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Real-time static potential in hot QCD

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Abstract

We derive a static potential for a heavy quark-antiquark pair propagating in Minkowski time at finite temperature, by defining a suitable gauge-invariant Green's function and computing it to first non-trivial order in Hard Thermal Loop resummed perturbation theory. The resulting Debye-screened potential could be used in models that attempt to describe the "melting" of heavy quarkonium at high temperatures. We show, in particular, that the potential develops an imaginary part, implying that thermal effects generate a finite width for the quarkonium peak in the dilepton production rate. For quarkonium with a very heavy constituent mass M, the width can be ignored for $T \lesssim g^2 M/12\pi$, where g^2 is the strong gauge coupling; for a physical case like bottomonium, it could become important at temperatures as low as 250 MeV. Finally, we point out that the physics related to the finite width originates from the Landau-damping of low-frequency gauge fields, and could be studied non-perturbatively by making use of the classical approximation.

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1. Introduction

Heavy quarkonium systems, $c\bar{c}$ and $b\bar{b}$, have turned out to provide extremely useful probes for QCD phenomenology [1]. On the theoretical side, the existence of a heavy mass scale makes these systems more susceptible to analytic treatments than hadrons made out of light quarks only, while on the experimental side, decays of heavy quarkonia lead to relatively clean signals which are precisely measured by now. It is therefore not surprising that the modifications that the quarkonium systems may undergo at high temperatures, are also among the most classic observables considered for heavy ion collision experiments [2].

Let us recall explicitly the conceptually clean connection that heavy quarkonium provides between thermal field theory and heavy ion phenomenology. Consider the production rate $\Gamma_{\mu^+\mu^-}$ of dileptons (for example, $\mu^+\mu^-$ pairs) with a total four-momentum $Q = P_{\mu^+} + P_{\mu^-}$ from a hot thermal medium. It can be shown that this rate is given by [3]

$$\frac{\mathrm{d}\Gamma_{\mu^+\mu^-}}{\mathrm{d}^4Q} = -\frac{e^2}{3(2\pi)^5 Q^2}\left(1+\frac{2m_\mu^2}{Q^2}\right)\left(1-\frac{4m_\mu^2}{Q^2}\right)^{\frac{1}{2}} \eta_{\mu\nu}\tilde{C}_<^{\mu\nu}(Q) \; , \tag{1.1}$$

where we assumed $Q^2 \ge (2m_\mu)^2$, e is the electromagnetic coupling, $\eta_{\mu\nu} = \mathrm{diag}(+{-}{-}{-})$, and $\tilde{C}_<$ is a certain two-point correlator of the electromagnetic current $\hat{\mathcal{J}}^\mu(x)$ in the Heisenberg picture, $\hat{\mathcal{J}}^\mu(x) = ... + \frac{2}{3}e\,\hat{\bar{c}}\,(x)\gamma^\mu\hat{c}(x) - \frac{1}{3}e\,\hat{\bar{b}}\,(x)\gamma^\mu\hat{b}(x)$:

$$\tilde{C}_<^{\mu\nu}(Q) \equiv \int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}\Big\langle \hat{\mathcal{J}}^\nu(0)\hat{\mathcal{J}}^\mu(x)\Big\rangle \; . \tag{1.2}$$

The expectation value refers to $\langle ... \rangle \equiv \mathcal{Z}^{-1}\mathrm{Tr}\,[\exp(-\hat{H}/T)(...)]$, where $\mathcal{Z}$ is the partition function, $\hat{H}$ is the QCD Hamiltonian operator, and T is the temperature.

Now, the heavy quark parts in the electromagnetic current induce a certain structure into the dilepton production rate. More precisely, they produce a *threshold* at around $Q^2 \simeq (2M)^2$, where M is the heavy quark mass, and a *resonance peak* (or peaks) near the threshold, with a certain height and width. The height and width could in principle be observed, as a function of the temperature T that is reached in the collision. Of course there are all kinds of practical limitations to this, related to non-equilibrium features, background effects, the energy resolution of the detector, etc, but at least some broad features like a total disappearance ("melting") of the quarkonium peak should ultimately be visible.

These circumstances have lead to a great number of studies of quarkonium physics at high temperatures. For instance, various types of potential models have been developed [4], with non-perturbative input taken from lattice simulations [5]. It is not quite clear, however, to which extent potential models are appropriate at finite temperatures, or which non-perturbative potentials should be used as input: at asymptotically large distances, the standard Polyakov loop correlator is known to fail to reproduce the expected Debye-screened potential [6], while many modified descriptions are afflicted by gauge ambiguities [7]. More recently, direct lattice determinations of the quarkonium spectral function have been attempted [8], given that in principle Euclidean data *can* be analytically continued to Minkowski spacetime [9] (though in practice model assumptions need again to be introduced, given the very finite number of points in time direction and the statistical nature of the data that are available for lattice studies). Finally, similar observables have been considered for strongly-coupled $\mathcal{N} = 4$ Super-Yang-Mills theory, through the AdS/CFT correspondence [10].

All of the studies mentioned, however, either resort to some degree of modelling, or attempt to tackle the complete problem by "brute force" on the lattice. It is our philosophy here to rather make some more use of the existence of the heavy mass M and the high temperature T that characterise the system, and the property of asymptotic freedom of QCD. Indeed, the strict weak-coupling expansion does appear to be applicable (within 10–20% accuracy) to hot QCD at temperatures as low as a few hundred MeV, once worked out to a high enough order and supplemented possibly by numerically determined non-perturbative *coefficients*, rather than complete functions; evidence for this has been obtained through precision studies with a large number of independent observables, such as spatial correlation lengths [11, 12], the spatial string tension [13, 14], quark number susceptibilites [15, 16, 17], the 't Hooft loop tension [18, 19], and perhaps also the equation-of-state [20].

More precisely, our goal here is to derive, through a resummed perturbative computation, a static potential that a heavy quark-antiquark pair propagating in Minkowski time at a finite temperature feels. This relatively simple computation illustrates a few interesting phenomena that have not been exhaustively addressed before, as far as we know. Eventually, our goal is to take the lessons brought by this study to a practical level, by carrying out a more extensive numerical investigation of the properties of heavy quarkonium [21].

The outline of this note is as follows. In Sec. 2 we set up the observable to be determined. The actual computation is discussed in some detail in Sec. 3. We elaborate on the main results in Sec. 4, and conclude in Sec. 5. There are two appendices summarising well-known formulae from the literature with our conventions.

2. Basic setting

Rather than Eq. (1.2), we prefer to concentrate in the following on the correlator

$$\tilde{C}_{>}(Q) \equiv \eta_{\mu\nu} \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \hat{\mathcal{J}}^{\mu}(x) \hat{\mathcal{J}}^{\nu}(0) \Big\rangle \,. \tag{2.1}$$

As we recall in Appendix A, the two time orderings are related to each other by $\tilde{C}_{<}(Q) = \exp(-\beta q^0)\tilde{C}_{>}(Q)$; for $q^0 \gtrsim 2M$, $\tilde{C}_{>}(Q)$ is of order unity while $\tilde{C}_{<}(Q)$ is exponentially suppressed. Furthermore, let us for simplicity set the spatial momentum to zero, $\mathbf{q} = \mathbf{0}$, and consider the correlator before taking the Fourier transform with respect to time:

$$C_{>}(t) \equiv \int \mathrm{d}^3\mathbf{x} \Big\langle \hat{\mathcal{J}}^{\mu}(t, \mathbf{x}) \hat{\mathcal{J}}_{\mu}(0, \mathbf{0}) \Big\rangle \,. \tag{2.2}$$

Note that translational invariance guarantees that $C_>(-t) = C_<(t)$.

Now, the determination of $\tilde{C}_>(Q)$ around the threshold requires a resummation of the perturbative series (this is the case even at zero temperature). A way to implement this is to deform the correlator suitably so that we obtain a Schrödinger-type equation, which can then be solved "non-perturbatively", implementing an all-orders resummation. To arrive at a Schrödinger equation, we define a point-splitting, by introducing a vector $\mathbf{r}$, and consider an extended interpolating operator for the electromagnetic current, rather than a local one (we also drop the electromagnetic couplings, and denote $\hat{c}, \hat{b}$ by a generic quark field $\hat{\psi}$):

$$\check{C}_>(t,\mathbf{r}) \equiv \int \mathrm{d}^3\mathbf{x} \left\langle \hat{\bar{\psi}}\,(t,\mathbf{x}+\frac{\mathbf{r}}{2})\gamma^\mu\, W_\mathbf{r}[(t,\mathbf{x}+\frac{\mathbf{r}}{2});(t,\mathbf{x}-\frac{\mathbf{r}}{2})]\,\hat{\psi}(t,\mathbf{x}-\frac{\mathbf{r}}{2})\;\hat{\bar{\psi}}\,(0,\mathbf{0})\gamma_\mu\hat{\psi}(0,\mathbf{0})\right\rangle\ . \quad (2.3)$$

Here $W_\mathbf{r}[x_1, x_0]$ is a Wilson line from x_0 to x_1, along a straight path in the direction of $\mathbf{r}$, inserted in order to keep the interpolating operator gauge-invariant. Once the solution for $\check{C}_>(t,\mathbf{r})$ is known, one can return back to the original situation $\mathbf{r} = \mathbf{0}$. We stress that only the $\mathbf{r} = \mathbf{0}$ limit corresponds to the physical current-current correlator in Eq. (2.2) that we are interested in. Indeed, the way the point-splitting is carried out may not be unique, but these ambiguities should disappear once we set $\mathbf{r} = \mathbf{0}$ in the solution.

Let us inspect $\check{C}_>(t,\mathbf{r})$ in the limit that the heavy quark mass, M, becomes very large. Then the heavy quarks are essentially non-relativistic, and should also be weakly interacting. Ignoring interactions altogether, a straightforward diagrammatic computation shows that the correlator in this limit indeed satisfies a Schrödinger-type equation,

$$\left[i\partial_t - \left(2M - \frac{\nabla_\mathbf{r}^2}{M} + \mathcal{O}\Big(\frac{1}{M^3}\Big)\right)\right]\check{C}_>(t,\mathbf{r}) = 0\ , \quad (2.4)$$

with the initial condition $\check{C}_>(0,\mathbf{r}) = -6N_\mathrm{c}\,\delta^{(3)}(\mathbf{r})$. Here the factor 6 corresponds physically to two heavy spin-1/2 degrees of freedom times a sum over the spatial components of the current. For $\check{C}_<(t,\mathbf{r})$, an analogous computation in the same limit yields

$$\left[i\partial_t + \left(2M - \frac{\nabla_\mathbf{r}^2}{M} + \mathcal{O}\Big(\frac{1}{M^3}\Big)\right)\right]\check{C}_<(t,\mathbf{r}) = 0\ , \quad (2.5)$$

with the same initial condition; this form is consistent with the symmetry $C_>(-t) = C_<(t)$.

Once interactions are switched on, we might expect (apart from the renormalization of the parameter M and of the initial condition) to generate a potential, $V(\mathbf{r})$, into Eqs. (2.4), (2.5), appearing in a form familiar from non-relativistic quantum mechanics. To be precise, we define the potential to be a function inside the round parentheses in Eq. (2.4) which scales as M^0 in the large-M expansion. To simplify the situation a bit, we would like to be able to extract the potential directly, without needing to bother about gradients like $\nabla_\mathbf{r}^2/M$. One possibility would be to consider the modified object

$$\begin{aligned}\bar{C}_>(t,\mathbf{r}) \quad \equiv \quad & \left\langle \hat{\bar{\psi}}\,(t,\frac{\mathbf{r}}{2})\gamma^\mu\, W_\mathbf{r}[(t,\frac{\mathbf{r}}{2});(t,-\frac{\mathbf{r}}{2})]\,\hat{\psi}(t,-\frac{\mathbf{r}}{2})\,\times\right.\\ & \left.\times\,\hat{\bar{\psi}}\,(0,-\frac{\mathbf{r}}{2})\gamma_\mu\, W_{-\mathbf{r}}[(0,-\frac{\mathbf{r}}{2});(0,\frac{\mathbf{r}}{2})]\,\hat{\psi}(0,\frac{\mathbf{r}}{2})\right\rangle\ . \end{aligned} \quad (2.6)$$

Here the quarks can be considered infinitely heavy, such that they do not move in space; the integral over $\mathbf{x}$ has consequently been dropped; and the quark fields at $t = 0$ point-split.

To simplify the situation even further, one notes that the infinitely heavy quark propagators are themselves represented by Wilson lines, apart from the "trivial" phase factor $\exp(-iMt)$. Thereby we arrive at a Wilson loop. It should be noted, though, that time ordering still plays a role, and the precise specification of the object considered is not as simple as in Euclidean field theory. In fact, the Minkowskian Wilson loop is most naturally defined just as an appropriate analytic continuation of the Euclidean object.

To summarise, we can imagine at least two ways of defining what we might call the static potential in real time. On one hand, one can start by computing the standard Wilson loop in Euclidean spacetime, and then carry out the analytic continuation that leads to the time ordering in Eq. (2.6). This is the computation that will be described below. On the other hand, one can consider directly Eq. (2.6), just replacing the quarks by (almost) infinitely heavy static ones: this amounts to NRQCD [22] with the omission of terms of order $\mathcal{O}(1/M)$. In the heavy-quark limit, $\bar{C}_{>}(t, \mathbf{r})$ is dominated by the forward-propagating part, with the phase factor $\sim \exp(-2iMt)$; the static potential can then be extracted from an equation like Eq. (2.4). We have carried out this computation as well, and do indeed obtain the same result as from the analytic continuation of the Euclidean Wilson loop; however, given that the NRQCD computation is quite a bit more involved than the Wilson loop one, we concentrate on the latter in the following.

As indicated above, the calculation of a static potential necessarily involves a point splitting and thus should be regarded as an intermediate step towards the computation of a current-current correlator. This feature is obviously shared by potential model approaches. However, we believe that our procedure offers several advantages over the latter. Since our calculation is performed in a well-defined field theoretic setting, the connection with the physical real-time observable of interest remains manifest, while non-perturbative gauge invariance as well as the correct perturbative limit of our static potential are ensured.

3. Details of the computation

Since we make use both of Minkowskian and Euclidean metrics, let us start by introducing some notation to keep them apart. Minkowskian four-momenta are denoted by capital symbols, Q, with components q^μ, while Euclidean ones are denoted by $\tilde{Q}$, with components $\tilde{q}_\mu$. The indices are kept down in the latter case, and our convention is $\tilde{q}_\mu \equiv (\tilde{q}_0, \tilde{q}_i) \equiv (\tilde{q}_0, -q^i)$. Spacetime coordinates are denoted by $x, \tilde{x}$, and here our convention is $\tilde{x}_\mu \equiv (\tilde{x}_0, x^i)$. The Euclidean scalar product is thereby naturally defined as $\tilde{x} \cdot \tilde{Q} \equiv \tilde{x}_\mu \tilde{q}_\mu = \tilde{x}_0 \tilde{q}_0 - x^i q^i$, the four-volume integral as $\int_{\tilde{x}} \equiv \int_0^\beta \mathrm{d}\tilde{x}_0 \int \mathrm{d}^3\mathbf{x}$, and the thermal sum-integral as $\SumInt_{\tilde{Q}} = T \sum_{\tilde{q}_0} \int \mathrm{d}^3\mathbf{q}/(2\pi)^3$. Wick rotation amounts to $\tilde{x}_0 \leftrightarrow ix^0$, $\tilde{q}_0 \leftrightarrow -iq^0$. All Matsubara frequencies we will meet are bosonic: i.e. $\tilde{q}_0 = 2\pi n T$, $n \in \mathbb{Z}$. The temperature is often

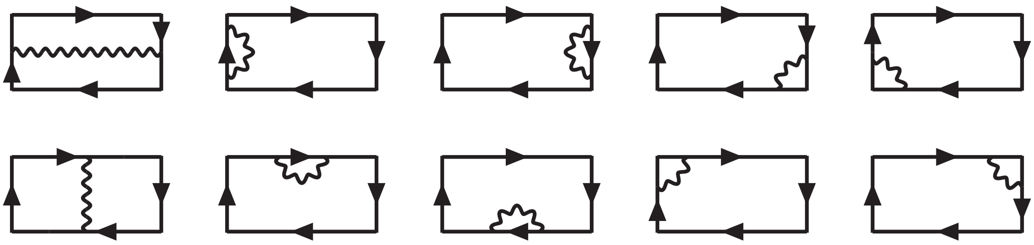


Figure 1: The graphs contributing to the static potential at $\mathcal{O}(g^2)$. Arrows indicate heavy quarks or Wilson lines, and wiggly lines stand for gluons.

expressed as $\beta \equiv T^{-1}$.

3.1. Wilson loop with Euclidean time direction

Let again $W[\tilde{z}_1; \tilde{z}_0]$ be a Wilson line from point $\tilde{z}_0$ to point $\tilde{z}_1$:

$$W[\tilde{z}_1; \tilde{z}_0] = \mathbb{1} + ig \int_{\tilde{z}_0}^{\tilde{z}_1} \mathrm{d}\tilde{x}_\mu A_\mu(\tilde{x}) + (ig)^2 \int_{\tilde{z}_0}^{\tilde{z}_1} \mathrm{d}\tilde{x}_\mu \int_{\tilde{z}_0}^{\tilde{x}} \mathrm{d}\tilde{y}_\nu \, A_\mu(\tilde{x}) A_\nu(\tilde{y}) + \dots \,, \tag{3.1}$$

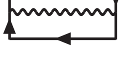
where $A_\mu = A_\mu^a T^a$ and T^a are the Hermitean generators of $\mathrm{SU}(N_\mathrm{c})$, normalised as $\mathrm{Tr}\,[T^a T^b] = \delta^{ab}/2$. The Euclidean correlation function considered is then defined as

$$C_E(\tau, \mathbf{r}) \equiv \frac{1}{N_\mathrm{c}} \mathrm{Tr} \Big\langle W[(0,\mathbf{r});(\tau,\mathbf{r})] \; W[(\tau,\mathbf{r});(\tau,\mathbf{0})] \; W[(\tau,\mathbf{0});(0,\mathbf{0})] \; W[(0,\mathbf{0});(0,\mathbf{r})] \Big\rangle \,, \tag{3.2}$$

where we have for convenience shifted the origin by $\mathbf{r}/2$ with respect to Eq. (2.6). The prefactor $1/N_\mathrm{c}$ has been inserted as a normalization, guaranteeing that $C_E(0,\mathbf{r}) = 1$.

We can formally expand C_E in a power series in the coupling constant g^2, understanding of course that the infrared problems of finite-temperature field theory necessitate the use of resummed propagators in order for this procedure to be valid (cf. Appendix B): $C_E = C_E^{(0)} + C_E^{(2)} + \dots$, where the superscript indicates the power of g appearing as a prefactor. The leading order result is trivial, $C_E^{(0)} = 1$. We now turn to the computation of $C_E^{(2)}$. The graphs entering at this order are shown in Fig. 1.

The computation of the graphs in Fig. 1 is not quite trivial. The problem is that in order to avoid ambiguous expressions (of the type "0/0"), one needs to treat the Matsubara zero-modes very carefully. In fact, we treat them separately from the non-zero modes. For the $\mathcal{O}(g^2)$-contribution to the first graph in Fig. 1, for instance, we obtain



$$\begin{aligned} &= g^2 C_F \int_0^\tau \mathrm{d}\tilde{x}_0 \int_0^\tau \mathrm{d}\tilde{y}_0 \, \sum\!\!\!\!\!\!\!\int_{\tilde{Q}} e^{i\tilde{q}_0(\tilde{x}_0 - \tilde{y}_0) - iq^3 r} \left[\frac{P_{00}^E(\tilde{Q})}{\tilde{Q}^2 + \Pi_E(\tilde{Q})} + \xi \frac{(\tilde{q}_0)^2}{(\tilde{Q}^2)^2} \right] \\ &= g^2 C_F \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \Big(e^{-iq^3 r} \Big) \bigg\{ \tau^2 T \, \frac{P_{00}^E(0,\mathbf{q})}{\mathbf{q}^2 + \Pi_E(0,\mathbf{q})} + \end{aligned}$$

$$+ \quad T \sum_{\tilde{q}_0 \neq 0} \frac{2 - e^{i\tilde{q}_0 \tau} - e^{-i\tilde{q}_0 \tau}}{\tilde{q}_0^2} \left[\frac{P_{00}^E(\tilde{Q})}{\tilde{Q}^2 + \Pi_E(\tilde{Q})} + \xi \frac{(\tilde{q}_0)^2}{(\tilde{Q}^2)^2} \right] \Bigg\} , \tag{3.3}$$

where we have inserted the gluon propagator from Eq. (B.3); defined $C_F \equiv (N_c^2 - 1)/2N_c$; and chosen the vector $\mathbf{r}$ to point in the x^3-direction: $\mathbf{r} \equiv (0, 0, r)$. It is perhaps appropriate to note that if one would rather attempt to keep all the Matsubara modes together, as an expression of the type on the latter row in Eq. (3.3), and then try to carry out the sum with the usual contour trick [23, 24], the zero-mode $\tilde{q}_0 = 0$ is not treated correctly, because of the ambiguity of the expression under $\tau \to \tau + n\beta$, $n \in \mathbb{Z}$.

Summing all the graphs together, the parts proportional to ξ, as well as the "longitudinal parts" $\propto \tilde{q}_\mu \tilde{q}_\nu$ from the terms multiplied by $P_{\mu\nu}^E$ (cf. Eq. (B.2)), cancel explicitly. Inserting finally the expressions of the projection operators from Eqs. (B.1), (B.2), we obtain

$$\begin{aligned} C_E^{(2)}(\tau, r) &= g^2 C_F \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{iq_3 r} + e^{-iq_3 r} - 2}{2} \Bigg\{ \frac{\tau^2 T}{\mathbf{q}^2 + \Pi_E(0, \mathbf{q})} + T \sum_{\tilde{q}_0 \neq 0} \left(2 - e^{i\tilde{q}_0 \tau} - e^{-i\tilde{q}_0 \tau} \right) \times \\ &\times \left[\left(\frac{1}{\tilde{q}_0^2} + \frac{1}{\mathbf{q}^2} \right) \frac{1}{\tilde{q}_0^2 + \mathbf{q}^2 + \Pi_E(\tilde{q}_0, \mathbf{q})} + \left(\frac{1}{q_3^2} - \frac{1}{\mathbf{q}^2} \right) \frac{1}{\tilde{q}_0^2 + \mathbf{q}^2 + \Pi_T(\tilde{q}_0, \mathbf{q})} \right] \Bigg\} . \end{aligned} \tag{3.4}$$

We have kept the spatial exponents in a form which makes it clear that the divergent-looking term $1/q_3^2$ is in fact harmless.

In order to carry out the remaining sum in Eq. (3.4), it is useful to write the resummed propagators in a spectral representation. For any function $\Delta(i\tilde{q}_0)$ which behaves regularly enough at infinity, we can define the spectral density $\rho(q^0)$ through

$$\rho(q^0) \equiv \frac{1}{2i} \left[\Delta(i\tilde{q}_0 \to q^0 + i0^+) - \Delta(i\tilde{q}_0 \to q^0 - i0^+) \right] , \tag{3.5}$$

which leads to the inverse transform

$$\Delta(i\tilde{q}_0) = \int_{-\infty}^{\infty} \frac{dq^0}{\pi} \frac{\rho(q^0)}{q^0 - i\tilde{q}_0} . \tag{3.6}$$

In particular,

$$\frac{1}{\tilde{q}_0^2 + \mathbf{q}^2 + \Pi_E(\tilde{q}_0, \mathbf{q})} = \int_{-\infty}^{\infty} \frac{dq^0}{\pi} \frac{\rho_E(q^0)}{q^0 - i\tilde{q}_0} , \tag{3.7}$$

and similarly with Π_T. The properties of the spectral functions ρ_T, ρ_E are described in Eqs. (B.7)–(B.13) of Appendix B.

Inserting the spectral representations into Eq. (3.4), and making use of Eq. (A.18), we can carry out the remaining sums: for $0 < \tau < \beta$,

$$\begin{aligned} T \sum_{\tilde{q}_0 \neq 0} \frac{2 - e^{i\tilde{q}_0 \tau} - e^{-i\tilde{q}_0 \tau}}{\tilde{q}_0^2 (q^0 - i\tilde{q}_0)} &= \frac{8T}{q^0} \sum_{n=1}^{\infty} \sin^2(\pi n T) \left[\frac{1}{(2\pi n T)^2} - \frac{1}{(2\pi n T)^2 + (q^0)^2} \right] \\ &= \frac{\tau(\beta - \tau)}{q^0 \beta} - \frac{n_B(q^0)}{(q^0)^2} \left[1 + e^{\beta q^0} - e^{\tau q^0} - e^{(\beta - \tau) q^0} \right] , \end{aligned} \tag{3.8}$$

$$T\sum_{\tilde{q}_0\neq 0}\frac{2-e^{i\tilde{q}_0\tau}-e^{-i\tilde{q}_0\tau}}{q^0-i\tilde{q}_0} = 8Tq^0\sum_{n=1}^{\infty}\sin^2(\pi nT)\frac{1}{(2\pi nT)^2+(q^0)^2}$$
$$= n_{\rm B}(q^0)\Big[1+e^{\beta q^0}-e^{\tau q^0}-e^{(\beta-\tau)q^0}\Big] , \tag{3.9}$$

where $n_{\rm B}(q^0)\equiv 1/[\exp(\beta q^0)-1]$. Noting furthermore that (cf. Eq. (3.7))

$$\int_{-\infty}^{\infty}\frac{{\rm d}q^0}{\pi}\frac{\rho_E(q^0)}{q^0}=\frac{1}{\mathbf{q}^2+\Pi_E(0,\mathbf{q})} , \tag{3.10}$$

we arrive at

$$\begin{aligned} C_E^{(2)}(\tau,r) &= g^2C_F\int\frac{{\rm d}^3\mathbf{q}}{(2\pi)^3}\frac{e^{iq_3r}+e^{-iq_3r}-2}{2}\bigg\{\frac{\tau}{\mathbf{q}^2+\Pi_E(0,\mathbf{q})}+\\ &+\int_{-\infty}^{\infty}\frac{{\rm d}q^0}{\pi}n_{\rm B}(q^0)\Big[1+e^{\beta q^0}-e^{\tau q^0}-e^{(\beta-\tau)q^0}\Big]\times\\ &\times\bigg[\bigg(\frac{1}{\mathbf{q}^2}-\frac{1}{(q^0)^2}\bigg)\rho_E(q^0,\mathbf{q})+\bigg(\frac{1}{q_3^2}-\frac{1}{\mathbf{q}^2}\bigg)\rho_T(q^0,\mathbf{q})\bigg]\bigg\} . \end{aligned}\tag{3.11}$$

It is useful to note that both the second and the third row in Eq. (3.11) are odd in q^0, so that the product is even, and the integral could also be restricted to positive values of q^0.

3.2. Equation of motion in Minkowski time

Eq. (3.11) has a form which can directly be analytically continued to Minkowski time, $\tau\to it$. Thereby we arrive at the object we are interested in, $C_>(t,r)$ (cf. Eqs. (A.2), (A.9)):

$$C_>^{(0)}(t,r) = 1 , \tag{3.12}$$

$$\begin{aligned} C_>^{(2)}(t,r) &= g^2C_F\int\frac{{\rm d}^3\mathbf{q}}{(2\pi)^3}\frac{e^{iq_3r}+e^{-iq_3r}-2}{2}\bigg\{\frac{it}{\mathbf{q}^2+\Pi_E(0,\mathbf{q})}+\\ &+\int_{-\infty}^{\infty}\frac{{\rm d}q^0}{\pi}n_{\rm B}(q^0)\Big[1+e^{\beta q^0}-e^{iq^0t}-e^{\beta q^0}e^{-iq^0t}\Big]\times\\ &\times\bigg[\bigg(\frac{1}{\mathbf{q}^2}-\frac{1}{(q^0)^2}\bigg)\rho_E(q^0,\mathbf{q})+\bigg(\frac{1}{q_3^2}-\frac{1}{\mathbf{q}^2}\bigg)\rho_T(q^0,\mathbf{q})\bigg]\bigg\} . \end{aligned}\tag{3.13}$$

We are now in a position to identify the heavy-quark potential. We define it as the quantity which determines the time behaviour of our correlator, in accordance with the Schrödinger equation:

$$i\partial_t C_>(t,r)\equiv V_>(t,r)C_>(t,r) . \tag{3.14}$$

Note that this $V_>(t,r)$ depends, in general, on the time t and on the time-ordering that is used for defining the correlator $C_>(t,r)$.

In perturbation theory, Eq. (3.14) needs to be fulfilled order by order. This implies that

$$i\partial_t C_>^{(0)}(t,r) = 0 , \tag{3.15}$$

$$i\partial_t C_>^{(2)}(t,r) = V_>^{(2)}(t,r)C_>^{(0)}(t,r) . \tag{3.16}$$

Eqs. (3.12), (3.13) then immediately lead to the identification

$$
\begin{aligned}
V^{(2)}_{>}(t,r) &= g^2 C_F \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{2 - e^{iq_3 r} - e^{-iq_3 r}}{2} \biggl\{ \frac{1}{\mathbf{q}^2 + \Pi_E(0,\mathbf{q})} + \\
&+ \int_{-\infty}^{\infty} \frac{\mathrm{d}q^0}{\pi} n_\mathrm{B}(q^0) q^0 \Bigl[e^{\beta q^0} e^{-iq^0 t} - e^{iq^0 t} \Bigr] \times \\
&\times \biggl[\biggl(\frac{1}{\mathbf{q}^2} - \frac{1}{(q^0)^2} \biggr) \rho_E(q^0,\mathbf{q}) + \biggl(\frac{1}{q_3^2} - \frac{1}{\mathbf{q}^2} \biggr) \rho_T(q^0,\mathbf{q}) \biggr] \biggr\} . \qquad (3.17)
\end{aligned}
$$

For $V^{(2)}_{<}(t,r)$, the analogue of Eq. (3.14) reads (cf. Eq. (2.5))

$$
i\partial_t C_{<}(t,r) \equiv -V_{<}(t,r) C_{<}(t,r) \; , \qquad (3.18)
$$

where the symmetry mentioned below Eq. (2.2), or an explicit computation with NRQCD, yield $V^{(2)}_{<}(t,r) = V^{(2)}_{>}(-t,r)$.

4. Discussion of the main results

We have shown in the previous section that the real-time "Wilson loop" $C_{>}(t,r)$, characterising the propagation of two infinitely heavy quarks separated by a distance r at a finite temperature T, satisfies Eq. (3.14), where the potential $V_{>}(t,r)$ is given in Eq. (3.17) to first non-trivial order in g^2. Let us now discuss the behaviour of $V_{>}(t,r)$ in various limits.

Perhaps the most interesting limit is what happens at large times, $t \to \infty$. In this limit, the exponential functions $\exp(\pm iq^0 t)$ average to zero, unless they are multiplied by a singular prefactor, $\sim 1/q^0$: then

$$
\lim_{t\to\infty} \frac{e^{iq^0 t} - e^{-iq^0 t}}{q^0} = 2\pi i \; \delta(q^0) \; . \qquad (4.1)
$$

Given that $n_\mathrm{B}(q^0) \approx 1/\beta q^0$ for $|q^0| \ll T$, and that $\rho_E(q^0,\mathbf{q})$ has a term linear in q^0, cf. Eq. (B.13), such a δ-function indeed emerges from the part containing $\rho_E(q^0,\mathbf{q})$. We obtain

$$
\begin{aligned}
\lim_{t\to\infty} V^{(2)}_{>}(t,r) &= g^2 C_F \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \Bigl(1 - e^{iq_3 r} \Bigr) \biggl\{ \frac{1}{\mathbf{q}^2 + m_\mathrm{D}^2} - \frac{i\pi m_\mathrm{D}^2}{\beta} \frac{1}{|\mathbf{q}|(\mathbf{q}^2 + m_\mathrm{D}^2)^2} \biggr\} \qquad (4.2) \\
&= -\frac{g^2 C_F}{4\pi} \biggl[m_\mathrm{D} + \frac{\exp(-m_\mathrm{D} r)}{r} \biggr] - \frac{ig^2 T C_F}{4\pi} \phi(m_\mathrm{D} r) \; , \qquad (4.3)
\end{aligned}
$$

where we have also made use of Eq. (B.11), and m_D is the Debye mass parameter. The r-independent constant in Eq. (4.3) has been evaluated in dimensional regularization, while the dimensionless function $\phi(x)$,

$$
\phi(x) = 2 \int_0^\infty \frac{\mathrm{d}z \, z}{(z^2+1)^2} \biggl[1 - \frac{\sin(zx)}{zx} \biggr] \; , \qquad (4.4)
$$

is finite and strictly increasing, with the limiting values $\phi(0) = 0$, $\phi(\infty) = 1$.[1]

We observe, thus, that apart from a standard Debye-screened static potential, the potential $V_>(t,r)$ also has an imaginary part. The sign is such that Eq. (3.14) leads to exponential decay for $C_>(t,r)$. The imaginary part obtained for $V_<(t,r)$ has the same magnitude but opposite sign, so that Eq. (3.18) leads to exponential decay as well. We will refer to the imaginary part as the "decay width".

In order to get a rough impression of the numerical orders of magnitude, let us solve the time-independent Schrödinger equation with the real part of Eq. (4.2):

$$\left[-\frac{1}{M}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} \right) - g^2 C_F \frac{\exp(-m_\mathrm{D} r)}{4\pi r} \right] \psi(r) = E\,\psi(r) \;, \tag{4.5}$$

where E denotes the binding energy. If M is very large, $g^2 C_F M/8\pi \gg m_\mathrm{D}$, Debye screening can be ignored, and we get the regular hydrogen atom solution, with an inverse Bohr radius $1/r_0 = g^2 C_F M/8\pi$, and a binding energy $|E_0| = M(g^2 C_F/8\pi)^2$. Treating the imaginary part, Γ, as a perturbation, $\Gamma \simeq (g^2 T C_F/4\pi) \int \mathrm{d}^3\mathbf{r}\, |\psi(r)|^2\, \phi(m_\mathrm{D} r)$, it is clear that $|\Gamma| \le g^2 T C_F/4\pi$, because $0 \le \phi(m_\mathrm{D} r) \le 1$; in fact, for $r_0 m_\mathrm{D} \ll 1$ as was the assumption, $|\Gamma| \ll g^2 T C_F/4\pi$. We thus note that for a very heavy quark mass M, the decay width can equal the binding energy only at temperatures $T \gg g^2 C_F M/16\pi$.

Lowering towards more realistic quark masses at a fixed temperature, r_0 increases, and at some point the Debye radius $1/m_\mathrm{D}$ becomes important. Consequently the binding energy decreases fast and, simultaneously, the width Γ increases, because the wave function spreads out, whereby $\phi(m_\mathrm{D} r)$ obtains non-zero values.

Unfortunately, it is non-trivial to make these arguments precise: to fix the scale parameter appearing in g^2 and m_D^2, another order in the perturbative expansion would be needed; once the width is substantial, it can no longer be treated as a perturbation; and the full time-dependence of $V_>^{(2)}(t,r)$ should be considered in order to determine the width reliably. Nevertheless, for illustration, we may fix the scales inside g^2 and m_D^2 by making use of another context where several orders are available, namely that of dimensionally reduced effective theories [25, 26, 27]. Concretely, employing simple analytic expressions that can be extracted from Ref. [28],

$$g^2 \simeq \frac{8\pi^2}{9\ln(9.082\,T/\Lambda_{\overline{\mathrm{MS}}})} \;, \quad m_\mathrm{D}^2 \simeq \frac{4\pi^2 T^2}{3\ln(7.547\,T/\Lambda_{\overline{\mathrm{MS}}})} \;, \qquad \text{for } N_\mathrm{c} = N_\mathrm{f} = 3 \;; \tag{4.6}$$

solving numerically the Schrödinger-equation in Eq. (4.5); and treating the imaginary part still as a perturbation, we obtain Fig. 2 for the bottomonium system. (For charmonium the relevant mass $M \simeq 1.25$ GeV is quite a bit smaller, and moves the patterns to smaller temperatures, where our methods are no longer reliable.) We stress that this figure is not meant to be quantitatively accurate, but it does illustrate something about the qualitative behaviour of the system: namely, that the width increases with temperature; the binding

[1] The latter part of the integral can be expressed as Meijer's G-function, with certain arguments.

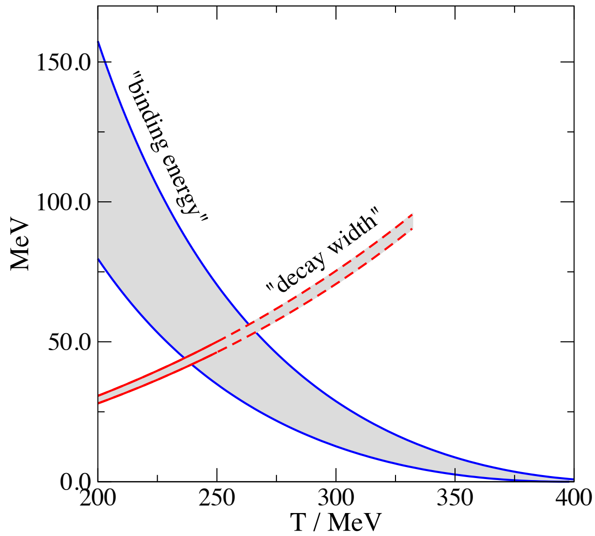


Figure 2: Very rough estimates for the binding energy and decay width of the bottomonium system ($M \simeq 4.25$ GeV), as a function of the temperature, based on Eqs. (4.2)–(4.6). The error bands have been obtained by varying $\Lambda_{\overline{\rm MS}}^{(3)}$ from 300 MeV (lower edges) to 450 MeV (upper edges). We expect the true errors to be much larger than these bands, for reasons discussed in the text.

energy decreases; and the width exceeds the binding energy already much before the latter disappears (i.e. bottomonium "melts").

Let us at this point inspect how the results would change in the *classical limit*.[2] For this, we need to reintroduce $\hbar$. After the standard rescaling of the fields, $\hbar$ appears in connection with the coupling constant, as $g^2\hbar$, as well as in the extent of the Euclidean time direction, as $\beta\hbar$. In the classical limit, therefore, $n_{\rm B}(q^0) \to 1/\beta\hbar q^0$.

We now note that $\hbar$ cancels from the imaginary part of Eq. (4.3), where the combination $g^2\hbar/\beta\hbar = g^2 T$ appears. This means that the decay width exists also in the classical limit. In fact, nothing but the decay width exists in the classical limit! Moreover, time-ordering has no meaning in the classical limit, which explains why the imaginary parts have the same magnitude in $V^{(2)}_{>}(t,r)$ and $V^{(2)}_{<}(t,r)$. That a decay width / damping rate should be classical, need not be surprising: particularly detailed demonstrations of this have been provided for scalar field theory [38].

Finally we remark that for $T \to 0$, i.e. $\beta \to \infty$, we can restrict to $q^0 > 0$ in Eq. (3.17) because of the symmetry, and then $n_{\rm B}(q^0) \to 0$, $n_{\rm B}(q^0)\exp(\beta q^0) \to 1$, whereby the structure of Eq. (3.17) simplifies. In particular, there is no imaginary part any more for $t \to \infty$, only the standard Coulomb-potential.

[2] This limit is of course singular in many respects, but can still be formally defined [29], and allows to address numerically many important non-perturbative problems in weakly-coupled gauge theories [30]–[35], in spite of the presence of significant discretization artifacts [36, 37] in the classical lattice gauge theory simulations [30] that are employed in the thermal context.

5. Conclusions and Outlook

The purpose of this note has been to present a derivation of a "static potential" that a heavy quark-antiquark pair propagating in a thermal medium feels. An integral representation of the result is shown in Eq. (3.17). This potential could then be inserted into Eq. (2.4), in order to estimate a certain real-time Green's function for the vector current, whose Fourier-transform determines the quarkonium contribution to the dilepton production rate. A detailed numerical evaluation is postponed to future work.

It is the first term in Eq. (3.17) which represents the standard time-independent Debye-screened potential. We note that this potential behaves as $-g^2 C_F \exp(-m_\mathrm{D} r)/4\pi r$ as a function of r. Thus it differs from the potential extracted from the correlator of two Polyakov loops in Euclidean spacetime, which behaves as $\sim -g^4 \exp(-2m_\mathrm{D} r)/4\pi r$ at phenomenologically relevant temperatures [11], and as $\sim -g^{4+n} \exp(-m_\mathrm{G} r)/4\pi r$, $n > 0$, at asymptotically high temperatures [6], where m_G is the lightest glueball mass of three-dimensional pure Yang-Mills theory [39], with the gauge coupling $g_3^2 = g^2 T[1 + \mathcal{O}(g)]$ [13, 40].

Motivated by our perturbative analysis, one can however envisage ways of defining a modified (real part of the) static potential, which would be gauge-invariant, possess the correct perturbative limit, and allow for a direct non-perturbative measurement with lattice simulations. Inspecting Eq. (3.11), the correct term is seen to sit on the first row. The corresponding functional dependence on τ differs from that on the second row in a qualitative way: for instance, it is linear in τ at $\tau \ll \beta$, while the second row is quadratic; and it is non-periodic in $\tau \to \beta - \tau$, while the second row is periodic. Both of these functional features can in principle also be isolated from non-perturbative data for the Euclidean Wilson loop; whether they lead to useful practical recipes for determining the real part of the static potential beyond perturbation theory, remains however to be tested.

Apart from the standard potential, we note that Eq. (3.17) also has another term, the second one, with a fairly rich structure. In the limit $t \to \infty$, the second terms amounts to a thermal decay width, induced by Landau damping of the low-frequency gauge fields that mediate interactions between the two heavy quarks. The thermal width induces a certain width also to the quarkonium peak in the dilepton production rate, thus making the peak much wider than at zero temperature. Physically, the Landau damping underlying this phenomenon originates from an energy transfer from low-frequency gauge fields that are responsible for the static interaction, to the "hard" particles (which have momenta of the order of the temperature) existing in the thermal plasma; technically, it originates from the "cut contribution" to the gluon spectral function at $|q^0| < |\mathbf{q}|$, cf. Eqs. (B.12), (B.13). Restricting to a very rough estimate, we have shown that this part leads to physically sensible qualitative structures, namely a width which increases with the temperature, exceeding the binding energy already quite a bit before the latter disappears and quarkonium "melts".

It is worth stressing, however, that apart from the limiting value at $t \to \infty$, the potential has a non-trivial time dependence, which also plays a role in the solution of Eq. (2.4) and, consequently, for the quarkonium contribution to the dilepton production rate.

In order to define the significance of these findings for heavy quarkonium, its finite mass M has to be taken into account systematically. For this one can make use of NRQCD [22], allowing to account analytically for all exponential effects like $\exp(-2\beta M)$, so that one can concentrate on the softer dynamics around the threshold. Perturbatively, this amounts to the solution of Eq. (2.4) with $V^{(2)}_{>}(t,r)$ inserted inside the round brackets. However, one might also be able to go beyond perturbation theory, by making use of the classical approximation for the gauge fields appearing in NRQCD, provided that genuinely quantum effects, such as the first term in Eq. (3.17), are properly taken into account. Hopefully this recipe could be formalised such that one would be able to account for the orders $(g^2\hbar)^0$ and $(g^2\hbar)^1$ in the "zero-temperature" expansion, but in each case for all orders in the possibly larger finite-temperature expansion parameters, g^2T/m_D and g^2T/m_G.

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Appendix A. Real-time Green's functions

For completeness and to fix the notation, we list in this Appendix some common definitions and relations that apply to two-point correlation functions built out of bosonic operators; for more details see, e.g., Refs. [23, 24].

We use the notation introduced at the beginning of Sec. 3, with $t \equiv x^0$ and $\tau \equiv \tilde{x}_0$. Arguments of operators denote implicitly whether we are in Minkowskian or Euclidean space-time. In particular, Heisenberg-operators are defined as

$$\hat{O}(t,\mathbf{x}) \equiv e^{i\hat{H}t}\hat{O}(0,\mathbf{x})e^{-i\hat{H}t} \;, \quad \hat{O}(\tau,\mathbf{x}) \equiv e^{\hat{H}\tau}\hat{O}(0,\mathbf{x})e^{-\hat{H}\tau} \;. \tag{A.1}$$

The thermal ensemble is defined by the density matrix $\hat{\rho} = \mathcal{Z}^{-1}\exp(-\beta\hat{H})$. We denote the operators which appear in the two-point functions by $\hat{\phi}^\mu(x)$, $\hat{\phi}^\nu(x)$; they could either be elementary field operators or composite operators.

We can now define various classes of correlation functions. The "physical" correlators are defined as

$$\tilde{C}^{\mu\nu}_{>}(Q) \equiv \int\!\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}\Big\langle \hat{\phi}^\mu(x)\hat{\phi}^\nu(0)\Big\rangle \;, \tag{A.2}$$

$$\tilde{C}^{\mu\nu}_{<}(Q) \equiv \int\!\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}\Big\langle \hat{\phi}^\nu(0)\hat{\phi}^\mu(x)\Big\rangle \;, \tag{A.3}$$

$$\rho^{\mu\nu}(Q) \equiv \int\!\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}\Big\langle \frac{1}{2}\Big[\hat{\phi}^\mu(x),\hat{\phi}^\nu(0)\Big]\Big\rangle \;, \tag{A.4}$$

$$\tilde{\Delta}^{\mu\nu}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \frac{1}{2} \Big\{ \hat{\phi}^\mu(x), \hat{\phi}^\nu(0) \Big\} \Big\rangle , \tag{A.5}$$

where $\rho^{\mu\nu}$ is called the spectral function, while the "retarded"/"advanced" correlators can be defined as

$$\tilde{C}^{\mu\nu}_R(Q) \equiv i \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \Big[\hat{\phi}^\mu(x), \hat{\phi}^\nu(0) \Big] \theta(t) \Big\rangle , \tag{A.6}$$

$$\tilde{C}^{\mu\nu}_A(Q) \equiv i \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle - \Big[\hat{\phi}^\mu(x), \hat{\phi}^\nu(0) \Big] \theta(-t) \Big\rangle . \tag{A.7}$$

On the other hand, from the computational point of view one is often faced with "time-ordered" correlation functions,

$$\tilde{C}^{\mu\nu}_T(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \hat{\phi}^\mu(x)\hat{\phi}^\nu(0)\theta(t) + \hat{\phi}^\nu(0)\hat{\phi}^\mu(x)\theta(-t) \Big\rangle , \tag{A.8}$$

which appear in time-dependent perturbation theory, or with the "Euclidean" correlator

$$\tilde{C}^{\mu\nu}_E(\tilde{Q}) \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\, e^{i\tilde{Q}\cdot\tilde{x}} \Big\langle \hat{\phi}^\mu(\tilde{x})\hat{\phi}^\nu(0) \Big\rangle , \tag{A.9}$$

which appears in non-perturbative formulations. Note that the Euclidean correlator is also time-ordered by definition, and can be computed with Euclidean functional integrals.

Now, all of the correlation functions defined can be related to each other. In particular, all correlators can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. In order to do this, we may first insert sets of energy eigenstates, to obtain the Fourier-space version of the so-called Kubo-Martin-Schwinger (KMS) relation: $\tilde{C}^{\mu\nu}_<(Q) = e^{-\beta q^0}\tilde{C}^{\mu\nu}_>(Q)$. Then $\rho^{\mu\nu}(Q) = [\tilde{C}^{\mu\nu}_>(Q) - \tilde{C}^{\mu\nu}_<(Q)]/2$ and, conversely,

$$\tilde{C}^{\mu\nu}_>(Q) = 2[1+n_\mathrm{B}(q^0)]\rho^{\mu\nu}(Q) , \quad \tilde{C}^{\mu\nu}_<(Q) = 2n_\mathrm{B}(q^0)\rho^{\mu\nu}(Q) . \tag{A.10}$$

Moreover, $\tilde{\Delta}^{\mu\nu}(Q) = [1+2n_\mathrm{B}(q^0)]\rho^{\mu\nu}(Q)$. Inserting the representation

$$\theta(t) = i\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i0^+} \tag{A.11}$$

into the definitions of $\tilde{C}_R$, $\tilde{C}_A$, we obtain

$$\tilde{C}^{\mu\nu}_R(Q) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho^{\mu\nu}(\omega,\mathbf{q})}{\omega - q^0 - i0^+} , \quad \tilde{C}^{\mu\nu}_A(Q) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho^{\mu\nu}(\omega,\mathbf{q})}{\omega - q^0 + i0^+} . \tag{A.12}$$

Doing the same with $\tilde{C}_T$ and making use of

$$\frac{1}{\Delta \pm i0^+} = P\Big(\frac{1}{\Delta}\Big) \mp i\pi\delta(\Delta) , \tag{A.13}$$

produces

$$\tilde{C}_T^{\mu\nu}(Q) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{i\rho^{\mu\nu}(\omega,\mathbf{q})}{q^0-\omega+i0^+} + 2\rho^{\mu\nu}(q^0,\mathbf{q})n_\mathrm{B}(q^0) \; . \tag{A.14}$$

Finally, writing the argument inside the τ-integration in Eq. (A.9) as a Wick rotation of the integrand in Eq. (A.2), which in turn is expressed as an inverse Fourier transform of $\tilde{C}_>^{\mu\nu}(Q)$, for which Eq. (A.10) is inserted, and changing orders of integration, we get

$$\tilde{C}_E^{\mu\nu}(\tilde{Q}) \;=\; \int_0^\beta \mathrm{d}\tau \, e^{i\tilde{q}_0\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-\omega\tau} \tilde{C}_>^{\mu\nu}(\omega,\mathbf{q}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho^{\mu\nu}(\omega,\mathbf{q})}{\omega - i\tilde{q}_0} \; . \tag{A.15}$$

This relation can formally be inverted by making use of Eq. (A.13),

$$\rho^{\mu\nu}(q^0,\mathbf{q}) = \frac{1}{2i}\Big[\tilde{C}_E^{\mu\nu}(-i[q^0+i0^+],\mathbf{q}) - \tilde{C}_E^{\mu\nu}(-i[q^0-i0^+],\mathbf{q})\Big] \; . \tag{A.16}$$

We also recall that bosonic Matsubara sums can be carried out through

$$T\sum_{\tilde{q}_0} \frac{i\tilde{q}_0 c + d}{\tilde{q}_0^2+E^2} e^{i\tilde{q}_0\tau} \;\equiv\; (c\partial_\tau + d) T\sum_{\tilde{q}_0} \frac{e^{i\tilde{q}_0\tau}}{\tilde{q}_0^2+E^2} \tag{A.17}$$

$$= \frac{n_\mathrm{B}(E)}{2E}\Big[(-cE+d)e^{(\beta-\tau)E} + (cE+d)e^{\tau E}\Big] \; , \tag{A.18}$$

where $\tilde{q}_0 = 2\pi n T$, with n an integer, and we assumed $0 < \tau < \beta$. This equation can be used as a starting point for determining the sums in Eqs. (3.8), (3.9).

Appendix B. Resummed gluon propagator

Introducing the projection operators (see, e.g., Refs. [23, 24])

$$P_{00}^T(\tilde{Q}) \;=\; P_{0i}^T(\tilde{Q}) = P_{i0}^T(\tilde{Q}) \equiv 0 \; , \quad P_{ij}^T(\tilde{Q}) \equiv \delta_{ij} - \frac{\tilde{q}_i\tilde{q}_j}{\tilde{\mathbf{q}}^2} \; , \tag{B.1}$$

$$P_{\mu\nu}^E(\tilde{Q}) \;\equiv\; \delta_{\mu\nu} - \frac{\tilde{q}_\mu\tilde{q}_\nu}{\tilde{Q}^2} - P_{\mu\nu}^T(\tilde{Q}) \; , \tag{B.2}$$

the Euclidean gluon propagator can be written as

$$\langle A_\mu^a(\tilde{x})A_\nu^b(\tilde{y})\rangle = \delta^{ab} \SumInt_{\tilde{Q}} e^{i\tilde{Q}\cdot(\tilde{x}-\tilde{y})} \left[\frac{P_{\mu\nu}^T(\tilde{Q})}{\tilde{Q}^2+\Pi_T(\tilde{Q})} + \frac{P_{\mu\nu}^E(\tilde{Q})}{\tilde{Q}^2+\Pi_E(\tilde{Q})} + \xi\frac{\tilde{q}_\mu\tilde{q}_\nu}{(\tilde{Q}^2)^2}\right] , \tag{B.3}$$

where ξ is the gauge parameter. The Hard Thermal Loop [41, 42] contributions read

$$\Pi_T(\tilde{Q}) \;=\; \frac{m_\mathrm{D}^2}{2}\left\{\frac{(i\tilde{q}_0)^2}{\tilde{\mathbf{q}}^2} + \frac{i\tilde{q}_0}{2|\tilde{\mathbf{q}}|}\left[1 - \frac{(i\tilde{q}_0)^2}{\tilde{\mathbf{q}}^2}\right]\ln\frac{i\tilde{q}_0+|\tilde{\mathbf{q}}|}{i\tilde{q}_0-|\tilde{\mathbf{q}}|}\right\} , \tag{B.4}$$

$$\Pi_E(\tilde{Q}) \;=\; m_\mathrm{D}^2\left[1 - \frac{(i\tilde{q}_0)^2}{\tilde{\mathbf{q}}^2}\right]\left[1 - \frac{i\tilde{q}_0}{2|\tilde{\mathbf{q}}|}\ln\frac{i\tilde{q}_0+|\tilde{\mathbf{q}}|}{i\tilde{q}_0-|\tilde{\mathbf{q}}|}\right] , \tag{B.5}$$

where $\tilde{q}_0$ denotes bosonic Matsubara frequencies, and

$$m_{\rm D}^2 = g^2 T^2 \left(\frac{N_{\rm c}}{3} + \frac{N_{\rm f}}{6} \right) . \tag{B.6}$$

In the limit $i\tilde{q}_0 \to 0$ but with $|\tilde{\mathbf{q}}| \neq 0$, $\Pi_T \to 0$, $\Pi_E \to m_{\rm D}^2$, while for $|\tilde{\mathbf{q}}| \to 0$ with $i\tilde{q}_0 \neq 0$, Π_T, $\Pi_E \to m_{\rm D}^2/3$.

After analytic continuation, $i\tilde{q}_0 \to q^0 + i0^+$, the propagators become

$$\frac{1}{\tilde{Q}^2 + \Pi_{T(E)}(\tilde{q}_0, \tilde{\mathbf{q}})} \to \frac{1}{-(q^0+i0^+)^2 + \mathbf{q}^2 + \Pi_{T(E)}(-i(q^0+i0^+), \mathbf{q})} , \tag{B.7}$$

where

$$\Pi_T(-i(q^0+i0^+), \mathbf{q}) = \frac{m_{\rm D}^2}{2} \left\{ \frac{(q^0)^2}{\mathbf{q}^2} + \frac{q^0}{2|\mathbf{q}|} \left[1 - \frac{(q^0)^2}{\mathbf{q}^2} \right] \ln \frac{q^0 + i0^+ + |\mathbf{q}|}{q^0 + i0^+ - |\mathbf{q}|} \right\} , \tag{B.8}$$

$$\Pi_E(-i(q^0+i0^+), \mathbf{q}) = m_{\rm D}^2 \left[1 - \frac{(q^0)^2}{\mathbf{q}^2} \right] \left[1 - \frac{q^0}{2|\mathbf{q}|} \ln \frac{q^0 + i0^+ + |\mathbf{q}|}{q^0 + i0^+ - |\mathbf{q}|} \right] . \tag{B.9}$$

For $|q^0| > |\mathbf{q}|$, Π_T, Π_E are real. For $|q^0| < |\mathbf{q}|$, they have an imaginary part. In particular, for $|q^0| \ll |\mathbf{q}|$, we get

$$\Pi_T \approx \frac{m_{\rm D}^2}{2} \left\{ -i\pi \frac{q^0}{2|\mathbf{q}|} + 2 \frac{(q^0)^2}{\mathbf{q}^2} + ... \right\} , \tag{B.10}$$

$$\Pi_E \approx m_{\rm D}^2 \left\{ 1 + i\pi \frac{q^0}{2|\mathbf{q}|} - 2 \frac{(q^0)^2}{\mathbf{q}^2} + ... \right\} . \tag{B.11}$$

The Hard Thermal Loop resummation of course only describes the behaviour correctly for $|\mathbf{q}| \sim gT$; for instance, for $|\mathbf{q}| \sim g^2 T$, loop corrections within the Hard Thermal Loop effective theory are large, and the small-frequency behaviour of the full Π_T is determined by a "colour conductivity" rather than Eq. (B.10): $\Pi_T \sim -i\sigma q^0$, where $\sigma \sim T/\ln(1/g)$ [43, 44].

The spectral functions ρ_T, ρ_E that are needed in the text follow by taking the imaginary part, or discontinuity (cf. Eq. (3.5)), of the right-hand side of Eq. (B.7). In particular,

$$\rho_T(q^0, \mathbf{q}) = \left\{ \begin{array}{ll} \pi \, {\rm sign}(q^0) \delta\Big((q^0)^2 - \mathbf{q}^2 - {\rm Re}\, \Pi_T \Big) , & |q^0| > |\mathbf{q}| \\ \pi m_{\rm D}^2 \dfrac{q^0}{4|\mathbf{q}|^5} , & |q^0| \ll |\mathbf{q}| \end{array} \right. , \tag{B.12}$$

$$\rho_E(q^0, \mathbf{q}) = \left\{ \begin{array}{ll} \pi \, {\rm sign}(q^0) \delta\Big((q^0)^2 - \mathbf{q}^2 - {\rm Re}\, \Pi_E \Big) , & |q^0| > |\mathbf{q}| \\ -\pi m_{\rm D}^2 \dfrac{q^0}{2|\mathbf{q}|(\mathbf{q}^2 + m_{\rm D}^2)^2} , & |q^0| \ll |\mathbf{q}| \end{array} \right. . \tag{B.13}$$

Again, "soft" loop corrections to these expressions are large for ultrasoft momenta, $|\mathbf{q}| \sim g^2 T$. Moreover, loop corrections are also significant around the plasmon poles, i.e. the δ-functions in Eqs. (B.12), (B.13), where a finite plasmon decay width gets generated [45].

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