi(\tau,x)\,\sigma_i\right] e^{-S[U]}$

Quarkonium at rest $D(\tau) = \int d^3x \, D(\tau,\vec{x})$

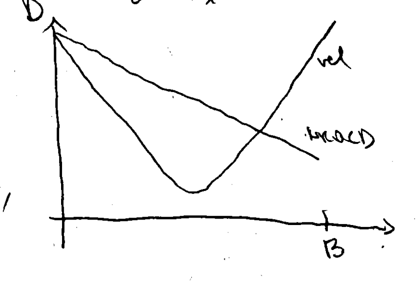
Strategy: compute the evolution of single heavy quark then combine into quarkonium correlator. (In Euclidean time $K_\psi = K_\chi$)

<u>At T=0</u> $$D = \sum_x \langle 0| J(x,\tau) J^+(x,0)|0\rangle \overset{\text{NRQCD}}{\underset{\text{time evol}}{=}} \sum_{x,n} \langle 0| J(x,0)\, e^{-H\tau} \underset{\text{state with } Q\bar{Q}}{|n\rangle} \langle n| J^+(x,0)|0\rangle$$

$$= \sum_n \underbrace{\sum_x |\langle 0|J(0,x)|n\rangle|^2}_{A_n} e^{-E_n\tau}$$ In general even at T>0 $D(\tau) = \int d\omega \, e^{-\omega\tau} \rho(\omega)$



- Beware E_0 still contains an Energy shift, since we subtracted $2M$ in L^{NRQCD}. Need to calibrate for each lattice spacing by fixing e.g. $^3S_1$ mass to PDG value.
- Positive: - Absence of backward travelling mode -> no symmetry in τ -> all data points usable.
  - Due to energy shift, exponential decay weaker, better S/N ratio.



What is the meaning of A_n? $\quad$ VIII

$\psi_n(t) = \langle 0 | \chi^+(x/2)\, \psi(-x/2) | n \rangle$
Bethe-Salpeter-Wavefunction

Related to the decay of heavy quarkonium, e.g. into dileptons:

$$\Gamma(Q\bar{Q} \to \ell^+\ell^-) = \sum_n \frac{2\,\mathrm{Im}(f_n(\Lambda))}{M^{d_n-4}} \langle Q\bar{Q} | O_n(\Lambda) | QQ \rangle$$

which in NRQCD is mediated via contact interactions, since it describes physics at the scale M which was integrated out. (not explicit in the EFT)

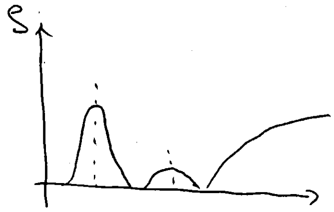
$$\Gamma(\Upsilon/\psi \to e^+e^-) = \frac{2\,\mathrm{Im}\, f_{e^+e^-}(^3S_1)}{M^2} |\langle 0 | \chi^+ \sigma \psi | \Upsilon/\psi \rangle|^2 + \frac{2\,\mathrm{Im}\, g_{e^+e^-}(^3S_1)}{M^4} \mathrm{Re}\left[\langle \Upsilon/\psi | \psi^+ \sigma \chi | 0 \rangle \langle 0 | \chi^+ \sigma (-\tfrac{i}{2}\vec{D})^2 \psi | \Upsilon/\psi \rangle \right]$$

other channels can also contain color octet contributions.

Decay rates can be systematically related to non-relativistic wavefunctions at the origin $\quad A_n(^1S_0) = |\psi_n(0)|^2$ via Bethe-Salpeter wavefunction.

(Current research interest: Use BSW to derive nuclear force from lattice QCD)

What is the situation at finite temperature?



Spectrum gives information about dilepton emission of thermal $Q\bar{Q}$ in a medium:
$$R_{e^+e^-} \propto \int dp_0 \int d^3p\, \frac{\rho(P)}{P^2}\, n_B(p_0)$$

reduces to the $T=0$ formula if one assumes $\rho(P) = \rho(\omega = \sqrt{p^2+M^2})$ and $\rho(\omega) = \delta(\omega - E_0)$

Lattice NRQCD at finite temperature:

The derivation of the lattice action proceeds equally if $T \neq 0$, i.e. for a finite Euclidean time extent. As long as $T \ll M$ the Wilson coefficients only receive small radiative corrections, which is neglected in current $T>0$ NRQCD studies. Solve diffusion equation along finite τ-axis.

Have a look at the extremes $T \to \infty$, free theory

remember: $K_\psi(\tau) = \frac{\Theta(\tau)}{(2\pi)^2} \frac{\boxed{1}}{2} \left(\frac{2M}{\tau}\right)^{3/2} e^{-\frac{M}{\tau} r^2} \qquad K_\chi^+ = \frac{\Theta(\tau)}{(2\pi)^2} \frac{\boxed{1}}{2} \left(\frac{2M}{\tau}\right)^{3/2} e^{-\frac{M}{\tau} r}$

$$D(\tau) = \int d^3x\, K_\psi K_\chi^+ \propto \left(\frac{2M}{\tau}\right)^3 \int d^3x\, e^{-\frac{2M}{\tau} r^2} \propto \left(\frac{2M}{\tau}\right)^3 \left(\frac{2M}{\tau}\right)^{-3/2} \propto \left(\frac{2M}{\tau}\right)^{3/2}$$

$$\underline{D(\tau) \propto \int d^3p\, e^{-2E_p \tau}} \quad \text{which with} \quad \underline{\rho(\omega) = \int d\omega\, e^{-\omega\tau} \rho(\omega)}$$

leads to $\quad \rho(\omega) = \int d^3p\, \delta(\omega - 2E_p) = \frac{N_c}{\pi} M^{3/2} \omega^{1/2} \Theta(\omega)$

Comparison to relativistic free spectrum

$$\rho(\omega) = \Theta(\omega^2 - 4M^2)\, \frac{N_c}{8\pi\omega} \sqrt{\omega^2 - 4M^2}\, \left(1 - 2n_F(\omega/2)\right) \left(\omega^2 \kappa + 4M^2 \kappa'\right) + 2\pi\omega\, \delta(\omega)\, N_c\, \gamma$$

$\overset{\omega \to \infty}{\approx} \omega^2$

Now the lattice NRQCD version: We get the dispersion relation from our time evolution operator $K_{x,\tau+1} = \left(1 - a\frac{p^2}{2M}\right) K_{x,\tau} \Rightarrow aE_p = -\log\left(1 - \frac{p^2}{2Ma}\right)$

C++ exercise: Compute the NRQCD free spectral function for $Ma = 5$

Let us have a look at some actual real-world NRQCD data:

HotQCD $N_f = 2+1$ HISQ ensembles $48^3 \times 12$ $(T>0)$ $48^3 \times 32, 48, 64$ $(T \approx 0)$

$M_b a \in [2.8 \ldots 0.9]$ $n = 4$ $\boxed{\text{Fig 8}}$ $\boxed{\text{Fig 9}}$

⇒ Notice mass difference between different lattice spacings → NRQCD energy shift

⇒ First hint at in-medium modification from correlator ratios. $\boxed{\text{Fig 10}}$ $\boxed{\text{Fig 11}}$

Correlators only give us information about global in-medium changes but we are interested in changes of individual states. Need spectral information (mass & width @ $T>0$). Once peaks acquire thermal width effective mass analysis fails (no plateau!)

Spectral function reconstruction using Bayesian inference

In general we have the relation $D(\tau) = \int d\omega\, K(\tau,\omega)\, \rho(\omega)$ which needs to be inverted. Since ρ can contain many features needs to be well resolved along ω: $D_i = \sum_{\ell=1}^{N_\omega} \Delta\omega_\ell\, K_{i\ell}\, \rho_\ell$

Relativistic $Q\bar{Q}$: $K = \frac{\cosh(\omega(\tau - \beta/2))}{\sinh(\omega\beta/2)}$

NRQCD $Q\bar{Q}$: $K = e^{-\omega\tau}$

Inversion is $\underbrace{\text{ill-posed}}_{N_\tau \ll N_\omega}$ and $\underbrace{\text{ill-conditioned}}_{D_i = D_i^{\text{ideal}} + \eta}$: Naive inversion in the case $N_\tau = N_\omega$ leads to exponential increase of noise.

∞ solutions to χ^2 fit.

Question: How to systematically regularize the problem?

Personal answer: Bayesian inference that allows to incorporate additional prior information on the spectrum. Subjective view of probability, assign $P[X]$ even if X is not random variable.

Multiplication law for prob.

$$P(\rho, D, I) = P[\rho|D,I]\, P[D|I] = P[D|\rho,I]\, P[\rho|I] \quad > \quad \underbrace{P[\rho|D,I]}_{\text{Posterior}} = \frac{\overbrace{P[D|\rho,I]}^{\text{Likelihood}}\ \overbrace{P[\rho|I]}^{\text{Prior}}}{\underbrace{P[D|I]}_{\text{Evidence}}}$$

↑ test function

Likelihood: How compatible is the data with current spectrum?
Prior: How compatible is ρ with prior information.

- For sampled data sets, due to the law of large numbers we can assume Gaussian distributed data: $P[D|\rho,I] = e^{-L}$ $L = \sum_{i,j} (D_i - D^\rho_i)\, C^{-1}_{ij}\, (D_j - D^\rho_j)$

$$C_{ij} = \frac{1}{N_m(N_m-1)} \sum_{e=1}^{N_m} (D^e_i - D_i)(D^e_j - D_j)$$

- The second ingredient is the prior probability
$P[\rho|I] = e^{-\alpha S[\rho,m]}$ α is a so called hyperparameter that weighs influence of data vs. prior information.

Prior information enters in two ways: By the functional form of S and by the choice of a function m, which fullfills $\frac{\delta S}{\delta \rho}\Big|_{\rho=m} = 0$ the so called default model. By definition it corresponds to the correct spectrum in the absence of data.

$\Rightarrow$ Eventually $\frac{\delta}{\delta \rho} P[\rho|D,I]\Big|_{\rho=\rho^{Bayes}} = 0$ defines the most probable spectrum in the Bayesian sense. Competition btw L & S.

On the market different implementations that differ in $P[\rho|I]$ and handling of α.

- Tikhonov: $S = \int d\omega\, (\rho - m)^2$ tune α such that $L = N_\tau$ (correct spectrum sampled with Gaussian noise gives $L = N_\tau$)

  issue: pulls the result very sharply towards m.

- Maximum Entropy Method (MEM): $S_{SJ} = \int d\omega \left(\rho - m - \rho \log \frac{\rho}{m}\right)$ tune α such that $P[D|I]$ extremal (positive definite spectra, Shannon-Jaynes entropy derived from arguments in 2d image reconstruction)

  issue: flat directions make numerical search difficult.

- Bayesian reconstruction method (BR): $S_{BR} = \int d\omega \left(1 - \frac{\rho}{m} + \log\left(\frac{\rho}{m}\right)\right)$ assume no knowledge of α and integrate out using $P[\alpha]$, enforce $L = N_\tau$ in addition.

<u>How to derive the BR regulator</u>: 4 Axioms

① <u>Subset independence</u>: Consistently combine prior information in different frequency regimes. (Probabilities multiply)

$S \propto \int d\omega\, s[\rho(\omega), m(\omega)]$

② Scale invariance:

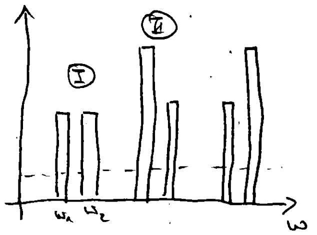
. Contrary to statements in the literature, ρ itself does NOT need to behave like a probability distribution. Scaling depends on the corresponding correlator

$$\rho_{QQ}^{NRQCD} \sim \sqrt{\omega} \qquad \rho_{QQ}^{rel.} \sim \omega^2 \qquad \rho_{WL} \sim \frac{1}{\omega} \qquad \rho_{EMT} \sim \omega^4$$

To make the results independent of the units used we only allow ratio of quantities to enter with some scaling. $\Rightarrow$ $S \propto \tilde{\alpha} \int d\omega\, s[\frac{\rho}{m}]$

Introduce a positive hyperparameter with units $[\tilde{\alpha}] = \frac{1}{[d\omega]}$.

③ Smoothness: Where the data does not encode peaked structures the spectrum shall remain smooth. Use here maximally uninformative default model $m(\omega) = m$.



Penalize deviations between neighbouring ρ values: $\nu_1 = \frac{\rho_1}{\omega_1}$ $\nu_2 = \frac{\rho_2}{\omega_2}$.

I vs. II $\quad 2s(\nu) - s(\nu(1+\varepsilon)) - s(\nu(1-\varepsilon)) = \varepsilon^2 C_2$

since $\nu > 0$ as multiplicative factor

same penalty for I and III

$$\Rightarrow \quad -\nu^2 s''(\nu) = C_2$$

Solution: $S = \tilde{\alpha} \int d\omega \left(C_0 - C_1 \frac{\rho}{m} + C_2 \log\left(\frac{\rho}{m}\right)\right)$

④ Bayesian meaning of S: $\quad S'(\nu=1) = 0, \quad S''(\nu=1) < 0 \quad$ (convention $S(\nu=1) = 0$)

after absorbing overall positive factor into $\tilde{\alpha}$

$$S_{BR} = \alpha \int d\omega \left(1 - \frac{\rho}{m} + \log\frac{\rho}{m}\right)$$

is strictly concave $S < 0$ $\frac{\delta^2 S}{\delta \rho^2} < 0$, i.e. $-L + \alpha S$ has unique extremum.

How to handle α: Make its role explicit and marginalize

$$P(\rho, D, \alpha, m) = P[D|\rho,\alpha,m]\, P[\rho|\alpha,m]\, P[\alpha|m]\, P[m]$$
$$= P[\alpha|\rho,D,m]\, P[\rho|D,m]\, P[D|m]\, P[m]$$

$P[D|\rho,\alpha,m] = P[D|\rho,m]$

$P[\alpha|m] = P[\alpha] \doteq 1$

$$\Rightarrow \quad P[\rho|D,m] = \frac{P[D|\rho,I]}{P[D|I]} \int d\alpha\, P[\rho|\alpha,m]$$

In addition we also enforce $L = N_\tau$ as further constraint.

Show example of spectral reconstruction mock test $\boxed{\text{Fig 12}}$ $\boxed{\text{Fig 13}}$

Different Non-Bayesian methods available

① Backus-Gilbert: Basic idea: if the Kernel is a δ-function the correlator equals the spectrum

⇒ Find linear combination of data-points that maximizes "peakedness" of the kernel.

issue: Implicit prior information in the optimization criterion.

② Padé-approximation: Basic idea: Project the data onto a set of rational basis functions on which the inversion is computed analytically.

issue: Data needs to be extremely precise, otherwise large oscillations when inverting analytically do not cancel.

Show results from actual NRQCD studies [Fig 14] [Fig 15]

<u>Caveats</u>: Temperature changes from different # of data points, need to disentangle method systematics from in-medium physics.

in NRQCD: Since Kernel T independent, truncate T=0 data set to $N_\tau^{T>0}$ and redo reconstruction (As if same spectrum is encoded at higher T)

relativistic: T-dependent kernel, need to compute "reconstructed correlator"

$$G^{rec}(\tau, \underset{\text{rws}}{T}, \underset{\text{ainp}}{T'}) = \sum_{\tau'=\tau,\, \Delta\tau'=N_\tau}^{N'_\tau - N_\tau + \tau} G(\tau', T')$$

<u>Error analysis</u>:

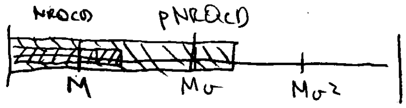
statistical: Jackknife resampling; repeat the reconstruction with a subset of simulated correlators.

Systematic: Change the default model and observe which parts of the reconstructed spectrum remain robust.

BOTH NEED TO BE CARRIED OUT!

Up to now we have looked at $Q\bar{Q}$ properties in thermal equilibrium. To move closer to experiment, we need to also understand real-time evolution. One possible starting point: Derivation of an in-medium potential for $Q\bar{Q}$ from QCD.

A new effective field theory: pNRQCD



For NRQCD one integrated out physics at the scale M, for pNRQCD also at the scale Mv. Explicitly captures physics at $E \sim E_b$

Relevant degrees of freedom: color singlet and octet wavefunctions

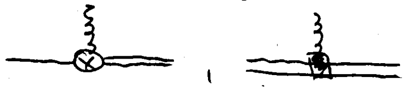
$$\mathcal{L}^{pNRQCD} = S^+(r,t)\Big(\underbrace{i\partial_t - \frac{p^2}{2M_{red}} - V_S(r)}_{h_S}\Big)S(r,t) + O^+(r,t)\Big(\underbrace{i\partial_t - \frac{p^2}{2M_{red}} - V_O(r)}_{h_O}\Big)O(r,t)$$
$$+\, V_A(r)\,\mathrm{Tr}\big[O^+\vec{r}\cdot g\vec{E}S + S^+\vec{r}\,g\vec{E}O\big] + \dots \quad -\frac{1}{4}F^{\mu\nu}_{US}F^{US}_{\mu\nu} \leftarrow \text{ultrasoft gluons}$$

Combines $1/M$ expansion with multipole expansion of cov. derivatives

$$V_S = V_S^{(0)} + \frac{V_S^{(1)}}{M_{red}} + \frac{V_S^{(2)}}{M_{red}^2} + \dots \quad \text{systematically improvable.}$$

Feynman rules:

singlet propagator —— $\dfrac{1}{E-h_S}$ octet propagator $=$ $\dfrac{1}{E-h_O}$



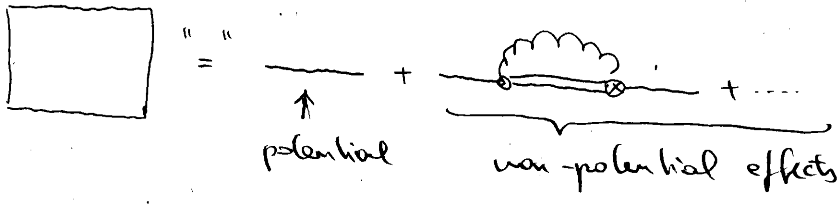
Potential evolution is tree level approximation of EFT.

In pNRQCD the non-local potentials are the Wilson matching coefficients. Fix via comparison of correlator to QCD: Here in the static case

pNRQCD : $\langle S^+(r,t)\, S(r,0)\rangle$ quarkonium singlet correlator point split

NRQCD : $\langle \chi^+(x_1)\, W(x_1,x_2)\, \psi(x_2)\, \psi^+(y_2)\, W^+(y_2,y_1)\, \chi(y_1)\rangle \quad W = e^{ig\int_{x_1}^{x_2} dx^\mu A_\mu}$

$$= \langle K^+_\chi(y_1,x_1)\, W(x_1,x_2)\, K_{\psi}(x_2,y_2)\, W^+(y_2,y_1)\rangle$$
$$\overset{m\to\infty}{=} \exp\Big[ig\oint_{\square} dx^\mu A_\mu\Big] \quad \text{real-time Wilson loop}$$

$\Rightarrow$ 

How can we establish the validity of the potential description from QCD?
$\Rightarrow$ Spectral functions of the Wilson loop are accessible on the lattice. XIII

From matching we see $\quad i\partial_t W_\square(r,t) = \Phi(r,t)\, W_\square(r,t) \qquad (*)$

and remember from NRQCD that $\quad W_\square(r,\tau) = \int d\omega\, e^{-\omega\tau} \rho_\square(r,\omega) \Leftrightarrow W_\square(r,t) = \int d\omega\, e^{-i\omega t} \rho_\square(r,\omega)$

We now know how to extract $\rho_\square$ from $W_\square(\tau)$ via Bayesian inference.

If the potential picture is applicable eventually: $\Phi(r,t) \xrightarrow{t\to\infty} V(r)$

$$\Rightarrow \quad V(r) = \lim_{t\to\infty} \frac{i\partial_t W_\square(r,t)}{W(r,t)} = \lim_{t\to\infty} \frac{\int d\omega\, \omega\, e^{-i\omega t} \rho_\square(r,\omega)}{\int d\omega\, e^{-i\omega t} \rho_\square(r,\omega)} \in \mathbb{C}$$

$\leftarrow$ first discovered by M. Laine in 2007.

only the lowest lying structure contributes to $V(r)$

How does the relevant spectral structure look like?

Decompose: $\Phi(r,t) = \phi(r,t) + V(r)$ and solve $(*)$ $\qquad \phi(r,t) = 0 \quad t > t_{QQ}$

$$W_\square(r,t) = \exp\Big[-i\big\{\mathrm{Re}V\cdot t + \mathrm{Re}[\sigma](r,t)\big\} - |\mathrm{Im}V|\,t + \mathrm{Im}[\sigma](r,t)\Big]$$

with $\sigma = \int_0^t dt'\, \phi(r,t)$ and for later $\sigma_\infty(r) = \sigma(t > t_{QQ}, r) = \int_0^\infty dt'\, \phi(r,t')$

Carrying out the Fourier transformation using $W^*(r,t) = W(r,-t)$

$$\rho_\square(\omega,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, W_\square(r,t) = \frac{1}{2\pi} e^{\mathrm{Im}[\sigma_\infty]} \int_{-\infty}^{\infty} dt\, \exp\Big[i(\omega - \mathrm{Re}V)t - |\mathrm{Im}V||t| - i\,\mathrm{Re}[\sigma_\infty]\,\mathrm{sign}(t)\Big]$$

$$+ \frac{1}{2\pi}\int_{-t_{QQ}}^{t_{QQ}} dt\, \exp\Big[i(\omega-\mathrm{Re}V)t - |\mathrm{Im}V||t|\Big]\left(e^{-i\,\mathrm{Re}\sigma\,\mathrm{sign}(t) + \mathrm{Im}\sigma} - e^{-i\,\mathrm{Re}[\sigma_\infty]\,\mathrm{sign}(t) + \mathrm{Im}(\sigma_\infty)}\right)$$

$$= \frac{1}{\pi} e^{\mathrm{Im}\,\sigma_\infty} \frac{|\mathrm{Im}V|\cos(\mathrm{Re}\,\sigma_\infty) - (\mathrm{Re}V - \omega)\sin(\mathrm{Re}(\sigma_\infty))}{\mathrm{Im}V^2 + (\mathrm{Re}V - \omega)^2} + c_0(r) + c_1(r)\,t_{QQ}(\mathrm{Re}V - \omega) + \dots$$

Skewed Lorentzian embedded in polynomial background. Position and width from potential, skew from non-potential effects. I.e. if we can identify such a structure in the Wilson loop spectrum the potential picture is applicable. (Technical detail: since $W_\square$ contains cusp divergences due to 90° angles, its s/n ratio is very low. Instead we use Wilson line correlator in Coulomb gauge.)

Show results from quenched QCD $32^3\times N_\tau$ $\beta=6.1$ $\xi_r=4$ Fig 16 / Fig 17 / Fig 18 / Fig 19

At $T=0$ we recover the textbook potential which agrees very well with the Cornell potential. $V_{T=0}(r) = \lim_{\tau\to\infty} \frac{1}{\tau} \log [W_\square(r,\tau)] \in \mathbb{R}$ [Fig. 20]

At $T=0$ the Wilson loop has a well separated lowest lying δ-like state. Otherwise the above definition does NOT give the correct potential

How to interpret the in-medium modification of $V_{Q\bar{Q}}(r) \in \mathbb{C}$?

At high temperatures $T>T_c$ the $Q\bar{Q}$ is surrounded by a QGP with free color charge carriers. First proposals were related to Debye screening.
Original idea: Use Gauss law for $V(r) = -\frac{\alpha}{r}$, i.e. $-\nabla^2 V = 4\pi\alpha\,\delta(r)$
and introduce background charge which is Boltzmann distributed

$$-\nabla^2 V = 4\pi\alpha\left(\delta + \langle\rho(r)\rangle\right) \qquad \rho(r) = \left(n_0 e^{-\beta V} - n_0\right) - \left(n_0 e^{+\beta V} - n_0\right) \overset{V/T \ll 1}{\approx} -2\beta n_0 V(r)$$

$$\Rightarrow -\nabla^2 V + \underbrace{8\pi\alpha n_0 \beta}_{A^2} V = 4\pi\alpha\delta \qquad \Rightarrow \quad V(r) = -\alpha\, e^{-Ar}/r$$

At this point no route for imaginary part, no confining potential @ $T>0$?

Modern approach: Use a generalized Gauss-law $\nabla\left(\frac{-\nabla V}{r^{a+1}}\right) = 4\pi q\,\delta(r)$
$a=-1$, $q=\alpha$ Coulomb, $a=1$, $q=\sigma$ string like potential.

Medium effects are introduced via in-medium permittivity of a weakly coupled gas of quarks and gluons. Disentangle non-perturbative $T=0$ physics from medium effects.

See what happens for Coulombic part of the potential: FT

$$p^2 V_c(p) = 4\pi\alpha_s \quad \Rightarrow \quad p^2 V_c(p) = \frac{4\pi\alpha_s}{\varepsilon(p)} \qquad \underbrace{\varepsilon_p^{-1}(p,m_D) = \frac{p^2}{p^2+m_D^2} - i\pi T\frac{p\,m_D^2}{(p^2+m_D^2)^2}}_{\text{from hard-thermal loop PT.}}$$

$$\Rightarrow -\nabla^2 V_c(r) + m_D^2 V_c(r) = \alpha_s\left(4\pi\delta(r) - i\,T m_D^2\, g(m_D r)\right)$$

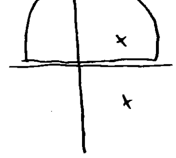
$$g(x) = 2\int_0^\infty dp\, \frac{\sin(px)}{px}\, \frac{p}{p^2+1}$$

Exercise: Show that this equation reproduces the results by Laine.

$$\mathrm{Re}\, V_c(r) = \frac{4\pi\alpha_s}{(2\pi)^3} \int_0^\infty dp \int_0^\pi d\theta \int_0^{2\pi} d\varphi\, \sin\theta\, \frac{p^2}{p^2+m_D^2}\, e^{ipr\cos\theta}$$

$$= \frac{2(2\pi)^2\alpha_s}{(2\pi)^3} \int_0^\infty dp \int_{-1}^{1} dy\, e^{ipry}\, \frac{p^2}{p^2+m_D^2} = \frac{2\alpha_s}{(2\pi)} \int_0^\infty dp \left(\frac{e^{ipr}}{ipr}\frac{p^2}{p^2+m_D^2} - \frac{e^{-ipr}}{ipr}\frac{p^2}{p^2+m_D^2}\right)$$

$$= \frac{2\alpha_s}{(2\pi)} \int_{-\infty}^{\infty} \frac{e^{ipr}}{ir} \, \frac{p}{p^2+m_D^2} \qquad \frac{p}{(p+im_D)(p-im_D)} \to \mathrm{Res}(p=im_D)\gamma = -\frac{im_D}{2im_D}$$



$$= -\alpha_s \, \frac{e^{-m_D r}}{r} \quad ✓$$

$\mathrm{Im}\, V_c$: $\mathrm{Im}\, V_c(p) = 4\pi\alpha_s \, \pi T \, \dfrac{m_D^2}{p\,(p^2+m_D^2)^2}$

$$\mathrm{Im}\, V_c(r) = 4\pi\alpha_s \pi T \int \frac{d^3p}{(2\pi)^3} \, e^{ipr\cos\vartheta} \, \frac{m_D^2}{p\,(p^2+m_D^2)^2} \qquad \text{with } z = \frac{p}{m_D}$$

$$= \alpha_s T \int dp \; 2 \, \frac{\sin(z m_D r)}{z m_D r} \, \frac{z}{(z^2+1)^2} \quad ✓$$

Now a similar line of argument can be constructed for the in-medium potential contribution coming from the string-like vacuum part.

$$\mathrm{Re}\, V_S(r) = -\frac{\Gamma(\frac{1}{4})}{2^{3/4}\sqrt{\pi}} \, \frac{\sigma}{\mu} \, D_{-\frac{1}{2}}(\sqrt{2}\,\mu r) + \frac{\Gamma(\frac{1}{4})}{2\,\Gamma(\frac{3}{4})} \, \frac{\sigma}{\mu} \qquad \mu^4 = m_D^2 \, \frac{\alpha}{\sigma}$$

$\mathrm{Im}\, V_S(r) = -\dfrac{i\sigma m_D^2}{\mu^4} \, T \, \psi(\mu r)$ $\quad$ $\psi(\mu r)$ integral expression arising from a Wronskian construction.

Show that the analytic formula with a single T-dependent parameter excellently reproduces the lattice ReV and ImV. All in-medium effects on the scales investigated here are summarized in $m_D(T)$. Around T_c the values of ImV from the lattice and Gauss-law deviate more than at high T.

Note: Need to be careful with the use of the word potential, since $V_{Q\bar{Q}} \in \mathbb{C}$ is NOT the usual potential for the wave function of $Q\bar{Q}$ but it governs the time evolution of the quarkonium propagator

$$D^>(r,t) = \langle \psi(r,t)\, \psi^*(r,0) \rangle$$

Solving the Schrödinger equation for $D^>$ gives access to quarkonium spectral functions $\rho_{Q\bar{Q}}(\omega) = \lim_{r\to 0} \int dt \, e^{-i\omega t} D^>(r,t)$. Show $\boxed{\text{Fig 24}}$ $\boxed{\text{Fig 25}}$

$\Rightarrow$ Plus: These spectra are much more precise than those obtained from direct NRQCD reconstructions.

Minus: Since only the static potential was used to evolve $D^>$, possibly systematic errors present. (question of accuracy)

Nevertheless we can use these spectra to estimate the ratio of Υ/ψ to ψ' at the phase boundary @ $T = 155$ MeV and find good agreement with experiment. [Fig 26]

XVI

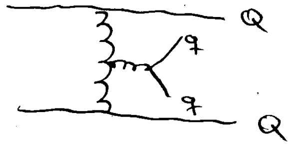
Last step: Towards a real time evolution of the $Q\bar{Q}$ wave function

Issue: Potential derived in the $M \to \infty$ limit, where heavy quarks cannot annihilate and singlet to octet transition weak. Using $V_{Q\bar{Q}} \in \mathbb{C}$ for the wave function leads to $|\psi_{Q\bar{Q}}| \propto e^{-|\mathrm{Im}\, V| t}$ heavy quarks are lost.

Resolution: Damping of the correlator on the other hand has clear physical meaning. $\langle \psi(r,t)\, \psi^*(r,0)\rangle \propto e^{-|\mathrm{Im}\, V| t}$ signals loss of memory of initial conditions, i.e. the system decoheres over time as it approaches thermal equilibrium.

Current research interest: How to realize $Q\bar{Q}$ evolution that incorporates decoherence.

One intuitive proposal: Imaginary part is related to kicks by medium partons on the gluon holding the $Q\bar{Q}$ together (Landau damping). Implement as stochastic perturbation of a real-valued potential.



$$\psi(t) = \exp\left[i \int dt \left(-\frac{\nabla^2}{2M} + \overset{\mathbb{R}}{V(r)} + \eta \right)\right] \psi(0) \qquad \langle \eta \rangle = 0 \quad \langle \eta \eta' \rangle = 2|\mathrm{Im}\, V|\, \delta(t-t')$$

$$\text{discretized} \qquad = \frac{2|\mathrm{Im}\, V|}{\Delta t}\, \delta_{tt'}$$

Expand evolution for infinitesimal time step:

$$\psi(t+\Delta t) \approx \left[1 + i\Delta t \left(-\frac{\nabla^2}{2M} + V(r) + \eta \right) - \underbrace{\frac{\Delta t^2}{2}\eta^2}_{\mathcal{O}(\Delta t)\,!} \right] \psi(t)$$

This stochastic Schrödinger equation implements unitary time evolution and reproduces the correct correlator

$$i\partial_t \langle \psi(r,t)\, \psi^*(r,0)\rangle = \left(-\frac{\nabla^2}{2M} + V(r) - i\,|\mathrm{Im}\, V| \right) \langle \psi(t)\, \psi^*(0)\rangle$$

Ongoing work to understand limitations of the stochastic Schrödinger equation and how to generalize it in the context of so called Open Quantum Systems (key words: Master equation, Lindblad equation)

Summary:

① Heavy quarkonium is a unique system that allows us to test our understanding of microscopic QCD with well established experimental data both in vacuum (masses; lifetimes) as well as at high temperatures in heavy-ion collision (production yields)

② Effective field theories allow us to efficiently compute quarkonium correlators in thermal equilibrium using standard lattice QCD simulations as input. From these correlators in-medium spectral properties can be extracted using Bayesian inference of spectral functions.

③ Understanding of $Q\bar{Q}$ real-time dynamics is emerging based on the concept of an in-medium potential whose T-dependence as found in lattice QCD can be understood from a phenomenological picture of a sharply coupled $Q\bar{Q}$ being immersed in a bath of weakly coupled quarks and gluons