

# **REAL- AND IMAGINARY-TIME FIELD THEORY AT FINITE TEMPERATURE AND DENSITY**

**N.P. LANDSMAN and Ch.G. van WEERT**

*Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam,  
The Netherlands*



**NORTH-HOLLAND – AMSTERDAM**

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**Abstract:**

This report gives a detailed account of relativistic quantum field theory in the grand canonical ensemble. Three approaches are discussed: traditional Euclidean Matsubara, and two recently developed real-time methods, namely, Minkowskian time-path and thermo field dynamics. The first two formulations are derived in a unified manner from the path-integral representation for the contour-ordered generating functional. Fields with spin and gauge fields, in particular, are included. Zero-temperature renormalizability is shown to imply UV finiteness at any temperature and density. Thermo field dynamics, which is basically an operator theory, is presented in a  $C^*$ -algebraic context. Relevant parts of the HHW formalism of quantum statistical mechanics, and the Tomita-Takesaki theory are explained. The next chapter contains an analysis of the structure and analytic properties of the self-energy and its relation to the full propagator. In this connection the concept of a statistical quasiparticle is briefly described. This is followed by a discussion of thermal WT identities. Results are applied to discuss transversality of the  $SU(N)$  gluon polarization tensor. The final chapter deals with the diagrammatic rules for evaluating the pressure and energy density. The energy-momentum tensor is analyzed as a composite operator, and a renormalized virial theorem is established to provide the link with the thermodynamic potential. The pressure of the  $SU(N)$  chromoplasma is calculated up to third order.

## 1. Introduction

Since the first development of relativistic many-body theory [2.1–2.8] the subject has now reached maturity, and has grown into what is known as field theory at finite temperature and density (FTFT), or statistical field theory. The present interest in this amalgamation of field theory and statistical mechanics springs from the realization that many problems encountered experimentally and theoretically in particle physics have many-body aspects [6.21]. A pertinent example is the deconfinement phase transition that is expected to occur in hadronic matter at a density of about  $1\text{ fm}^{-3}$  [6.1–6.24]. This quark-liberating transition is predicted by lattice QCD calculations to occur at temperatures in the range 150–250 MeV, or at lower temperatures provided the chemical potential is sufficiently large [6.16, 6.18, 6.22]. In nature the quark-gluon plasma may have existed in the early universe, and the phase transition may have played a role in the formation of dark matter [6.12, 6.13, 6.19]. It is conceivable that it is formed in the collapse of heavy stars [6.1, 6.3, 6.4]. The hope even exists that small droplets of short-lived quark-gluon plasma may be created in heavy-ion collision experiments in the near future [6.4, 6.8, 6.14].

Seen from a more general perspective, it has been the possibility of phase transitions in spontaneously broken gauge theories at finite temperature [2.3, 2.6, 2.7, 7.3, 7.19] that has been responsible for the increased interest in FTFT since the early seventies. This observation lies at the basis, for example, of the inflationary scenario [7.17, 7.20], and recently much discussion has concerned superstring cosmology in which the old idea of a limiting temperature has been revived [7.25, 7.27]. Furthermore, it may well be that thermal effects have a profound influence on the stability structure of field theory [7.7, 7.26, 7.28, 7.31]. Chern-Simons mass terms [7.12, 7.23] may be induced by thermal effects, and the remarkable analogy with curved space effects has been appreciated for some time now [9.21, 12.23, 8.13]. Should things come to unfathomed depths, 't Hooft's density-matrix theory of black holes may be quoted [12.24].

In the present report we shall be concerned with the basic formal content of FTFT and its perturbative structure from a functional point of view. On this level two modes of description may be distinguished. Historically the oldest, and most often used, is the imaginary-time (Euclidean) formalism. It is based on the formal analogy, first noticed by F. Bloch [1.1], between inverse temperature and imaginary time, and it leads to so-called temperature Green functions with purely imaginary time arguments. That is, one works in Euclidean space. The formalism was developed by many authors [1.3–1.33, 2.1, 2.2], but is usually named after Matsubara, who was the first to set up a diagrammatic perturbation theory for the grand partition function on a field-theoretic basis [1.3]. Important steps in

the further development of this many-body theory at finite temperature were the introduction of a Fourier representation [2.1, 1.9, 1.12] (cf. [2.22] for history) and the functional formulation of the theory [1.2, 1.26, 2.5, 2.17].

In Fourier language, the Matsubara formalism involves discrete complex energies, the so-called Matsubara frequencies [2.1], which appear both on internal and external lines of Feynman diagrams. The internal energies have to be summed over for which purpose several standard algorithms are available [1.20, 1.22, 1.23, 6.2–6.4, 2.8, 2.14, 2.15, 10.10]. The external energies, on the other hand, pose a problem because they define the Green functions at a discrete set of points in the complex energy plane. But in order to answer dynamical questions, a knowledge of real-time Green functions is usually indispensable. This implies that the temperature Green functions have to be extended from the discrete energies to the real axis. In principle [1.9, 1.12, 1.18, 1.30, 7.15], this extension can be obtained by a process of analytic continuation, but in the case of several complex variables this is a mathematically difficult task. Moreover, even in the case of a single external energy, one is confronted with the problem that such an analytic extension is not unique without further delimitations. This should not be entirely entered on the debit side, as the inherent freedom of analytic continuation can be exploited to define new concepts such as that of a statistical quasiparticle [10.4, 10.5, 10.10, 10.13, 10.14, 10.15]. Furthermore, the formalism is well suited to the evaluation of static thermodynamic properties, i.e. the thermodynamic potential [1.3, 1.25, 1.30, 2.8, 2.14, 2.15]. The basic disadvantage of the Matsubara formalism lies in the unphysical representation of time and energy. Moreover, systematically performing the frequency sums one finds a proliferation of so-called vertex-ordered diagrams [1.20, 1.23, 1.25, 2.8].

Analytic continuation can be avoided by a different approach to FTFT which is known as the Minkowski-space or time-path formalism. This approach can be traced back to the early work of Schwinger [3.1], Keldysh [3.2] and others [3.3–3.7] on non-equilibrium quantum statistics. It is based on the concept of a closed time contour in the complex plane running parallel to the real-time axis and back. It involves the use of both time- and anti-time-ordered Green functions and gives rise to an effective doubling of the degrees of freedom. Specializing the time-path method to equilibrium, Niemi and Semenoff [3.11, 3.12] have recently developed a transparent path-integral formulation of FTFT allowing the computation of Green functions directly as functions of continuous real energy variables. Perturbative calculations and Feynman integrals are very similar to those in vacuum field theory [4.13, 3.12], and one of the virtues of the formalism is that the whole renormalization machinery [4.13, 3.12, 3.16], and the renormalization group equations [8.5, 8.15, 4.40, 8.21, 8.23], in particular, are easily incorporated. The doubling of degrees of freedom alluded to above leads to a  $2 \times 2$  matrix structure of propagators and self-energies. This feature implies that in higher orders a much larger number of diagrams has to be taken into account as compared to the vacuum theory; cf. [4.13, 4.12, 8.8, 8.9–8.11, 8.19]. However, we noticed already that a similar proliferation occurs in a systematic treatment of the Matsubara formalism. Perhaps a more serious drawback of the real-time method lies in the occurrence of certain singularities in intermediate stages of calculation. If one is careful enough, however, and properly regularizes the theory, these pathologies are guaranteed to cancel out among themselves in the end [4.13, 3.12, 4.12, 8.8]. Early attempts at real-time techniques ignored the aforementioned matrix structure, which led to hopeless difficulties [2.6].

The time-path approach to FTFT is tailor-made for the calculation of Green functions. Although in principle the full theory may be reconstructed from these quantities, an operator formulation of FTFT might be valuable for both practical and theoretical purposes. In the first place, operator techniques like canonical transformations would then become available, and secondly it might lead to a deeper

understanding of FTFT. In particular, it might explain why a doubling of the degrees of freedom is necessary in the real-time formalism but not in the Matsubara formalism. Such an operator formulation was developed by Umezawa in the early seventies [4.1, 4.2] (cf. [4.17, 4.31]). The first complete version was laid down in a seminal paper by Takahashi and Umezawa [4.3], who coined the name thermo field dynamics (TFD). The central idea is the doubling of the Hilbert space of states. This permits thermal averages, which are traces over Fock space in the ordinary formulation, to be written as expectation values with respect to a single state, called the thermal vacuum. The operators on this doubled space are effectively doubled in number as well. Within this framework quantum field theory can naturally be extended to finite temperature without having to consider imaginary time.

Subsequently, thermo field dynamics was developed extensively [4.4–4.18] and applied to problems in condensed matter physics [4.7] as well as high-energy physics [8.1–8.32]. Later it also became clear [4.4, 4.6] that a close relationship exists with the so-called Haag–Hugenholtz–Winnink (HHW) formulation of quantum statistical mechanics [5.3]. Here the doubling of the degrees of freedom, appearing for the first time in the work of Araki and Woods on the ideal Bose gas [5.1], naturally emerges. Moreover, it could be shown that TFD leads to the same Feynman rules as a particular variant of the time-path method [4.9, 4.14].

In presenting FTFT in this report we shall not follow the historical development as sketched above. Instead, we opt for a unified derivation of both imaginary- and real-time formalisms based on the contour-dependent Feynman–Matthews–Salam path-integral formula [12.1, 12.2, 1.26, 3.11, 3.16]. Since the various topics, as listed in the table of contents, are concisely discussed at the beginning of each section, we will be brief here. In chapter 2 the rules for writing down the generating functional will be derived. Higher-spin and massless gauge fields will be examined, and special attention will be paid to some subtleties that have not always been fully appreciated in the literature. In the last section of this chapter we shall derive thermo field dynamics from the HHW formalism, emphasizing the role of the thermodynamic limit, and relevant parts of the underlying Tomita–Takesaki theory [5.5, 5.7] will be presented. (One may entertain the hope that this conceptual framework could have some bearing on 't Hooft's theory mentioned earlier; indeed, the possible relevance of TFD to black-hole thermodynamics has already been recognized as early as 1976 by Israel [12.15].)

As it will turn out, in chapter 3, the full many-body propagator and self-energy obtained in the real-time method, on the one hand, and in the imaginary-time method, on the other, are intimately related. Having discussed this relationship, we give some illustrative examples of real-time calculations for the  $SU(N)$  gauge theory. For this case we also discuss symmetry, BRS invariance, and Slavnov–Taylor identities.

Chapter 4, which is mainly technical in content, is devoted to the calculation of the thermodynamic potential, that is, the thermodynamic pressure. As is often stated, the Matsubara formalism is well suited for this purpose. However, in the case of an infrared (IR) singular Yang–Mills theory the trustworthiness of the perturbative expressions is open to doubt. As a case in point we discuss the plasmon effect in  $SU(N)$  [10.11, 10.15]. In the real-time method the thermodynamic potential is less directly accessible, but in perturbation theory the interaction part can be computed straightforwardly enough [4.21]. Identification with the hydrostatic pressure necessitates a detailed discussion of the renormalized thermal energy–momentum tensor, and the virial theorem [3.16].

A number of important topics had to be left out for lack of space, although we do provide basic references. The first is the effective potential at finite temperature and density, which determines the phase structure of a (gauge) field theory. Literature in which the Matsubara technique is employed is abundant [2.6, 7.3, 7.5, 7.6, 7.20]. It mostly uses Jackiw's elegant method of computing the effective

action [12.13, 12.12, 12.20]. This procedure is not directly applicable in real time, yet a slightly modified version [8.10] can be shown [4.35, 4.34] to do the job [8.8–8.11]. In addition, the somewhat less direct tadpole method [12.10, 2.7, 12.14] can be used [3.12, 8.10, 8.24]. Furthermore, we refer to the literature for the field-theoretic derivation of kinetic equations [1.21, 1.29, 3.2, 3.9, 3.10, 11.14, 3.13, 3.21], and the recently developed subject of non-equilibrium thermo field dynamics [4.19, 4.24, 4.26–4.30, 4.36–4.41] which may shed new light on the origin of dissipative structures [4.27] even at zero temperature [4.43].

Our references have been grouped according to subject first, then to year of publication, and finally to alphabet. Completeness has not been strived for, except for papers on foundations and relativistic applications of thermo field dynamics, and on the equilibrium time-path method for which as yet no bibliography was available.

### Notation

$\hbar = c = k_B = 1$	
$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$	Minkowski metric
$\tilde{f}$	Fourier transform
$n_{\pm}(p_0)$	(anti-)particle distribution function, eq. (2.2.34)
$N(k_0)$	Bose distribution function, eq. (2.3.24)
$\rho_0$	spectral density, eq. (2.2.35)
$d_{\alpha\beta}$	Klein–Gordon divisor, eq. (2.2.30)
Bare propagators:	
$G_{\text{in}}^{(\alpha\beta)}$ , $\alpha, \beta = \pm$	Keldysh, section 2.1.3
$D^{(c)}, D_g^{(c)}, D_{\mu\nu}^{(c)}, S^{(c)}, D_{\alpha\beta}^{(c)}$	contour (scalar, ghost, vector, Dirac, generic), section 2.2.2
$D^{(E)}$	Euclidean, section 2.3.1
$\Delta$	Matsubara, section 2.3.1
$D^{(rs)}$ , etc., $r, s = 1, 2$	real time, section 2.4.2
$\Delta_F$	vacuum, section 2.4.2
Full propagators (Green functions):	
$G^{(\alpha\beta)}$ , $\alpha, \beta = \pm$	Keldysh, section 2.1.3
$G^{(c)}$	contour, section 2.1.2
$G^{(E)}$	Euclidean, section 2.3.2
$\mathcal{G}$	temperature, section 3.1.1
$G^{(rs)}$ , $r, s = 1, 2$	real time, section 3.2.1
$\Delta'$	analytic, section 3.1.1
$\Delta'_F$	Feynman, section 3.1.2.

## 2. Finite-temperature field theory

### 2.1. Green functions

The Green function method for treating many-particle systems was developed in the fifties in close analogy to the formalism of relativistic quantum field theory; a key reference in this context is the

article of Martin and Schwinger [1.14]. With regard to large systems, the physical quantities of interest naturally fall into two categories. The first one concerns the behaviour of extensive thermodynamic variables whose bulk changes are measured. The second one refers to basic dynamical properties of the system such as the local response to external disturbances. Although the Green functions are most suited to a study of the latter kind of problems, they can also be used to obtain the thermodynamics of the system in perturbation theory by an integration with respect to the coupling constant [1.6, 1.11].

Before describing the Green function formalism, we find it useful to review some thermodynamic relations in the context of the grand-canonical ensemble. We then define the thermal Green functions and introduce the associated generating functional which will form the basis for the further development of the theory.

Finally, we shall make an excursion and briefly discuss the non-equilibrium Green function techniques initiated by Schwinger [3.1], Kadanoff and Baym [1.21] and Keldysh [3.2], and further developed by many others [1.29, 3.3–3.9, 11.14, 3.21]; for an extensive review see ref. [3.13]. The non-equilibrium theory has many features in common with the real-time formulation for equilibrium systems, as will become clear in later sections. Also, it may be helpful in elucidating the distinction between the statistical and dynamical aspects of the theory.

### 2.1.1. Grand-canonical ensemble

We shall consider a dynamical system characterized by a time-independent Hamiltonian  $\hat{H}$ , and a number of conserved and mutually commuting charges  $\hat{Q}_A$ ,  $A = 1, 2, \dots$ . (The circumflex indicates a quantum-mechanical operator in the Heisenberg picture.) The equilibrium state of the system at rest in a large volume  $V$  is described by the grand-canonical density operator

$$\hat{\rho} = \exp\left(-\Phi - \sum_A \alpha_A \hat{Q}_A - \beta \hat{H}\right), \quad (2.1.1)$$

where

$$\Phi(\alpha_A, \beta, V) = \log \text{Tr} \exp\left(-\sum_A \alpha_A \hat{Q}_A - \beta \hat{H}\right) \quad (2.1.2)$$

is a Massieu function, that is, a Legendre transform of the entropy [11.1]. The Lagrange multipliers  $\alpha_A$ ,  $\beta$  are related to the temperature  $T$  and the independent chemical potentials  $\mu_A$  through  $\beta = T^{-1}$ ,  $\alpha_A = -\beta \mu_A$ , and are conjugate to the charge densities  $N_A$  and the energy density  $E$  defined as

$$N_A = \frac{1}{V} \langle \hat{Q}_A \rangle = -\frac{1}{V} \frac{\partial \Phi}{\partial \alpha_A}, \quad (2.1.3)$$

$$E = \frac{1}{V} \langle \hat{H} \rangle = -\frac{1}{V} \frac{\partial \Phi}{\partial \beta}, \quad (2.1.4)$$

where the brackets indicate the grand-canonical average.

Since the Massieu function (2.1.2) is an extensive quantity we may define the potential density

$$P(T, \mu_A) = \frac{1}{\beta V} \Phi(\alpha_A, \beta, V), \quad (2.1.5)$$

which may be identified as the thermodynamic pressure on account of the variational relation

$$\delta(\beta P) = - \sum_A N_A \delta\alpha_A - E \delta\beta , \quad (2.1.6)$$

which is an immediate consequence of the above formulae. The additivity of the entropy  $SV = -\langle \log \hat{\rho} \rangle$  provides us with the Euler relation

$$S = \beta P + \beta E + \sum_A \alpha_A N_A , \quad (2.1.7)$$

which expresses the entropy density  $S$  in terms of the other thermodynamic variables. (The Euler relation may not hold for gravitational and other long-range forces [11.15], but this case will be excluded here.) By combining (2.1.6) and (2.1.7) we get the Gibbs–Duhem relation

$$\delta P = S \delta T + \sum_A N_A \delta\mu_A , \quad (2.1.8)$$

which shows that the thermodynamic pressure, with  $T$  and  $\mu_A$  as independent variables, is a characteristic function from which other thermodynamic quantities may be obtained by differentiation [11.9].

The rules of thermodynamics do not rely on non-relativistic conditions. If so desired the various thermodynamic identities may be cast in covariant form by treating the macroscopic four-velocity  $U^\mu$  of the system on a par with the temperature  $T$  and the chemical potentials  $\mu_A$  [11.11, 7.11, 11.18, 3.12].

### 2.1.2. Generating functional

For simplicity's sake we first consider the case of a Hermitian scalar Bose field  $\hat{\phi}(x)$ , carrying no conserved charges, whose dynamics is governed by the Hamiltonian  $\hat{H}$ . We define

$$\hat{\phi}(x) = e^{i\hat{H}\tau} \phi(0, x) e^{-i\hat{H}\tau} , \quad (2.1.9)$$

where the time coordinate  $x^0 = t$  may be complex. Observable properties of the system can be extracted from a knowledge of the thermal Green functions

$$G^{(c)}(x_1, x_2, \dots, x_n) := \langle T_c \hat{\phi}(x_1) \hat{\phi}(x_2) \cdots \hat{\phi}(x_n) \rangle , \quad (2.1.10)$$

defined as the statistical average of a product of Heisenberg fields. The ordering instruction  $T_c$  orders the operators along a given path  $C$  in the complex time plane [3.1, 3.7]. That is, it prescribes that the operators it is applied to be rearranged in the order in which their arguments lie along the oriented contour  $C$ , with those nearest to the beginning at the right and those nearest to the end to the left. If the contour  $C$  is parametrically given as a function  $t = z(\tau)$  with  $\tau$  real and monotonically increasing, the  $T_c$  operation is nothing but standard time ordering with respect to  $\tau$ .

Contour ordering can be formalized by introducing a contour step function and a contour Dirac delta function according to [3.11]

$$\theta_c(t - t') = \theta(\tau - \tau') , \quad (2.1.11)$$

$$\delta_c(t - t') = \left( \frac{\partial z}{\partial \tau} \right)^{-1} \delta(\tau - \tau'). \quad (2.1.12)$$

With these definitions we may write, for example,

$$T_c \hat{\phi}(x) \hat{\phi}(x') = \theta_c(t - t') \hat{\phi}(x) \hat{\phi}(x') + \theta_c(t' - t) \hat{\phi}(x') \hat{\phi}(x), \quad (2.1.13)$$

$$\partial_t T_c \hat{\phi}(x) \hat{\phi}(x') = \delta_c(t - t') [\hat{\phi}(x), \hat{\phi}(x')] + T_c \partial_t \hat{\phi}(x) \hat{\phi}(x'). \quad (2.1.14)$$

Functional differentiation can also be extended in a straightforward manner,

$$\frac{\delta j(x')}{\delta j(x)} = \delta_c(t - t') \delta(x - x'), \quad (2.1.15)$$

for c-number functions  $j(x)$  living on the contour.

We now seek a generating functional  $Z[j]$  such that the thermal Green functions (2.1.10) are reproduced by functional differentiation with respect to the c-number sources  $j(x)$ :

$$G^{(c)}(x_1, \dots, x_n) = \frac{1}{Z[0]} \frac{\delta^n Z[j]}{i \delta j(x_1) \cdots i \delta j(x_n)} \Big|_{j=0}. \quad (2.1.16)$$

Obviously, a generating functional that has this property is

$$Z[j] = Z[0] \left\langle T_c \exp \left[ i \int_C d^4x j(x) \hat{\phi}(x) \right] \right\rangle. \quad (2.1.17)$$

The contour  $C$  must go through the time arguments of all Green functions we wish to know. The arbitrary normalization factor will be set equal to the partition function:  $Z[0] = \exp \Phi(\beta, V) = \text{Tr} \exp -\beta \hat{H}$ .

At this point we note that not all contours  $C$  are allowed if we require the Green functions to be analytic with respect to their time arguments [3.7]. Consider, for example, the two-point Green function

$$G^{(c)}(x, x') = \theta_c(t - t') C^>(x, x') + \theta_c(t' - t) C^<(x, x'), \quad (2.1.18)$$

where

$$C^>(x, x') = \langle \hat{\phi}(x) \hat{\phi}(x') \rangle = C^<(x', x), \quad (2.1.19)$$

is the two-point correlation function. Evaluating the statistical average with respect to a complete set  $|n\rangle$  of the energy eigenstates,  $\hat{H}|n\rangle = E_n|n\rangle$ , one gets the spectral form [3.7, 1.30]

$$C^>(t, t') = e^{-\Phi} \sum_{m,n} |\langle n | \hat{\phi}(0) | m \rangle|^2 \exp[-iE_n(t - t')] \exp[iE_m(t - t' + i\beta)], \quad (2.1.20)$$

where all reference to the spatial coordinates has been suppressed. If it is assumed that the exponentials

dominate the convergence of the the sum, the condition  $-\beta < \text{Im}(t - t') < 0$  guarantees the existence of  $C^>(t, t') \equiv C^>(t - t')$  as an analytic function, and the condition  $0 < \text{Im}(t - t') < \beta$  the existence of  $C^<(t - t')$ . The limit of an analytic function on the boundary of its domain of definition, where it is still continuous, is a generalized function. This implies that the two-point Green function (2.1.18) is well defined on the strip

$$-\beta \leq \text{Im}(t - t') \leq \beta , \quad (2.1.21)$$

if we have  $\theta_c(t - t') = 0$  for  $\text{Im}(t - t') \geq 0$ . This imposes the restriction on  $C$  that a point which moves along  $C$  must have a monotonically decreasing or constant imaginary part. This condition is sufficient as well for the existence of all higher-order thermal Green functions with time arguments on the contour  $C$  [1.14, 3.7].

Let us now calculate the generating functional for a free field obeying the equal-time commutation relation

$$[\hat{\phi}(x), \partial'_i \hat{\phi}(x')] \delta_c(t - t') = i \delta_c(x - x') , \quad (2.1.22)$$

and the field equation

$$K_x \hat{\phi}(x) := -(\partial_t^2 - \nabla^2 + m^2) \hat{\phi}(x) = 0 . \quad (2.1.23)$$

The time derivative must be understood here as the directional derivative along the contour. Recalling (2.1.13), (2.1.14), one arrives at the Dyson–Schwinger equation for the free generating functional

$$K_x \frac{\delta Z_0[j]}{i \delta j(x)} + j(x) Z_0[j] = 0 , \quad (2.1.24)$$

which has the same form as in ordinary quantum field theory [12.4, 12.19]. In the present case the solution is

$$Z_0[j] = Z_0[0] \exp \left[ -\frac{i}{2} \int_C d^4x \int_C d^4x' j(x) D^{(c)}(x - x') j(x') \right] , \quad (2.1.25)$$

in terms of a one-particle propagator which satisfies the inhomogeneous differential equation

$$K_x D^{(c)}(x - x') = \delta_c(x - x') . \quad (2.1.26)$$

On account of (2.1.16) we have

$$G_0^{(c)}(x, x') = i D^{(c)}(x - x') , \quad (2.1.27)$$

which identifies the propagator as the free two-point Green function (apart from a conventional factor  $i$ ).

In the derivation of the above results no use whatsoever has been made of the specifics of the averaging procedure, and in fact the eqs. (2.1.24) and (2.1.26) only summarize the dynamical

information contained in the field equation (2.1.23). A unique solution can only be obtained by imposing appropriate boundary conditions. It is at this stage that the information that the Green functions in question are thermal averages of path-ordered fields is used as input.

Instead of relying on boundary conditions we can also go back to the defining equation (2.1.18) of the two-point Green function. For a free system the fields may be expanded in the usual manner in terms of annihilation and creation operators

$$\hat{\phi}(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} [a_k \exp(-ik \cdot x) + a_k^\dagger \exp(ik \cdot x)], \quad (2.1.28)$$

with  $k_0 = \omega_k = (\mathbf{k}^2 + m^2)^{1/2}$ . In the case of an ideal gas in equilibrium one has [1.3, 1.16, 7.3]

$$\langle a_k^\dagger a_{k'} \rangle_0 = (2\pi)^3 2\omega_k N(\omega_k) \delta(\mathbf{k} - \mathbf{k}'), \quad (2.1.29)$$

$$\langle a_k a_{k'}^\dagger \rangle_0 = (2\pi)^3 2\omega_k [1 + N(\omega_k)] \delta(\mathbf{k} - \mathbf{k}'), \quad (2.1.30)$$

where  $N(k_0) = [\exp(\beta k_0) - 1]^{-1}$  is the Bose distribution function. Combinations of two creation or two annihilation operators vanish.

Substitution of these results into definition (2.1.19) of the correlation functions leads to

$$C_0^{\geqslant}(x, x') = \int \frac{d^4 k}{(2\pi)^4} \exp[-ik \cdot (x - x')] \tilde{C}_0^{\geqslant}(k). \quad (2.1.31)$$

The Fourier transforms are given by

$$\tilde{C}_0^>(k) = \rho_0(k) [1 + N(k_0)] = \exp(\beta k_0) \tilde{C}_0^<(k) \quad (2.1.32)$$

in terms of the spectral density

$$\rho_0(k) = [\tilde{C}_0^>(k) - \tilde{C}_0^<(k)] = 2\pi\varepsilon(k_0) \delta(k^2 - m^2), \quad (2.1.33)$$

where  $\varepsilon(k_0) = \theta(k_0) - \theta(-k_0)$ . In view of the defining eqs. (2.1.18) and (2.1.27), the propagator may be written in the spectral form

$$iD^{(c)}(x - x') = \int \frac{d^4 k}{(2\pi)^4} \rho_0(k) \exp[-ik \cdot (x - x')] [\theta_c(t - t') + N(k_0)]. \quad (2.1.34)$$

This spectral representation of the contour-ordered propagator was derived by Mills [3.7] for non-relativistic particles. In the case of ordinary time ordering it reduces to the result of Dolan and Jackiw [2.6]. We also note that a similar spectral form holds true for interacting fields because eq. (2.1.32) remains valid in that case on account of the Kubo–Martin–Schwinger (KMS) boundary condition [1.4, 1.14], cf. section 2.2.

### 2.1.3. Non-equilibrium

Problems in non-equilibrium statistical mechanics concerned with the response of a many-particle

system to an imposed disturbance very often can be reduced to that of calculating a Green function for a system in thermal equilibrium. This case will be our main concern here, but it is of some interest to point out that there is also the possibility of using diagram techniques for calculating Green functions of systems not necessarily in equilibrium. The method is due to Keldysh [3.2], and is based on the use of a closed contour for time ordering running from minus infinity to plus infinity and then back to minus infinity. This time-path technique, originally suggested by Schwinger [3.1], historically predates the real-time equilibrium Green function method which we shall discuss in the next sections. Since the two approaches have a number of features in common, we like to briefly summarize the basic reasoning underlying the closed time-path Green function formalism. More details and further references can be found in recent review articles [3.13, 3.21]. We also mention that the same formalism may be obtained by an analytic continuation of the imaginary-time Green function equations of Kadanoff and Baym to real times [1.21, 1.29, 11.14].

The Green functions for a non-equilibrium system are defined as in the equilibrium case, cf. (2.1.10), except that the average is now taken over any ensemble of quantum states of the system. In perturbation theory it should be possible to express these Green functions of the interacting system in terms of those of an ideal gas. To that end we assume that the system has been prepared in some initial state at time  $t_i$  in the past. (Eventually we shall take the limit  $t_i \rightarrow -\infty$ .) At this instant the interaction is switched on and the system allowed to evolve to time  $t$ . Then there exists a unitary time-evolution operator which transforms the free field  $\hat{\phi}_{in}(x)$  in the interaction picture into the Heisenberg field operator  $\hat{\phi}(x)$  according to

$$\hat{\phi}(x) = U(t_i, t) \hat{\phi}_{in}(x) U(t, t_i). \quad (2.1.35)$$

If the interaction does not contain derivative fields, we have

$$U(t_2, t_1) = T_c \left[ \exp i \int_{t_1}^{t_2} d^4x \hat{\mathcal{L}}_I(x) \right]_{in}, \quad (2.1.36)$$

where the interaction Lagrangian density  $\hat{\mathcal{L}}_I$  is expressed in terms of interaction-picture fields. The time integration is along a contour connecting the two points  $t_1$  and  $t_2$  in the complex time plane.

Let us now pick a time  $t_f$  as some latest time after which the interaction is switched off. Equation (2.1.35) may then also be written as

$$\hat{\phi}(x) = U(t_i, t_f) U(t_f, t) \hat{\phi}_{in}(x) U(t, t_i). \quad (2.1.37)$$

The successive contours connecting  $t_i$  to  $t$ ,  $t$  to  $t_f$ , and  $t_f$  to  $t_i$ , may be joined together to form a single contour  $C$  running from  $t_i$  to  $t_f$ , and then back to the initial time; see fig. 2.1. We can then rewrite the whole expression (2.1.37) in the compact form

$$\hat{\phi}(x) = T_c \left[ \hat{\phi}(x) \exp i \int_C d^4x' \hat{\mathcal{L}}_I(x') \right]_{in}, \quad (2.1.38)$$

where all fields at the right-hand side are in the interaction picture. The generalization to  $T_c$ -ordered products of Heisenberg fields leads to the following perturbative expression for the generating

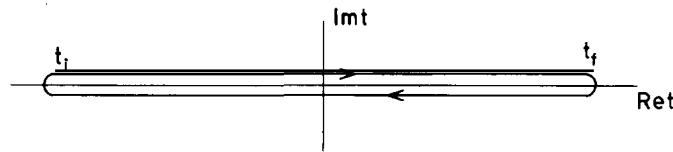


Fig. 2.1. Closed time-path contour.

functional of the non-equilibrium Green functions:

$$Z[j] = Z[0] \left\langle T_c \exp i \int_C d^4x [\hat{\mathcal{L}}_I(x) + j(x) \hat{\phi}(x)]_{in} \right\rangle. \quad (2.1.39)$$

The angular brackets indicate averaging with respect to some initial ensemble.

The initial state, which up to now has been left unspecified, demands some consideration. Suppose that we expand the exponential in (2.1.39) in terms of  $T_c$ -ordered products of interaction-picture fields. These products may be reworked into normal-ordered products with the help of Wick's theorem which also applies to  $T_c$  ordering [12.1, 3.5, 3.8, 3.21]. Hence, ultimately the generating functional (2.1.39) is expressible in terms of an infinite set of correlation functions generated by the functional

$$C[j] = \left\langle : \exp i \int_C d^4x j(x) \hat{\phi}_{in}(x) : \right\rangle, \quad (2.1.40)$$

which describes the statistical properties of the initial state. This means that the Green functions at arbitrary times can only be calculated after a complete specification of the initial state. This is not surprising, of course, as the perturbation series is nothing but a particular solution of the equation of motion.

Although the present scheme allows all initial correlations to be taken into account, for many systems of interest the initial correlations have little effect on the behaviour of the system for times long compared to a characteristic interaction time. For such systems one may adopt the initial condition of Bogoliubov [11.2] which consists in ignoring all correlations which may be present initially. This initial condition of vanishing correlations in the remote past may be formulated as

$$\left\langle T_c \exp i \int_C d^4x j(x) \hat{\phi}_{in}(x) \right\rangle = \exp - \frac{1}{2} \int_C \int_C d^4x d^4x' j(x) G_{in}^{(c)}(x, x') j(x'), \quad (2.1.41)$$

where

$$G_{in}^{(c)}(x, x') = \left\langle T_c \hat{\phi}_{in}(x) \hat{\phi}_{in}(x') \right\rangle \quad (2.1.42)$$

is a two-point non-interacting Green function. Alternatively, this factorization rule may be obtained by assuming that in the remote past the state of the system corresponds to the equilibrium state of free particles with the interaction switched off adiabatically [3.2]. In the literature the prescription (2.1.41) is sometimes called the generalized thermodynamic Wick theorem. Actually it is not a theorem as our reasoning shows [3.6, 3.8, 3.9] but a statistical assumption about the initial state of the system. Of course, for the vacuum state the factorization rule holds rigorously.

For some applications it may be necessary to take initial correlations into account, for example, when the system is prepared in an equilibrium state at a finite time. Such a state may be obtained from a non-correlated one through an imaginary time evolution as discussed in ref. [11.14].

The Feynman rules which result from an application of the Wick decomposition (2.1.41) are similar to those of the standard vacuum Feynman rules. The main difference is that all time integrations run along the contour C. In practical calculations it is often convenient to break up the time contour into two parts:

$$\int_C dt = \int_{-\infty}^{\infty} dt_+ - \int_{-\infty}^{\infty} dt_-, \quad (2.1.43)$$

so that the exponent at the right-hand side of (2.1.41) becomes

$$\int_C d^4x d^4x' j(x) G_{\text{in}}^{(c)}(x, x') j(x') = \int_{-\infty}^{\infty} d^4x d^4x' j_\alpha(x) G_{\text{in}}^{(\alpha\beta)}(x, x') j_\beta(x'), \quad (2.1.44)$$

where  $j_\alpha(x) = \alpha j(t_\alpha, x)$ ,  $\alpha = \pm$ , are to be regarded as two independent source functions. The matrix propagator has as diagonal components the real-time Green functions

$$G_{\text{in}}^{(++)}(x, x') = \langle T_+ \hat{\phi}_{\text{in}}(x) \hat{\phi}_{\text{in}}(x') \rangle, \quad (2.1.45)$$

$$G_{\text{in}}^{(--)}(x, x') = \langle T_- \hat{\phi}_{\text{in}}(x) \hat{\phi}_{\text{in}}(x') \rangle, \quad (2.1.46)$$

where  $T_+$  and  $T_-$  indicate chronological and anti-chronological ordering, respectively, and as off-diagonal components the real-time correlation functions

$$G_{\text{in}}^{(+-)}(x, x') = \langle \hat{\phi}_{\text{in}}(x') \hat{\phi}_{\text{in}}(x) \rangle, \quad (2.1.47)$$

$$G_{\text{in}}^{(-+)}(x, x') = \langle \hat{\phi}_{\text{in}}(x) \hat{\phi}_{\text{in}}(x') \rangle. \quad (2.1.48)$$

The resulting perturbation expansion may be represented by Feynman diagrams in the usual manner, except that propagator lines have + or - attached to each end corresponding to the four possible propagator forms (2.1.45) through (2.1.48). The full list of rules may be found in refs. [3.10, 3.13, 3.21] together with some useful computational lemmas.

Because the diagrams contributing to the full Green function, which may be written in matrix form

$$G^{(\alpha\beta)}(x, x') = \langle T_c \hat{\phi}^\alpha(x) \hat{\phi}^\beta(x') \rangle \quad (2.1.49)$$

like the non-interacting one, have the same structure as in the vacuum theory, the perturbation expansion leads to a set of Dyson equations

$$G^{(\alpha\beta)}(x, y) = G_{\text{in}}^{(\alpha\beta)}(x, y) - i \int d^4x' d^4y' G_{\text{in}}^{(\alpha\gamma)}(x, x') \Sigma^{(\gamma\delta)}(x', y') G^{(\delta\beta)}(y', y). \quad (2.1.50)$$

After a resummation, the particle lines in the proper self-energy  $\Sigma^{(\alpha\beta)}(x, x')$  represent exact Green functions (2.1.49). Hence, the Dyson equation may be regarded as a self-consistent integral equation for the exact Green function were it not for the explicit appearance of the two non-interacting Green functions in (2.1.50). The latter are rather artificial elements in the theory, because their definition involves the unspecified initial state. This difficulty may be overcome by using their equation of motion

$$K_x G_{\text{in}}^{(\alpha\beta)}(x, x') = i(\tau_3)^{\alpha\beta} \delta(x - x'), \quad (2.1.51)$$

where  $\tau_3 = \text{diag}(1, -1)$ . Applying the operator  $K_x$  to eq. (2.1.50), one obtains a set of four integro-differential equations which no longer involve the initial state; in this way all reference to the initial state in the remote past is erased. These equations constitute a complete description, in principle, of the behaviour of a non-equilibrium system. The solution is arbitrary to the same extent as that of the Boltzmann equation, of which they are a generalization and to which they can be reduced in the appropriate quasiclassical and quasistationary limits [11.14, 3.21].

## 2.2. Path-integral formulation

In this section we shall extend the time-path method developed above to fields of any spin and to gauge fields in particular. Although for the greater part this is a straightforward generalization of the scalar theory, it is of some value to formulate the theory in the language of path integrals [3.11]. This will make available the powerful tools of the path-integral formalism to field theory at finite temperature and density. At this point we shall lose contact with the parallel development of the non-equilibrium theory, because essential use will be made of the form of the grand canonical ensemble and the corresponding Kubo–Martin–Schwinger (KMS) boundary conditions for the propagators [1.4, 1.14, 4.7]. To be sure, a formal derivation of the generating functional as a path integral for a non-equilibrium system can be given [3.13], but useful calculational methods have so far only been implemented for the equilibrium case.

### 2.2.1. Feynman–Matthews–Salam formula

First we consider a scalar field theory like before, and a single field operator  $\hat{\phi}(x)$  in the Heisenberg picture. Let  $|\phi(x); t\rangle$  be the state vector which at time  $t$  is an eigenstate of the field operator  $\hat{\phi}(x)$  with eigenvalue  $\phi(x)$ :\*

$$\hat{\phi}(x)|\phi(x); t\rangle = \phi(x)|\phi(x); t\rangle. \quad (2.2.1)$$

The time evolution (2.1.9) of the Heisenberg field induces the time evolution

$$|\phi(x); t\rangle = e^{i\hat{H}t}|\phi(x), 0\rangle \quad (2.2.2)$$

of the state vector. By taking  $t$  complex we extend the support of the field variables and state vectors to the entire complex plane.

We assume that at any given time the above Heisenberg states form a complete set. Let us now evaluate the statistical average (2.1.17) with respect to such a complete set at time  $t_i$ . We then get for

\* As noticed recently [12.26] eq. (2.2.1) does not seem to have a solution in Fock space, so the subsequent argument (but not the conclusion) is heuristic. A more rigorous derivation should use coherent states [12.18] from the outset.

the generating functional of the Green functions

$$Z[j] = \int d\phi' \langle \phi'; t_i | e^{-\beta \hat{H}} T_c \exp \left[ i \int_C d^4x j(x) \dot{\phi}(x) \right] | \phi'; t_i \rangle, \quad (2.2.3)$$

with  $\phi' = \phi'(x)$ . Here a product over all space points  $x'$  is understood. The action of the canonical operator may be translated into an imaginary time shift

$$\langle \phi'; t_i | e^{-\beta \hat{H}} = \langle \phi'; t_i - i\beta | \quad (2.2.4)$$

by applying formula (2.2.2).

Wishing to write  $Z[j]$  as a path integral, we recall the Feynman–Matthews–Salam (FMS) formula [1.26, 12.2], which expresses the matrix elements of a time-ordered functional of the fields  $\dot{\phi}(x)$  in terms of a path integral. It reads

$$\langle \phi'; t' | T F[\dot{\phi}] | \phi''; t'' \rangle = \mathcal{N}' \int [d\phi][d\pi] F[\phi] \exp \int_{t''}^{t'} dt \int_V d^3x [\pi(x) \dot{\phi}(x) - \mathcal{H}(x)], \quad (2.2.5)$$

where the normalization factor depends on the time difference  $t' - t''$ , and the volume  $V$ , but not on the functional  $F$ . In this equation the time is real and the path integral is over all c-number fields  $\phi(x)$  that satisfy the boundary conditions

$$\phi(t', x) = \phi'(x), \quad \phi(t'', x) = \phi''(x). \quad (2.2.6)$$

The integration with respect to  $\pi(x)$ , the canonical momentum conjugate to  $\phi(x)$ , is not subject to any boundary condition. Finally, the time arguments occurring in  $F[\phi]$  must lie in the interval  $(t'', t')$ .

To proceed we observe that the FMS formula retains its validity for complex times if we replace the time-ordering instruction by our contour ordering  $T_c$ . More precisely, we allow  $t'$  and  $t''$  to be complex, and we let the time integration in the exponent of eq. (2.2.5) go over some contour  $C$  that starts at  $z = t''$  and ends at  $z = t'$ . The only restriction is that  $C$  must go monotonically downward or parallel to the real axis as discussed in section 2.1.2. Of course, for formula (2.2.5) to be valid in that case, the time arguments in  $F[\phi]$  should all lie on  $C$ . To show that this reasoning is viable it suffices to point out that any contour allows a decomposition into pieces, each parametrized by a real parameter, such that contour ordering coincides with ordinary time-ordering in this real parameter. The original FMS formula can then be applied piecewisely.

To establish contact with formulae (2.2.3) and (2.2.5), we let the contour start at  $z = t_i$  and end at  $z = t_i - i\beta$ , subject to the monotonousness condition, but otherwise arbitrary. The generating functional can then be written as the path integral

$$Z[j] = \mathcal{N}' \int [d\phi][d\pi] \exp i \int_C d^4x [\pi(x) \dot{\phi}(x) - \mathcal{H}(x) + j(x) \phi(x)], \quad (2.2.7)$$

where the dot denotes the directional derivative on  $C$ . The path integration is now over all fields satisfying the periodicity condition

$$\phi(t_i - i\beta, x) = \phi(t_i, x), \quad (2.2.8)$$

on account of (2.2.4), (2.2.6) and the trace integration in (2.2.3).

If we now specialize to a field theory without derivative couplings, the Gaussian path integral over the canonical momentum variables can be performed to yield

$$Z[j] = \mathcal{N} \int_C [d\phi] \exp i \int d^4x [\mathcal{L}(x) + j(x) \phi(x)] \quad (2.2.9)$$

with

$$\mathcal{L}(x) = \frac{1}{2} \phi(x) (-\partial^2 - m^2) \phi(x) + \mathcal{L}_i(x) \quad (2.2.10)$$

the Lagrangian density. Expression (2.2.9) may be understood as a statistical average in which the trace over a complete set of states has been replaced by a functional integration over c-number fields:

$$Z[j] = Z[0] \left\langle \exp i \int_C d^4x j(x) \phi(x) \right\rangle. \quad (2.2.11)$$

The unspecified and ill-defined normalization factor  $\mathcal{N}$ , which depends on the temperature as well as the volume, has been absorbed into the normalization

$$Z[0] = \mathcal{N}(\beta, V) \int [d\phi] \exp i \int_C d^4x \mathcal{L}(x) \quad (2.2.12)$$

of the generating functional. It is irrelevant for the calculation of the Green functions. However, it does play a role if we wish to calculate  $Z[0]$  which, for the proper choice of  $\mathcal{N}$ , should be equal to the canonical partition function; cf. the starting eq. (2.2.3).

In principle, the normalization factor may be calculated by a careful analysis of the measure factors in the path integral [1.26, 2.5, 2.20]. Fortunately, however, its contribution to the partition function can also be ascertained by a simple consistency argument. To that end we calculate the energy density

$$\begin{aligned} E &= -\frac{1}{V} \frac{1}{Z[0]} \frac{\partial Z[0]}{\partial \beta} \\ &= -\frac{1}{V} \left\langle i \frac{\partial}{\partial \beta} \int_C d^4x \mathcal{L}(x) \right\rangle. \end{aligned} \quad (2.2.13)$$

As we shall now demonstrate, the term written out is indeed the energy density, implying that the terms coming from the  $\beta$  dependence of  $\mathcal{N}$  and  $[d\phi]$  must cancel. To that purpose we parametrize the contour  $C$  such that it is given by

$$z(\lambda) = t_i - i\beta f(\lambda), \quad (2.2.14)$$

with  $f(\lambda)$  a monotonic complex function having boundary values  $f(0) = 0$  and  $f(1) = 1$  at the endpoints

of the interval  $0 \leq \lambda \leq 1$ . Furthermore, we define fields  $\chi(\lambda, x) := \phi(z(\lambda), x)$  which are periodic on the unit interval. After expressing the action in terms of these fields and replacing  $t$  by  $\lambda$  as integration variable, one may easily calculate the  $\beta$  derivative in (2.2.13). The result is

$$E = \frac{i}{\beta V} \int_C dt \int_V d^3x \langle [\partial_0 \phi(x)]^2 - \mathcal{L}(x) \rangle. \quad (2.2.15)$$

Since the statistical average does not depend on the space-time coordinates, we get

$$E = \langle \mathcal{H}(x) \rangle, \quad (2.2.16)$$

where  $\mathcal{H}(x)$  is the Hamiltonian density of the system. We conclude that (2.2.12) is indeed the canonical partition function, and that the normalization factor  $\mathcal{N}$  is there to cancel any unwanted (usually infinite) contributions of the path integral which would invalidate (2.2.16).

Having established the path-integral representation for the generating functional, we now rewrite it in the form that will be used later to derive the Feynman rules. In the usual manner [12.4, 12.8, 12.18, 12.19] we replace the fields in the interaction Lagrangian by a functional differentiation with respect to the source functions to obtain:

$$Z[j] = \exp i \int_C d^4x \mathcal{L}_I \left[ \frac{\delta}{i \delta j(x)} \right] Z_0[j]. \quad (2.2.17)$$

Here  $Z_0[j]$  is just eq. (2.2.9) with  $\mathcal{L}_0$  instead of  $\mathcal{L}$ .

It is possible to extend the above considerations to fermion field theories [12.18, 12.19]. In spite of the complications introduced by the anticommuting nature of the fermion fields, the final answer for the path integral is, mutatis mutandis (cf. (2.2.21)), the same as (2.2.17). Therefore, we will turn directly to the determination of the free generating functional of fields with any spin [3.19].

## 2.2.2. Propagators

We consider the general case of a multi-component covariant complex field  $\hat{\psi}_\alpha^i(x)$  which transforms under some representation  $\mathcal{D}_{\alpha\beta}$  of the Lorentz group. This field may carry an arbitrary number of charges  $q_A^{ij}$  such that

$$[\hat{Q}_A, \hat{\psi}_\alpha^i(x)] = -q_A^{ij} \hat{\psi}_\alpha^j(x), \quad (2.2.18)$$

with  $q_A^{ij}$  an Hermitian matrix in the space of the internal degrees of freedom labelled by the indices  $i, j$ . In the following these internal indices will be suppressed. Let now the free Lagrangian density be given by

$$\mathcal{L}_0(x) = \bar{\psi}_\alpha(x) \Lambda_{\alpha\beta}(i\partial) \psi_\beta(x). \quad (2.2.19)$$

Here  $\bar{\psi}_\alpha = (\psi^\dagger A)_\alpha$  is the Pauli-adjoint field and  $\Lambda(i\partial)$  a differential operator of finite order satisfying  $[A\Lambda(i\partial)]^\dagger = A\Lambda(-i\partial)$ ; the matrix  $A_{\alpha\beta}$  intertwines the representations  $\mathcal{D}^\dagger$  and  $\mathcal{D}^{-1}$ . The mass spectrum and the spin content of the field  $\psi$  can be inferred from  $\Lambda$  [12.5, 12.16].

For this case the path-integral representation of the free generating functional takes the form

$$Z_0[\bar{j}, j] = \mathcal{N} \int_C [d\bar{\psi}] [d\psi] \exp i \int d^4x [\bar{\psi} \Lambda \psi + \bar{j} \psi + \bar{\psi} j], \quad (2.2.20)$$

where the  $\psi$  and  $\bar{\psi}$  are independent integration variables. In the exponent a contraction of all discrete indices is understood, e.g.  $\bar{j}\psi = \bar{j}_\alpha(x)\psi_\alpha^i(x)$ . Green functions are obtained by functional differentiation from the left with respect to  $j_\alpha(x)$  and from the right with respect to  $\bar{j}_\alpha(x)$ . The fields have to satisfy the quasiperiodicity conditions

$$\psi(t_i, x) = \eta e^{\beta\mu} \psi(t_i - i\beta, x), \quad (2.2.21)$$

where  $\eta = \pm 1$  for bosons and fermions, respectively, and where we have set  $\mu = \sum_A q_A \mu_A$ . The fermionic factor  $-1$  is a consequence of the trace definition of the partition function and the anticommuting nature of the fermion fields [11.4–11.6, 6.24], while the occurrence of  $e^{\beta\mu}$  stems from the identity  $\langle \psi | \exp(\beta \sum_A \mu_A \hat{Q}_A) = \langle e^{\beta\mu} \psi |$  [3.14].

In passing we mention that it may sometimes be convenient to work with fields which are purely (anti-)periodic in the time variable. This can easily be achieved by defining

$$\tilde{\psi}(x) = e^{i\mu} \psi(x), \quad (2.2.22)$$

and by expressing the path integral in terms of these new fields. It has as a consequence that all time derivatives must be shifted according to the rule:  $i\partial_0 \rightarrow i\partial_0 + \mu$ .

The Gaussian integral (2.2.20) may be calculated in the usual manner by applying the shift

$$\psi_\alpha(x) \rightarrow \psi_\alpha(x) - \int_C d^4x' D_{\alpha\beta}^{(c)}(x - x') j_\beta(x') \quad (2.2.23)$$

(and the corresponding transformation on the adjoint field) with the propagator satisfying the equation

$$\Lambda_{\alpha\beta}(i\partial) D_{\beta\gamma}^{(c)}(x - x') = \delta_{\alpha\gamma} \delta_c(x - x'). \quad (2.2.24)$$

We now require invariance of the path integral under the shift (2.2.23), i.e. the shifted field must satisfy the same boundary condition (2.2.21) as the original field. This demand on  $D_{\alpha\beta}^{(c)}$  uniquely defines  $iD_{\alpha\beta}^{(c)}$  as the free two-point Green function, since we obtain

$$Z_0[\bar{j}, j] = Z_0[0, 0] \exp \left[ -i \int_C d^4x d^4x' \bar{j}_\alpha(x) D_{\alpha\beta}^{(c)}(x - x') j_\beta(x') \right]. \quad (2.2.25)$$

Had the path integral not been invariant under the shift (2.2.23), then the coefficient of the exponential in (2.2.25) would have been  $j$ -dependent [3.17]. Now, however, we have according to (2.1.16) and (2.2.25)

$$iD_{\alpha\beta}^{(c)}(x - x') = G_{0\alpha\beta}^{(c)}(x, x') = \langle \psi_\alpha(x) \bar{\psi}_\beta(x') \rangle_0, \quad (2.2.26)$$

where the average is the one defined in formula (2.2.11).

Suppressing spatial variables, we write the propagator in terms of contour step functions like in (2.1.18):

$$D_{\alpha\beta}^{(c)}(t - t') = \theta_c(t - t') D_{\alpha\beta}^>(t - t') + \theta_c(t' - t) D_{\alpha\beta}^<(t - t'). \quad (2.2.27)$$

Upon substitution of this expression into (2.2.23), we find that the boundary conditions (2.2.21), when imposed on the shifted fields, lead to

$$\int_C dt' D_{\alpha\beta}^>(t_i - i\beta - t') j_\beta(t') = \eta e^{-\beta\mu} \int_C dt' D_{\alpha\beta}^<(t_i - t') j_\beta(t'), \quad (2.2.28)$$

where we made use of the fact that  $\theta_c(t_i - i\beta - t') = \theta_c(t_f - t') = 1$  for any  $t'$  on the contour. Since the above equation must hold true for any  $j(t')$ , we require

$$D_{\alpha\beta}^>(t - i\beta) = \eta e^{-\beta\mu} D_{\alpha\beta}^<(t). \quad (2.2.29)$$

This is nothing but the well-known KMS boundary condition [1.4, 1.14, 1.21] for two-point equilibrium correlation functions, which can be derived in the operator formalism by using (2.1.1) and the cyclic invariance of the trace. Conversely, had we imposed the KMS condition, we would have been led to the quasiperiodicity conditions on the fields. This illustrates the internal consistency of the scheme. We note that the interacting two-point Green functions generated by (2.2.17) with (2.2.25) now satisfy the appropriate KMS condition by construction [4.11].

Let us now assume that there exists a multi-mass Klein–Gordon divisor  $d(i\partial)$  satisfying [12.5, 12.16]

$$d_{\alpha\beta}(i\partial) \Lambda_{\beta\gamma}(i\partial) = \Lambda_{\alpha\beta}(i\partial) d_{\beta\gamma}(i\partial) = \delta_{\alpha\gamma} \prod_l (-\partial^2 - m_l^2), \quad (2.2.30)$$

where the  $m_l$ 's represent the mass spectrum of  $\Lambda$ . In that case the solution of the propagator equation (2.2.24) may be written as

$$D_{\alpha\beta}^{(c)}(x) = d_{\alpha\beta}(i\partial) D^{(c)}(x) \quad (2.2.31)$$

in terms of the solution of the Klein–Gordon propagator equation

$$\prod_l (-\partial^2 - m_l^2) D^{(c)}(x - x') = \delta_c(x - x') \quad (2.2.32)$$

satisfying the appropriate boundary conditions.

In section 2.1.2 we have already studied the Klein–Gordon propagator for a single type of scalar bosons. It is not difficult to guess that the solution of eq. (2.2.32) in the general case has the form

$$iD^{(c)}(x - x') = \int \frac{d^4 p}{(2\pi)^4} \rho_0(p) \exp[-ip \cdot (x - x')] [\theta_c(t - t') + \eta n_+(p_0)]. \quad (2.2.33)$$

The statistical factors

$$n_\pm(p_0) = \{\exp[\beta(p_0 \mp \mu)] - \eta\}^{-1} \quad (2.2.34)$$

are the Fermi–Dirac or Bose–Einstein distribution functions for (anti-)particles having a non-zero chemical potential. It is obvious that the form (2.2.33) is compatible with the boundary conditions (2.2.29). For the spectral density we obtain

$$\rho_0(p) = i \operatorname{Disc} \prod_l 1/(p^2 - m_l^2), \quad (2.2.35)$$

where the discontinuity of an arbitrary function of the energy has been defined as  $\operatorname{Disc} f(p_0) = f(p_0 + i\epsilon) - f(p_0 - i\epsilon)$ , cf. appendix A.1. We mention that in this construction of the propagator, contour step functions are to be regarded as commuting with the Klein–Gordon divisor [12.5]. Another technical point is that the charge matrices  $q_A^{ij}$  commute both with  $\Lambda(i\partial)$  and  $d(i\partial)$ , if it is assumed that the charge operators commute with  $\mathcal{L}_0$ .

### 2.2.3. Gauge fields

The presence of gauge fields modifies the above reasoning only insofar that the gauge freedom has to be taken into account. To be definite, we consider the  $SU(N)$  gauge theory characterized by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}, \quad (2.2.36)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \quad (2.2.37)$$

with  $a = 1, 2, \dots, N^2 - 1$  colour indices. As first noted by Bernard [2.5], one cannot naively write down a path-integral formula for this system because the sum over the states is restricted to one over physical states. To remove this restriction one inserts a projection operator [2.13, 4.6] which involves the integration over a Lagrange multiplier field [6.5]. One then applies the standard Faddeev–Popov gauge-fixing procedure [12.8, 12.18] to arrive at the partition function

$$Z[0] = \mathcal{N} \int [dA_\mu][d\bar{\omega}][d\omega] \exp i \int_C d^4x \mathcal{L}_{\text{eff}}, \quad (2.2.38)$$

where  $\mathcal{L}_{\text{eff}}$  is the well-known effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \lambda (\partial^\mu A_\mu^a)^2 - (\partial^\mu \bar{\omega}^a) \partial_\mu \omega^a - g f^{abc} (\partial^\mu \bar{\omega}^a) A_\mu^b \omega^c. \quad (2.2.39)$$

The path integral (2.2.38) is independent of the gauge parameter  $\lambda$ , apart from a field-independent factor which can be absorbed in  $\mathcal{N}$ .

The gauge fields, being bosonic, have to satisfy the periodic boundary condition  $A_\mu^a(t_i, x) = A_\mu^a(t_i - i\beta, x)$ . The ghost fields  $\omega^a$ , on the other hand, must be treated as anticommuting pseudo fermion fields, but subject to the same periodic boundary condition as the gauge fields [2.5, 2.9, 2.13]. This is a consequence of the fact that the Faddeev–Popov determinant is defined on the space of periodic functions, namely the gauge fields. The periodic boundary conditions of the ghosts are also found [2.13, 4.6], (also cf. [4.8, 8.27]) in the covariant canonical operator formalism of gauge theories [12.17]. For these boundary conditions the contour-dependent action in (2.2.38) is BRS invariant which ensures that the Slavnov–Taylor identities can be easily generalized to finite temperature, see section 3.3.1.

In passing we mention that it would be possible, in principle, to assign antiperiodic boundary conditions to the unphysical components of  $A_\mu^a$ . In that case the ghost fields would likewise be antiperiodic as if they were actual fermion fields. However, in practice, this procedure is rather awkward and not free of computational difficulties. Moreover, the ghosts would still not represent physical particles that can come into thermal equilibrium with a heat bath, as their only role is to subtract unphysical contributions due to spurious degrees of freedom of the gauge fields  $A_\mu^a$  (cf. [8.7, 8.14]). On the other hand, it is certainly possible to choose a non-covariant axial gauge and to eliminate the spurious degrees of freedom altogether. At finite temperature and/or density, where Lorentz covariance is broken by the presence of a privileged frame of reference, this last procedure enjoys a certain popularity [6.4, 6.15].

We close this discussion with the remark that in an Abelian gauge theory, or in a non-Abelian theory in an axial gauge, the ghost fields completely decouple from the physical fields. The resulting multiplicative constant in the generating functional can be ignored as far as expectation values are concerned. Of course, its contribution should be retained when calculating the partition function. Both in Abelian and non-Abelian theories the inclusion of fermions does not affect the Faddeev–Popov ghost fields introduced to implement a path-integral representation for the generating functional.

The quadratic part of the effective Lagrangian (2.2.39) may be cast in the form

$$\mathcal{L}_0 = \frac{1}{2} A_a^\mu A_{\mu\nu} (i\partial) A_a^\nu + \bar{\omega}_a \partial^2 \omega_a , \quad (2.2.40)$$

similar to (2.2.19). It is convenient to add a small mass  $\mu$ , and to write the differential operator as it appears in the Stuckelberg Lagrangian [12.18]:

$$\Lambda_{\mu\nu}(i\partial) = g_{\mu\nu}(\partial^2 + \mu^2) + (\lambda - 1)\partial_\mu \partial_\nu . \quad (2.2.41)$$

It possesses a twofold mass spectrum  $m_1^2 = \mu^2$ ,  $m_2^2 = \mu^2/\lambda$ , and has a two-mass Klein–Gordon divisor

$$d_{\mu\nu}(i\partial) = g_{\mu\nu}(\partial^2 + \mu^2/\lambda) - (1 - \lambda^{-1})\partial_\mu \partial_\nu . \quad (2.2.42)$$

Having established this formula we may set  $\mu = 0$ .

The procedure of the preceding subsection may now be invoked to construct the gauge-boson propagator satisfying the equation

$$\Lambda^{\mu\nu}(i\partial) \delta^{ab} D_{\nu\sigma}^{(c)bd} = g_\sigma^\mu \delta^{ad} \delta_c(x - x') , \quad (2.2.43)$$

and the appropriate boundary conditions. From (2.2.31) and (2.2.33) we get immediately

$$\begin{aligned} D_{\mu\nu}^{(c)ab}(x - x') &= -\delta^{ab} \int \frac{d^4 k}{(2\pi)^4} \rho_0(k) \exp[-ik \cdot (x - x')] [g_{\mu\nu} k^2 - (1 - \lambda^{-1}) k_\mu k_\nu] \\ &\times [\theta_c(t - t') + N(k_0)] \end{aligned} \quad (2.2.44)$$

with the spectral density, cf. (A.3) and (2.2.35),

$$\rho_0(k) = 2\pi\varepsilon(k_0) \delta'(k^2) , \quad (2.2.45)$$

and the Bose distribution function  $N(k_0)$ . The method is also applicable to the vacuum theory, and somewhat simpler than the one described in textbooks [12.18].

It is easy to check the result (2.2.44) in the special case of the Feynman gauge  $\lambda = 1$ . Then all indices “go along for a free ride” in eq. (2.2.43) and we simply have

$$D_{\mu\nu}^{(c)ab}(x - x') = -g_{\mu\nu}\delta^{ab}D^{(c)}(x - x'), \quad (2.2.46)$$

where  $D^{(c)}(x - x')$  is the now familiar Klein–Gordon propagator (2.1.34) of the scalar boson field with  $m = 0$ . Because the ghost fields obey Bose statistics, the ghost propagator

$$D_g^{(c)ab} = -\delta^{ab}D^{(c)}(x - x') \quad (2.2.47)$$

is proportional to this same Klein–Gordon propagator.

As long as one sticks to the grand canonical ensemble, no new techniques are required for the treatment of the matter part of a gauge (or BRS) invariant Lagrangian. For the use of the canonical ensemble in this context we refer to the literature [2.21].

### 2.3. Matsubara contour

Formally the path integral of finite-temperature field theory is very similar to that of the vacuum theory. New features are that the time integrations are to be performed along a contour  $C$  from an arbitrary time  $t_i$  down to  $t_i - i\beta$ , and that propagators involve the distribution functions of the particles. The next step in the development of finite-temperuture field theory is an analysis in terms of Feynman diagrams. Since the lines in these graphs represent contour-dependent propagators, a specific choice of the contour  $C$  will lead to a specific formulation of quantum field theory [4.14] and a corresponding specific set of Feynman rules.

In quantum statistics the traditional choice for the contour has been the most obvious one, namely, a straight line entirely on the imaginary axis. It is named after Matsubara [1.3] who was the first to set up a perturbation theory based on this contour. On this specific contour, the general contour-ordering  $T_c$  defined in section 2.1.2 is equal to the so-called temperature-ordering  $T_\tau$  [1.9, 1.22, 1.30]. This means that fields are just time-ordered in terms of the real variable  $\tau = it$ ,  $t$  on  $C$ . Then the generating functional (2.1.39) generates temperature Green functions defined by the path-integral expectation values of imaginary-time ordered Euclidean fields.

The well-known advantage of the Matsubara formulation is that it generates a perturbation expansion which is represented by the same diagrams as in the vacuum theory. The major difficulty that arises is that one has to deal with Euclidean propagators and Green functions with imaginary-time arguments. In principle, real-time quantities can be obtained by analytic continuation to the real axis, but in practice the proper procedure is not always self-evident. As a case in point we shall consider the plasmon effect in section 4.4 later on.

#### 2.3.1. Euclidean propagators

The Matsubara contour  $E$  generates Green functions with imaginary-time arguments lying in the interval  $0 \leq it \leq \beta$ . In the propagator given by (2.2.33) we set  $t - t' = -i\tau$ ,  $-\beta \leq \tau \leq \beta$ , and define

$$\Delta(\tau, x) = i e^{\mu\tau} D^{(E)}(-i\tau, x). \quad (2.3.1)$$

The periodicity properties of this Euclidean propagator may be inferred from the discussion in section 2.2.2, or from the explicit form

$$\Delta(\tau, \mathbf{x}) = \int \frac{d^4 p}{(2\pi)^4} \exp(i\mathbf{p} \cdot \mathbf{x}) \rho_0(p) \exp[-\tau(p_0 - \mu)] [\theta(\tau) + \eta n_+(p_0)]. \quad (2.3.2)$$

If  $0 \leq \tau \leq \beta$  we have  $\Delta(\tau - \beta) = \eta\Delta(\tau)$ , and if  $-\beta \leq \tau \leq 0$  we have  $\Delta(\tau + \beta) = \eta\Delta(\tau)$ . Hence, this propagator is (anti-)periodic in the time variable  $\tau$  with period  $\beta$ .

It follows that the Fourier transform

$$\tilde{\Delta}(i\omega_n, \mathbf{p}) = \int_{-\beta}^{\beta} d\tau \int d^3x \exp[i\omega_n \tau - i\mathbf{p} \cdot \mathbf{x}] \Delta(\tau, \mathbf{x}), \quad (2.3.3)$$

defined for discrete frequencies  $\omega_n$ , is independent of  $a$ ,  $0 \leq a \leq \beta$ ; the bosonic frequencies  $\omega_n = 2\pi n \beta^{-1}$  are restricted to the even integers, and the fermionic frequencies  $\omega_n = (2n+1)\pi\beta^{-1}$  to the odd integers. The  $\omega_n$  are known as the Matsubara frequencies, although they were introduced in [2.1, 1.9, 1.12]. Inserting (2.3.2), we find that the distribution function drops out, and that the Fourier coefficient has the simple spectral form

$$\tilde{\Delta}(i\omega_n, \mathbf{p}) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{\rho_0(p)}{p_0 - \mu - i\omega_n} \quad (2.3.4)$$

for both statistics, with  $\rho_0(p)$  the spectral density (2.2.35). A simple calculation yields the multi-mass Matsubara propagator

$$\tilde{\Delta}(i\omega_n, \mathbf{p}) = \prod_l \frac{1}{\mathbf{p}^2 + m_l^2 - (i\omega_n + \mu)^2}. \quad (2.3.5)$$

For bosons one must impose  $\mu < m$  in order that the Matsubara propagator is non-singular. This constraint arises from the demand that the occupation numbers be positive [11.7, 11.10].

The Fourier inversion theorem permits the Euclidean propagator (2.3.2) to be written as the Fourier sum

$$\Delta(\tau, \mathbf{x}) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \exp[i\mathbf{p} \cdot \mathbf{x} - i\omega_n \tau] \tilde{\Delta}(i\omega_n, \mathbf{p}). \quad (2.3.6)$$

This expression exists for any real  $\tau$ , and is (anti-)periodic with period  $\beta$ . On the interval  $-\beta \leq \tau \leq \beta$  it coincides with the propagator (2.3.2), and outside this interval it may be regarded as the continuation. (In this connection one may recall that in general a two-point Green function is only well-defined on the strip  $-\beta \leq \text{Im}(t - t') \leq \beta$  of the complex time plane.)

It is possible to maintain a useful formal analogy with the vacuum theory by defining a four-momentum  $p^\mu = (p_0, \mathbf{p})$  with complex energy component  $p_0 = i\omega_n + \mu$ . In view of (2.3.1), (2.3.5) and (2.3.6) we may then write the Euclidean propagator as

$$iD^{(E)}(x) = \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} \exp(-ip \cdot x) \prod_l \frac{1}{m_l^2 - p^2}. \quad (2.3.7)$$

This expression may be thought of as being obtained from the corresponding vacuum expression [12.16] by the substitution  $\int dp_0/2\pi i \rightarrow \beta^{-1}\Sigma$ , with the understanding that the real continuous energy variable must be replaced by the discrete set of complex values  $p_0 = i\omega_n + \mu$ .

For fields characterized by a Klein-Gordon divisor  $d(i\partial)$ , see section 2.2.2, the appropriate Euclidean propagator is

$$iS^{(E)}(x) = \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} d(p) \exp(-ip \cdot x) \prod_l \frac{1}{m_l^2 - p^2}. \quad (2.3.8)$$

For spin  $\frac{1}{2}$  Dirac fields, in particular, we have  $d(p) = \gamma \cdot p + m$ , hence

$$iS^{(E)}(x) = \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} \frac{\exp(-ip \cdot x)}{m - \gamma \cdot p}. \quad (2.3.9)$$

The  $\gamma$ 's are the ordinary Dirac matrices, and the metric is Minkowskian as throughout this report.

Gauge fields may be treated in a similar manner. However, there is no need to go through all calculations in detail because the aforementioned substitution rule applies here too. In this way one immediately arrives at the Euclidean gauge boson and ghost propagators; their Fourier transforms are given in eqs. (2.3.20) and (2.3.21) below.

In passing it may be mentioned that in the vacuum limit  $\beta \rightarrow \infty$ ,  $\mu \rightarrow 0$ , the energy summations can be replaced by integrations:  $\beta^{-1}\Sigma \rightarrow \int dk_4/2\pi$ ,  $k_0 = ik_4$ . In this limit the Matsubara formalism reduces to the corresponding Euclidean vacuum theory as it is obtained by an ordinary Wick rotation [12.18].

### 2.3.2. Feynman rules

In the imaginary-time formalism, the Feynman rules take their simplest form in momentum space. In virtue of the periodicity conditions, the time dependence of all fields may be represented by a Fourier sum over discrete frequencies. In general we write

$$f(x) = \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} \exp(-ip \cdot x) \tilde{f}(p), \quad (2.3.10)$$

where  $p_0$  takes on the discrete values  $p_0 = i2n\pi\beta^{-1} + \mu$  or  $p_0 = i(2n+1)\pi\beta^{-1} + \mu$  depending on the periodicity properties of the field  $f(x)$ .

Let us first consider a scalar field theory. The Fourier-transformed Green functions

$$\tilde{G}^{(E)}(k_1, \dots, k_n) = \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_n) \rangle \quad (2.3.11)$$

are generated by functional differentiation of the generating functional

$$Z[\tilde{j}] = Z[0] \left\langle \exp \frac{1}{\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \tilde{j}(k) \tilde{\phi}(k) \right\rangle, \quad (2.3.12)$$

with respect to the c-number functions  $\tilde{j}(k) = \tilde{j}^*(-k)$ . Because of translational invariance these Green functions are proportional to  $\beta(2\pi)^3 \delta_{n,0} \delta(\mathbf{k})$ , where  $\mathbf{k} = \sum_i \mathbf{k}_i$  and  $n = \sum_i n_i$ . Hence, total four-momentum  $k^\mu = (i2n\pi\beta^{-1}, \mathbf{k})$  is globally conserved.

In terms of the Fourier-transformed fields the action becomes

$$i \int_0^{-i\beta} d^4x \mathcal{L}(x) = \frac{-1}{2\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(k) \tilde{\Delta}^{-1}(k) \tilde{\phi}(k) + \mathcal{S}_I[\tilde{\phi}], \quad (2.3.13)$$

where  $\tilde{\Delta}(k) = (m^2 - k^2)^{-1}$  is the Fourier transform of the Euclidean propagator; cf. (2.3.7). The perturbation expansion with respect to the interaction is then obtained by manipulating the generating functional (2.3.12) into the form

$$Z[\bar{j}] = Z_0[0] \exp\{\mathcal{S}_I[\delta/\delta\bar{j}]\} \exp\left\{\frac{1}{2\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \bar{j}(k) \tilde{\Delta}(k) \bar{j}(k)\right\}. \quad (2.3.14)$$

The Green functions as well as the partition function  $Z[0]$  can then be found to arbitrary orders in perturbation theory by expanding the exponentials. To be explicit we consider a  $\phi^l$  theory with

$$\mathcal{S}_I = -\frac{\lambda}{l!} \prod_{i=1}^l \frac{1}{\beta} \sum_{n_i} \int \frac{d^3k_i}{(2\pi)^3} \tilde{\phi}(k_i) \beta(2\pi)^3 \delta_{n,0} \delta(\mathbf{k}). \quad (2.3.15)$$

On account of the locality of the interaction, energy and momentum are conserved at each vertex.

The Feynman rules may now be stated as follows:

- (i) Draw diagrams and determine symmetry factors as in the vacuum theory.
- (ii) Assign a propagator  $\tilde{\Delta}(k)$  to each line, and a factor  $-\lambda$  to each vertex.
- (iii) Conserve energy and momentum at each vertex according to the prescription  $\beta(2\pi)^3 \delta_{n,0} \delta(\mathbf{k})$ ; global energy and momentum conservation may be separated off by excluding one arbitrarily chosen vertex.
- (iv) Integrate and sum over all internal momenta and energies according to  $\beta^{-1} \sum \int d^3k / (2\pi)^3$ .

By a comparison with the Feynman rules of the vacuum theory [12.18], it is seen that the finite-temperature Green functions may be obtained by making the following substitutions in the Minkowski-space Feynman integrals [2.5]:

$$\tilde{G}(k_1, \dots, k_n) \rightarrow (-i)^n \tilde{G}^{(E)}(k_1, \dots, k_n), \quad (2.3.16)$$

$$\int \frac{d^4k}{(2\pi)^4 i} \rightarrow \beta \sum_n \int \frac{d^3k}{(2\pi)^3}, \quad (2.3.17)$$

$$i(2\pi)^4 \delta(k) \rightarrow \beta(2\pi)^3 \delta_{n,0} \delta(\mathbf{k}). \quad (2.3.18)$$

The first prescription takes care of a factor of  $(-i)$  for each external propagator.

The same substitution rules apply in more complicated cases such as a  $SU(N)$  gauge theory with fermions. The last substitution rule implies that at each vertex effectively a factor  $i$  disappears, and the first two together imply that propagators loose a factor of  $(-i)$ . It follows that for spin- $\frac{1}{2}$  fields the

propagator becomes

$$\tilde{S}(k) = 1/(m - \gamma \cdot k), \quad (2.3.19)$$

and that for gauge and ghost fields the Matsubara propagators are

$$\tilde{D}_{\mu\nu}^{ab}(k) = \delta^{ab}[g_{\mu\nu}/k^2 - (1 - \lambda^{-1})k_\mu k_\nu/k^4], \quad (2.3.20)$$

$$\tilde{D}_g^{ab}(k) = \delta^{ab}/k^2. \quad (2.3.21)$$

These expressions are the Fourier transforms of the corresponding Euclidean propagators; see e.g. (2.3.9). Because of the periodic boundary conditions, discussed at length in section 2.2.3, the Matsubara frequencies of the ghost propagator are even, like those of the gauge boson propagator:  $k_0 = i2n\pi\beta^{-1}$ .

In actual computations the above rules are supplemented with the usual prescriptions such as a minus sign for every fermion and ghost loop. Moreover, one may take over the renormalization prescriptions of the vacuum theory to eliminate all ultraviolet divergencies of the corresponding field theory at finite temperature and density [2.8, 2.10, 6.3, 9.22] (see also section 2.4.6). In practice, a convenient way of regulating the perturbation theory is to continue dimensionally [9.22, 9.24].

### 2.3.3. Analytic continuation

The energy summations that occur in the Matsubara formalism may be performed by an analytic extension away from the discrete complex energies down to the real axis. The method is standard [1.30, 1.33] and involves the replacement of energy sums by contour integrals. For a summation over even bosonic energies  $k_0 = i2n\pi\beta^{-1}$  the rule is [6.3]:

$$\frac{1}{\beta} \sum_n f(k_0) = \int_{C_+} \frac{dz}{2\pi i} N(z)[f(z) + f(-z)] + \int_{-\infty}^{i\infty} \frac{dz}{2\pi i} f(z), \quad (2.3.22)$$

and for odd fermionic frequencies  $p_0 = i(2n+1)\pi\beta^{-1} + \mu$  [10.10],

$$\frac{1}{\beta} \sum_n f(p_0) = - \int_{C_+} \frac{dz}{2\pi i} [n_+(z)f(z) + n_-(z)f(-z)] + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z). \quad (2.3.23)$$

These formulae are valid for any function  $f(z)$  which is analytic in the neighbourhood of the imaginary axis, and which has the property that the product  $f(z)\exp -\beta|z|$  vanishes sufficiently fast at infinity. The contour  $C_+$  circumscribes *clockwise* all singularities of the functions  $f(z)$  and  $f(-z)$  in the right half plane, but none of the poles of the Fermi–Dirac and Bose–Einstein distribution functions

$$n_{\pm}(z) = [e^{\beta(z \mp \mu)} + 1]^{-1}, \quad N(z) = [e^{\beta z} - 1]^{-1} \quad (2.3.24)$$

at the Matsubara frequencies.

One may notice that no use is made of the convergence factor which plays a crucial role in

non-relativistic many-body theory [1.19–1.22, 1.30, 1.33, 11.9]. This convergence factor enables one to close the vacuum contour in the left half plane. In a theory without antiparticles this amounts to a specific renormalization prescription to eliminate vacuum terms.

Let us consider the application of these formulae to the analytic continuation of a closed connected graph  $\mathcal{B}$ , called a bubble diagram. (For clarity's sake we shall discuss a scalar theory.) Leaving out numerical and vertex factors, we may write the typical contribution coming from such a graph with  $I$  internal lines and  $V$  vertices as

$$\mathcal{B}(I, V) = \text{Tr}_{i=1}^I \tilde{\Delta}(k_1) \cdots \tilde{\Delta}(k_I) \delta(k_1, \dots, k_I), \quad (2.3.25)$$

where the trace symbol denotes both integrations and summations over the internal momenta and energies, respectively. The  $\delta$ -factor enforces energy and momentum conservation at  $V - 1$  vertices. Using for the Kronecker delta its integral representation, we have

$$\delta(k_1, \dots, k_I) = \prod_{v=1}^{V-1} (2\pi)^3 \delta(\epsilon_{vi} k_i) \int_0^\beta du_1 \cdots \int_0^\beta du_{V-1} \exp(u_v \epsilon_{vi} k_i^0), \quad (2.3.26)$$

where  $\epsilon_{vi}$ ,  $v = 1, 2, \dots, V$ ,  $i = 1, 2, \dots, I$  is the incidence matrix [12.7, 12.18] of the graph  $\mathcal{B}$ .

The most straightforward approach to the problem of evaluating the multiple frequency sums in (2.3.25) is to introduce  $L = I - V + 1$  independent loop variables  $l_j$  by making use of energy-momentum conservation at the vertices. Then (2.3.25) reduces to

$$\mathcal{B}(I, V) = \text{Tr}_{l_1, \dots, l_L} \prod_{i=1}^I \tilde{\Delta}(k_i), \quad (2.3.27)$$

where the  $k$ 's are now linear combinations of the  $l$ 's with coefficients  $\pm 1$  or 0. The remaining  $L$  frequency summations can then be done one after another with the help of the contour formulae (2.3.22) and (2.3.23). Singularities occur when the internal momenta are on their mass shell. If none of these singularities overlap, the analytic continuation of the propagator product in (2.3.27) has simple poles which generate branch cuts when the momentum integrations are carried out. The residues must be calculated as the discontinuities across these cuts; see fig. 2.2.

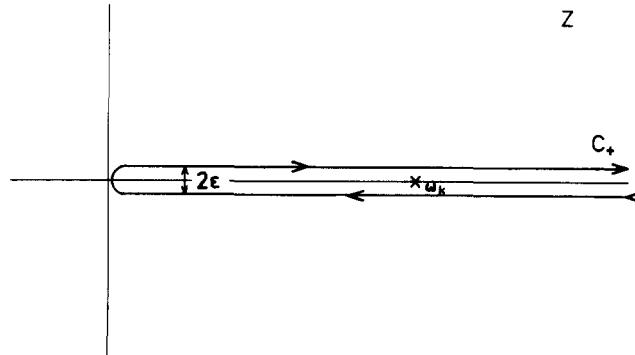


Fig. 2.2. The contour  $C_+$  encircling a branch cut.

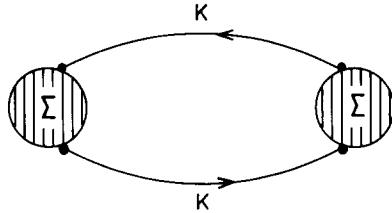


Fig. 2.3. A 2-cycle of lines.

In general, one encounters in higher-order calculations direct products  $\tilde{\Delta}^m(k) = [\tilde{\Delta}(k)]^m$  of propagators with identical denominators. In diagrammatic language such products are associated with a cycle of lines of multiplicity  $m$  [1.24, 1.25]. This is a set of  $m$  lines in a bubble diagram such that the diagram separates into two disconnected pieces if any two of the lines of the set are cut. Removal of all  $m$  lines separates the diagram into  $m$  disconnected parts which are at least one-particle irreducible (1PI); that is, they are self-energy insertions. The lines in a bubble diagram can unambiguously be classified as belonging to cycles with multiplicities  $m = 2, 3, \dots$ , or to no cycle at all, which is called a 1-cycle for convenience. On account of energy-momentum conservation all lines of an  $m$ -cycle carry the same energy and momentum. This gives rise to the occurrence of higher-order poles when the associated propagator products are analytically continued to the complex plane. Nevertheless, the above reasoning goes through almost unchanged [10.10, 10.15]. Again the residue may be computed as the discontinuity across the cut by making use of the formula

$$\int_{C_+} \frac{dz}{2\pi i} \frac{g(z)}{(z-a)^m} = \int_{-\infty}^{\infty} \frac{dx}{2\pi i} g(x) \operatorname{Disc} \frac{1}{(x-a)^m} \quad (2.3.28)$$

pertaining to a pole of order  $m$  at the real axis.

A direct analytic continuation of Matsubara diagrams without contour integration has been proposed by Dzyaloshinskii [1.20], and independently by Balian and de Dominicis [1.15] and by Baym and Sessler [1.23]. The starting point is formula (2.3.25), but before doing the frequency sums one substitutes the following time representation for the propagators:

$$\tilde{\Delta}(i\omega_n, k) = \int_0^\beta d\tau \exp(i\omega_n \tau) \Delta(\tau, k), \quad (2.3.29)$$

$$\Delta(\tau, k) = \int \frac{dk_0}{2\pi} \rho_0(k) \exp(-\tau k_0) [\theta(\tau) + N(k_0)], \quad (2.3.30)$$

obtained from (2.3.2) and (2.3.3) by reverting from frequency to time. This has the advantage that the frequency summations can now be trivially performed, and one finds

$$\mathcal{B}(I, V) = \int \prod_{i=1}^I \frac{d^3 k_i}{(2\pi)^3} \prod_{v=1}^{V-1} (2\pi)^3 \delta(\epsilon_{v,i} k_i) \int_0^\beta du_1 \cdots \int_0^\beta du_{V-1} \prod_{i=1}^I \Delta(\tau_i, k_i), \quad (2.3.31)$$

where the times

$$\tau_i = - \sum_{v=1}^{V-1} \epsilon_{vi} u_v \quad (2.3.32)$$

are related to the integration variables through the incidence matrix.

The multiple integral (2.3.31) may be regarded as the sum of  $(V-1)!$  integrals, where each of the latter refers to a contribution from a region distinguished by a definite ordering of the  $u$ 's. Take, for example, the contribution coming from the region  $u_1 \geq u_2 \geq \dots \geq u_{V-1}$ . This contribution may be associated with a time-ordered sequence of the vertices. (The one vertex not represented plays the unique role of being prior to the time  $u_{V-1}$  and after the time  $u_1$ .) A graphic representation is obtained by arranging the vertices in the diagram  $\mathcal{B}$  such that the time coordinates  $u_v$  corresponding to the vertices decrease from right to left [1.20]. Now the important observation is that the times  $\tau_i$  have a definite sign; on an internal line directed from right to left the sign is positive, and on a line with the opposite direction negative. In the one case the propagator (2.3.30) contributes a factor  $1 + N(k_0)$ , and in the other a factor  $N(k_0)$ . The time integrations can then be performed, and it is easy to verify that one gets exponentials and denominators containing partial sums of the energies corresponding to all intermediate states. Finally, the sum has to be taken over all possible time orderings. For sample calculations, which may clarify the reasoning, we refer to sections 3.1.3 and 4.2.2.

Norton and Cornwall [2.8] have developed a variant of the Dzyaloshinskii algorithm based on the analytic extension of the  $\delta$  symbol (2.3.26) as an entire function to the whole complex plane. This clearly exhibits the special feature of this method which is that the only singularities of the summand in (2.3.25) are the simple poles of the propagators at the real axis. It leads to the representation of a single bubble graph by a sum of real phase-space integrals over forward scattering Feynman amplitudes [2.14, 2.15, 10.14]. The external lines of these amplitudes are on-mass-shell and weighted with the appropriate Bose and Fermi distribution functions. Unfortunately, however, this physically attractive representation develops mass-shell divergencies in higher orders in perturbation theory. These divergencies are associated with propagator products having denominators which all vanish at the same momentum and energy, i.e. self-energy insertions.

The occurrence of these pathologies indicates that it may be profitable to first analyze a given diagram in terms of the  $m$ -cycles it contains. To an  $m$ -cycle one should assign the appropriate  $m$ -fold propagator product for which one may derive the following time representation:

$$\tilde{\Delta}^m(i\omega_n, k) = \int_0^\beta d\tau \exp(i\omega_n \tau) \Delta^{(m)}(\tau, k), \quad (2.3.33)$$

$$\Delta^{(m)}(\tau, k) = \int \frac{dk_0}{2\pi} \rho_0^{(m-1)} \exp(-\tau k_0) [\theta(\tau) + N(k_0)]. \quad (2.3.34)$$

The spectral density in the last expression is the  $(m-1)$ -fold derivative, i.e.,

$$\rho_0^{(m)}(k) = \frac{1}{(m)!} \left( \frac{\partial}{\partial k_0^2} \right)^m \rho_0(k), \quad (2.3.35)$$

of the single-propagator spectral density.

The Dzyaloshinskii algorithm seems to give a unique answer, whereas the result of the loop procedure apparently depends on the choice of the loop variables, and the order in which the loop

summations are performed. In the literature there has been some confusion about this issue, and it has even been suggested that the loop procedure is mathematically ill defined [6.2]. The point is that there is no unique analytic continuation of a function which is defined only on the integers. In general, different analytic continuations lead to formally different phase-space integral representations, that is, to functionally different expressions

$$\mathcal{B}(I, V) = F[n_{\pm}, N] \quad (2.3.36)$$

as a real functional of Fermi–Dirac and Bose–Einstein distribution functions.

Some of these functional representations may be particularly convenient for special purposes, such as the derivation of quasiparticle behaviour, see section 4.2.2, or the aforementioned reduction to phase-space integrals over Feynman amplitudes. However, ultimately all these representations are physically equivalent on account of the unique pole structure of a bubble diagram. Hence, as a function of the temperature and chemical potentials,  $\mathcal{B}(I, V)$  is uniquely defined. The non-uniqueness of the functional (2.3.36) may then be understood as the inherent freedom to algebraically manipulate the functional with the help of identities satisfied by the Fermi and Bose distribution functions [10.5, 10.15] as well as the generalized functions that occur. One may compare this with the freedom of renormalizing the vacuum divergences; whatever procedure is chosen, ultimately the numerical value of  $\mathcal{B}(I, V)$  must be independent of the renormalization point.

## 2.4. Real-time contours

The Matsubara formalism cannot yield real-time Green functions without a non-trivial process of analytic continuation (see section 2.3.3). A direct evaluation of real-time Green functions is possible, however, if a time contour is used which contains the real axis. The simplest family of such “real-time” contours is depicted in fig. 2.4. The path goes from  $t_i$  to  $t_f$  along the real axis, drops vertically from  $t_f$  to  $t_f - i\sigma$ ,  $0 \leq \sigma \leq \beta$ , returns parallel to the real axis to  $t_i - i\sigma$ , and ends at  $t_i - i\beta$ . By varying  $\sigma$ , an equivalence class of quantum field theories at finite temperature is generated [4.9, 4.14]. A special case is  $\sigma = 0$ , which leads to the Keldysh perturbation expansion (see section 2.1.3). The choice  $\sigma = \beta/2$  will be favoured here for reasons to become clear as we go along.

A more general family of real-time contours is obtained by going back and forth  $N$  times instead of only once [4.14]. It can be shown, however, that the contributions to the real-time Green functions of the additional sweeps  $N > 1$  cancel out. Hence, the use of these contours makes not much sense. There are also more fundamental reasons for discarding this family [4.14]. A path that goes around the real-time axis several times may be useful, however, if one is interested in the more general problem of calculating correlations between chronologically or antichronologically ordered operator products [3.22]. Also the  $N = 4$  contour has found an application in curved spacetime [8.13].

### 2.4.1. Factorization

We start from eq. (2.2.17) for the generating functional, with  $C$  the real-time contour of fig. 2.4. Confining ourselves for the moment to a single scalar field, we have

$$Z_0[j] = \mathcal{N} \exp -\frac{i}{2} \left[ \int_{C_{12}} \int_{C_{12}} + \int_{C_{12}} \int_{C_{34}} + \int_{C_{34}} \int_{C_{12}} + \int_{C_{34}} \int_{C_{34}} \right] d^4x d^4x' j(x) D^{(c)}(x - x') j(x') . \quad (2.4.1)$$

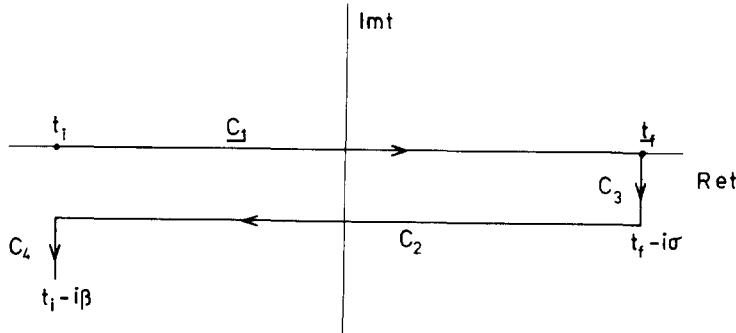


Fig. 2.4. Real-time contours.

The integration contour has been split into four pieces, as indicated in the figure, and  $C_{rs} = C_r \cup C_s$ . Now consider the case  $t$  on  $C_1$ , and  $t'$  on  $C_3$ . We parametrize  $C_3$  by  $t' = t_f - i\lambda$ ,  $0 \leq \lambda \leq \sigma$ , and use the fact that  $\theta_c(t - t') = 0$  to write the corresponding propagator as, see (2.2.33),

$$iD^{(13)}(x - x') = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \exp[ip_0(t_f - t)] \tilde{F}(p_0), \quad (2.4.2)$$

with

$$\tilde{F}(p_0) = \eta \int \frac{d^3 p}{(2\pi)^3} \exp[i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')] \rho_0(p) n_+(p_0) \exp(\lambda p_0), \quad (2.4.3)$$

where only the relevant  $p_0$  dependence has been indicated. As long as we have  $0 < \lambda < \beta$ , the combination  $n_+(p_0) \exp(\lambda p_0)$  damps for large  $|p_0|$ , so that  $D^{(13)}(x - x')$  is analytical for these values of  $\lambda$ , in agreement with the discussion in subsection 2.1.2.

Now, in order that the contour piece  $C_3$  can be ignored for large  $t_f$ , we have to show that the right-hand side of eq. (2.4.2) vanishes as  $t_f$  approaches infinity. The Riemann–Lebesgue lemma [5.23] states that the Fourier transform maps the function space  $L^1$  onto the space of continuous functions which vanish at infinity. Regarding the propagator (2.4.2) as a Fourier transform of  $\tilde{F}(p_0)$ , we see that we can apply the Riemann–Lebesgue lemma if: (i)  $\tilde{F}(p_0)$  is integrable, i.e. the spectral density  $\rho_0(p)$  in (2.4.3) is an ordinary function, and (ii)  $t \neq t_f$ .

The explicit form (2.2.35) makes it clear that the spectral density becomes a generalized function in the limit  $\varepsilon \rightarrow 0$ . Since this invalidates condition (i), we conclude that we must hold  $\varepsilon$  uniformly finite throughout, and take the limit  $\varepsilon \rightarrow 0$  last of all. In other words, delta functions should always be understood as regularized. One may easily verify the necessity of this  $\varepsilon$  regularization by taking  $\tilde{F}(p_0) = \delta(p_0)$  in (2.4.2). It is important to point out that this prescription does not violate the KMS condition provided the argument  $p_0$  of the distribution function in (2.4.3) is left as it is, and is not replaced by its mass-shell value  $\pm \omega_p$  [4.11, 3.23].

Condition (ii) is not satisfied for  $t \rightarrow \infty$ . However, we still obtain the desired result

$$\lim_{t_f \rightarrow \infty} \int \int_{C_1 C_3} d^4 x d^4 x' j(x) D^{(c)}(x - x') j(x') = 0, \quad (2.4.4)$$

if we have in addition

$$\lim_{t \rightarrow \pm\infty} j(x) = 0. \quad (2.4.5)$$

Imposing this condition on the sources is equivalent to restricting the fields occurring in the path integral to those satisfying free-field equations at  $t = t_i \rightarrow -\infty$  and  $t = t_f \rightarrow \infty$ . Physically, this condition plays the same role as the Bogoliubov condition [11.2] in the Keldysh formalism; cf. section 2.1.3. For relativistic local-field theories with the usual canonical commutation relations, this extra boundary condition is compatible with the KMS boundary condition. Therefore, in our case we can consistently impose (2.4.5), and conclude that (2.4.4) holds true.

A similar reasoning applied to the contour pieces  $C_2$  and  $C_4$  leads to the conclusion that the second as well as the third contour contributions to (2.4.1) vanish in the limit  $t_i \rightarrow -\infty$ ,  $t_f \rightarrow \infty$ . Moreover, we obviously have

$$\frac{\delta j(x')}{\delta j(x)} = 0, \quad t \in C_{12}, \quad t' \in C_{34}, \quad (2.4.6)$$

or vice versa. This brings us to the important result [3.7, 3.11] that the full generating functional factorizes into a contribution from  $C_{12}$  and a contribution from  $C_{34}$ :

$$Z[j] = \mathcal{N} Z_{12}[j] Z_{34}[j] \quad (2.4.7)$$

with the two partial generating functionals given by

$$Z_{12}[j] = \exp \left\{ i \int_{C_{12}} d^4x \mathcal{L}_1 \left[ \frac{\delta}{i \delta j} \right] \right\} \exp \left\{ -\frac{i}{2} \int_{C_{12}} d^4x d^4x' j(x) D^{(c)}(x - x') j(x') \right\}, \quad (2.4.8)$$

and a similar expression for  $Z_{34}[j]$ .

For the computation of real-time Green functions the factor  $Z_{34}$  is a multiplicative constant like  $\mathcal{N}$ , which entirely drops out. For this purpose the vertical contour pieces  $C_{34}$  can be ignored completely, and we shall do so from now on. On the other hand, according to eq. (2.1.2), (2.1.5) and (2.2.3), the pressure is given by

$$P = \frac{1}{\beta V} (\log Z_{12}[0] + \log Z_{34}[0] + \log \mathcal{N}), \quad (2.4.9)$$

and  $Z_{34}$  contributes. As a consequence, the computation of the pressure requires the introduction of special techniques (see chapter 4).

For completeness' sake, we mention that recently a class of non-relativistic theories has been discovered for which the asymptotic conditions (2.4.5) cannot be imposed, and the factorization (2.4.7) does not occur [4.23, 4.31, 4.32]. In other words, Bogoliubov's initial condition is not applicable. Among such models are the spin system [4.36–4.38] and the Anderson model [4.22, 4.39], wherein the asymptotic condition is violated by certain discrete zero-energy modes. It is very difficult to construct a Feynman diagram method for these models, even at zero temperature, but they have successfully been treated in the formalism of thermo field dynamics.

Another problematic case concerns massless field theories, especially in lower dimensions [4.2]. For

these it may be impossible to use the Riemann–Lebesgue lemma because the integral (2.4.3) is infra-red divergent. The ultimate cancellation of infra-red divergences only takes place after a certain resummation of terms which also involve  $Z_{34}$  [3.16]. In the sequel we shall only discuss theories having the factorization property (2.4.7).

### 2.4.2. Real-time propagators

In the preceding subsection we have shown that the contour pieces  $C_3$  and  $C_4$  can be discarded entirely if one is interested in real-time Green functions. Then one only needs to calculate  $Z_{12}[j]$  as given in (2.4.8). Omitting spatial variables for brevity, we can write the first exponential as

$$\exp i \int_{C_{12}} dt \mathcal{L}_l \left[ \frac{\delta}{i \delta j} \right] = \exp i \int_{-\infty}^{\infty} dt \left( \mathcal{L}_l \left[ \frac{\delta}{i \delta j_1} \right] - \mathcal{L}_l \left[ \frac{\delta}{i \delta j_2} \right] \right), \quad (2.4.10)$$

where we defined:

$$j_1(t) = j(t), \quad j_2(t) = j(t - i\sigma), \quad (2.4.11)$$

which are to be regarded as independent sources. The minus sign in (2.4.10) is a consequence of the negative orientation of  $C_2$ . Similarly, the second exponential is rewritten as

$$\exp -\frac{i}{2} \int_{C_{12}} \int dt dt' j(t) D^{(c)}(t - t') j(t') = \exp -\frac{i}{2} \int_{-\infty}^{\infty} dt dt' j_r(t) D^{(rs)}(t - t') j_s(t'), \quad (2.4.12)$$

which involves the following four real-time propagators:

$$D^{(11)}(t - t') = D^{(c)}(t - t'), \quad (2.4.13)$$

$$D^{(22)}(t - t') = D^{(c)}((t - i\sigma) - (t' - i\sigma)) = -[D^{(c)}(t' - t)]^*, \quad (2.4.14)$$

$$D^{(12)}(t - t') = D^{(c)}(t - (t' - i\sigma)) = D^<(t - t' + i\sigma), \quad (2.4.15)$$

$$D^{(21)}(t - t') = D^{(c)}((t - i\sigma) - t') = D^>(t - t' - i\sigma). \quad (2.4.16)$$

The right-hand sides of (2.4.15) and (2.4.16) follow by observing that times on  $C_2$  are always “later” than those on  $C_1$ . The last member of (2.4.14) reflects the antichronological ordering on  $C_2$ . To avoid later confusion we explicitly state here that complex conjugation of Green functions only refers to explicit factors of  $i$  and  $i\varepsilon$ , and not to possible group matrices, Klein–Gordon divisors, etc., cf. section 3.2.1. (This is the same convention as used in unitary equations [12.9].)

In view of eq. (2.4.10), one would perhaps have expected a minus sign in eqs. (2.4.15) and (2.4.16). However, in eqs. (2.4.10) through (2.4.16) it is understood that

$$\frac{\delta j_r(t)}{\delta j_s(t')} = \delta_{rs} \delta(t - t'), \quad r, s = 1, 2, \quad (2.4.17)$$

in contravention of what is prescribed by rule (2.1.15) which implies a minus sign on  $C_2$  due to the negative orientation. From now on we will adopt the new sign convention (2.4.17). (Also cf. [3.17].)

Equations (2.4.10) and (2.4.12) imply that the generating functional  $Z_{12}$ , henceforth denoted as  $Z$ , takes the form [4.13, 4.11, 3.11, 3.12]

$$Z[j] = \exp\left\{ i \int d^4x \left( \mathcal{L}_I \left[ \frac{\delta}{i \delta j_1} \right] - \mathcal{L}_I \left[ \frac{\delta}{i \delta j_2} \right] \right) \right\} \exp\left\{ -\frac{i}{2} \int_{-\infty}^{\infty} \int d^4x d^4x' j_r(x) D^{(rs)}(x - x') j_s(x') \right\}. \quad (2.4.18)$$

A summation over the repeated indices  $r, s$  is understood.

The equivalent path-integral representation is

$$Z[j] = \int [d\phi_1][d\phi_2] \exp i \int d^4x [\phi_r(D^{-1})^{(rs)} \phi_s + \mathcal{L}_I(\phi_1) - \mathcal{L}_I(\phi_2) + j_r \phi_r]. \quad (2.4.19)$$

The boundary conditions are unimportant here because they are only necessary for the determination of the propagators which we already know. From the generating functional, real-time Green functions are obtained by functional differentiation

$$G(x_1, \dots, x_n) = \frac{1}{Z[0]} \frac{\delta}{i \delta j_1(x_1)} \cdots \frac{\delta}{i \delta j_1(x_n)} Z[j] \Big|_{j_1=j_2=0}, \quad (2.4.20)$$

with respect to the c-number sources  $j_1(x)$ . This implies that only type-1 fields can appear on external lines. We remark, that in the operator formalism the fields  $\phi_1$  and  $\phi_2$  can be understood as the original real-time field and its shifted counterpart, respectively, analogous to (2.4.11) for the sources. This interpretation is possible in the path integral (2.4.19) too [3.11, 3.12]. However, it is not compulsory there because the  $\phi$ 's are just dummy integration variables.

By this construction it appears that a doubling of the degrees of freedom is necessary in order to be able to calculate real-time Green functions. This doubling is absent in the Matsubara formalism, and thence could be a mathematical artifact [8.18, 4.42]. However, such a doubling of the field degrees of freedom also appears in the axiomatic formulation of quantum statistical mechanics; see section 2.5. This indicates that a two-component extension is the prerequisite for a consistent Minkowski-space field theory at finite temperature and density.

We will now consider the general case of an arbitrary covariant field. Mutatis mutandis the above derivation goes through unchanged, and we can give the explicit forms of the real-time propagators defined in (2.4.13) through (2.4.16). Since the general contour propagator is known, cf. (2.2.33), it is a straightforward exercise to determine the Fourier-transformed expressions:

$$i\tilde{D}_{\alpha\beta}^{(11)}(p) = [i\theta(p_0)\tilde{\Delta}_F(p) + i\theta(-p_0)\tilde{\Delta}_F^*(p) + \eta\rho_0(p)n_+(p_0)]d_{\alpha\beta}(p), \quad (2.4.21)$$

$$i\tilde{D}_{\alpha\beta}^{(22)}(p) = [i\tilde{D}_{\alpha\beta}^{(11)}(p)]^*, \quad (2.4.22)$$

$$i\tilde{D}_{\alpha\beta}^{(12)}(p) = \eta\rho_0(p)\exp(p_0\sigma)n_+(p_0)d_{\alpha\beta}(p), \quad (2.4.23)$$

$$i\tilde{D}_{\alpha\beta}^{(21)}(p) = \eta e^{-\beta\mu}\exp[(\beta - 2\sigma)p_0]i\tilde{D}_{\alpha\beta}^{(12)}(p). \quad (2.4.24)$$

The function

$$\tilde{\Delta}_F(p) = \prod_l \frac{1}{p^2 - m_l^2 + i\epsilon} \quad (2.4.25)$$

is the vacuum multi-mass Feynman propagator in terms of which the spectral density (2.2.35) is given by

$$\rho_0(p) = i\epsilon(p_0)[\tilde{\Delta}_F(p) - \tilde{\Delta}_F^*(p)]. \quad (2.4.26)$$

The  $\epsilon$ -parameter which regulates these expressions, has been kept uniformly finite during the calculations, in conformity with the remarks made in the preceding subsection. Substituting eq. (2.4.26) into the propagator expression (2.4.21), the latter can be cast in the form

$$\tilde{D}_{\alpha\beta}^{(11)}(p) = \{[1 + \eta n(p_0)]\tilde{\Delta}_F(p) - \eta n(p_0)\tilde{\Delta}_F^*(p)\}d_{\alpha\beta}(p), \quad (2.4.27)$$

where the positive- and negative-energy distribution functions have been combined into the single distribution function

$$n(p_0) = \theta(p_0)n_+(p_0) + \theta(-p_0)n_-(-p_0), \quad (2.4.28)$$

which is never negative and vanishes in the zero temperature and density limit.

The off-diagonal propagators (2.4.23) and (2.4.24) explicitly depend on the parameter  $\sigma$  which labels the family of contours illustrated in fig. 2.4. It is rather obvious, and we shall prove this in section 2.4.3, that the real-time Green functions will be independent of  $\sigma$ . So one is free to adjust  $\sigma$  at one's convenience. If one takes  $\sigma = 0$ , one recovers Keldysh's finite-temperature field theory of section 2.1.3 as it applies to equilibrium systems. A particularly elegant formulation is obtained by setting  $\sigma = \beta/2$ , and following refs. [4.9, 3.11, 3.12] we shall adopt this choice here. Using (2.4.26) and (2.4.28) we may then write the propagator (2.4.23) as

$$\tilde{D}_{\alpha\beta}^{(12)}(p) = e^{\beta\mu/2}\sqrt{n(p_0)}\sqrt{1 + \eta n(p_0)}[\eta\theta(p_0) + \theta(-p_0)][\tilde{\Delta}_F(p) - \tilde{\Delta}_F^*(p)]d_{\alpha\beta}(p). \quad (2.4.29)$$

Note that in this case the connection (2.4.24), which is a consequence of definitions (2.4.15) and (2.4.16), and the KMS condition, is particularly simple.

The special feature of the two propagators (2.4.27) and (2.4.29), together with the ones implied via (2.4.22) and (2.4.24), is that they can be assembled into the single  $2 \times 2$  matrix

$$\tilde{D}_{\alpha\beta}(p) = M_\eta \begin{pmatrix} d_{\alpha\beta}(p)\tilde{\Delta}_F(p) & 0 \\ 0 & -d_{\alpha\beta}(p)\tilde{\Delta}_F^*(p) \end{pmatrix} M_\eta. \quad (2.4.30)$$

All thermal information is contained in the transformation matrix [3.19]

$$M_\eta = \begin{pmatrix} \cos(h)\Theta_p & \eta e^{\beta\mu/2} \sin(h)\Theta_p \\ e^{-\beta\mu/2} \sin(h)\Theta_p & \cos(h)\Theta_p \end{pmatrix}, \quad (2.4.31)$$

where the angle  $\Theta_p$  is implicitly given by

$$\sin(\hbar) \Theta_p = \sqrt{n(p_0)}, \quad (2.4.32)$$

$$\cos(\hbar) \Theta_p = [\theta(p_0) + \eta\theta(-p_0)]\sqrt{1 + \eta n(p_0)}. \quad (2.4.33)$$

The goniometric (hyperbolic) functions arise in the fermionic (bosonic) case. The possibility of representing thermal propagators in this way was first discovered by Matsumoto in the context of thermo field dynamics [4.5, 4.7]. It is to be remarked that the matrix (2.4.31) has the alternative representation [3.24, 4.31]

$$M_\eta = \exp \Theta_p \begin{pmatrix} 0 & \eta e^{\beta\mu/2} \\ e^{-\beta\mu/2} & 0 \end{pmatrix}. \quad (2.4.34)$$

The above formulae contain the real-time propagators for fermion and gauge fields derived in refs. [4.8] and [3.14] as special cases.

It is easily seen what happens in the vacuum limit  $\mu = 0, \beta \rightarrow \infty$ . In this limit the distribution function  $n(p_0)$ , given in (2.4.28), is zero. As a consequence the cross propagators are zero too, and  $M_\eta$  reduces to a trivial diagonal matrix which can be omitted. This means that type-1 and type-2 fields decouple, and that the generating functional completely factorizes:  $Z_{12} = Z_1 Z_2$ ,  $Z_2$  dropping out in the evaluation of vacuum Green functions.

In the case of a degenerate Fermi system at zero temperature the distribution function (2.4.28) reduces to

$$n(p_0) = \theta(p_0) \theta(\mu - p_0) + \theta(-p_0) \theta(p_0 - \mu). \quad (2.4.35)$$

This implies again that the cross propagators vanish and that the generating functional factorizes. In the single-mass case the relevant propagator follows from (2.4.27) and (2.4.35) as

$$\tilde{D}_{\alpha\beta}^{(11)}(p) = \frac{d_{\alpha\beta}(p)}{p^2 - m^2 + i\epsilon p_0(p_0 - \mu)}, \quad (2.4.36)$$

which exhibits indeed the correct pole prescription [6.4].

### 2.4.3. Real-time Feynman rules

We are now in a position to state the Feynman rules. In the expansion of the generating functional (2.4.18) two types of vertices occur, one type describing the interactions of the original real-time field  $\phi_1$  and the other the interactions of the “thermal ghost field”  $\phi_2$ . The Feynman rules for these two types of vertices differ by a sign only. There is no direct coupling between the two types of fields, but they can propagate into each other because of the non-diagonal elements of the propagator matrix. In virtue of (2.4.20), external lines of physical Green functions are always of type-1. Therefore, to find a particular Green function  $\tilde{G}^{(11)}(p_1, \dots, p_n)$  in momentum space, we must draw all diagrams with  $n$  external points of type-1, and an arbitrary number of vertices of type  $r = 1, 2$ . These two types of vertices are connected to each other by (directed) propagator lines representing  $\tilde{D}_{\alpha\beta}^{(rs)}(p)$ . (The lattice so obtained is reminiscent of the Wheeler–Widom construction [11.3].) As in the vacuum theory, disconnected blobs

without external legs are divided out by the denominator in (2.4.20), and can be ignored. (In fact, they equal unity, cf. (2.4.44).)

The propagators can be directly read off from eqs. (2.4.27), (2.4.29). For later applications we shall give the explicit Feynman rules for a scalar boson theory with interaction Lagrangian density

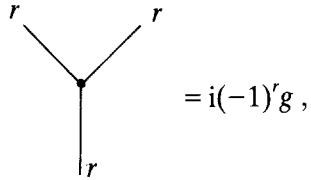
$$\mathcal{L}_I = -\frac{g}{3!} \phi^3 - \frac{\lambda}{4!} \phi^4. \quad (2.4.37)$$

In fig. 2.5 we define the propagator and vertices for this theory. Following [4.13] we decompose propagator lines into a sum of the vacuum part ————— and the thermal part —————. This graphical notation [3.11, 3.12] stems from the formal similarity between the Feynman rule for the thermal propagator and the cut propagator used in unitarity equations [12.9]. The two vacuum propagator parts read explicitly

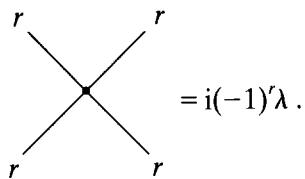
$$1 \xrightarrow[k]{\text{———}} 1 = \frac{i}{k^2 - m^2 + i\epsilon}, \quad (2.4.38)$$

$$2 \xrightarrow[k]{\text{———}} 2 = \frac{-i}{k^2 - m^2 - i\epsilon}. \quad (2.4.39)$$

$$r \xrightarrow[k]{\text{———}} s = i\tilde{D}^{(rs)}(k), \quad r, s = 1, 2.$$



$$= i(-1)^r g,$$



$$= i(-1)^r \lambda.$$

Fig. 2.5. Feynman rules for a theory characterized by the Lagrangian density (2.4.37).

In the scalar boson case there is no chemical potential and the thermal propagator is symmetric with equal diagonal elements. The two independent elements are

$$1 \xrightarrow[\text{———}]{\text{———}} 1 = 2\pi\delta(k^2 - m^2) \sinh^2 \Theta_k, \quad (2.4.40)$$

$$1 \xrightarrow[\text{———}]{\text{———}} 2 = \pi\delta(k^2 - m^2) \sinh 2\Theta_k. \quad (2.4.41)$$

The angle  $\Theta_k$  is given by, cf. (2.4.32), (2.4.33),

$$\sinh^2 \Theta_k = N(|k_0|) \quad (2.4.42)$$

in terms of the Bose function. We stress once more here that the delta functions are to be regarded as an abbreviation for their regularized representation; only in the simplest diagrams the delta functions can be taken literally. Also the distribution function has to be kept off-shell; cf. the discussion in section 2.4.1.

Another comment concerns the relation with the Matsubara formalism (to be more fully discussed in section 3.1). It so happens that the Euclidean propagator (2.3.2) is nothing but the Fourier transform of  $\tilde{D}^{(11)}(k)$  extended to the imaginary-time axis. In the early days of relativistic finite-temperature field theory [2.6] this was interpreted to mean that  $\tilde{D}^{(11)}(k)$  was the “analytic continuation” of the discrete-energy Matsubara propagator (2.3.4) to the real axis. However, using only the single propagator  $\tilde{D}^{(11)}(k)$  in a real-time formulation quickly leads to pathologies, such as the occurrence of products of delta functions. These difficulties do not arise when the full matrix structure of the propagator is taken into account; a detailed proof will be deferred to section 3.2.2.

Let us now demonstrate explicitly that the parameter  $\sigma$  which labels the family of contours depicted in fig. 2.4, completely drops out of the real-time Green functions. For the proof, which we take from ref. [4.9] (also cf. [4.14]), we need the observation that the  $\tilde{D}^{(11)}$  and  $\tilde{D}^{(22)}$  propagators are independent of  $\sigma$ , whereas  $\tilde{D}^{(12)}$  and  $\tilde{D}^{(21)}$  are proportional to  $\exp(p_0\sigma)$  and  $\exp(-p_0\sigma)$ , respectively; see (2.4.21) through (2.4.24). The external legs of all diagrams contributing to real-time Green functions are of type-1. Any such diagram can be decomposed into subdiagrams which only contain type-1 vertices (I) and those that only contain type-2 vertices (II); see fig. 2.6. The propagator lines in the blobs represent either  $\tilde{D}^{(11)}$  or  $\tilde{D}^{(22)}$  propagators. Hence, the blobs are  $\sigma$ -independent. However, the external legs of II represent either  $\tilde{D}^{(12)}$  or  $\tilde{D}^{(21)}$ . Since the  $\sigma$ -dependence is a multiplicative factor, we conclude that diagram II is proportional to  $\exp(\sigma(p_1^0 + \dots + p_n^0))$  if all momenta are taken as outgoing. On account of global energy-momentum conservation we may conclude that diagram II is  $\sigma$ -independent like diagram I. This completes the proof.

As an interesting corollary to this result, it may be shown that the sum of all bubble diagrams, with the proper weight factors and with vertices occurring once as type-1 and once as type-2, vanishes. For the proof we set  $\sigma = \beta$  in fig. 2.4, and observe that  $C_4 = 0$  in that case. This means that  $C_3$  becomes the Matsubara contour shifted to  $t = t_f$ . Since the starting point of the Matsubara contour is arbitrary,  $C_3$  generates the full Matsubara perturbation theory. Let now  $t_f$  approach infinity. Then the factorization property (2.4.7) of the generating functional implies

$$\log Z[0] = \log(\mathcal{N} Z_{12}[0] Z_3[0]). \quad (2.4.43)$$

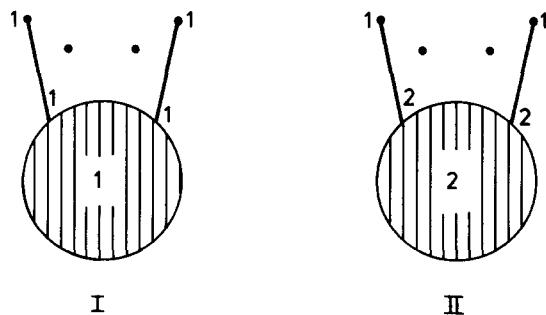


Fig. 2.6. Subdiagrams which only contain type-1 and type-2 vertices, respectively.

Since  $\mathcal{N}Z_3[0]$  gives already the complete result, we must have

$$\log Z_{12}[0] = 0, \quad (2.4.44)$$

which is the concise formulation of the theorem stated in words at the beginning of the paragraph. This theorem was obtained earlier by Kobes and Semenoff [3.15] who used the largest-time equation [12.9], and by Matsumoto et al. [4.14] in the operator formalism. However, to set the record straight, it must be mentioned that the identity  $Z_{12}[0] = 1$  was already discovered long ago in the context of the Keldysh formalism [3.5]. The implication is that disconnected diagrams exactly cancel and that Green functions (2.4.20) are automatically normalized.

#### 2.4.4. Renormalization

On the basis of the Feynman rules stated above, it is possible to prove that thermal Green functions are ultraviolet (UV) finite if the theory has been renormalized at zero temperature. This was first shown to be true in thermo field dynamics [4.13] and later in the time-path method [3.12]. A rigorous proof, involving Zimmermann's forest formula [12.7] for Feynman diagrams, was given in ref. [3.16] and will be reproduced below. The result could have been expected on physical grounds [2.2]. Indeed, the UV divergences arise from the singular short-distance behaviour of the theory, which is quite unaffected by the presence of a heat bath. In other words, in the UV limit the temperature and chemical potentials can effectively be treated as being zero. In the real-time formalism this property is immediately obvious because in the propagators (2.4.40), (2.4.41) the vacuum part is explicitly separated from the temperature-dependent part. The latter contains the thermodynamic parameters only through the distribution function, or through factors directly proportional to the latter. This is true for any field theory, as can be seen from the general structure of the propagator (2.4.27), (2.4.29). In the Matsubara formalism, before analytic continuation, this feature is less obvious. Nevertheless, the same theorem has been proved there [2.8, 2.10, 6.3, 9.22].

We shall use a dimensionally regularized version [9.22] of the original BPHZ scheme [12.7, 12.18]. Let us start with some technicalities. We first discuss the notion of a superficially divergent diagram. A given diagram  $\Gamma$  with  $n$  external legs corresponds to a Feynman integral  $F_\Gamma$ , constructed according to the Feynman rules of the theory in question in momentum space, over  $L$  loop momenta  $l_i$ . In dimensional regularization the value of this integral is defined in  $d$  dimensions. The superficial degree of divergence  $d(\Gamma)$  of the diagram is defined as the total power of all momenta in the integral, i.e. the sum of the powers of the momenta in the numerator, including  $d^d l_1 \cdots d^d l_L$ , minus the sum of the powers of the momenta in the denominator. If  $d(\Gamma) \geq 0$  for  $d = 4$ , the diagram is said to be superficially divergent. If  $\Gamma$  is also proper (or one-particle irreducible), it is called a renormalization part. If  $d(\Gamma) < 0$  for  $d = 4$ , the diagram is called superficially convergent, although the integral corresponding to it may in fact be divergent for  $d = 4$ .

We now define a quantity  $R(\Gamma)$  defined in  $d \neq 4$  dimensions. If  $\Gamma$  is not a renormalization part then  $R(\Gamma)$  is defined to be equal to the value of  $F_\Gamma$ . If  $\Gamma$  is a renormalization part  $R(\Gamma)$  is defined to be the minimally subtracted value of  $F_\Gamma$  [9.22]. In general,  $R(\Gamma)$  exists for  $d = 4$  only if  $\Gamma$  is a renormalization part. We stress that  $R(\Gamma)$  is not the renormalized value of  $\Gamma$ , which will be defined below, except when  $\Gamma$  is a renormalization part.  $R(\Gamma)$  is a function of the  $n - 1$  independent external momenta belonging to  $\Gamma$ .

Next, we define a contracted diagram. Let  $\lambda$  be a subdiagram of  $\Gamma$ . Then the contracted diagram  $\Gamma/\lambda$  is obtained by contracting  $\lambda$  to a point, that is, a vertex with  $m$  external lines carrying the momenta  $k_1, \dots, k_m$  external to  $\lambda$ . This new vertex is given the Feynman rule  $R(\lambda)$ .

We are now in a position to state Zimmermann's forest formula [12.7, 9.22]: the renormalized diagram  $r(\Gamma)$  corresponding to  $\Gamma$  is obtained by means of the forest formula

$$r(\Gamma) = \sum_U \prod_{\lambda \in U} \Gamma/\lambda \quad (2.4.45)$$

The BPHZ theorem [12.7, 12.18] then asserts that  $F_{r(\Gamma)}$  is finite for  $d = 4$ . The meaning of eq. (2.4.45) is as follows. The sum is over all  $\Gamma$ -forests  $U$ ; this is a set of subdiagrams of  $\Gamma$  satisfying the conditions:

(i) The elements of  $U$  are renormalization parts.

(ii) Two elements of the same forest  $U$  must be non-overlapping, i.e., they must be either disjoint or nested.

(iii) A forest may be empty.

If two contracted diagrams are disjoint, their order in the product is immaterial; if they are nested (one diagram being a subdiagram of the other) then the subdiagram appears on the right in the product.

For our purpose it is most important to be aware that the subdiagrams  $\lambda$  are all renormalization parts, i.e., they are all superficially divergent. Superficially convergent diagrams do not occur in eq. (2.4.45). Furthermore, we may mention that in practice the renormalized diagram is obtained by means of the introduction of counterterms.

We can now show that a diagram renormalized at  $T = 0, \mu = 0$ , and hence finite at  $T, \mu = 0$ , remains finite at any  $T$  and  $\mu$  for  $d = 4$ . Use of the forest formula in combination with the real-time Feynman rules makes the proof almost trivial. To see this we note that the thermal propagators (2.4.40), (2.4.41) go as  $\exp -\beta |k_0|$  for large  $k$ . Hence, their power-counting dimension is clearly infinite. Therefore, any  $T, \mu$ -dependent subdiagram of a given diagram  $\Gamma$  is superficially convergent (although in fact it may be divergent), and cannot be a renormalization part. But this means that the contractions in eq. (2.4.45) are all independent of the temperature and chemical potentials, and so are the counterterms. This proves the theorem.

Although it is sufficient to renormalize the theory at  $T, \mu = 0$  in order to have a finite theory at any  $T, \mu$ , it may be physically convenient to choose a renormalization prescription at a given  $T, \mu \neq 0$  [4.13, 8.5]. The renormalized masses and coupling constants then become  $T, \mu$ -dependent. Because the renormalized theory is finite at any  $T, \mu$ , quantities renormalized at different renormalization points differ by a finite renormalization. Indeed, the whole machinery of the renormalization group can be extended to the case of finite temperature and density [8.5, 8.15, 4.40, 8.21, 8.23].

## 2.5. Thermo field dynamics

We have seen that there is a certain freedom in the choice of contour in the complex time plane. In the preceding section we worked out a particular formulation of quantum field theory at finite temperature, and were led to propagator expressions in terms of a unimodular transformation matrix (section 2.4.2). In the derivation given these formulae seem to appear almost by accident. However, there exists another way of deriving them which also sheds some light on the doubling of the degrees of freedom at finite temperature. This is the algebraic operator approach known as thermo field dynamics (TFD), which we shall now discuss in some detail.

One way of understanding TFD [4.4, 4.6, 4.31] is to place this theory in the context of the  $C^*$ -algebra formulation of quantum statistical mechanics and, in particular, the so-called Haag–Hugenholtz–Winnink (HHW) formalism [5.3, 5.8], which runs parallel with a mathematical development known as the Tomita–Takesaki modular theory. Each of these topics will be briefly reviewed in

the following, although most mathematical technicalities will be left out. We mention that our presentation does not follow the historical development of TFD [4.1–4.5]; in fact, the founders of TFD independently rediscovered some basic aspects of the HHW formalism. The relationship between TFD, on the one hand, and the algebraic method, on the other, was laid down in a fundamental paper by Ojima [4.6].

### 2.5.1. GNS construction

The  $C^*$ -algebra approach to quantum statistical mechanics and quantum field theory (for a survey see refs. [5.9, 5.12, 5.20]) grew out of a general dissatisfaction with the standard formalism of quantum mechanics involving a Hilbert space of states fixed once and for all. As we shall see, for various physical as well as mathematical reasons it is more flexible to start with the algebra of operators describing the observables of the system and let the Hilbert space depend on the particular pure or mixed state of the system under consideration. In this context a state  $\psi$  is defined as a positive linear functional on the operator algebra, rather than as a vector in a given Hilbert space, or a density operator acting on the latter in the case of a mixed state. The value  $\langle \psi; A \rangle$  of the functional is interpreted as the expectation of  $A$  in the state  $\psi$ ; we shall always consider normalized states, i.e.  $\langle \psi; \mathbb{1} \rangle = 1$ , where  $\mathbb{1}$  is the unit operator in the algebra.

For physical purposes it would be sufficient to equip the operator algebra with the structure of a Jordan algebra, in which only the symmetrized operator product  $\{A, B\}$  is defined. Mathematically it is convenient to assume that the operator algebra can be embedded in a so-called  $C^*$ -algebra [5.23]. Then addition and multiplication of operators is defined with each other and with complex scalars. Moreover, a norm  $\|\cdot\|$  and involution  $*$  exist such that  $A^{**} = A$ ,  $\|A^*\| = \|A\|$ , and  $\|A\|^2 = \|A^* A\|$ .

In order to regain the ordinary structure of quantum mechanics one constructs a Hilbert space  $\mathcal{H}$  and represents the  $C^*$ -algebra  $\mathfrak{A}$  by a subset of the collection of all bounded operators  $\mathfrak{B}(\mathcal{H})$  on  $\mathcal{H}$ . The crucial point is that, in general,  $\mathcal{H}$  depends on the state  $\psi$  chosen for its construction; once a state is chosen, the GNS construction [5.9, 5.20] of the Hilbert space  $\mathcal{H}_\psi$  and the representation  $\pi_\psi(\mathfrak{A})$  induced by  $\psi$  proceeds in the following steps:

- (i) Form the set  $A_\psi$  consisting of all  $A \in \mathfrak{A}$  satisfying  $\langle \psi; A^* A \rangle = 0$ .
- (ii) Form equivalence classes  $E(A)$  in  $\mathfrak{A}$  such that  $A \sim B \Leftrightarrow A - B \in A_\psi$ .
- (iii) Construct a vector space  $V_\psi$  with elements  $E(A)$ ; addition and multiplication by scalars is defined by  $\lambda E(A) + \mu E(B) = E(\lambda A + \mu B)$ .
- (iv) One may now define a non-degenerate inner product in  $V_\psi$  by  $(E(A), E(B)) = \langle \psi; A^* B \rangle$ , which is independent of the choice of  $A \in E(A)$ ,  $B \in E(B)$ .
- (v) Complete  $V_\psi$  into a Hilbert space  $\mathcal{H}_\psi$ , which is the carrier space of the representation.
- (vi) The GNS representation  $\pi_\psi$  is then given by

$$\pi_\psi(A) E(B) = E(AB), \quad (2.5.1)$$

where  $\pi_\psi(A) \in \mathfrak{B}(\mathcal{H})$ . This is again independent of the choice of  $B \in E(B)$ . It can be shown that  $\pi_\psi$  is indeed a representation of  $\mathfrak{A}$ , that is,

$$\pi_\psi(\lambda A + \mu B) = \lambda \pi_\psi(A) + \mu \pi_\psi(B), \quad (2.5.2)$$

$$\pi_\psi(AB) = \pi_\psi(A) \pi_\psi(B), \quad (2.5.3)$$

$$\pi_\psi(A^*) = \pi_\psi(A)^\dagger, \quad (2.5.4)$$

where the dagger denotes the adjoint in  $\mathcal{H}$ .

The Hilbert space  $\mathcal{H}_\psi$  thus constructed naturally contains the vector  $\Omega := E(\mathbb{1})$ . By definition of the inner product (step (iv)) and of the representation (2.5.1), this vector satisfies  $(\Omega, \pi_\psi(A)\Omega) = \langle \psi; A \rangle$  for all  $A \in \mathfrak{A}$ . Moreover,  $\Omega$  is cyclic, i.e.  $\pi_\psi(\mathfrak{A})\Omega$  is dense in  $\mathcal{H}$ , as follows immediately from (2.5.1). We see that the vector  $\Omega$  plays the role of a state vector of the system.

Two representations  $\pi_1: \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_1)$  and  $\pi_2: \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_2)$  are said to be unitarily equivalent if there exists an isomorphism  $U: \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $\pi_2(A) = U\pi_1(A)U^{-1}$  for all  $A \in \mathfrak{A}$ . Not all representations  $\pi_\psi$  induced by the states  $\psi$  on  $\mathfrak{A}$  are unitarily equivalent. This fact is one of the main advantageous features of the  $C^*$ -algebra approach, because it avoids paradoxes encountered in the ordinary formalism where symmetries such as time translations cannot be unitarily implemented. Broken symmetry is another case in point since the different vacuum states cannot be connected by *unitary* representatives of the symmetry group in question [5.9]. What happens there is that each vacuum state induces a truly different representation of the operator algebra. A further relevant example [4.2, 4.7] is provided by the *unitary* inequivalence of GNS representations induced by two Gibbs states at different temperatures [5.20]. (Note, however, that there exists the notion of *weak* equivalence of representations in which sense all faithful representations of  $\mathfrak{A}$  are equivalent [5.9, 5.20].)

An important theorem [5.9, 5.20, 5.23], which we shall need later on, states that any representation  $\pi$  with a cyclic vector  $\Omega$  such that  $(\Omega, \pi(A)\Omega) = \langle \psi; A \rangle$  for all  $A \in \mathfrak{A}$  is unitarily equivalent to the GNS representation  $\pi_\psi$ .

A further concept of interest is the irreducibility of a representation: a representation  $\pi$  of  $\mathfrak{A}$  is called irreducible if there is no subspace  $S \subset \mathcal{H}$  so that  $\pi(\mathfrak{A})S \subseteq S$ , except  $\mathcal{H}$  itself and its zero vector. Here the relevant theorem [5.9, 5.20] is the one stating that the following conditions are equivalent:

- (i) the GNS representation  $\pi_\psi$  of  $\mathfrak{A}$  is irreducible;
- (ii) the commutant  $\pi_\psi(\mathfrak{A})'$  is trivial (i.e. equal to  $\{\lambda I\}$ ,  $\lambda \in \mathbb{C}$ );
- (iii)  $\psi$  is a pure state on  $\mathfrak{A}$ .

Here the commutant of  $\pi(\mathfrak{A})$  is the set of all operators in  $\mathcal{B}(\mathcal{H})$  that commute with all elements of  $\pi(\mathfrak{A})$ ; this notion will play a role in the sequel. A pure state  $\psi$  is a normalized state which cannot be decomposed as  $\psi = \sum \lambda_i \psi_i$ , with  $\sum \lambda_i = 1$  and all  $\psi_i$  normalized; this class contains the pure (vector) states as defined in the usual formulation of quantum mechanics [5.8, 5.20].

### 2.5.2. HHW formalism and Tomita–Takesaki theorems

Now that we do not have an a priori representation of a state as a vector in a fixed Hilbert space, the question arises how we should characterize a given state. In the case of  $\psi_\beta$  being a state so that  $\langle \psi_\beta; A \rangle$  is the expectation value of  $A$  in equilibrium at temperature  $\beta^{-1}$  (for the incorporation of the chemical potential in the present context see [5.18, 5.20]) the answer is obvious: we use the KMS condition. This presupposes a time evolution of the operators  $A \in \mathfrak{A}$ , which we denote by  $\alpha_t[A]$ . The KMS condition in its most precise formulation then reads [5.3, 5.6, 5.8]:

$$\int_{-\infty}^{\infty} dt f(t - i\beta) \langle \psi_\beta; A \alpha_t[B] \rangle = \int_{-\infty}^{\infty} dt f(t) \langle \psi_\beta; \alpha_t[B] A \rangle, \quad (2.5.5)$$

where  $f(t)$  is the Fourier transform of an arbitrary (Schwartz) test function with compact support. Equation (2.5.5) implies both the flip-flop property  $\langle \psi_\beta; \alpha_{t-i\beta}[B]A \rangle = \langle \psi_\beta; A\alpha_t[B] \rangle$ , and the fact that  $\langle \psi_\beta; A\alpha_t[B] \rangle$  is analytic and uniformly bounded in the strip  $0 < \text{Im } t < \beta$ , and continuous on its boundary. Moreover, the form (2.5.5) allows taking the thermodynamic limit [5.3, 5.8], for it avoids the operation  $\alpha_{t+i\beta}[B]$  which, in that limit, maps a bounded operator into an unbounded one.

The remarkable feature of eq. (2.5.5) is the interplay between the properties of a given state and the time evolution of the *operator*-algebra. Mathematically, the existence of such a structure and its consequences, namely the doubling of degrees of freedom in a certain sense, are the subject of the so-called Tomita–Takesaki theory of von Neumann-algebras [5.5, 5.7, 5.13, 5.15, 5.20]. The important features were independently discovered by Haag, Hugenholtz and Winnink [5.3] from a physical point of view. As this theory helps understanding thermo field dynamics, we shall briefly discuss its chief results. We begin by describing the abstract mathematical setting, which later on will be placed in the relevant physical context.

We suppose that the  $C^*$ -algebra  $\mathfrak{A}$  has been represented by bounded operators on a certain Hilbert space  $\mathcal{H}$ , for example, by means of the GNS construction. The enveloping von Neumann-algebra  $\mathfrak{M}_\mathfrak{A}$  of  $\pi(\mathfrak{A})$  is defined as its double commutant  $\pi(\mathfrak{A})''$ , which is larger than  $\pi(\mathfrak{A})$ , in general. Clearly,  $\mathfrak{M}_\mathfrak{A}'' = \mathfrak{M}_\mathfrak{A}$ , which is, in fact, the defining property of a von Neumann-algebra. We assume that the cyclic vector  $\Omega$  of this representation (see section 2.5.1) is *separating* for  $\mathfrak{M}_\mathfrak{A}$ , i.e.  $A\Omega = 0$  implies  $A = 0$  for  $A \in \mathfrak{M}_\mathfrak{A}$ .

We mention that this property is automatically satisfied if  $\pi$  is the GNS representation induced by a KMS state satisfying (2.5.5) [5.8, 5.10, 5.21]. In mathematical literature, a von Neumann-algebra thus affiliated with a cyclic and separating vector is called  $\sigma$ -finite [5.20].

Since  $\Omega$  is cyclic, every vector  $\xi$  in  $\mathcal{H}$  can be written (or arbitrarily well approximated) as  $\xi = A\Omega$  for some  $A \in \mathfrak{M}_\mathfrak{A}$ . We now define the antilinear mapping  $S: \mathcal{H} \rightarrow \mathcal{H}$  by

$$S_1: \quad SA\Omega = A^\dagger\Omega . \quad (2.5.6)$$

Just as any ordinary matrix can be written as the product of a unitary and a Hermitian matrix,  $S$  admits a unique polar decomposition

$$S_2: \quad S = J\Delta^{1/2} = \Delta^{-1/2}J , \quad (2.5.7)$$

$$S_3: \quad S^\dagger = J\Delta^{-1/2} = \Delta^{1/2}J , \quad (2.5.8)$$

with  $\Delta = S^\dagger S$  positive so that it can be raised to any power. From  $S_1 - S_3$  it is not difficult to show that  $\Delta$  and  $J$  satisfy

$$\Delta_1: \quad \Delta^\dagger = \Delta , \quad J_1: \quad J = J^\dagger , \quad (2.5.9)$$

$$\Delta_2: \quad \Delta > 0 , \quad J_2: \quad J^2 = 1 , \quad (2.5.10)$$

$$\Delta_3: \quad \Delta\Omega = \Omega , \quad J_3: \quad J\Omega = \Omega . \quad (2.5.11)$$

The main theorem of the Tomita–Takesaki theory is that  $\mathfrak{M}_\mathfrak{A}$  and its commutant  $\mathfrak{M}'_\mathfrak{A}$  are in 1-1 correspondence:

$$J_4: \quad J\mathfrak{M}_{\mathfrak{A}} J = \mathfrak{M}'_{\mathfrak{A}}. \quad (2.5.12)$$

This, in fact, gives rise to the doubled degrees of freedom in that  $\mathfrak{M}'_{\mathfrak{A}}$  is a copy of  $\mathfrak{M}_{\mathfrak{A}}$ . This is not all: let us define a one-parameter evolution in  $\mathfrak{M}_{\mathfrak{A}}$

$$\sigma_t[A] = \Delta^{-it/\beta} A \Delta^{it/\beta}, \quad A \in \mathfrak{M}_{\mathfrak{A}}. \quad (2.5.13)$$

Another important theorem asserts that

$$\Delta_4: \quad \sigma_t[\mathfrak{M}_{\mathfrak{A}}] = \mathfrak{M}_{\mathfrak{A}}, \quad (2.5.14)$$

$$\Delta_5: \quad (\Omega, A\sigma_{t+i\beta}[B]\Omega) = (\Omega, \sigma_t[B]A\Omega). \quad (2.5.15)$$

The last equality is the KMS condition! In the following subsection,  $\sigma_t$  will be identified with the physical time evolution in the representation space; in view of  $\Delta_4$  and  $\Delta_5$  this is reasonable. We see that the delicate interplay between states and evolution of operators is a consequence of the single assumption of the existence of a cyclic and separating vector  $\Omega$  in  $\mathcal{H}$ , which will be identified, of course, with a Gibbs state.

In the literature  $\Delta$  is referred to as the modular operator, and  $J$  is called the modular conjugation. To be complete, we also mention that the argument can be reversed in that  $S_1 - S_3$  follow from  $\Delta_1 - \Delta_5$  and  $J_1 - J_5$ , where

$$J_5: \quad (AJA\Omega, \Omega) \geq 0, \quad A \in \mathfrak{M}_{\mathfrak{A}}, \quad (2.5.16)$$

is a property which is not of immediate importance to us.

Below, an explicit example of the theory sketched above will be constructed.

### 2.5.3. Explicit representation

Let us now illustrate the abstract scheme of the preceding subsection. Following [5.3, 5.6, 5.8, 5.12, 5.16], we shall construct an explicit representation of the von Neumann-algebra and the operators  $J$  and  $\Delta$  considered above. We shall assume that the physical system our operator algebra  $\mathfrak{A}$  pertains to is enclosed in a fixed, finite volume  $V$ . In the end, the modifications introduced by the thermodynamic limit will be indicated.

We start with the Gibbs state  $\psi_\beta$  characterized by the KMS condition (2.5.5). We recall that the cyclic vector  $\Omega$  of the GNS representation  $\pi_\psi$  is also separating, which we need in view of section 2.5.2. Instead of directly constructing the GNS representation  $\pi_\psi$ , we shall construct a representation, also denoted as  $\pi_\psi$ , which is unitarily equivalent to it, and somewhat more manageable. To do so we start from the ordinary Fock-space representation of  $\mathfrak{A}$ , on which a Hamiltonian  $H$  and a density operator  $\rho = \exp(-\beta H)/\text{Tr exp}(-\beta H)$  are defined. (We assume that  $H$  has a discrete spectrum with finite multiplicity.) As the carrier space of  $\pi_\psi$  we take the Hilbert–Schmidt-algebra  $\mathfrak{L}(\mathcal{F})$  of all bounded operators  $A$  on the Fock space  $\mathcal{F}$  which satisfy  $\text{Tr } A^\dagger A < \infty$ . With the inner product  $(A, B) = \text{Tr } A^\dagger B$ ,  $\mathfrak{L}(\mathcal{F})$  can be completed to a Hilbert space. We may identify the operator algebra  $\mathfrak{A}$  with the algebra  $\mathfrak{B}(\mathcal{F})$  of all operators on the Fock space. Its representation  $\pi_\psi$  into operators acting on  $\mathfrak{L}(\mathcal{F})$ , regarded as a Hilbert space for this purpose, is defined by

$$\pi_\psi(A)K = AK \quad (2.5.17)$$

for any  $A \in \mathfrak{A} = \mathfrak{B}(\mathcal{F})$  and  $K \in \mathfrak{L}(\mathcal{F})$ ;  $AK$  is also Hilbert–Schmidt if  $A$  is bounded, hence  $AK \in \mathfrak{L}(\mathcal{F})$ . We remark that  $\Omega = \rho^{1/2}$ , where  $\rho$  is the density operator, is cyclic and separating in  $\mathfrak{L}(\mathcal{F})$  for the representation  $\pi_\psi$  [5.3], and in accordance with the inner product defined above we have

$$(\Omega, \pi_\psi(A)\Omega) = \text{Tr } \rho\pi_\psi(A) = \langle \psi_\beta; A \rangle. \quad (2.5.18)$$

The theorem mentioned in section 2.5.1 then implies that  $\pi_\psi$  thus constructed is unitarily equivalent to the GNS representation induced by  $\psi_\beta$ .

We shall now construct the commutant  $\pi_\psi(\mathfrak{A})'$  of  $\pi_\psi(\mathfrak{A})$ , and the operators  $\Delta$  and  $J$  of section 2.5.2. In order to find  $\pi_\psi(\mathfrak{A})'$ , we consider the antilinear representation  $\nu_\psi$  of  $\mathfrak{A}$  on  $\mathfrak{L}(\mathcal{F})$  defined by

$$\nu_\psi(A)K = KA^\dagger \quad (2.5.19)$$

for  $K \in \mathfrak{L}(\mathcal{F})$ ,  $A \in \mathfrak{B}(\mathcal{H})$ . It is then trivial to show that

$$\nu_\psi(A) \pi_\psi(B) = \pi_\psi(B) \nu_\psi(A), \quad (2.5.20)$$

and we have [5.3, 5.9]

$$\pi_\psi(\mathfrak{A})' = \nu_\psi(\mathfrak{A})''; \quad \nu_\psi(\mathfrak{A})' = \pi_\psi(\mathfrak{A}''), \quad (2.5.21)$$

where  $\nu_\psi(\mathfrak{A}'')$  and  $\pi_\psi(\mathfrak{A})'$  are both von Neumann-algebras with cyclic and separating vector  $\Omega$ , so that we can reconstruct the results of the Tomita–Takesaki theory with  $J$  and  $\Delta$  given by

$$JK = K^\dagger \quad (2.5.22)$$

for all  $K \in \mathfrak{L}(\mathcal{F})$ , and

$$\Delta = \exp - \beta \hat{H}, \quad (2.5.23)$$

with the so-called Liouville operator

$$\hat{H} = \pi_\psi(H) - \nu_\psi(H). \quad (2.5.24)$$

Indeed,  $\Delta_1$ ,  $\Delta_2$ ,  $J_1$ ,  $J_2$  are trivially satisfied;  $J_3$  follows from the operator equation  $\rho^\dagger = \rho$ , and the computation

$$\Delta\Omega = \exp[-\beta\pi_\psi(H)] \exp[\beta\nu_\psi(H)] \rho^{1/2} = e^{-\beta H} \rho^{1/2} e^{\beta H} = \rho^{1/2} = \Omega, \quad (2.5.25)$$

or, equivalently,

$$\hat{H}\Omega = 0, \quad (2.5.26)$$

establishes  $\Delta_3$ . Here we used the Campbell–Baker–Hausdorff formula and the fact that  $\pi_\psi(H)$  and  $\nu_\psi(H)$  commute. Furthermore, for any  $K \in \mathcal{L}(\mathcal{F})$  and  $A \in \mathfrak{A}$ , we have

$$J\pi_\psi(A)JK = JAK^\dagger = KA^\dagger = \nu_\psi(A)K, \quad (2.5.27)$$

so with (2.5.21)  $J_4$  also follows. To show that  $\sigma_t$  as defined in eq. (2.5.13) indeed generates time translations on both  $\pi_\psi(\mathfrak{A})$  and  $\nu_\psi(\mathfrak{A})$ , we compute [5.8]

$$\begin{aligned} \sigma_t[\pi_\psi(A)]K &= \Delta^{-it/\beta}\pi_\psi(A)\Delta^{it/\beta}K \\ &= e^{-iHt}Ae^{iHt}K = \pi_\psi(\alpha_t[A])K, \end{aligned} \quad (2.5.28)$$

where time evolution of the set  $\mathfrak{B}(\mathcal{F})$  generated by the Hamiltonian is denoted by  $\alpha_t$ ; hence  $\Delta_4$  holds true. The KMS condition follows in the usual manner. The proof for  $\nu_\psi(A)$  is similar.

One may notice that the operator  $U_t = \exp[-it\pi_\psi(H)]$  instead of  $\Delta^{-it/\beta}$  also generates time evolution in  $\pi_\psi(\mathfrak{A})$ , but it does not do so in  $\nu_\psi(\mathfrak{A})$ , nor does it leave the cyclic vector  $\Omega$  invariant as  $\Delta$  does; see  $\Delta_3$ . Therefore,  $\Delta^{-it/\beta}$  is the correct time-evolution operator in the full space  $\pi_\psi(\mathfrak{A}) \cup \nu_\psi(\mathfrak{A})$ , and the prominent role played by  $\Delta$  in the Tomita–Takesaki theory was just an indication for this.

Let us now discuss the thermodynamic limit. We should first stress that neither its existence, nor the continued validity of the KMS condition is a trivial issue, as can be seen from the literature on this subject [5.9, 5.20]. Assuming these two properties to hold, we shall indicate which features of the above representation need to be modified.

Strictly speaking, the entire construction based on the density matrix on Fock space loses its meaning in the thermodynamic limit, because the total Hamiltonian of an infinite system cannot be defined. Nevertheless, certain properties of the representation supersede the way in which they were explicitly constructed. It will be clear that those features which are direct consequences of the Tomita–Takesaki theory are preserved in the thermodynamic limit. In particular, the operators  $J$  and  $\Delta$  with the properties (2.5.25)–(2.5.27) remain well-defined, like the crucial commutant relation (2.5.21). Also, if the KMS state  $\psi$  describes a single thermodynamic phase (is extremal), the representation  $\pi_\psi$  is still primary, i.e.  $\pi_\psi(\mathfrak{A}) \cap \pi_\psi(\mathfrak{A})' = \mathbb{C}1$  (trivial centre) [5.10, 5.20]. For our purpose, the major modification concerns the Liouville operator  $H$  (2.5.24). Since  $H$ , and therefore  $\pi_\psi(H)$  and  $\nu_\psi(H)$ , have no meaning for an infinite system, the Liouvillian  $\hat{H}$  obviously can no longer be decomposed as in (2.5.24). On the other hand, the right-hand side of (2.5.24) has a finite limit when both terms tend to infinity, and  $\hat{H}$  remains well-defined (although unbounded) in the thermodynamic limit. Although it is then no longer an observable, it remains the generator of time evolution in the doubled representation space, and its action on the cyclic vector  $\Omega$  still vanishes according to (2.5.26). Mathematically, the non-decomposability of  $\hat{H}$  is related to the fact that the von Neumann-algebra  $\pi_\psi(\mathfrak{A})''$  in the thermodynamic limit is of the so-called type-III (purely infinite), as can be shown directly from the KMS condition [5.4, 5.6, 5.20]. In that limit, the KMS state  $\psi_\beta$  is no longer normal but only locally normal [5.10, 5.20].

These observations will deepen our understanding of thermo field dynamics in the sequel.

Finally, we mention that a notable feature of the Gibbs state  $\psi_\beta$  is easily derived in the above formalism, namely that it is not a pure state. Indeed, the commutant  $\pi_\psi(\mathfrak{A})'$  is non-trivial, hence, by the last theorem of section 2.5.1,  $\psi_\beta$  cannot be a pure state.

### 2.5.4. Structure of thermo field dynamics

As shown by Ojima [4.6], thermo field dynamics can be satisfactorily described in terms of the concepts introduced in the preceding subsection. As earlier, we begin by enclosing our system in a finite box, postponing the discussion of the infinite volume until section 2.5.6. For notational clarity, however, we shall write expressions like (2.5.36), (2.5.45) and (2.5.47) with a continuous momentum spectrum. Also circumflexes on Heisenberg operators will be deleted here.

We start with the scalar bosonic case. The operator algebra describing the field-theoretic system consists of the operator fields  $\phi(x)$ . (All operators are treated as bounded; this may be achieved, e.g., by introducing a suitable cutoff [5.9].) We consider the GNS representation  $\pi_\psi$  induced by the KMS state  $\psi_\beta$  at temperature  $\beta^{-1}$ . We reintroduce a bra-ket notation, and denote the cyclic vector  $\Omega$  satisfying (2.5.15) by  $|0(\beta)\rangle$ ; in the thermo field literature it is appropriately called the “thermal vacuum” [4.3]. The representative  $\pi_\psi(\phi(x))$  is simply written  $\phi(x)$ , and its modular conjugate field  $J\phi(x)J$  is denoted by  $\tilde{\phi}(x)$ , and called the tilde field [4.3]. Equation (2.5.22) implies that the tilde operator is antilinear and involutive:

$$(\lambda_1 \phi_1 + \lambda_2 \phi_2)^\sim = \lambda_1^* \tilde{\phi}_1 + \lambda_2^* \tilde{\phi}_2 ; \quad \tilde{\tilde{\phi}} = \phi . \quad (2.5.29)$$

Equation (2.5.18) implies that the thermal (grand-)canonical expectation value of an ordinary (non-tilde) operator  $A(\phi)$  is given by the “vacuum” matrix element

$$\langle A(\phi) \rangle_\beta = \langle 0(\beta) | A(\phi) | 0(\beta) \rangle . \quad (2.5.30)$$

According to eqs. (2.5.23), (2.5.24), and (2.5.28), time translations in the set of  $\phi$ 's and  $\tilde{\phi}$ 's are generated by (2.5.24):

$$\hat{H} = H - \tilde{H} \quad (2.5.31)$$

as shown in eq. (2.5.28).

In the new notation, (2.5.26) reads [4.3, 4.7]

$$\hat{H}|0(\beta)\rangle = 0 , \quad (2.5.32)$$

while invariance of the thermal vacuum under tilde conjugation is expressed by  $J_3$  in (2.5.11). Properties  $S_1$ – $S_3$  of section 2.5.2 give the so-called tilde substitution rule, also called the thermal state condition [4.3, 4.7] (for a general discussion cf. [4.3, 4.32, 4.39]),

$$e^{\beta \hat{H}/2} \tilde{\phi}(x) |0(\beta)\rangle = \phi^\dagger(x) |0(\beta)\rangle \quad (2.5.33)$$

on which many thermo field computations are based. The commutant relation (2.5.12) with (2.5.21) implies [4.3, 4.7]

$$[\phi(x), \tilde{\phi}(y)] = [\phi(x), \tilde{\phi}^\dagger(y)] = 0 . \quad (2.5.34)$$

It should be remarked that all these properties hold true irrespective of whether  $\phi(x)$  is a free or an interacting field. This is the strength of the formalism.

For a first application of the tilde substitution rule, let us assume there is no interaction, that is,

$$[\hat{H}, a_k] = [H, a_k] = -\omega_k a_k , \quad (2.5.35)$$

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 2\omega_k \delta(\mathbf{k} - \mathbf{k}') . \quad (2.5.36)$$

With the help of (2.5.32)–(2.5.35) we compute

$$\begin{aligned} \langle 0(\beta) | a_k^\dagger a_k | 0(\beta) \rangle &= \langle 0(\beta) | \tilde{a}_k e^{\beta \hat{H}} \tilde{a}_k^\dagger | 0(\beta) \rangle \\ &= e^{-\beta \omega_k} \langle 0(\beta) | a_k a_k^\dagger | 0(\beta) \rangle . \end{aligned} \quad (2.5.37)$$

This shows that the tilde substitution rule gives us the type of flip-flop formulae which are ordinarily obtained from the KMS condition; in fact, eq. (2.5.33) is nothing but a translation of the KMS condition into tilde language. Indeed, the KMS condition can directly be derived from the tilde substitution rule by using that the time evolution is generated by  $\hat{H}$ , and that tilde and non-tilde operators commute [4.7].

From (2.5.35) and the definition (2.5.31) of  $\hat{H}$ , it follows that we have

$$e^{\beta \hat{H}} a_k e^{-\beta \hat{H}} = \exp(-\beta \omega_k) a_k , \quad e^{\beta \hat{H}} \tilde{a}_k e^{-\beta \hat{H}} = \exp(\beta \omega_k) \tilde{a}_k , \quad (2.5.38)$$

$$e^{\beta \hat{H}} a_k^\dagger e^{-\beta \hat{H}} = \exp(\beta \omega_k) a_k , \quad e^{\beta \hat{H}} \tilde{a}_k^\dagger e^{-\beta \hat{H}} = \exp(-\beta \omega_k) \tilde{a}_k^\dagger . \quad (2.5.39)$$

These formulae, in conjunction with the tilde substitution rule (2.5.33) and eq. (2.5.32), imply that the “thermal annihilation operator”

$$a_k(\beta) := \sqrt{1 + N(\omega_k)} a_k - \sqrt{N(\omega_k)} \tilde{a}_k^\dagger , \quad (2.5.40)$$

with  $N(\omega_k)$  the Bose distribution function, annihilates the thermal vacuum:

$$a_k(\beta) |0(\beta)\rangle = 0 . \quad (2.5.41)$$

This suggests the definition of the thermal doublet [4.5]

$$a_k = \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} , \quad (2.5.42)$$

and the Bogoliubov transformation

$$a_k(\beta) = \begin{pmatrix} \cosh \Theta_k & -\sinh \Theta_k \\ -\sinh \Theta_k & \cosh \Theta_k \end{pmatrix} a_k , \quad (2.5.43)$$

where the Bogoliubov parameter is given by  $\sinh^2 \Theta_k = N(\omega_k)$ . The inverse of the transformation (2.5.43) contains the same transformation matrix as the time-path result (2.4.31).

Alternatively, the definition of thermal annihilation and creation operators, and their tilde counterparts can be understood as a unitary transformation [4.3, 4.7]

$$a_k(\beta) = e^{-iG} a_k e^{iG} \quad (2.5.44)$$

with  $G$  the Hermitian operator

$$G = i \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \Theta_k (\tilde{a}_k^\dagger a_k^\dagger - a_k \tilde{a}_k). \quad (2.5.45)$$

Therefore, the transformation (2.5.43) preserves the commutation relations, and is indeed a Bogoliubov transformation in the usual sense [4.7]. A physical picture [4.7] behind this Bogoliubov transformation is that  $a_k^\dagger$  creates thermally unstable quanta, whereas the quasiparticles created by  $a_k^\dagger(\beta)$  are stable in view of (2.5.41). It is a remarkable feature of thermo field dynamics that for any operator  $A$  there exists a combination of  $A$  and  $\tilde{A}^\dagger$  that annihilates the thermal vacuum as in (2.5.41) [4.16, 4.23, 4.31, 4.32, 4.39].

It is now possible to set up a perturbation theory at finite temperature in terms of the thermal vacuum and the thermal fields in the interaction picture [4.9, 4.11], exactly as at  $T=0$ , but with a doubling of the degrees of freedom. In accordance with eq. (2.5.31) the thermal Lagrangian is  $\hat{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}}$ , which leads to two types of interaction vertices, namely, type-1 for couplings of the ordinary fields, and type-2 for those of the tilde fields, differing by signs only. Defining

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \tilde{\phi}^\dagger \end{pmatrix}, \quad (2.5.46)$$

we infer from eq. (2.5.35) that  $\boldsymbol{\phi}$  and  $\boldsymbol{\phi}^\dagger$  can be expanded in annihilation and creation operators  $a_k$  and  $a_k^\dagger$ . After eq. (2.5.43) has been inverted, it is a simple exercise to derive the propagator

$$\langle 0(\beta) | T\boldsymbol{\phi}(x)\boldsymbol{\phi}^\dagger(0) | 0(\beta) \rangle = i \int \frac{d^4 p}{(2\pi)^4} \exp(-ip \cdot x) M_+ \begin{pmatrix} \Delta_F(p) & 0 \\ 0 & -\Delta_F^*(p) \end{pmatrix} M_+. \quad (2.5.47)$$

This expression is identical to the time-path result (2.4.30). Therefore, it is quite obvious that the perturbation theory thus obtained in TFD entails the same Feynman rules as the time-path method [4.9, 4.11, 4.14, 4.20].

### 2.5.5. Fermion fields

In the case of fermionic fields  $\psi(x)$  a problem arises if we define the tilde field as  $\tilde{\psi} = J\psi J$ , like for bosonic fields, because then  $\psi$  and  $\tilde{\psi}$  commute rather than anticommute. This problem has been resolved by Ojima [4.6] by using a Klein transformation, which is well known in the context of the spin-statistics theorem [5.2]. Let us define operators  $\theta$  and  $\tilde{\theta} = J\theta J$  so that

$$\theta\psi\theta = -\psi, \quad \tilde{\theta}\tilde{\psi}\tilde{\theta} = -\tilde{\psi}, \quad (2.5.48)$$

and  $\theta^2 = \tilde{\theta}^2 = 1$ . We can combine these transformation formulae into

$$\bar{\theta}\Psi\bar{\theta} = -\Psi, \quad (2.5.49)$$

where  $\Psi$  is either  $\psi$  or  $\tilde{\psi}$ , or one of their adjoints, and  $\bar{\theta} = \theta\tilde{\theta}$ . We will assume that the non-vanishing

Green functions of the theory always contain an even number of fermion fields, or, in other words, that  $\bar{\theta}$  is conserved:

$$[\hat{H}, \bar{\theta}] = 0. \quad (2.5.50)$$

Hence, the action of  $\bar{\theta}$  on any vector  $|N\rangle = \Psi_1 \dots \Psi_N |0(\beta)\rangle$  yields

$$\bar{\theta}|N\rangle = (-1)^N|N\rangle, \quad (2.5.51)$$

and  $\bar{\theta}$  is completely defined by a linear extension of (2.5.51) because the thermal vacuum is cyclic with respect to the operator algebra of the  $\Psi$ 's.

The operator  $\bar{\theta}$  may now be used to define a “true” tilde operator

$$\tilde{\psi}^\dagger = i\tilde{\psi}\bar{\theta} = iJ\psi J\theta\bar{\theta} \quad (2.5.52)$$

in terms of  $\tilde{\psi}$  by a Klein transformation. The factor  $i$  has been included so as to ensure that the true tilde operation and Hermitian conjugation commute. The modified tilde operation is still antilinear and involutive. (Neither  $\sim$  nor  $\sim t$  are the tilde operations for fermion fields as originally employed in the thermo field literature; for a comparison see ref. [4.6, 4.20]; the recent literature [4.31, 4.32] uses (2.5.52).) What is achieved by the Klein transformation (2.5.52) is that  $\tilde{\psi}^\dagger$  and  $\psi$  are now anticommuting. To show this we calculate

$$\begin{aligned} \{\psi, \tilde{\psi}^\dagger\}|N\rangle &= [i\psi\tilde{\psi}(-1)^N + i\tilde{\psi}\psi(-1)^{N+1}]|N\rangle \\ &= i(-1)^N[\psi, \tilde{\psi}]|N\rangle = 0. \end{aligned} \quad (2.5.53)$$

The proof involving adjoint field operators is similar.

From (2.5.52) we see that the tilde substitution rule for fermionic operators  $\phi$  gets a factor  $i$  on the right-hand side if the modified tilde operation is used.

Like in the boson case, we may now define the thermal annihilation operator [4.6]

$$b_p(\beta) = b_p \cos \Theta_p + i(\tilde{b}_p^\dagger)^\dagger \sin \Theta_p, \quad (2.5.54)$$

where  $\sin^2 \Theta_p = n(\varepsilon_p)$ , which annihilates the thermal vacuum. This equation can be understood as the unitary Bogoliubov transformation

$$b_p(\beta) = \exp(-iG_F) b_p \exp(iG_F), \quad (2.5.55)$$

$$G_F = \int \frac{d^3 p}{(2\pi)^3 2\omega_p} \Theta_p [b_p^\dagger (\tilde{b}_p^\dagger)^\dagger + \tilde{b}_p^\dagger b_p]. \quad (2.5.56)$$

If we define the thermal doublet [4.6]

$$\mathbf{b}_p = \begin{pmatrix} b_p \\ i(\tilde{b}_p^\dagger)^\dagger \end{pmatrix}, \quad (2.5.57)$$

we may write

$$\mathbf{b}_p(\beta) = \begin{pmatrix} \cos \Theta_p & \sin \Theta_p \\ -\sin \Theta_p & \cos \Theta_p \end{pmatrix} \mathbf{b}_p . \quad (2.5.58)$$

The matrix is the inverse of the transformation matrix  $M_-$  already encountered in section 2.4.2. It will come as no surprise that the free-field propagators turn out to be those of the time-path method; for a more detailed proof we refer to the literature [4.6, 3.14].

### 2.5.6. Discussion

The conclusion we draw is that thermo field perturbation theory coincides with the time-path perturbation theory, both for fermions and bosons. Nevertheless, it is not correct to directly equate the tilde fields of TFD with the type-2 fields, as defined in the time-path method by shifting the time axis downward and reversing its direction. The point is that different representations of the algebra of operator fields are involved. In thermo field dynamics this is the GNS representation induced by KMS states, whereas the time-path fields represent the operator algebra in ordinary Fock space. Therefore, the time-path type-1 and type-2 fields do not in general commute as do the tilde and non-tilde fields. A close formal correspondence does exist, however, in the sense that the algebra of both the type-2 and tilde fields is obtained by an anti-unitary transformation. Indeed, the modular conjugation operator  $J$  in TFD and the operation of time-reversal in the time-path method are both anti-unitary. Therefore, we can go only so far as to say that both theories are equivalent with regard to the perturbative calculation of thermal Green functions; for a detailed discussion of this point see [4.34].

In view of this correspondence, one might wonder what the freedom of contour choice in the time path method (cf. section 2.4) corresponds to in TFD. This freedom may be mimicked in TFD by redefining the tilde fields as  $\tilde{\phi}_\gamma = \exp(-\gamma \hat{H}) J \phi J \exp(\gamma \hat{H})$ . Note that  $\gamma = 0$  is the unique choice that makes tilde and Hermitian conjugation commute [4.14]. The tilde substitution rule (2.5.33) now becomes  $\tilde{\phi}_\gamma(t - i\sigma)|0(\beta)\rangle = \phi^\dagger(t)|0(\beta)\rangle$ , i.e.  $\beta/2$  is replaced by  $\sigma := \gamma + \beta/2$ . By following the same reasoning as in section 2.5.4 one may then derive  $\sigma$ -dependent free propagators which turn out to be of the form (2.4.21)–(2.4.24), cf. [4.14]. Hence,  $\sigma$  as defined above corresponds to the parameter  $\sigma$  used in section 2.4. In the special case of a system with conserved operators there is an additional freedom in the use of phase factors in the definition of the tilde operation. This modifies the second member of (2.5.29) and (2.5.33), and can also be understood as a certain redefinition of the thermal vacuum satisfying (2.5.30) and (2.5.32). For a complete analysis, see [4.20]. Choosing  $\gamma = 0$ , and the tilde operation to be involutive for both bosons and fermions in addition to using the GNS-induced cyclic vector for the thermal vacuum, as in the preceding text, seems to be the simplest and most natural choice. It corresponds to the time contour with  $\sigma = \beta/2$  of section 2.4, which also turned out to lead to the most convenient propagators.

From the discussion at the end of section 2.5.3 we can infer what consequences the thermodynamic limit has for thermo field dynamics. We shall take the existence of this limit and the KMS condition for granted, although this problem has not yet been studied for interacting field-theoretic systems. Then, the key features of thermo field dynamics, namely the tilde substitution rule (2.5.33) and the commutation relations (2.5.34) and (2.5.53), remain valid. The thermo field Hamiltonian  $\hat{H}$ , identical to the Liouville operator in the HHW formalism, retains its meaning as a well-defined generator of time evolution of the ordinary as well as the tilde fields. Also it still annihilates the thermal vacuum according to (2.5.32), but it can no longer be decomposed as in (2.5.31), because neither  $H$  nor  $\tilde{H}$  can

be defined for an infinite system. The free-field Bogoliubov transformation (2.5.40), (2.5.42), (2.5.43) (or (2.5.54), (2.5.57), (2.5.58)) preserves its proper meaning too, including the “vacuum annihilation” property (2.5.41), the latter being a consequence of the tilde substitution rule (2.5.33) and the Liouvillian time evolution (2.5.38)–(2.5.39), which both survive the thermodynamic limit. Even so, the representation (2.5.44) (or (2.5.55)) of the Bogoliubov transformation loses its meaning for an (infinite) thermodynamic system, in as much as the operator  $\exp iG$  occurring in it has no domain on the representation space involved [4.7]. This fact, however, has no direct bearing on thermo field dynamics which uses the well-defined form (2.5.43) (or (2.5.58)). It just states that the operator-algebra representation which is used in thermo field dynamics (and in the HHW formalism) is unitarily inequivalent to the Fock-space representation. In other words, the thermal vacuum  $|0(\beta)\rangle$  and the ordinary vacuum  $|0\rangle$  do not belong to the same Hilbert space.

In principle, these observations would imply that in any (non-)perturbative calculation based on thermo field dynamics one should take the thermodynamic limit last of all, for the TFD effective thermal Lagrangian is based on the decomposition (2.5.31) of the TFD Hamiltonian; in perturbation theory the latter gives the two types of interaction vertices. Holding the volume finite throughout a particular (diagrammatic) computation is obviously rather cumbersome, although this procedure gives well-defined expressions in every step of the calculation. Fortunately, in perturbation theory one can take the thermodynamic limit at once in the expression for the action, writing the latter formally as the effective Lagrangian  $\mathcal{L} - \tilde{\mathcal{L}}$  integrated over all Minkowski space. But one must simultaneously put the arguments of the distribution functions figuring in the Bogoliubov matrix (2.5.43) (or (2.5.58)) off-shell, that is, one must replace  $\omega_k$  by  $|k_0|$ , and hold the  $\varepsilon$ -regulator of the distributions involved uniformly finite until the very end. These prescriptions are identical to those in the time-path method, cf. section 2.4.1, where they were seen to be necessary in order to apply the Riemann–Lebesgue lemma which ensures factorization of the real-time path integral. We now see that they are actually related to the difficulty of taking the thermodynamic limit in a theory based on the decomposed Hamiltonian (2.5.31). The price one pays is the occurrence of potentially singular terms in intermediate steps of most calculations, but as shown in section 3.2.2 the final result is always regular, and the two stipulations mentioned above then ensure that the correct analyticity properties of the time- (energy-)dependent quantities computed, implied by the KMS condition [4.11], are automatically satisfied.

For applications of the operator formulation of TFD we refer to the book by Umezawa et al. [4.7], and to the body of work on relativistic field theory written after 1980 [8.1–8.30]. A recent summary is ref. [4.31], in which the density matrix and the super operator (Liouville space) approaches to TFD are considered as well. Other reviews are [4.17, 4.32, 4.39, 8.20].

### 3. Self-energy and full propagator

#### 3.1. Matsubara formalism

One of the simplest, and at the same time most important, quantities one would like to calculate at finite temperature, is the one-particle propagator, that is to say, the self-energy. Its real part describes the excitation spectrum, while its imaginary part, which may be non-vanishing due to thermal effects, is related to the dissipative properties of the system. As we have seen, there are two field-theoretic formalisms for doing calculations at finite temperature. The Matsubara formalism yields the temperature self-energy which, in momentum space, is defined at a discrete set of points in the complex energy

plane. One must then extend the result away from the discrete points down to the real axis to describe particles with real energy. This is not a unique operation without further delimitation. In the real-time method, on the other hand, the energies are real from the outset.

Below we shall first investigate the analytic properties of the self-energy with the help of a Lehmann spectral decomposition. This will allow us to establish the precise relationship between the temperature and real-time self-energies. We shall also briefly discuss a third type of self-energy, called statistical, obtained by a different rule of analytic continuation, which plays an important role in the Landau theory of Fermi liquids.

### 3.1.1. Analytic many-body propagator

We recall that the two-point Green function in the Matsubara formalism is defined as

$$\tilde{G}^{(E)}(k, k') = \frac{\delta^2 \log Z[\tilde{j}]}{\delta \tilde{j}(k) \delta \tilde{j}(k')} \Big|_{\tilde{j}=0}. \quad (3.1.1)$$

The generating functional as given in (2.3.14) permits the Green function to be expressed as a perturbation expansion in terms of free propagators and vertices. The diagrammatic representation consists of the sum of all connected diagrams with two external legs and with the proper combinatoric coefficients. The Feynman rules pertaining to the scalar theory were already enumerated in section 2.3.2. It is convenient to split off global energy-momentum conservation, and to write

$$\tilde{G}^{(E)}(k, k') = (2\pi)^3 \beta \delta_{n+n',0} \delta(k + k') \tilde{\mathcal{G}}(k), \quad (3.1.2)$$

which defines the full, or interacting, many-body propagator  $\tilde{\mathcal{G}}$ , also called the temperature Green function, as a function of momentum  $k$  and discrete imaginary energy  $k_0 = i\omega_n = i2n\pi\beta^{-1}$ . The Feynman rules are the same as those for the Green functions  $\tilde{G}^{(E)}$  with the understanding that the two external legs are to carry the same energy-momentum, and that the corresponding Kronecker and Dirac deltas are to be deleted.

Perturbation theory yields the full propagator as a set of Fourier coefficients. In principle, one may obtain the corresponding real-time Green function by summing the Fourier series and performing an analytic continuation in the time plane. However, one may also do the analytic continuation in the energy plane, by extending the result away from the discrete imaginary frequencies so that

$$\tilde{\mathcal{G}}(i\omega_n, k) = \Delta'(z, k) \Big|_{z=i\omega_n}. \quad (3.1.3)$$

Here  $\Delta'(z, k)$  is the analytic extension which can be chosen to be any of the infinite number of analytic continuations which coincide with the Matsubara Fourier coefficients at the discrete set of frequencies  $z = i\omega_n$ . Conventionally, one resolves this ambiguity by imposing the requirements: (i)  $\Delta'(z, p)$  goes to zero as  $|z|$  approaches infinity, and (ii)  $\Delta'(z, p)$  is analytic off the real axis. These requirements imply that this particular analytic extension has the unique [1.12, 1.18, 1.22, 1.30, 1.32] representation

$$\Delta'(z, k) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\rho(k)}{k_0 - z}. \quad (3.1.4)$$

We compare with (2.3.4) and see that the real quantity  $\rho(k)$ , defined as the discontinuity across the real axis

$$i\rho(k) = \text{Disc } \Delta'(k), \quad (3.1.5)$$

may be regarded as the spectral density of the interacting system. Indeed, this very same spectral density appears in the general Euclidean and contour propagators for interacting fields; they are of the forms (2.3.2) and (2.1.34), respectively, but with  $\rho(k)$  instead of the free-field spectral density (2.1.33).

With the help of a Lehmann spectral analysis [1.22, 1.30], it may be shown that  $\rho(k)$  is an odd function of energy with the positivity property

$$k_0 \rho(k) \geq 0, \quad (3.1.6)$$

and normalized so that

$$\int_0^\infty \frac{dk_0^2}{2\pi} \rho(k) = 1. \quad (3.1.7)$$

The first property allows us to write

$$\Delta'(z, k) = \int_0^\infty \frac{dk_0^2}{2\pi} \frac{\rho(k)}{k_0^2 - z^2}. \quad (3.1.8)$$

The positivity property then implies that  $\Delta'(z, k)$  has neither zeroes nor poles off the real axis. Furthermore, on the imaginary axis the propagator is real, positive and monotonically decreasing as  $|z|$  increases.

One implication of the fact that the analytic propagator function (3.1.4) has no complex zeroes, is that the inverse propagator also exists as an analytic function. The sum rule (3.1.7) furnishes the asymptotic behaviour of both  $\Delta'$  and its inverse at infinity:

$$\Delta'(z, k) = -\frac{1}{z^2} \left[ 1 + \frac{1}{z^2} \int \frac{dk_0^2}{2\pi} k_0^2 \rho(k) + \mathcal{O}(z^{-4}) \right], \quad (3.1.9)$$

$$\Delta'^{-1}(z, k) = -z^2 + \int \frac{dk_0^2}{2\pi} k_0^2 \rho(k) + \mathcal{O}(z^{-2}). \quad (3.1.10)$$

For the inverse propagator we write

$$\Delta'^{-1}(z, k) = \Delta^{-1}(z, k) + \Sigma(z, k), \quad (3.1.11)$$

which defines the analytic self-energy;  $\Delta(z, k) = [m^2 - z^2 + |\mathbf{k}|^2]^{-1}$  is the analytic continuation of the free propagator. Separating off the limit at infinite frequency, the remainder is analytic and approaches zero as  $|z| \rightarrow \infty$ .

Hence, we may write

$$\Sigma(z, \mathbf{k}) = \int_0^\infty \frac{dk_0^2}{2\pi} (k^2 - m^2) \rho(k) - \int_0^\infty \frac{dk_0^2}{2\pi} \frac{\sigma(k)}{k_0^2 - z^2}, \quad (3.1.12)$$

(cf. [4.11]). It can easily be shown that  $\sigma(k)$  must be odd and non-negative,  $k_0 \sigma(k) \geq 0$ , like  $\rho(k)$ .

On the imaginary axis the self-energy monotonically approaches its high-frequency limit from below. Therefore, in order that the inverse propagator has no zero on the imaginary axis, the static self-energy has to satisfy the necessary and sufficient condition

$$|\mathbf{k}|^2 + m^2 + \Sigma(0, \mathbf{k}) > 0 \quad (3.1.13)$$

for all values of  $\mathbf{k}$ . In a fully consistent theory this should come out automatically. However, one should notice that  $\Sigma$  can have either sign. Especially in a massless theory, there is the danger, therefore, that some approximate expression for  $\Sigma$  may in fact be in conflict with the positivity criterion (3.1.13). This is what actually happens in QED and QCD where the one-loop transverse polarization function violates the positivity requirement [10.11, 10.15]. This situation is the exact opposite to that for the longitudinal case: a single zero on the imaginary axis of the longitudinal propagator cannot be ruled out a priori [1.28], but the standard perturbative expressions for the longitudinal polarization function are positive definite and in agreement with (3.1.13), which excludes such a zero.

### 3.1.2. Feynman many-body propagator

To establish the connection between the real-time Green functions and the analytic propagator, we bring again to mind the generic form of the contour propagator which is (2.1.34) with  $\rho_0$  replaced by  $\rho$ . It then follows that the time-ordered Green function has the spectral form [3.7]

$$G^{(11)}(x, x') = \int \frac{d^4 k}{(2\pi)^4} \rho(k) \exp[-ik \cdot (x - x')] [\theta(t - t') + N(k_0)]. \quad (3.1.14)$$

After a Fourier transformation we get [2.6]

$$\tilde{G}^{(11)}(k) = i \int_{-\infty}^{\infty} \frac{dk'_0}{2\pi} \frac{\rho(k'_0, \mathbf{k})}{k_0 - k'_0 + i\varepsilon} + \rho(k) N(k_0). \quad (3.1.15)$$

Comparing with (3.1.4) and (3.1.5) we see that we may express the time-ordered Green function in terms of the analytic propagator as follows:

$$i\tilde{G}^{(11)}(k) = [1 + N(k_0)] \Delta'(k_0 + i\varepsilon, \mathbf{k}) - N(k_0) \Delta'(k_0 - i\varepsilon, \mathbf{k}). \quad (3.1.16)$$

The boundary values of  $\Delta'$  as  $z$  approaches the real axis from above and below may be identified with the retarded and advanced Green functions, respectively [1.22, 1.27, 1.30, 1.32, 7.4].

A quantity of interest in the present context is the (thermal) Feynman propagator

$$\tilde{\Delta}'_F(k) = \int_0^\infty \frac{dk'_0}{2\pi} \frac{\rho(k'_0, k)}{k_0^2 - k'^2 + i\epsilon}. \quad (3.1.17)$$

It can be obtained from the analytic propagator by taking the limit  $z \rightarrow k_0 + ik_0\epsilon$  corresponding to a Feynman contour in the complex energy plane:

$$\tilde{\Delta}'(k) = -\Delta'(k_0 + ik_0\epsilon, k). \quad (3.1.18)$$

The difference with the advanced and retarded propagators is the  $i\epsilon$ -prescription which now gets a sign factor. Then, by treating positive and negative energies separately, we can easily rewrite (3.1.16) as

$$\tilde{G}^{(11)}(k) = i[1 + N(|k_0|)] \tilde{\Delta}'_F(k) - iN(|k_0|) \tilde{\Delta}'_F^*(k), \quad (3.1.19)$$

with  $N(|k_0|) = \theta(k_0) N(k_0) + \theta(-k_0) N(-k_0)$ , cf. (2.4.28). It is obvious that the same answer would have been obtained by a real-time calculation; cf. (2.4.27). We conclude that the real-time interacting Green function is related to the analytic continuation of the Matsubara propagator via (3.1.18) and (3.1.19).

By a similar analysis one may verify that all real-time Green functions  $\tilde{G}^{(rs)}(k)$  may be expressed in terms of the Feynman propagator (3.1.18). In fact, the result can be represented in the same matrix form (2.4.30) as already derived for free propagators [4.13, 3.12]:

$$\tilde{G}(k) = iM_+ \begin{pmatrix} \tilde{\Delta}'_F(k) & 0 \\ 0 & -\tilde{\Delta}'_F^*(k) \end{pmatrix} M_+. \quad (3.1.20)$$

A compact spectral representation may be obtained by writing the Feynman propagator as

$$\tilde{\Delta}'_F(k) = \int_0^\infty \frac{d\omega^2}{2\pi} \rho(\omega, k) \tilde{\Delta}_F(k_0, \omega), \quad (3.1.21)$$

where in the free Feynman propagator we have replaced  $|k|^2$  by  $\omega^2 - m^2$ :

$$\tilde{\Delta}_F(k_0, \omega) := \frac{1}{k_0^2 - \omega^2 + i\epsilon}. \quad (3.1.22)$$

We then have the matrix equation [4.5, 4.13, 3.12]

$$\tilde{G}(k) = i \int_0^\infty \frac{d\omega^2}{2\pi} \rho(\omega, k) \tilde{D}(k_0, \omega) \quad (3.1.23)$$

due to the identical matrix structures of the interacting and non-interacting propagators.

Finally, we show that the matrix expression (3.1.20) must have an inverse. For that purpose we return to eq. (3.1.11) which defines the analytic self-energy. Writing analogously

$$\tilde{\Delta}'^{-1}(k) = k^2 - m^2 - \tilde{\Sigma}_F(k), \quad (3.1.24)$$

we find with the help of (3.1.18)

$$\tilde{\Sigma}_F(k) = \Sigma(k_0 + ik_0\epsilon, \mathbf{k}) \quad (3.1.25)$$

in terms of the analytic self-energy. This implies the spectral representation

$$\tilde{\Sigma}_F(k) = \int_0^\infty \frac{dk'_0}{2\pi} \left[ (k'^2 - |\mathbf{k}|^2 - m^2)\rho(k'_0, \mathbf{k}) + \frac{\sigma(k'_0, \mathbf{k})}{k_0^2 - k'^2 + i\epsilon} \right] \quad (3.1.26)$$

from which the properties

$$\tilde{\Sigma}_F(k_0, \mathbf{k}) = \tilde{\Sigma}_F(-k_0, \mathbf{k}), \quad (3.1.27)$$

$$\text{Im } \tilde{\Sigma}_F(0, \mathbf{k}) = 0, \quad (3.1.28)$$

are deduced immediately.

Let us now invert (3.1.20), and define a  $2 \times 2$  self-energy matrix so that

$$\tilde{G}^{-1}(k) = -i(k^2 - m^2)\tau_3 + i\tilde{\Sigma}(k), \quad (3.1.29)$$

where  $\tau_3$  is the third Pauli matrix. Substituting (3.1.24) into the inverted expression we discover that the matrix self-energy has the form [4.13, 3.12]

$$\tilde{\Sigma}(k) = M_+^{-1} \begin{pmatrix} \tilde{\Sigma}_F(k) & 0 \\ 0 & -\tilde{\Sigma}_F^*(k) \end{pmatrix} M_+^{-1}. \quad (3.1.30)$$

In deriving this result we used  $M_\eta^{-1}\tau_3M_\eta^{-1} = \tau_3$ . This uniquely identifies the matrix self-energy in terms of the boundary value (3.1.25) of the analytic self-energy.

### 3.1.3. Lowest-order self-energies

If one knows the temperature self-energy  $\tilde{\Sigma}(i\omega_n, \mathbf{k})$  at the Matsubara frequencies, one can in principle obtain the real-time self-energy by an analytic continuation from the discrete set of points to the whole complex plane so that

$$\tilde{\Sigma}(i\omega_n, \mathbf{k}) = \Sigma(z, \mathbf{k})|_{z=i\omega_n}. \quad (3.1.31)$$

The procedure can be made unique by imposing that  $\Sigma(z, \mathbf{k})$  be bounded as  $|z| \rightarrow \infty$ , and be analytic off the real axis.

To illustrate what is involved mathematically and physically, we first take a look at the simple example of a  $\phi^3$  theory. The lowest-order self-energy

$$\tilde{\Sigma}(k) = -\frac{g^2}{2\beta} \sum_i \int \frac{d^3 k_1}{(2\pi)^3} \tilde{\Delta}(k - k_1) \tilde{\Delta}(k_1), \quad (3.1.32)$$

graphically depicted in fig. 3.1, is defined at the discrete frequencies  $k_0 = i\omega_n$ . Now, the analytic self-energy is not simply obtained by formally considering  $k_0 = i\omega_n$  as a continuous variable  $z$ . The analytic continuation

$$\tilde{\Sigma}(z, k) = -\frac{g^2}{2\beta} \sum_i \int \frac{d^3 k_1}{(2\pi)^3} \tilde{\Delta}(z - k_1^0, k - k_1) \tilde{\Delta}(k_1) \quad (3.1.33)$$

constructed in this manner, although bounded, has cuts in the complex plane. Nevertheless, at the real axis this trivial analytic continuation is well-defined; see below. Moreover, this quantity is real there, in contrast to the proper analytic extension which we shall construct next.

The analytic self-energy is obtained by first doing the frequency summation in (3.1.32) and then setting  $i\omega_n = z$ . It is worth noting the order of these operations as compared to (3.1.33). The summation is straightforward and can be done by either the contour formula or the Dzyaloshinskii algorithm of section 2.3.3. Using the latter we get

$$\tilde{\Sigma}(i\omega_n, k) = -\frac{1}{2} g^2 \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} (2\pi)^3 \delta(k + k_1 + k_2) \int_0^\beta du \exp(i\omega_n u) \Delta(u, k_1) \Delta(u, k_2), \quad (3.1.34)$$

where the propagator is the one given in (2.3.30). The time integration is trivial and yields

$$\tilde{\Sigma}(i\omega_n, k) = \frac{1}{2} g^2 \prod_{i=1}^2 \frac{d^4 k_i}{(2\pi)^4} \rho_0(k_i) (2\pi)^3 \delta(k + k_1 + k_2) \frac{N_1 N_2 - (1 + N_1)(1 + N_2)}{i\omega_n + k_1^0 + k_2^0}, \quad (3.1.35)$$

with  $N_i = N(k_i^0)$ . The two contributions correspond to the two vertex-ordered graphs of fig. 3.2 [1.20]. A factor of  $N(k_i^0)$  is assigned to each internal line carrying an energy  $k_i^0$  to the right and a factor of  $N(-k_i^0) = -[1 + N(k_i^0)]$  for each line whose frequency propagates to the left. The denominator is equal to the sum of the energies which are intersected by the vertical line; the energy is counted positive if the



Fig. 3.1. First-order self-energy diagrams.

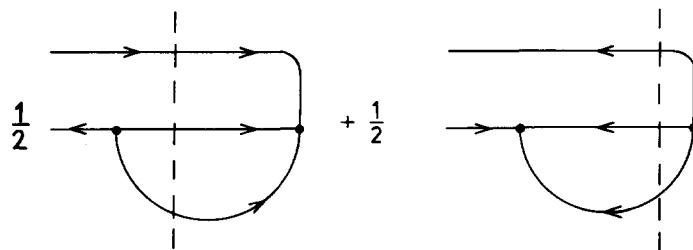


Fig. 3.2. Vertex-ordered diagrams.

line is directed to the right and negative if the line is directed to the left. There is also a spectral density  $\rho_0(k_i)$  and an integration for each internal line.

One can immediately extend the self-energy (3.1.35) to the full complex plane. This extension may be written in the spectral form

$$\Sigma(z, \mathbf{k}) = \int \frac{dk_0}{2\pi} \frac{\sigma(k)}{z - k_0}, \quad (3.1.36)$$

with spectral density

$$\sigma(k) = \frac{1}{2} g^2 \int \prod_{i=1}^2 \frac{d^4 k_i}{(2\pi)^4} \rho_0(k_i) (2\pi)^4 \delta^4(k + k_1 + k_2) [N_1 N_2 - (1 + N_1)(1 + N_2)]. \quad (3.1.37)$$

As it should be, this expression is positive definite for  $k_0 > 0$ . The various positive- and negative-energy contributions have an interpretation as a probability for particle creation and decay with statistical weight  $N_1 N_2$  in the initial state and  $(1 + N_1)(1 + N_2)$  in the final state [2.19]. The spectral density (3.1.37) also determines the Feynman self-energy (3.1.26). Conversely, one has

$$\sigma(k) = -2\varepsilon(k_0) \operatorname{Im} \tilde{\Sigma}_F(k). \quad (3.1.38)$$

With the aid of the real-time rules of section 2.4.3, one may verify that the real-time method (see also section 3.2.1) yields the same answer as obtained here by analytic continuation [3.15, 3.18, 8.22].

The fact that, in general, at finite temperature the Feynman self-energy has an imaginary part  $\operatorname{Im} \tilde{\Sigma}_F \neq 0$ , does not mean that the particles decay away. Even after an infinite time there still will be a heat bath with a thermal distribution of particles. Rather, what the imaginary part describes is the approach to equilibrium. In fact,  $\operatorname{Im} \tilde{\Sigma}_F$  can be related to the dissipative transport coefficients of viscosity and heat conductivity, either via the collision term of the Boltzmann equation [1.21, 3.13, 11.14], or the Green–Kubo expressions [7.15].

If the damping by collisions is small, i.e. the mean free path is large, the theory admits an interpretation in terms of “dynamical” quasiparticles defined by the weakly damped poles of the Feynman propagator [10.3]. In that case, the spectral density that determines this propagator has the approximate form

$$\rho(k) \approx 2 \operatorname{Im} \frac{1}{\varepsilon_k^2 - (k_0 + i\Gamma_k)^2}, \quad (3.1.39)$$

with the energy of the dynamical quasiparticle given by

$$\varepsilon_k^2 = \omega_k^2 + \operatorname{Re} \tilde{\Sigma}_F(k^2 = m^2). \quad (3.1.40)$$

The damping coefficient

$$2\omega_k \Gamma_k = -\operatorname{Im} \tilde{\Sigma}_F(k^2 = m^2), \quad (3.1.41)$$

which determines the lifetime, should be small in order that the very concept of a dynamical

quasiparticle has any meaning at all. To lowest order, one has  $\Gamma_k = 0$  in virtue of (3.1.38) and the delta-function constraints implied by the free spectral functions in (3.1.37). This feature is lost if higher-order corrections are taken into account, because in general the on-shell Feynman self-energy is a complex quantity due to thermal effects.

Many authors [2.18, 7.4, 7.9, 7.14, 7.21, 8.1, 8.6, 7.24] have concerned themselves with the perturbative calculation of dynamical quasiparticle properties, such as mass and magnetic moment shifts at finite temperature and density. In particular, one may consider the electron mass shift

$$\delta m^2 = \frac{1}{2} \text{tr}[(\gamma \cdot p + m) \bar{\Sigma}_F(p)]_{p^2=m^2} \quad (3.1.42)$$

in terms of the on-shell one-loop Feynman self-energy corresponding to the second diagram in fig. 3.1. It can be shown that this on-shell mass shift is gauge invariant despite the fact that the self-energy itself is a gauge dependent quantity. At zero temperature the one-loop mass shift is real [7.21, 7.24], but at finite temperature the infrared behaviour generates an imaginary part of order  $\alpha mT$  related to the possibility of soft photon emission [7.14]. However, this affects the observable quantities [7.21] only in higher orders.

For a clear physical interpretation it would be desirable if a mass shift could be defined with no imaginary part at all to arbitrary order. This can be accomplished if one abstains from an interpretation of quasiparticles in terms of dynamical propagators, but uses instead Landau's "statistical" quasiparticle definition [10.2]. An outline of this scheme is sketched in section 4.2.2. As shown in [10.10, 10.15], it leads to the definition of a fermionic "statistical" self-energy which is represented by the same set of diagrams as the ordinary temperature self-energy, but differs because the external energy is real. Its construction involves no more than the formal replacement of the external energy variable by a continuous one like in (3.1.33). For instance, the first-order statistical self-energy obtained in this manner reads:

$$\bar{\Sigma}(p) = -\frac{e^2}{\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \gamma_\mu \tilde{S}(p-k) \gamma_\nu \tilde{D}^{\mu\nu}(k), \quad (3.1.43)$$

in terms of the fermion and gauge-boson propagators (2.3.19) and (2.3.20), respectively, with  $p_0$  real and  $k_0 = i\omega_n$  imaginary.

One may verify by mere inspection that the statistical self-energy (3.1.43) is real. This becomes even more evident after the internal frequency sum has been performed, either by contour integration [10.10], or with the help of the Dzyaloshinskiĭ algorithm:

$$\begin{aligned} \bar{\Sigma}(p) &= 2e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 k'}{(2\pi)^4} \rho_0^F(p+k) \rho_0^B(k') (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \\ &\times (2m - \gamma \cdot p - \gamma \cdot k) \frac{1 + N(k_0) + N(k'_0)}{k_0 + k'_0}. \end{aligned} \quad (3.1.44)$$

The two consecutive spectral densities are those for massive and massless particles, respectively. Apart from the absence of an imaginary part, two major differences with the "dynamical" self-energy may be noted: (i) there is no dependence on the chemical potential (to this order) and (ii) there is a non-vanishing zero-temperature mass shift for which, after subtraction of the infinite vacuum mass

counter term, one obtains [10.10]:

$$\delta\bar{m}^2(T=0) = \frac{\alpha m^2}{\pi} [y\sqrt{y^2+1} - 3\log(y + \sqrt{y^2+1})], \quad (3.1.45)$$

with  $y = |\mathbf{p}|/m$ . The mass shift is negative for small values of the velocity  $y$ , but vanishes in the rest frame, as it should.

### 3.2. Real-time formalism

In the preceding section spectral forms for the full many-body propagator and self-energy have been derived for the scalar bosonic case. Notably, the matrix forms of the real-time thermal propagator and self-energy have come out of this analysis; cf. (3.1.20), (3.1.30). Since these representations lie at the very heart of thermo field dynamics, it is of obvious importance to generalize them to the case of any spin, and in addition, to translate them into diagrammatic language.

In presenting the derivation of the generalized formulae [3.19], we shall follow a route somewhat different from the one travelled in section 3.1. This will enable us to avoid the use of a spectral representation for the full propagator, which can be quite cumbersome to work with in the case of higher spins [12.5], and non-Abelian gauge theories in particular [12.6].

#### 3.2.1. Full thermal propagator

Let us start with the generalization of (3.1.20) for any spin. To do so, we define the full matrix propagator by functional differentiation of the generating functional

$$G_{\alpha\beta}^{(rs)}(x, x') = -\frac{\delta}{\delta j_\alpha^\tau(x)} \log Z[\bar{j}, j] \left. \frac{\delta}{\delta j_\beta^s(x')} \right|_{j=\bar{j}=0}. \quad (3.2.1)$$

As explained at length in chapter 2, the generating functional generates contour-ordered operator expectation values. In particular, the thermo field contour of fig. 2.4, consisting of a time-ordered path  $C_1$ , and an anti-time-ordered path  $C_2$ , gives rise to a matrix propagator which in operator language reads

$$G_{\alpha\beta}(x, x') = \begin{pmatrix} \langle T\hat{\phi}_\alpha(x) \hat{\phi}_\beta(x') \rangle & \eta \langle \hat{\phi}_\beta(t' - \frac{1}{2}i\beta, x') \hat{\phi}_\alpha(x) \rangle \\ \langle \hat{\phi}_\alpha(t - \frac{1}{2}i\beta, x) \hat{\phi}_\beta(x') \rangle & \langle T^* \hat{\phi}_\alpha(x) \hat{\phi}_\beta(x') \rangle \end{pmatrix}, \quad (3.2.2)$$

where  $T$  is the time- and  $T^*$  the anti-time-ordering instruction.

The elements of the matrix (3.2.2) are connected by a number of identities. First we use the CPT theorem [5.2, 12.18] in the form

$$\langle \hat{\phi}_\alpha^\theta(-x) \hat{\phi}_\beta^\theta(-x') \rangle^* = \langle \hat{\phi}_\alpha(x) \hat{\phi}_\beta(x') \rangle, \quad (3.2.3)$$

where the CPT-transformed field is denoted by

$$\hat{\phi}_\alpha^\theta(x) = \Theta \hat{\phi}_\alpha(-x) \Theta^{-1} \quad (3.2.4)$$

with  $\Theta$  the anti-unitary CPT operator. For example, we have for scalar, Dirac and vector fields, respectively,

$$\hat{\phi}^\theta(x) = \hat{\phi}^\dagger(x), \quad (3.2.5)$$

$$\hat{\psi}^\theta(x) = i\gamma^5 \hat{\psi}^\dagger(x), \quad \hat{\psi}^\theta(x) = -i\hat{\psi}(x) \gamma^5 \gamma^0, \quad (3.2.6)$$

$$\hat{A}_\mu^\theta(x) = -\hat{A}_\mu(x). \quad (3.2.7)$$

Equation (3.2.3), in conjunction with translational invariance, implies

$$G_{\alpha\beta}^{(22)}(x, x') = \langle T\hat{\phi}_\alpha^\theta(x') \hat{\phi}_\beta^\theta(x) \rangle^* =: [G_{\alpha\beta}^{(11)}(x', x)]^{\theta*}. \quad (3.2.8)$$

In momentum space this reads

$$\tilde{G}_{\alpha\beta}^{(22)}(p) = [\tilde{G}_{\alpha\beta}^{(11)}(p)]^{\theta*}. \quad (3.2.9)$$

When applied to diagrams, the operation “ $\theta^*$ ” boils down to taking the ordinary complex conjugate of factors of  $i$  and  $i\varepsilon$  in vertices and propagators, while leaving untouched Klein–Gordon divisors, etc. In fact, this is precisely the prescription for complex conjugation as stated under (2.4.16). We will adhere to this convention, and simply drop the suffix  $\theta$ .

A second identity is provided by the KMS condition which gives

$$\tilde{G}_{\alpha\beta}^{(21)}(p) = \eta e^{-\beta\mu} \tilde{G}_{\alpha\beta}^{(12)}(p). \quad (3.2.10)$$

As an aside we mention that in case  $\mu^{ij} = \sum q_A^{ij} \mu_A$  is a matrix in some internal space, the matrix  $\exp -\beta\mu$  commutes with the matrix  $G_{\alpha\beta}$ . This is a consequence of the integrated Ward identity for the Noether current associated with the symmetry generator  $Q_A$ ; cf. the remark after (2.2.35). Finally, the definition of time and anti-time ordering trivially leads to

$$\tilde{G}_{\alpha\beta}^{(11)}(p) + \tilde{G}_{\alpha\beta}^{(22)}(p) = \exp(\beta p_0/2) \tilde{G}_{\alpha\beta}^{(21)}(p) + \exp(-\beta p_0/2) \tilde{G}_{\alpha\beta}^{(12)}(p). \quad (3.2.11)$$

Let us now define a quantity  $\tilde{G}_{F\alpha\beta}(p)$  according to

$$\tilde{G}_{\alpha\beta}^{(11)}(p) = [1 + \eta n(p_0)] \tilde{G}_{F\alpha\beta}(p) + \eta n(p_0) \tilde{G}_{F\alpha\beta}^*(p), \quad (3.2.12)$$

where  $n(p_0)$  is the distribution function (2.4.28). Then eqs. (3.2.9) through (3.2.11) imply that the full matrix propagator may be written as

$$\tilde{G}_{\alpha\beta}(p) = M_\eta \begin{pmatrix} \tilde{G}_{F\alpha\beta}(p) & 0 \\ 0 & \tilde{G}_{F\alpha\beta}^*(p) \end{pmatrix} M_\eta, \quad (3.2.13)$$

where  $M_\eta$  is the Bogoliubov matrix (2.4.31). Obviously, the free-field propagator (2.4.30) is a special case of (3.2.13). For scalar fields  $\tilde{G}_F$  reduces to  $i\tilde{\Delta}'$  defined in section 3.1.2.

We may assume that the full propagator matrix satisfies the Dyson–Schwinger equation [12.4, 12.19, 4.13, 3.16], see fig. 3.3,

$$\tilde{G}_{\alpha\beta}^{(rs)} = i\tilde{D}_{\alpha\beta}^{(rs)} + (\tilde{D}\tilde{\Sigma}\tilde{G})_{\alpha\beta}^{(rs)}, \quad (3.2.14)$$

where  $\tilde{D}_{\alpha\beta}^{(rs)}$  is the free propagator (2.4.30). This equation defines the self-energy matrix  $-i\tilde{\Sigma}_{\alpha\beta}^{(rs)}$  as the sum of all one-particle irreducible (1PI) two-point diagrams with one external vertex of type  $r$ ,  $\alpha$  and the other of type  $s$ ,  $\beta$ . The formal solution of eq. (3.2.14) is

$$(\tilde{G}^{-1})_{\alpha\beta}^{(rs)} = -i(\tilde{D}^{-1})_{\alpha\beta}^{(rs)} + i\tilde{\Sigma}_{\alpha\beta}^{(rs)}. \quad (3.2.15)$$

The self-energy so defined coincides with the earlier definition (3.1.29) for the scalar case. One should note, however, that the  $i\varepsilon$  prescriptions (boundary conditions) for  $\tilde{G}$  and  $\tilde{D}$  are lost in (3.2.15). They may be inferred either by solving (3.2.14) iteratively, or from a spectral representation.

We write  $D_{F\alpha\beta}(p) = d_{\alpha\beta}(p)\Delta_F(p)$  and assume that there exists a “scalar self-energy”, defined as the boundary value of the analytic self-energy like in section 3.1.2, so that

$$\tilde{G}_{F\alpha\beta} = i\tilde{D}_{F\alpha\beta} + (\tilde{D}_F\tilde{\Sigma}_F\tilde{G}_F)_{\alpha\beta}. \quad (3.2.16)$$

Then (3.2.13), (3.2.14) and (3.2.15) lead to the matrix form

$$\tilde{\Sigma}_{\alpha\beta}(p) = M_\eta^{-1} \begin{pmatrix} \tilde{\Sigma}_{F\alpha\beta}(p) & 0 \\ 0 & -\tilde{\Sigma}_{F\alpha\beta}^*(p) \end{pmatrix} M_\eta^{-1}, \quad (3.2.17)$$

for the self-energy matrix, as an immediate generalization of (3.1.30). We see that the self-energy matrix has, in fact, only one independent component. In practice, one first calculates  $\tilde{\Sigma}_{\alpha\beta}^{(11)}$  in perturbation theory, and then uses (3.2.17) to solve for  $\tilde{\Sigma}_F$ , or for any other component of the self-energy matrix. The following identities, which are all direct consequences of (3.2.17), may be of help in this context (cf. [3.15] for the scalar case):

$$\text{Re } \tilde{\Sigma}_{F\alpha\beta}(p) = \text{Re } \Sigma_{\alpha\beta}^{(11)}(p), \quad (3.2.18)$$

$$\text{Im } \tilde{\Sigma}_{F\alpha\beta}(p) = \varepsilon(p_0)[1 + 2\eta n_+(p_0)]^{-1} \text{Im } \Sigma_{\alpha\beta}^{(11)}(p), \quad (3.2.19)$$

$$\tilde{\Sigma}_{\alpha\beta}^{(22)}(p) = -[\tilde{\Sigma}_{\alpha\beta}^{(11)}(p)]^*, \quad (3.2.20)$$

$$\begin{aligned} \tilde{\Sigma}_{\alpha\beta}^{(21)}(p) &= \eta e^{-\beta\mu} \tilde{\Sigma}_{\alpha\beta}^{(12)}(p) \\ &= -2i \exp(-\beta p_0/2) [1 + 2\eta n_+(p_0)]^{-1} [1 + \eta n_+(p_0)] \text{Im } \tilde{\Sigma}_{\alpha\beta}^{(11)}(p). \end{aligned} \quad (3.2.21)$$

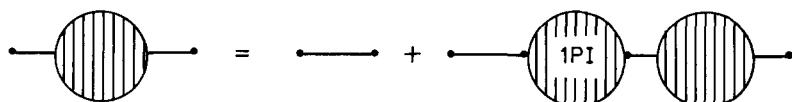


Fig. 3.3. Dyson–Schwinger equation.

In many cases of practical interest [1.21, 3.13, 3.21, 11.14], one wishes only to evaluate  $\text{Im } \tilde{\Sigma}_F$ . To use then eq. (3.2.19) can be quite cumbersome. However, for this specific purpose diagrammatic techniques have been developed by Kobes and Semenoff [3.15, 3.18], and by Fujimoto et al. [8.22], which allow a fast and elegant computation of the imaginary part.

### 3.2.2. Absence of pathologies

The matrix representations (3.2.13) and (3.2.17) of the full propagator and self-energy, respectively, allow a simple proof of the statement that the sum of all diagrams contributing to a Green function in a particular order is free of “pathologies”. In this context a pathology is a so-called pinch singularity [12.18]. Singularities of this type occur when an expression like

$$\tilde{\Delta}_F(k) \tilde{\Delta}_F^*(k) = \frac{1}{k_0^2 - \omega_k^2 + i\epsilon} \frac{1}{k_0^2 - \omega_k^2 - i\epsilon} \quad (3.2.22)$$

is integrated over  $k_0$ . In the limit  $\epsilon \downarrow 0$  the integration contour is then “pinched” between the poles on each side of the real axis, and the integral cannot be assigned a well-defined meaning. In the general expression (2.4.30) one clearly sees the danger that this will happen. In the simple example (2.4.40), (2.4.41) the pinch singularities manifest themselves as products of delta functions with the same argument.

Since we have stipulated that  $\epsilon$  should be kept finite till the end, the pinch singularities are regularized in intermediate stages of a particular calculation. It is now our aim to show that potentially dangerous terms will cancel after all relevant diagrams have been taken into account. This property was discovered to be true in special cases [4.12, 8.8, 3.11]. It was shown to be a consequence of the (thermo field) Feynman rules in refs. [4.13] and [3.12].

The proof proceeds in two related steps. We first observe that pinch singularities can only occur when bare propagators  $\tilde{D}^{(rs)}(k)$  carrying the same energy and momentum are multiplied with each other. (Lorentz indices etc. will be suppressed.) The corresponding lines in the diagram are separated by irreducible self-energy insertions  $\tilde{\Sigma}_n^{(rs)}(k)$ ,  $n = 1, 2, \dots, M$ . Together this structure builds a particular approximation

$$\tilde{G}_{\{n\}}^{(rs)}(k) = \left\{ \prod_{n=1}^M [\tilde{D}(k) \tilde{\Sigma}_n(k)] \tilde{D}(k) \right\}^{(rs)} \quad (3.2.23)$$

to the full propagator. Hence, the absence of pinch singularities ensures the existence of the thermal propagator (3.2.23), and vice versa.

The first step of the proof consists in showing that the propagator (3.2.23) is regular, given that the self-energy insertions are regular. The key is in recognizing that, even though many terms in the matrix product (3.2.23) will exhibit pinch singularities, the entire product can be represented as

$$\tilde{G}_{\{n\}}^{(rs)} = M_n \begin{pmatrix} \tilde{G}_{F\{n\}} & 0 \\ 0 & \tilde{G}_{F\{n\}}^* \end{pmatrix} M_n, \quad (3.2.24)$$

where we have

$$\tilde{G}_{F(n)} = \prod_{n=1}^M (\tilde{\Delta}_F \tilde{\Sigma}_{Fn}) \tilde{\Delta}_F \quad (3.2.25)$$

due to the matrix forms (3.2.17) of the self-energy, and (2.4.30) of the bare propagator. In these last expressions products of  $\tilde{\Delta}_F$ 's are completely separated from products of  $\Delta_F^*$ 's. Since self-energies are 1PI and do not contain internal lines with the same four-momentum as the external ones, this implies the absence of pinch singularities.

It is now a simple matter to complete the proof, that is, to show that the self-energy insertions itself are regular. We use mathematical induction and start with supposing that self-energy diagrams with  $V$  vertices are regular for  $V < V_1$ . Now consider a self-energy diagram with  $V_1$  vertices. On the basis of the preceding argument, we maintain that singularities can only come from a nested subdiagram of the kind discussed above. Its contribution is of the form (3.2.23). Now the various separate self-energy insertions certainly contain less vertices than the total self-energy diagram under consideration. Hence, all these self-energy insertions are regular by assumption. But we have already shown that if the separate self-energy insertions are regular, the whole chain (3.2.23) is regular. This then implies that the self-energy itself is regular, thus completing the proof.

We close by pointing out once again that the matrix structures (3.2.13) and (3.2.17) were instrumental in the proof. Keeping only the (11)-components of these matrices, one does not arrive at expressions like (3.2.24). This explains why in the early attempts at a real-time approach [2.6] one was faced with inadmissible singularities of the pinch type.

### 3.3. The case of $SU(N)$

The general theory developed in the preceding sections will now be specialized to the case of a  $SU(N)$  gauge theory (quarkless QCD with  $N$  colours) at finite temperature and density. The topics studied include the Slavnov–Taylor identity for the thermal gluon propagator, and the transversality of the gluon polarization tensor. The results will play a role in the discussion of the plasmon effect in section 4.4.

#### 3.3.1. Slavnov–Taylor identity

Our aim is the generalization of the Slavnov–Taylor identity implied by  $SU(N)$  gauge invariance to finite temperature. For preparation, we first study the general symmetry structure of real-time thermal Green functions, confining ourselves to global symmetries. This is sufficient for our purpose because, after gauge fixing, the local gauge invariance of the theory is effectively replaced by global Becchi–Rouet–Stora (BRS) invariance [12.17, 12.18, 9.24].

In general, symmetries of the action lead to Ward–Takahashi (WT) identities for the Green functions. (See ref. [4.7] for a discussion of WT identities at zero as well as finite temperature; also cf. [3.13, 8.12].) In the functional approach these identities are derived by performing a transformation of the field variables in the path integral [9.22, 12.19]. Taking the path integral (2.2.9) as an example, we recall that the fields are defined on a contour  $C$  in the time plane. Hence, a global field transformation affects the fields in the same way at every point of the contour. What we have to check is whether the transformed fields are still compatible with the original boundary conditions (2.2.21). (At zero temperature and density this is automatic because all fields are subject to the same Feynman boundary condition.) If this is the case, the standard derivation [9.22, 12.18] goes through unchanged, and one obtains the well-known integrated WT identity (no spontaneous symmetry breaking)

$$\delta \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0, \quad (3.3.1)$$

where the time arguments of the fields lie on the contour C.

In the real-time method as outlined in section 2.4, the two real-time fields  $\phi_1$  and  $\phi_2$  are identical to the original field  $\phi$  on the forward and backward pieces of the contour, respectively. Since  $\phi$  undergoes the same global transformation  $\phi \rightarrow \phi + \delta\phi$  at all points of the contour, it follows that both  $\phi_1$  and  $\phi_2$  are to be transformed:  $\phi_r \rightarrow \phi_r + \delta\phi_r$ ,  $r = 1, 2$ . This immediately yields the real-time version of (3.3.1):

$$\delta \langle \phi_{r_1}(x_1) \cdots \phi_{r_n}(x_n) \rangle = 0, \quad (3.3.2)$$

if it is assumed that the boundary conditions remain invariant under the transformation in question.

Alternatively, eq. (3.3.2) can be derived directly from the real-time path integral (2.4.19) by checking the invariance of the action under transformations of the real-time fields  $\phi_r$ . The boundary conditions are then contained in the matrix propagator (2.4.30), and its invariance should be established explicitly. This procedure is generally more laborious than the one starting from the contour path integral. However, for a discussion of supersymmetry at finite temperature the second, “pure thermo field” method has distinct advantages [8.3, 8.4, 8.18, 8.25, 8.26] (also cf. [7.29]). We may add that the incorporation of supersymmetry in the Matsubara formalism runs into the problems [7.2, 7.5, 7.10, 7.22], because the supersymmetric transformation changes bosonic fields into fermionic ones with antiperiodic instead of periodic boundary conditions.

Equation (3.3.2) will now be used to derive the Slavnov–Taylor identity for the full thermal gluon propagator [10.16]. It is well-known [9.24, 12.17] that this identity is generated by the invariance of the effective  $SU(N)$  Lagrangian (2.2.39) under the BRS transformations:

$$\delta_{BRS} A_\mu^a = \xi(\partial_\mu \omega^a + gf^{abc} A_\mu^b \omega^c), \quad (3.3.3)$$

$$\delta_{BRS} \omega^a = -\frac{1}{2} g \xi f^{abc} \omega^b \omega^c, \quad (3.3.4)$$

$$\delta_{BRS} \bar{\omega}^a = \lambda \xi \partial^\mu A_\mu^a, \quad (3.3.5)$$

where  $\xi$  is an anticommuting,  $x$ -independent parameter. On account of the fact that the ghost fields are subject to the same boundary conditions as the gauge fields (cf. the discussion in section 2.2.3), the boundary conditions are left invariant; this feature also survives the addition of fermionic quark fields. It permits us to conclude that the following WT identity holds for real-time fields:

$$\delta_{BRS} \langle A_r^{\mu a}(x) \bar{\omega}_s^b(x') \rangle = 0. \quad (3.3.6)$$

Here the indices  $r, s = 1, 2$  refer to time-path type- $r, s$  fields, respectively.

Using eqs. (3.3.3) and (3.3.5), we may write eq. (3.3.6) more explicitly as

$$\partial_\nu' \langle A_r^{\mu a}(x) A_s^{\nu b}(x') \rangle = -\lambda^{-1} [\partial^\mu \langle \omega_r^a(x) \bar{\omega}_s^b(x') \rangle + gf^{acd} \langle A_r^{\mu c}(x) \omega_r^d(x) \bar{\omega}_s^b(x') \rangle]. \quad (3.3.7)$$

We immediately see that the right-hand side vanishes in the Landau gauge. The full ghost propagator satisfies the Dyson–Schwinger equation [12.19]

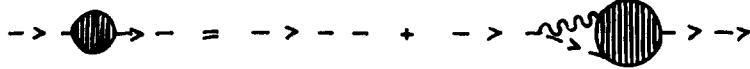


Fig. 3.4. Dyson–Schwinger equation for the ghost propagator.

$$\langle \omega_r^a(x) \bar{\omega}_s^b(x') \rangle = \langle \omega_r^a(x) \bar{\omega}_s^b(x') \rangle_0 + igf^{ecd} \sum_{t,u=1}^2 \int d^4y d^4y' d^4y'' \langle \omega_r^a(x) \bar{\omega}_t^e(y) \rangle_0 \\ \times (\tau_3)^{tu} \partial_\sigma \delta^4(y - y') \delta^4(y' - y'') \langle A_u^{\sigma c}(y'') \omega_u^d(y') \bar{\omega}_s^b(x') \rangle. \quad (3.3.8)$$

Here, the subscript 0 indicates the free-field expectation value. In the vacuum theory, the three-point function in (3.3.8) must be proportional to  $\partial/\partial y'_\sigma$ . At finite temperature there can be a term proportional to  $U^\sigma = (1, \mathbf{0})$  as well. To proceed, we operate with  $\partial/\partial x^\mu$  on (3.3.7), and use the equation

$$\partial^2 D_{gab}^{(rs)}(x - x') = \delta_{ab}(\tau_3)^{rs} \delta^4(x - x') \quad (3.3.9)$$

for the free ghost propagator, see (2.2.47), (2.4.30). We then find that the second terms at the right-hand sides of eqs. (3.3.8) and (3.3.7) cancel. Fourier transforming the remaining expression, we arrive at the finite-temperature real-time Slavnov–Taylor identity

$$k^\mu k^\nu \tilde{G}_{\mu\nu}^{(rs)ab}(k) = i\lambda^{-1} k^2 \tilde{D}_g^{(rs)ab}(k) \quad (3.3.10)$$

for the thermal gluon propagator.

Finally, we make use of the general representation (3.1.13) to rewrite the last identity in terms of Feynman propagator:

$$k^\mu k^\nu \tilde{G}_{F\mu\nu}^{ab}(k) = -i\lambda^{-1} \delta^{ab}. \quad (3.3.11)$$

This equation is formally identical to the one valid for the vacuum propagator [12.9, 12.19].

### 3.3.2. Polarization tensor

We shall now investigate how the Slavnov–Taylor identity (3.3.11) for the thermal gluon propagator constrains the real-time gluon polarization tensor. We use the notation  $\tilde{\Pi}_{\mu\nu}(k)$  for this quantity (dropping the index F), which is nothing but the irreducible gluon self-energy as it appears on the diagonal of the matrix (3.2.17). Colour indices are deleted; they can be trivially reinstated by multiplying with  $\delta^{ab}$ .

For the moment we shall work in a manifestly covariant manner. We introduce the hydrodynamic four-velocity of the heat bath  $U^\mu$ , normalized to unity, and define the projector orthogonal to  $U^\mu$  as  $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$ . All preceding formulae may now be made formally covariant by writing  $k \cdot U$  for  $k_0$ , and the space-like momentum vector  $\kappa^\mu = \Delta^{\mu\nu} k_\nu$  for the spatial momentum  $\mathbf{k}$ ; in the rest frame we have  $k_0 = k \cdot U$  and  $\kappa^\mu = (0, \mathbf{k})$ , but note that  $\kappa^\mu \kappa_\mu = -|\mathbf{k}|^2$ .

Let us now define the four independent symmetric tensors [6.2, 6.5, 6.15, 10.16]

$$A_{\mu\nu} = \Delta_{\mu\nu} - \kappa_\mu \kappa_\nu / \kappa^2 , \quad (3.3.12)$$

$$B_{\mu\nu} = U_\mu U_\nu + \kappa_\mu \kappa_\nu / \kappa^2 - k_\mu k_\nu / (k^2 + i\epsilon) , \quad (3.3.13)$$

$$C_{\mu\nu} = \frac{\sqrt{-\frac{1}{2}\kappa^2}}{U \cdot k} \left[ \frac{(U \cdot k)^2}{\kappa^2} U_\mu U_\nu - \frac{\kappa_\mu \kappa_\nu}{\kappa^2} + \frac{2\kappa^2 - k^2}{\kappa^2} \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right] , \quad (3.3.14)$$

$$D_{\mu\nu} = k_\mu k_\nu / (k^2 + i\epsilon) . \quad (3.3.15)$$

Here  $A$  is transverse with respect to the four-velocity, while both  $B$  and  $C$  are mixtures of space-like and time-like components. With respect to the four-momentum  $k_\mu$ , these tensors have the transversality properties

$$k_\mu A^{\mu\nu} = k_\mu B^{\mu\nu} = 0 , \quad k_\mu k_\nu C^{\mu\nu} = 0 . \quad (3.3.16)$$

Furthermore, the tensors  $A$ ,  $B$  and  $D$  are idempotent and mutually orthogonal, whereas the product of  $C$  with the other three tensors has vanishing trace. Any (causal) symmetric second-rank tensor can be decomposed as

$$T_{\mu\nu} = \alpha A_{\mu\nu} + \beta B_{\mu\nu} + \gamma C_{\mu\nu} + \delta D_{\mu\nu} , \quad (3.3.17)$$

with  $\alpha = \frac{1}{2} \text{tr } AT$ ,  $\beta = \text{tr } BT$ ,  $\gamma = -\text{tr } CT$  and  $\delta = \text{tr } DT$ .

From these relations we infer that the gluon propagator figuring in the Slavnov-Taylor identity (3.3.11), must have the tensor structure

$$\tilde{\Pi}_{F\mu\nu}(k) = -\frac{iA_{\mu\nu}(k)}{k^2 - \tilde{\Pi}_T(k) + i\epsilon} - \frac{iB_{\mu\nu}(k)}{k^2 - \tilde{\Pi}_L(k) + i\epsilon} - i\lambda^{-1} \frac{D_{\mu\nu}(k)}{k^2 + i\epsilon} - ic(k) \frac{C_{\mu\nu}(k)}{k^2 + i\epsilon} . \quad (3.3.18)$$

We shall ignore the last term in the sequel;  $c(k)$  vanishes in the Landau gauge [6.15] and in all known approximations in other gauges too. Then the self-energy equals

$$\tilde{\Pi}_{\mu\nu}(k) = -\tilde{\Pi}_T(k) A_{\mu\nu}(k) - \tilde{\Pi}_L(k) B_{\mu\nu}(k) . \quad (3.3.19)$$

Since both tensors are transverse with respect to  $k$ , we have

$$k^\mu \tilde{\Pi}_{\mu\nu}(k) = 0 . \quad (3.3.20)$$

Hence the polarization tensor is transverse, like in the vacuum theory. If  $c(k) \neq 0$  one still has  $k^\mu k^\nu \tilde{\Pi}_{\mu\nu}(k) = 0$  on account of (3.3.11).

### 3.3.3. Polarization functions

The scalar longitudinal and transverse polarization functions can be expressed in terms of the polarization tensor as

$$\tilde{\Pi}_L(k) = -\frac{k^2}{\kappa^2} U_\mu U_\nu \tilde{\Pi}^{\mu\nu}(k), \quad (3.3.21)$$

$$\tilde{\Pi}_T(k) = \frac{1}{2} \left( \frac{k^2}{\kappa^2} U_\mu U_\nu - g_{\mu\nu} \right) \tilde{\Pi}^{\mu\nu}(k), \quad (3.3.22)$$

by projecting out the longitudinal and transverse parts, and by making use of the transversality property (3.3.20). As can be seen from these connections, the polarization functions do not have a uniform limit for  $k \rightarrow 0$ . In fact, if we set  $k_0 = 0$  in the rest frame and then let  $k \rightarrow 0$ , we get

$$\lim_{k \rightarrow 0} \tilde{\Pi}_L(0, k) = -\tilde{\Pi}_{00}(0) =: \kappa_L^2, \quad (3.3.23)$$

$$\lim_{k \rightarrow 0} \tilde{\Pi}_T(0, k) = \frac{1}{2} [\tilde{\Pi}_{00}(0) - \tilde{\Pi}_\mu^\mu(0)] =: \kappa_T^2. \quad (3.3.24)$$

On the other hand, if we first set  $k_0 = |\mathbf{k}|$ , we find

$$\lim_{k \rightarrow 0} \tilde{\Pi}_L(|\mathbf{k}|, k) = 0, \quad (3.3.25)$$

$$\lim_{k \rightarrow 0} \tilde{\Pi}_T(|\mathbf{k}|, k) = -\frac{1}{2} \tilde{\Pi}_\mu^\mu(0) =: \omega_T^2. \quad (3.3.26)$$

To lowest order, the manner in which the limit  $k \rightarrow 0$  is taken in  $\tilde{\Pi}_{\mu\nu}(k)$  turns out to be immaterial.

Let us now show how to actually calculate these quantities. We shall first compute  $\tilde{\Pi}_{\mu\nu}^{(11)}(k)$ , and then we use (3.2.18), (3.2.19) to obtain  $\tilde{\Pi}_{\mu\nu}(k)$ . In fact, according to (3.1.28), the imaginary part will be zero in the static limit  $k_0 = 0$ , and what has to be calculated is therefore

$$-i\tilde{\Pi}_{\mu\nu}(0) = -i\tilde{\Pi}_{\mu\nu}^{(11)}(0) = \text{blob} \quad (3.3.27)$$

The blob stands for the sum of all 1PI diagrams, in this case with zero external four-momentum. To one-loop order, the temperature-dependent part of the polarization tensor is given by the diagrams in fig. 3.5.

In the Feynman gauge the propagators are almost the same as those for a massless scalar theory; cf. section 2.4.3. In fact, we only have to add a factor of  $-g_{\mu\nu}\delta^{ab}$  to get the gluon propagators, e.g.

$$\nu, a \quad \text{---} \quad \mu, b = -ig_{\mu\nu}\delta^{ab}/(k^2 + i\epsilon), \quad (3.3.28)$$

$$\nu, a \quad \text{---} \quad \text{ghost loop} \quad \mu, b = -2\pi g_{\mu\nu}\delta^{ab}\delta(k^2)\sinh^2\Theta_k, \quad (3.3.29)$$

and a factor of  $-\delta^{ab}$  to obtain the ghost propagators, e.g.

$$a \quad \text{---} \quad \text{ghost loop} \quad b = -i\delta^{ab}/(k^2 + i\epsilon), \quad (3.3.30)$$

$$a \quad \text{---} \quad \text{ghost loop} \quad b = -2\pi\delta^{ab}\delta(k^2)\sinh^2\Theta_k, \quad (3.3.31)$$

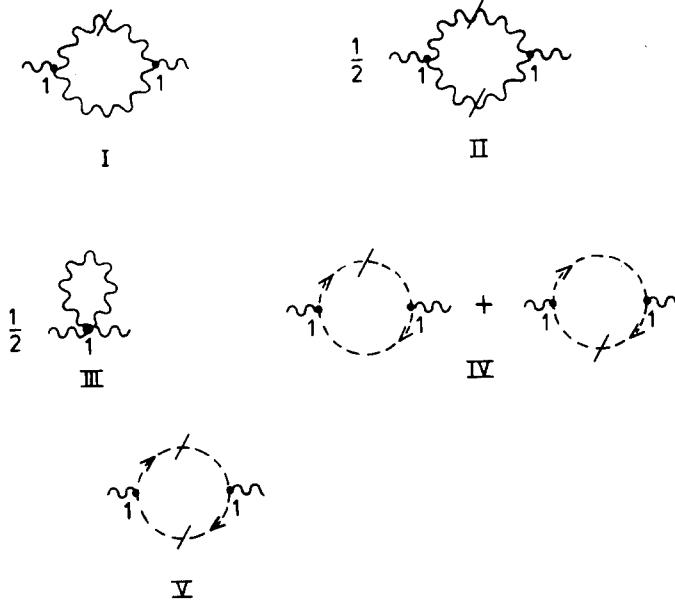


Fig. 3.5. The one-loop gluon polarization tensor.

with  $\sinh^2 \Theta_k$  given by (2.4.42). The type-1 vertices are those of the vacuum theory [12.8, 12.18, 9.24], while the type-2 vertices appear with an overall minus sign.

The first and second diagrams of fig. 3.5 now yield, respectively,

$$I = -ig^2 N \delta^{ab} \int \frac{d^4 q}{(2\pi)^3} \frac{1}{q^2 + i\epsilon} \delta((k+q)^2) \sinh^2 \Theta_{k+q} Q_{\mu\nu}, \quad (3.3.32)$$

$$II = \frac{1}{2} g^2 N \delta^{ab} \int \frac{d^4 q}{(2\pi)^3} 2\pi \delta(q^2) \delta((k+q)^2) \sinh^2 \Theta_q \sinh^2 \Theta_{k+q} Q_{\mu\nu}, \quad (3.3.33)$$

with

$$Q_{\mu\nu} = g_{\mu\nu} (5k^2 + 2k \cdot q + 2q^2) + 10q_\mu q_\nu - 2k_\mu k_\nu + 5(q_\mu k_\nu + k_\mu q_\nu). \quad (3.3.34)$$

The  $\delta$ -functions appearing in these expressions must be understood as  $\epsilon$ -regularized; cf. the remark to this effect in section 2.4.1. We see that both (3.3.32) and (3.3.33) display a pinch singularity with respect to the  $q_0$  integration if we set  $k = 0$ , and then let  $\epsilon \rightarrow 0$ . On account of the general argument of section 3.2.2, we feel confident that these singularities will cancel, and indeed they do. Using (A.7), we find for the sum of the first two diagrams in this limit

$$I + II = ig^2 N \delta^{ab} \int \frac{d^4 q}{(2\pi)^3} \sinh^2 \Theta_q (g_{\mu\nu} q^2 + 5q_\mu q_\nu) \frac{\partial}{\partial q^2} \delta(q^2), \quad (3.3.35)$$

where now the  $\delta$ -function can be taken literally. The contribution to  $\tilde{H}_{00}(0)$  and  $\tilde{H}_\mu^\mu(0)$  is easily extracted with a partial integration and use of (A.14), (A.21).

A similar cancellation of singularities occurs in diagrams IV and V, whereas diagram III is regular from the outset. The integrals are easy to write down, and we only give the final results:

$$\tilde{H}_{00}(0) = g^2 NT^2 \left( -\frac{1}{8} - \frac{1}{4} + \frac{1}{24} \right) = -\frac{1}{3} g^2 NT^2 , \quad (3.3.36)$$

$$\tilde{H}_\mu^\mu(0) = g^2 NT^2 \left( \frac{3}{4} - 1 - \frac{1}{12} \right) = -\frac{1}{3} g^2 NT^2 , \quad (3.3.37)$$

where the first, second, and third terms represent the contributions of diagrams I + II, III, and IV + V, respectively.

Finally, substituting these results into (3.3.23), (3.3.24) and (3.3.26), we obtain

$$\kappa_L^2 = \frac{1}{3} g^2 NT^2 , \quad \kappa_T^2 = 0 , \quad (3.3.38)$$

$$\omega_T^2 = \frac{1}{6} g^2 NT^2 . \quad (3.3.39)$$

These quantities have also been calculated in the Matsubara formalism for several gauges [6.2–6.5], and exactly the same result has been found. Their gauge independence points toward a physical significance. Indeed,  $\kappa_L^{-1}$  may be interpreted as the Debye screening length of the long-range chromo-electric force (*pace* [10.12]), while  $\omega_T$  plays the role of a dynamically generated gluon mass [6.4, 6.16, 10.11, 10.16]. The equation  $\kappa_T = 0$  just states that to this order of approximation there is no chromomagnetic screening; it is unknown whether this result holds generally [6.9, 10.9, 10.12, 6.25].

## 4. Pressure and energy

### 4.1. Energy-momentum tensor

An important quantity of physical interest that one may want to calculate in field theory, either at zero, or at finite temperature and density, is the energy-momentum tensor. One may think of the application to the cosmology of the early universe, or the more mundane question of calculating the energy density and hydrostatic pressure of some many-body system in condensed matter physics. The energy-momentum tensor is not a Green function itself but a composite operator, that is, a local product of fields and their derivatives. For that reason a separate discussion is required, and we shall begin with a short technical introduction to the subject of composite operators [12.7, 12.18, 9.22, 9.24].

The main ideas will be illustrated for scalar theories but, where necessary, generalizations will be indicated. It will be understood that all fields are defined for real time, i.e. they are either type-1 or type-2 in the sense of section 2.4, and expectation values are defined with respect to the real-time path integral. Most results are independent of the value of the thermal indices of the composite or elementary fields, and they will often be omitted. It is worth mentioning that most of the formal results are applicable to the imaginary-time formalism as well. For obvious reasons the exceptions are the construction of the Feynman rules, and the proof of renormalizability.

#### 4.1.1. Composite operators

Let  $c(x)$  be a local polynomial in the field  $\phi(x)$  and its derivatives, and

$$C(x) = \langle c(x) \rangle , \quad (4.1.1)$$

$$C(x|y^n) = \langle c(x) \phi(y_1) \cdots \phi(y_n) \rangle, \quad (4.1.2)$$

the expectation values we wish to calculate; to shorten the writing we will denote the sequence  $y_1, \dots, y_n$  as  $y^n$  where possible. The angular brackets denote path-integral expectation values; in the operator formalism (4.1.2) would be a time- (or contour-)ordered operator average. Suppose the composite operator has the form

$$c_{\{\mu\}}(x) = \frac{1}{m!} \phi_{(\mu_1)}(x) \cdots \phi_{(\mu_m)}(x), \quad (4.1.3)$$

where the index  $(\mu_i)$  stands for any combination of derivatives  $i\partial_\mu$  acting on the field carrying the index. Our strategy is based on the observation that the expectation value (4.1.2) may be obtained from the “lifted” Green function

$$G_{\{\mu\}}(x^m|y^n) := \frac{1}{m!} \langle \phi_{(\mu_1)}(x_1) \cdots \phi_{(\mu_m)}(x_m) \phi(y_1) \cdots \phi(y_n) \rangle \quad (4.1.4)$$

by simply setting  $x^m = x$ . If it is understood that the time derivatives also act on the time ordering, we may bring the differential operators out of the expectation value and write

$$C_{\{\mu\}}(x|y^n) = \frac{1}{m!} \mathcal{D}(i\partial_{\{\mu\}}) G(x^m, y^n)|_{x^m=x}, \quad (4.1.5)$$

where  $G$  is the ordinary (not necessarily connected) Green function, and  $\mathcal{D}$  stands for the total collection of differential operators.

The Feynman rules will be specified in momentum space. Substituting for the Green function its Fourier integral expression in terms of the Fourier transform  $\tilde{G}$ , and setting the space-time coordinates  $x_i$  equal to  $x$ , we easily obtain

$$\tilde{C}_{\{\mu\}}(p|q^n) = \frac{1}{m!} \int \prod_{i=1}^m \frac{d^4 p_i}{(2\pi)^4} (2\pi)^4 \delta(p - \sum_j p_j) \mathcal{D}(p_{\{\mu\}}) \tilde{G}(p^m, q^n), \quad (4.1.6)$$

where  $p^m$ ,  $q^n$  and  $p$  flow out of the Green function and the composite vertex, respectively. The diagrammatic representation of this formula is given in fig. 4.1. The new vertex  $\otimes$  has the value  $\mathcal{D}(p_{\{\mu\}})$ , which is equal to one if no derivatives are involved. The new feature is that this “composite” vertex can carry off energy-momentum.

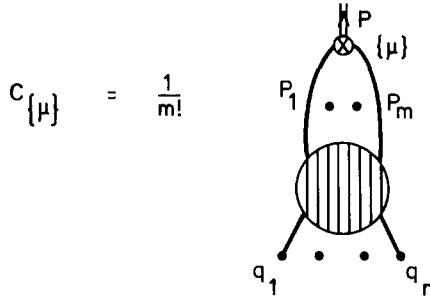


Fig. 4.1. Diagrammatic representation of eq. (4.1.6).

The advantage of the above construction is that the diagrams contributing to the expectation value  $\tilde{C}$  can be calculated with the help of the ordinary Feynman rules for Green functions. The typical contribution of a given diagram  $\Gamma$  will be of the form

$$F(p|q^n) = (2\pi)^4 \delta\left(p + \sum_j q_j\right) \int \prod_{i=1}^m \frac{d^4 p_i}{(2\pi)^4} \int \prod_{j=1}^l \frac{d^4 l_j}{(2\pi)^4} \delta^4\left(\sum_i p_i + \sum_j q_j\right) \mathcal{D}(p_{\{\mu\}}) I_{\Gamma}(p^m, q^n, l^l). \quad (4.1.7)$$

Here global energy-momentum conservation has been factored out. There is a problem, however: the appearance of the new vertex leads to new UV divergencies, even when the diagram  $\Gamma$  has been subject to ordinary renormalization. Fortunately, no new techniques are needed to handle these divergencies, as recognized by Zimmermann [12.7]. We only have to apply the forest formula (2.4.45) to make expression (4.1.7) finite. The proof of the absence of temperature-dependent divergencies goes through for composite-operator Green functions as well.

As in the ordinary case, the subtractions implied by the forest formula may be implemented by means of counterterms. In general the counterterms referring to the composite operator are not proportional to it. This leads to the subject of operator mixing [12.18, 9.22, 9.16, 9.17] which will not be discussed here. We just define Green functions with composite operators by demanding that contributing diagrams are renormalized by the forest formula. To make this explicit, we write  $N[c(x)]$  instead of simply  $c(x)$  in the Green functions;  $N$  is called a (Zimmermann) normal product.

There is the possibility that due to cancellations there is no need for additional counterterms. Then we have

$$N[c(x)] = c(x). \quad (4.1.8)$$

This may occur in cases that the composite operator is conserved.

#### 4.1.2. Conserved observables

If, as a consequence of an internal symmetry of the Lagrangian, the composite operator  $c(x)$  is a conserved current at the classical level, then the last equation (4.1.8) above nearly always holds true [9.22]. The basic idea may be illustrated for the simple example of the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi^*) \partial^\mu \phi - \mu^2 \phi^* \phi - \frac{1}{4} \lambda (\phi^* \phi)^2, \quad (4.1.9)$$

in which  $\phi$  is a complex scalar field. The Lagrangian is invariant under the U(1) transformation

$$\phi(x) \rightarrow \phi(x) + \delta\phi(x), \quad \delta\phi(x) = -i\varepsilon\phi(x). \quad (4.1.10)$$

Noether's theorem then implies that the current density

$$j^\mu(x) = i\phi^*(x) \overleftrightarrow{\partial}^\mu \phi(x) \quad (4.1.11)$$

is conserved. At the quantum level this gives rise to the Ward-Takahashi identity [4.7, 9.22]

$$\varepsilon \partial_\mu \langle j^\mu(x) \phi(y_1) \cdots \phi^*(y_n) \rangle = -i \sum_{i=1}^n \delta(x - y_i) \langle \phi(y_1) \cdots \delta\phi(y_i) \cdots \phi^*(y_n) \rangle. \quad (4.1.12)$$

Since  $\delta\phi$  is essentially just  $\phi$  itself, the right-hand side is a sum of ordinary Green functions. In momentum space we have

$$k^\mu \tilde{J}_\mu(k|q_1, \dots, q_n) = -i \sum_i \tilde{G}(q_1, \dots, k + q_i, \dots, q_n). \quad (4.1.13)$$

It is obvious that the right-hand side is UV-finite after renormalization according to the standard prescriptions. Formally expanding both left- and right-hand sides of (4.1.13) in powers of the momenta, we find that  $\tilde{J}_\mu$  itself must be finite. Hence no additional prescriptions are needed, and we have

$$N[j^\mu(x)] = j^\mu(x). \quad (4.1.14)$$

This result is valid for most currents which are conserved on account of internal symmetries. For exceptional cases see ref. [9.22].

In general this theorem is not true for external, or space-time, symmetries [9.2, 9.5, 9.6]. Let us consider an infinitesimal translation

$$\delta\phi(x) = \varepsilon^\mu \partial_\mu \phi(x). \quad (4.1.15)$$

The corresponding Noether current

$$t^{\mu\nu} = (\partial^\mu \phi^*)(\partial^\nu \phi) + (\partial^\nu \phi^*)(\partial^\mu \phi) - g^{\mu\nu} \mathcal{L} \quad (4.1.16)$$

is known as the canonical energy-momentum tensor. The conservation law

$$\partial_\mu t^{\mu\nu}(x) = 0 \quad (4.1.17)$$

for this current leads to a WT identity similar to (4.1.12). Hence we infer that

$$k^\mu \tilde{T}_{\mu\nu}(k|q_1, \dots, q_n) = k^\mu (\alpha g_{\mu\nu} + \beta U_\mu U_\nu + \gamma k_\mu k_\nu + \delta g_{\mu\nu} k^2 + \dots) \quad (4.1.18)$$

is finite. However, this only guarantees that, e.g.,  $\gamma + \delta$  is finite. It does not follow that  $\gamma$  and  $\delta$  are finite separately. Therefore, the composite operator  $t^{\mu\nu}$  has to be renormalized in general.

One may exploit the fact that a Noether current is not uniquely defined. Indeed, the current

$$\bar{j}^\mu = j^\mu + \partial_\nu \psi^{\nu\mu}, \quad \psi^{\nu\mu} = -\psi^{\mu\nu} \quad (4.1.19)$$

is conserved like the original current. Moreover, it produces the same global charges (generators). It has been shown that the “improved” energy-momentum tensor [9.5, 9.6, 9.18–9.20]

$$\theta^{\mu\nu} = t^{\mu\nu} - \frac{1}{d-1} \left[ \frac{d-2}{4} + f(d, \lambda) \right] (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2 \quad (4.1.20)$$

leads to finite matrix elements. Here  $f(d, \lambda)$  represents a series of pure poles in  $d=4$  [9.18]. The additional term is clearly of the required form (4.1.19).

Improvement in this sense is only necessary for theories with scalar fields, since the so-called

Belinfante energy-momentum tensor of higher spin fields, which can be obtained from the canonical one by a simple prescription [9.2, 9.5–9.7], has finite matrix elements already [9.2, 9.6]; we shall consider gauge fields below. We emphasize that solely the matrix elements of  $\theta^{\mu\nu}$ , i.e. Green functions, are finite. The expectation value  $\langle \theta^{\mu\nu} \rangle$  will, in general, be UV-divergent, and must be renormalized as a normal product:  $\Theta^{\mu\nu}(x) = \langle N[\theta^{\mu\nu}(x)] \rangle$ .

#### 4.1.3. Gauge theories

With regard to the energy-momentum tensor of a gauge theory, one faces the following situation. In classical electrodynamics, one can use the canonical prescription [9.10, 9.30]

$$t^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L}, \quad (4.1.21)$$

where the sum is over all fields in the Lagrangian, to find a non-symmetric, non-gauge-invariant energy-momentum tensor. Subsequently, the classical equations of motion and the freedom (4.1.19) can be used to show that the canonical tensor is equivalent to the manifestly symmetric and gauge invariant energy-momentum [9.10, 9.30]

$$\theta^{\mu\nu} = -F_\sigma^\mu F^{\nu\sigma} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \quad (4.1.22)$$

where  $F_{\mu\nu}$  is the field tensor. Note that  $\theta^{\mu\nu}$  is precisely the Belinfante tensor of the classical theory.

For quantized gauge theories the situation is more complicated. Applying the canonical prescription (4.1.21) to the effective quantum Lagrangian (2.2.39), we find

$$\theta_{\text{can}}^{\mu\nu} = -g^{\mu\nu}\mathcal{L}_{\text{eff}} - F_\sigma^{\mu a} \partial^\nu A^{\sigma a} - \lambda(\partial \cdot A^a) \partial^\nu A^{\mu a} - (\partial^\mu \bar{\omega}^a) \partial^\nu \omega^a - (\partial^\nu \bar{\omega}^a) \partial^\mu \omega^a - gf^{abc}(\partial^\nu \bar{\omega}^a) A^{\mu b} \omega^c. \quad (4.1.23)$$

This expression is non-symmetric, gauge noninvariant and not even BRS-invariant, and therefore  $\theta_{\text{can}}^{\mu\nu}$  cannot be an observable [12.17]. However, we may apply the standard Belinfante prescription to (4.1.23) and use the equation of motion for the field strength derived from (2.2.39) to find the Belinfante tensor corresponding to the Lagrangian (2.2.39) [9.24]

$$\begin{aligned} \theta_B^{\mu\nu} = & \theta^{\mu\nu} - (\partial^\mu \bar{\omega}^a) D^\nu \omega^a - (\partial^\nu \bar{\omega}^a) D^\mu \omega^a + g^{\mu\nu}[(\partial_\sigma \bar{\omega}^a) D^\sigma \omega^a - \frac{\lambda}{2}(\partial \cdot A^a)^2 - \lambda(\partial_\sigma \partial \cdot A^a) A^{\sigma a}] \\ & + \lambda(A^{\mu a} \partial^\nu \partial \cdot A^a + A^{\nu a} \partial^\mu \partial \cdot A^a). \end{aligned} \quad (4.1.24)$$

Here  $\theta^{\mu\nu}$  is the gauge-invariant tensor (4.1.22) (with a colour trace understood), and  $D^\mu \omega^a$  is the covariant derivative appearing in (3.3.3). The Belinfante tensor differs from the canonical one by a transformation of the form (4.1.19) only, and is, therefore, still conserved by construction. Moreover, it is manifestly symmetric. As in all other known cases, the Belinfante tensor (4.1.24) coincides with the gravitational energy-momentum tensor [9.10] derived from (2.2.39), cf. [9.8].

It is not difficult to show that  $\theta_B^{\mu\nu}$  (4.1.24) is BRS-invariant. Firstly,  $\theta^{\mu\nu}$  is gauge-invariant hence BRS-invariant. Secondly, the other terms in (4.1.24) all contain the BRS transformation of either  $A^{\mu a}$  (3.3.3) or  $\bar{\omega}^a$  (3.3.5). Using the anticommutating nature of the parameter  $\xi$  in (3.3.3) and (3.3.5), it is

then easy to verify that (4.1.24) can be written as (cf. [12.17], formula 5.63a)

$$\theta_B^{\mu\nu} = \theta^{\mu\nu} + \xi^{-1} \delta_{\text{BRS}} [(\partial^\mu \bar{\omega}^a) A^{va} + (\partial^\nu \bar{\omega}^a) A^{\mu a} - g^{\mu\nu} ((\partial_\sigma \bar{\omega}^a) A^{\sigma a} + \frac{1}{2} \bar{\omega}^a \partial \cdot A^a)]. \quad (4.1.25)$$

Here we used the fact that the BRS variation of  $\partial \cdot A^a$  vanishes on account of (3.3.3) and the ghost equation of motion derived from (2.2.39). Finally, the BRS transformation is nilpotent when the ghost equation of motion is imposed [12.17, 12.18]. Hence BRS invariance of  $\theta_B^{\mu\nu}$  follows immediately from (4.1.25).

Because both the gauge invariant tensor (4.1.22) and the Belinfante tensor (4.1.25) are BRS-invariant and in addition have vanishing ghost number, we may conclude that both are acceptable as a local observable [12.17]. The difference is that  $\theta_B^{\mu\nu}$  obviously enjoys all properties one would expect a physical energy-momentum tensor to have, whereas  $\theta^{\mu\nu}$  is not conserved and its interpretation could be doubtful. However, as first shown by Joglekar [9.13], the matrix elements of  $\theta^{\mu\nu}$  and  $\theta_B^{\mu\nu}$  between (on-shell) physical states coincide, so that, in the end,  $\theta^{\mu\nu}$  and  $\theta_B^{\mu\nu}$  are physically indistinguishable. In particular, the divergence of  $\theta^{\mu\nu}$  vanishes between physical states, and the expectation value of the charges  $\int d^3x \theta^{0\nu}(x)$  in a given physical state corresponds to the conserved total four-momentum of that state [9.13].

Joglekar's results, obtained by functional methods, find their rationale in the covariant operator formalism of gauge theories [12.17]. Namely, it can be shown that physical states are singled out by the requirement that they are annihilated by the so-called BRS charge (i.e. the generator of the BRS transformation (3.3.3)–(3.3.5)). For our purpose this can be translated into the condition that the BRS variation of any operator has vanishing matrix elements between physical states. Seen in this light the physical equivalence of  $\theta^{\mu\nu}$  and  $\theta_B^{\mu\nu}$  follows immediately from (4.1.25).

In addition, it can be shown that the Green functions of both  $\theta_B^{\mu\nu}$  [9.8] and  $\theta^{\mu\nu}$  [9.13] are finite (albeit different off-shell), i.e., there is no need for an “improvement term” as in the scalar case, cf. section 4.1.2. (This holds true in spite of the fact that the ghosts are formally scalars, and that the field virial of the gauge field is not a total divergence, as required in [9.6].)

These results allow us to employ either the Belinfante or the gauge invariant energy-momentum tensor, depending on which is the most convenient in a given situation. For example, in explicit computations it is clearly advantageous to use the gauge invariant tensor (4.1.22). On the other hand, the Belinfante tensor will turn up in the proof of the virial theorem in section 4.3.

As a technical detail we mention that the statements made above, in fact, were proved for a tensor  $\theta^{\mu\nu} + c^{\mu\nu}$ , rather than  $\theta^{\mu\nu}$  itself. The tensor  $c^{\mu\nu}$ , which vanishes by the equations of motion, plays a role in the renormalization process [9.13, 9.16, 9.17, 9.22]. For our purpose it can be totally ignored.

#### 4.1.4. Renormalization

In the following we will generically denote the energy-momentum tensor by  $t^{\mu\nu}$ . The Pauli transformation (4.1.19) has no effect on the equilibrium expectation value, so we may take  $t^{\mu\nu}$  to the “improved” tensor (4.1.20) for scalar fields, or the gauge field tensor (4.1.22), etc.

We recall from section 4.1.2 that the composite operator  $t^{\mu\nu}$ , being conserved, has the special property that its Green functions  $T_{\mu\nu}(x|y')$ , also called its matrix elements, are finite after ordinary coupling constant and wave function renormalization, but its expectation value  $\langle t^{\mu\nu} \rangle$  in general is not. Still we would like to define a finite expectation value at any temperature and density. It is possible, of course, to replace  $t^{\mu\nu}$  by the normal product  $N[t^{\mu\nu}]$ , but this could lead to operator mixing, rendering the interpretation of  $N[t^{\mu\nu}]$  as an energy-momentum tensor dubious. Moreover, we would have to

invoke finite-temperature renormalization prescriptions to ensure that the virial theorem of section 4.2 remains valid.

To avoid these problems we will simply define the renormalized expectation value of  $t^{\mu\nu}$  by

$$\langle t^{\mu\nu} \rangle_{\text{ren}} := \langle t^{\mu\nu} \rangle - \langle t^{\mu\nu} \rangle_{\text{vac}}. \quad (4.1.26)$$

In other words, we subtract the self-stress of the vacuum. This makes sense physically, but we still have to show that it is a consistent prescription, meaning that it does not lead to temperature- and density-dependent divergences after ordinary renormalization at  $T, \mu = 0$ . The point is that our earlier discussion of renormalization has been one in terms of the forest formula and Zimmermann's normal products. Only for free fields a prescription like (4.1.26) is equivalent to normal ordering. That it will come out allright though for the energy-momentum tensor, crucially depends on its being conserved; for an arbitrary composite operator, (4.1.26) would not be correct.

The proof that the right-hand side of (4.1.26) is finite, turns out to be not so difficult after the preparatory work in sections 2.4.4, 4.1.1 and 4.1.2. Let us give the argument for a theory with arbitrary cubic and quartic couplings. The kinetic part of the energy-momentum tensor gives rise to a composite two-point vertex independent of the coupling constants. The three- and four-point composite vertices, on the other hand, are proportional to the cubic and quartic coupling constants, respectively. Since the expectation value  $\langle t^{\mu\nu}(x) \rangle$  is independent of  $x$ , we may take the time to be real, which implies that the composite vertices are of type-1. Then the expectation value  $\langle t^{\mu\nu} \rangle$  can be graphically represented as in fig. 4.2. The blobs comprise the sum of all ordinary renormalized two-, three- and four-point (not necessarily connected) Green functions; the disjoint bubble diagrams and global energy-momentum conservation have been divided out. A summation over all directions (if any) of the internal lines is understood; the combinatorial factors then come out as indicated. Finally, thermal and other indices are suppressed; in the sequel also the arrow indicating zero four-momentum outflow on  $\langle t^{\mu\nu} \rangle$  will be deleted.

By inserting the graphical representation of the Dyson-Schwinger equation for the Green functions [12.19] and rearranging terms, one may derive the diagrammatic expansion of fig. 4.3. Now the crucial observation is that each of the sets of diagrams between parentheses can be factorized into the product of an ordinary Green function  $\tilde{G}$  and an (amputated) composite Green function  $\tilde{T}_{(i)\text{amp}}^{\mu\nu}$ , where the

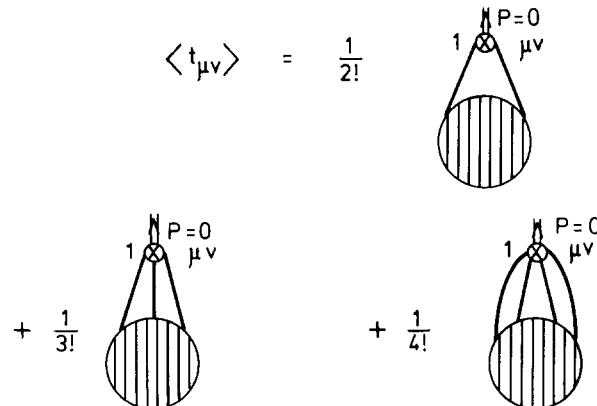


Fig. 4.2. The expectation value  $\langle t^{\mu\nu} \rangle$ .

$$\begin{aligned}
 \langle t_{\mu\nu} \rangle &= \frac{1}{2} \left( \text{Diagram 1} + \left( \frac{1}{2} \text{Diagram 2} + \frac{1}{3} \text{Diagram 3} \right) \right. \\
 &\quad \left. + \left( \frac{1}{4} \text{Diagram 4} + \frac{1}{8} \text{Diagram 5} \right) + \left( \frac{1}{8} \text{Diagram 6} + \frac{1}{12} \text{Diagram 7} \right) \right) \\
 &\quad + \left( \frac{1}{12} \text{Diagram 8} + \frac{1}{36} \text{Diagram 9} + \frac{1}{48} \text{Diagram 10} \right) \\
 &\quad + \left( \frac{1}{72} \text{Diagram 11} + \frac{1}{144} \text{Diagram 12} \right)
 \end{aligned}$$

(4.1.27)

Fig. 4.3. Dyson-Schwinger expansion of  $\langle t^{\mu\nu} \rangle$ .

index  $(i)$ ,  $i = 0, 1, 2$ , indicates that the latter is either bare, linear, or quadratic in the coupling constants:

$$\begin{aligned}
 \langle t^{\mu\nu} \rangle &= \tilde{T}_{(0)}^{\mu\nu}(0) + \sum_{m=1}^2 \int \prod_{i=1}^m \frac{d^4 k_i}{(2\pi)^4} \delta\left(\sum_{j=1}^m k_j\right) \tilde{T}_{(1)\text{amp}}^{\mu\nu}(0|k^m) \tilde{G}(k^m) \\
 &\quad + \sum_{m=4}^6 \int \prod_{i=1}^m \frac{d^4 k_i}{(2\pi)^4} \delta\left(\sum_{j=1}^m k_j\right) \tilde{T}_{(2)\text{amp}}^{\mu\nu}(0|k^m) \tilde{G}(k^m).
 \end{aligned} \tag{4.1.27}$$

The amputation refers to external bare propagators. Equation (4.1.27) follows by inspection of fig. 4.3, and subsequent use of the DS equations connecting the Green functions.

Now, save for the first term, the right-hand side of (4.1.27) is finite after standard renormalization, because both the ordinary Green functions and the composite energy-momentum Green functions have this property. (An example of the remarkable cancellation of infinities implied by (4.1.27) will be given in section 4.2.) Thus, the only divergence occurs in the free-field contribution, more specifically, in its bare vacuum part. But this term, together with other (finite) vacuum contributions, is explicitly subtracted in (4.1.26). We conclude that the prescription (4.1.26) leads to a satisfactory definition of the energy-momentum tensor. We also see that a mere subtraction of the bare vacuum part would have been sufficient. However, this prescription would lead to a nuisance in the proof of the virial theorem; cf. section 4.3.2.

## 4.2. Thermodynamic potential

The thermodynamic potential, or equivalently the thermodynamic pressure, as a function of temperature and chemical potentials, completely characterizes the thermodynamic state of a many-body system. It is of some interest, therefore, to state the Feynman rules which permit a perturbative calculation. Basically, there are two methods: either one calculates the thermodynamic potential directly as a sum of closed connected diagrams (bubbles), or one first calculates a Green function and then integrates with respect to the coupling constant or some other parameter [1.6, 1.11, 1.25, 1.30]. Both methods will be discussed below. In the Matsubara formalism they are entirely equivalent. As it will turn out, however, in the real-time formalism only the second method can be applied [4.21]. Still, the final Feynman rules will closely resemble those of the direct method.

### 4.2.1. Effective action

The Matsubara formalism provides a general diagram technique for evaluating the thermodynamic potential  $\Omega = -PV$ , or what amounts to the same, the thermodynamic pressure

$$P = \frac{1}{\beta V} \log Z[0]. \quad (4.2.21)$$

Here  $Z[0]$  is the partition function for which we have the path-integral representation (2.3.14). We first evaluate the free partition function by performing a Gaussian integral in Euclidean space:

$$\log Z_0[0] = \frac{1}{2}\eta \int_0^\beta d\tau \int_V d^3x \text{tr} \log \Delta(x, x) + \log \mathcal{N}. \quad (4.2.2)$$

The trace symbol implies contraction of all discrete indices; note that in the case of complex fields the factor one-half must be deleted.

The Euclidean propagator  $\Delta$  is to be regarded as an operator on the space of (anti)periodic functions with matrix elements  $\Delta(x, x') := \Delta(\tau - \tau', \mathbf{x} - \mathbf{x}')$  given by (2.3.6). The eigenfunctions of this operator are obviously  $\exp(ip \cdot x - i\omega_n \tau)$  with eigenvalues  $\tilde{\Delta}(i\omega_n, p)$ . Therefore, the logarithm in (4.2.2) has the matrix elements

$$\log \Delta(x, x') = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \exp[i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') - i\omega_n(\tau - \tau')] \log \tilde{\Delta}(i\omega_n, p). \quad (4.2.3)$$

In virtue of translational invariance the diagonal elements are independent of  $x$ . Hence we see that a factor of  $\beta V$  comes out of (4.2.2) in a natural manner, and that the extensivity of the free thermodynamic potential is a consequence of the translational invariance of the underlying field theory. Equations (4.2.1)–(4.2.3) then yield

$$P_0 = -\frac{1}{2}\eta \text{Tr} \log \tilde{\Delta}^{-1}(k) + \frac{1}{\beta V} \log \mathcal{N}. \quad (4.2.4)$$

The trace symbol now includes an integration and summation over the momentum and energy, respectively. The propagator  $\tilde{\Delta}(k)$  may be any free propagator, but we shall confine ourselves here to the scalar case.

The perturbation expansion of the interaction pressure  $P_1 = P - P_0$  follows by expanding the

representation (2.3.14), and by setting the sources equal to zero afterwards. This expansion generates  $P_I = \Sigma(\text{bubbles})$  [1.25, 2.17, 12.19], that is, the sum of all closed connected diagrams, multiplied by their combinatoric weight; see fig. 4.4. The Feynman rules, as stated in section 2.3.2, apply with the stipulation that the factor  $\beta(2\pi)^3\delta(\mathbf{0}) = \lim \beta V$  arising from global energy-momentum conservation should be deleted; cf. rule (iii). This shows, by the way, that the interaction part of  $\log Z[0]$ , like the free part, is extensive as a consequence of translation invariance.

In general, combinatoric factors of bubble diagrams are not derived by any simple rule [12.9, 12.19, 12.22]. Fortunately, they may also be inferred from some functional relationships involving the full propagator  $\tilde{\Delta}(k)$  (3.1.2). From the path-integral representation with the imaginary-time action (2.3.13) it immediately follows that one has [2.11]

$$\frac{\delta P}{\delta \tilde{\Delta}^{-1}(k)} = -\frac{1}{2}\eta \tilde{\mathcal{G}}(k), \quad (4.2.5)$$

a connection valid for fields of any spin. (For charged fields the factor one-half must be omitted.)

The formula can be visualized as follows. A closed graph may be considered as the closure of a self-energy graph. (Here a self-energy graph is understood to be an amputated full propagator graph; in general such graphs are reducible. A one-particle irreducible self-energy graph will be called proper in the sequel.) Conversely, a self-energy graph may be obtained by removing one of the lines in a closed graph. In fact, the class of self-energy graphs may be generated by opening up one line in all possible ways in the graphical expansion of the pressure:

$$\frac{\delta P_I}{\delta \tilde{\Delta}(k)} = \frac{1}{2}\eta \sum_{m=1}^{\infty} [-\tilde{\Sigma}(k) \tilde{\Delta}(k)]^m \tilde{\Delta}^{-1}(k). \quad (4.2.6)$$

Here  $\tilde{\Sigma}$  is the self-energy represented by minus the sum of all proper graphs. The extra minus sign for fermions appears because a closed fermion loop carries a minus sign. Using now the Dyson equation

$$\tilde{\mathcal{G}}^{-1}(k) = \tilde{\Delta}^{-1}(k) + \tilde{\Sigma}(k) \quad (4.2.7)$$

for the full propagator, one may easily rederive the functional relationship (4.2.5).

In the graphical analysis of  $P_I$  the concept of an  $m$ -cycle of lines plays an important role. As already explained in section 2.3.3, this is a set of  $m$  lines, all carrying the same energy and momentum, associated with an  $m$ -fold direct product  $\tilde{\Delta}^m(k) = [\tilde{\Delta}(k)]^m$  of propagators with the same denominator. Since each line in a closed diagram can be classified, without ambiguity, as belonging to a certain cycle, we may consider  $P_I$  as a functional of cycles with increasing multiplicity

$$P_I = P_I[\tilde{\Delta}^1, \tilde{\Delta}^2, \dots, \tilde{\Delta}^m, \dots]. \quad (4.2.8)$$



Fig. 4.4. First few bubble diagrams of a  $\phi^3$  theory.

If a line belonging to a 1-cycle is cut in the diagrammatic representation of  $P_1$ , the self-energy diagram so obtained cannot be separated into two disconnected parts by cutting one other single line. In other words, these diagrams are proper and represent the self-energy. The generalization of this reasoning to cycles of arbitrary multiplicity leads to the functional relationship [10.10, 10.15]

$$\frac{\delta P_1}{\delta \tilde{\Delta}^m(k)} = \eta \frac{1}{2m} [-\tilde{\Sigma}(k)]^m. \quad (4.2.9)$$

It algebraically expresses the topological property that the parts of a diagram connected by cycles are proper self-energy insertions; see fig. 2.3.

It is possible to carry out a partial resummation of the self-energy insertions, and to express the pressure in terms of the full propagator [10.3, 2.11]. The standard way to achieve this is by a Legendre transformation [1.24]. We define

$$\Gamma = P + \frac{1}{2}\eta \text{Tr } \tilde{\Delta}^{-1}\tilde{\mathcal{G}} - \frac{1}{2}\eta \text{Tr } \log \tilde{\mathcal{G}}. \quad (4.2.10)$$

The ancillary quantity  $\Gamma$ , which is called the effective action in this context, may be regarded as a functional of the full propagator. Indeed, upon varying (4.2.10) and using (4.2.5) and (4.2.7), we obtain

$$\delta\Gamma/\delta\tilde{\mathcal{G}}(k) = -\frac{1}{2}\eta\tilde{\Sigma}(k); \quad \delta\Gamma/\delta\tilde{\Delta}^{-1}(k) = 0. \quad (4.2.11)$$

This is equivalent to the requirement

$$\delta P/\delta\tilde{\mathcal{G}}(k) = 0 \quad (4.2.12)$$

that the full propagator be fixed at a local extremum of  $P$ ; cf. [2.11].

We conclude that the graphical representation of  $\Gamma$  consists only of those bubble diagrams that do not contain cycles with a multiplicity greater than one; e.g. the first and second diagrams in fig. 4.4. These diagrams cannot be split apart by cutting two lines. In other words, these bubble diagrams are two-particle irreducible (2PI) [12.11, 12.19].

The effective action has many applications in field theory. For example, the mechanism of dynamical symmetry breaking has been studied [12.11, 12.21] by looking for solutions of eq. (4.2.12). The effective action has also been used to obtain self-consistent approximations to the full Schwinger–Dyson equation for the gluon propagator. Recently, Kajantie and Kapusta [6.15] have calculated some non-perturbative contributions to the gluon self-energy at high temperature in this way.

#### 4.2.2. Statistical quasiparticles

Basic to the current understanding of the properties of interacting many-body systems is the quasiparticle concept introduced by Landau [10.2, 1.32]. Broadly speaking, these quasiparticles are elementary excitations of the system with a definite relation between energy and momentum, in terms of which the system is conveniently described. In the early work on many-body theory by Luttinger and Ward [10.3], and others [1.21, 1.22, 1.30–1.32], the functional expression (4.2.10) in terms of the full propagator served as a basis for the formal derivation of quasiparticle behaviour of a degenerate electron gas. This led to the identification of the low-lying excitations of a Fermi liquid at low

temperature with the weakly damped poles of the one-particle fermion propagator. The associated self-energy determines both the energy and the lifetime of these “dynamical” quasiparticles, which should be long in order that the very concept of a quasiparticle has any meaning. Since the conventional many-body self-energy usually possesses an imaginary part, except close to the Fermi surface [10.3, 1.32], the dynamical description is confined to Fermi systems at very low temperature. The extension of the dynamical quasiparticle picture to truly finite temperatures seems to be forbidden by a mathematical no-go theorem [5.24].

In view of eq. (4.2.5) one may say that dynamical quasiparticles are defined by functional differentiation of the pressure with respect to the bare propagator followed by analytic continuation. This is to be contrasted with Landau’s definition of a quasiparticle [10.2] which involves a functional differentiation with respect to the distribution function. These two prescriptions are quite different in general, because the operations of functional differentiation and analytic continuation do not commute.

To get acquainted with the main idea of Landau quasiparticles, we shall first calculate the ideal pressure (4.2.4) for a boson gas. The summation which has to be performed in that case is

$$\frac{1}{\beta} \sum_n \log[\beta^2 \tilde{\Delta}^{-1}(i\omega_n, k)] = \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dz}{2\pi i} [2N(z) + 1] \log[\beta^2 \Delta^{-1}(z, k)], \quad (4.2.13)$$

where  $\Delta^{-1}(z, k) = \omega_k^2 - z^2$  is the analytic continuation of the bare propagator. The argument of the logarithm has been made dimensionless by factors of  $\beta$  provided by the  $\log \mathcal{N}$  term at the right-hand side of (4.2.4), which is also used to cancel part of the term without the Bose distribution function; an infinite vacuum contribution is subtracted by renormalization [2.5, 2.20]. The logarithm may be analytically continued away from the imaginary axis to the complex plane where it has a cut along the real axis from  $k_0 = \pm \omega_k$  running to plus and minus infinity. Since the Bose function exponentially decreases in the right half plane, the contour running up the imaginary axis may be deformed into the contour  $C_+$  encircling the cut, as prescribed in (2.3.22). Hence, the frequency sum (4.2.13) may be written as an integral of the discontinuity across the real axis:

$$\frac{1}{\beta} \sum_n \log[\beta^2 \tilde{\Delta}^{-1}(i\omega_n, k)] = \int_0^\infty \frac{dk_0}{\pi i} N(k_0) \text{Disc} \log[\beta^2 \Delta^{-1}(k)]. \quad (4.2.14)$$

Substitution into (4.2.4) yields

$$P_0 = \int \frac{d^4 k}{(2\pi)^3} H_0(k) N(k_0). \quad (4.2.15)$$

We shall call

$$H_0(k) = -\frac{1}{\pi} \theta(k_0) \text{Im} \log \Delta^{-1}(k) = \theta(k_0 - \omega_k) \quad (4.2.16)$$

the spectral function. An integration by parts with respect to the energy yields the standard formula (4.3.24) for the pressure of an ideal Bose gas.

The spectral function (4.2.16) expresses the fact that a free system is composed of stable particles

possessing a fixed mass. The assumption which lies at the root of the Landau theory is that there exists a one-to-one correspondence between the excitation spectrum of an interacting system and that of a free quantum gas. We formulate this ansatz as follows [10.15, 10.17]. Suppose the pressure is given as a functional  $P = P[N]$  of the distribution function. Then the step which allows a single-particle description is to assume that the spectral function, generally defined as the functional derivative

$$\frac{\delta P}{\delta N(\mathbf{k})} = H(k) \quad (4.2.17)$$

takes the form

$$H(k) = -\frac{1}{\pi} \theta(k_0) \operatorname{Im} \log[\Delta^{-1}(k) + \bar{\Sigma}(k)]. \quad (4.2.18)$$

Here  $\bar{\Sigma}(k)$  is some real function of  $k_0$  and  $\mathbf{k}$  such that the dispersion relation

$$\Delta^{-1}(k) + \bar{\Sigma}(k) = 0, \quad k_0 > 0, \quad (4.2.19)$$

has a single solution  $\varepsilon_k$ . This defines the energy spectrum of the Landau quasiparticle, also given the adjective “statistical”, as opposed to “dynamical”, in the literature [10.4].

Since the entropy of a state is determined purely by combinatorial considerations, a viable single-particle description requires that the entropy density  $S = \partial P / \partial T$  of a quasiparticle system have the same form as that of a free quantum gas, i.e. [10.2, 1.32, 10.8],

$$S = - \int \frac{d^3 k}{(2\pi)^3} [N_k \log N_k - (1 + N_k) \log(1 + N_k)], \quad (4.2.20)$$

where  $N_k = N(\varepsilon_k)$  is the quasiparticle distribution function. By some straightforward algebraic manipulations, it is easy to show that the ansatz (4.2.18) is indeed consistent with this requirement [10.15]. In fact, it is eq. (4.2.20) that is the pivot of the whole scheme.

Now the problem of finite-temperature field theory is to justify the ansatz (4.2.18) for specific systems, and to give a meaning to the self-energy  $\bar{\Sigma}(k)$ . For illustration we consider the exchange pressure, i.e. the two-loop diagram in fig. 4.4. Applying the general formula (2.3.31) furnished by the Dzyaloshinskii algorithm, we obtain

$$\begin{aligned} P^{\text{ex}} &= \frac{1}{12} \mathcal{B}(3, 2) \\ &= \frac{g^2}{12} \int \prod_{i=1}^3 \frac{d^4 k_i}{(2\pi)^4} \rho_0(k_i) (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{(1 + N_1)(1 + N_2)(1 + N_3) - N_1 N_2 N_3}{k_1^0 + k_2^0 + k_3^0}, \end{aligned} \quad (4.2.21)$$

where  $N_i = N(k_i^0)$  is the Bose distribution function.

The first property to be noticed is that, on account of the identity  $1 + N(k_0) = N(k_0) \exp \beta_0 k_0$ , the integrand does not have a pole when the total energy of the intermediate state vanishes. However, a fundamental difficulty arises when we wish to take the functional derivative (4.2.17) because the resulting expression is undefined. In this simple case it is sufficient to add an infinitesimal imaginary part  $i\epsilon$  to the denominator of (4.2.21), because the symmetry properties of the integrand guarantee that the

whole expression is still real; recall that  $\rho_0(k)$  is an odd function of energy. Then functional differentiation yields:

$$\frac{\delta P^{\text{ex}}}{\delta N(k)} = -\frac{1}{2\pi} \theta(k_0) \rho_0(k) \operatorname{Re} \Sigma(k_0 + i\varepsilon, k), \quad (4.2.22)$$

where  $\operatorname{Re} \Sigma$  is the real part of the analytic self-energy (3.1.36) on the real axis. Comparing with (4.2.17) and (4.2.18) we see that to lowest order one may identify

$$\bar{\Sigma}(k) = \operatorname{Re} \Sigma(k_0 + i\varepsilon, k), \quad (4.2.23)$$

which is a valid but misleading result [2.8, 2.14].

The problem is that (4.2.23) ceases to be true in higher orders because infinitesimal imaginary parts  $i\varepsilon$  must be associated with frequency denominators in a consistent manner to ensure that functional differentiation is allowed [10.4]. It would take us too far afield to discuss this issue in any detail here [10.14]. But let us assume that it is possible, in principle, to do this in such a way that after analytic continuation by the Dzyaloshinskii algorithm the functional relationship (4.2.9) survives in the form

$$\left. \frac{\delta P_1}{\delta \tilde{\Delta}^m(i\omega_n, k)} \right|_{i\omega_n=k_0} = \frac{1}{2m} [-\bar{\Sigma}(k)]^m. \quad (4.2.24)$$

The self-energy  $\bar{\Sigma}(k)$  is represented by the same set of diagrams as the ordinary many-body self-energy, but the rules for its calculation must be such that it has no imaginary part. Functional differentiation with respect to the distribution function can now be translated into a functional differentiation with respect to cycles, cf. section 4.2.1, and we obtain the formal rule

$$\frac{\delta P_1}{\delta N(k)} = \frac{1}{\pi} \theta(k_0) \sum_{m=1}^{\infty} \rho_0^{(m-1)}(k) \left. \frac{\delta P_1}{\delta \tilde{\Delta}^m(i\omega_n, k)} \right|_{i\omega_n=k_0}. \quad (4.2.25)$$

Here  $\rho_0^{(m)}$  is the spectral density (2.3.35) of the cycle propagator (2.3.34). Combining the last two formulae, and using (A.3), we recognize the Taylor expansion of the logarithm (4.2.18).

The above statistical quasiparticle theory indicates that Landau theory is exact at finite temperature and is not limited by the lifetime of the quasiparticle states. The essential ingredient needed for this description is the real self-energy function  $\bar{\Sigma}(k)$  which determines the “statistical” energy spectrum. Contrary to the energy spectrum of the dynamical quasiparticles, the former is not directly related to the spectrum of the many-body Hamiltonian. For a Bose system it may be hard to actually construct this function in practice because of the delicate problem of vanishing energy denominators. Surprisingly enough, the construction of this self-energy for fermionic quasiparticles in a theory with fermions is almost trivial. In this case the energy denominators are regularized automatically, if all internal bosonic frequencies are kept imaginary [10.10, 10.15]. By this prescription one is led to the fermionic self-energy discussed in section 3.1.3.

#### 4.2.3. Real-time method

We have already seen in section 2.4.3 that the real-time formalism is not suitable to directly evaluate the partition function

$$Z[0] = e^{-\beta \Omega}. \quad (4.2.26)$$

Therefore, we shall approach the problem in an indirect manner and first write the thermodynamic potential as the grand-canonical expectation value of some real-time Green function. We can then use the thermo field rules of section 2.4.3. To achieve this goal one may proceed in two different ways. The one we shall discuss first is computationally the most convenient of the two, but is restricted to the evaluation of the interaction pressure. It has been developed by Matsumoto et al. [4.21] in the context of the operator formulation of thermo field dynamics. We shall give a functional derivation valid for any theory which has a path-integral representation of the form (2.2.9). This class includes all theories without derivative couplings, and gauge theories. (In other cases one should take (2.2.7) as a starting point.)

Let us return to the general, contour-dependent generating functional (2.2.11) and partition function (2.2.12). We now employ the well-known trick [1.6, 1.11] of multiplying  $\mathcal{L}_I$  in (2.2.10) by a constant  $\lambda$ , so that  $Z[0]$  is now a function of  $\lambda$ . Taking the derivative we find the differential equation

$$\frac{\partial \Omega}{\partial \lambda} = -\frac{i}{\beta} \int_C dt \int_V d^3x \langle \mathcal{L}_I(x) \rangle_\lambda \quad (4.2.27)$$

with the initial value  $\Omega(\lambda=0) = \Omega_0$ . The subscript  $\lambda$  indicates that the expectation value should be computed with interaction Lagrangian  $\lambda \mathcal{L}_I$ . On account of translation invariance the expectation value at the right-hand side of (4.2.27) is independent of  $x$ . Hence, we can trivially perform the integration. We are also free to take  $x_0 = t$  to be real, that is, to choose the fields in  $\mathcal{L}_I$  as type-I; cf. subsections 2.4.2 and 3.2.1. Then the solution to (4.2.27) yields the interaction pressure as

$$P_I = \int_0^1 d\lambda \langle \mathcal{L}_I(\phi_1) \rangle_\lambda. \quad (4.2.28)$$

We remark that this reasoning cannot be applied to give the total pressure. Indeed, had we multiplied  $\mathcal{L}$  by  $\lambda$  then the initial condition would have been  $\Omega(\lambda=0) = N \int [d\phi]$  which is a meaningless expression.

The expectation value at the right-hand side of (4.2.28) can now be evaluated with the help of the real-time diagram technique of section 2.4. The interaction Lagrangian  $\mathcal{L}_I$  plays the role of a composite operator giving rise to one or more composite vertices; see section 4.1, especially fig. 4.1. Numerically these vertices are equal to the type-1 vertices of the theory divided by a factor of  $i$ , but topologically they must be regarded as distinct. For any given diagram contributing to  $\langle \mathcal{L}_I \rangle$  which contains  $V$  vertices (including the composite vertex), the  $\lambda$  integration in (4.2.28) is trivial and simply gives a factor of  $1/V$ . Finally, it should be remarked that  $\langle \mathcal{L}_I \rangle$  is finite after ordinary mass, coupling constant, and wave-function renormalization. This is just a consequence of the fact that, except for the combinatorics, the diagrams of the theory are constructed in the usual manner.

Let us now summarize the Feynman rules by which the interaction pressure can be determined in the real-time method:

- (i) Draw all bubble diagrams containing one distinct type-1 vertex.
- (ii) Determine combinatorial factors by examining the permutation group symmetry of the diagrams; permutation of the special type-1 vertex is not allowed.
- (iii) Divide by  $iV$ , where  $V$  is the total number of vertices.

The other rules like the assignment of propagators are the ones detailed in section 2.4.

For certain simple diagrams it may happen that the same Feynman integral results, irrespective of the position of the special type-1 vertex. In that case, one may put it at any place in the diagram and determine the combinatorial factor as if there were no special vertex; the factor  $V^{-1}$  of rule (iii) must then be omitted [4.12, 4.21]. However, this is not the rule in general.

It is worth mentioning that formula (4.2.28) can also be applied in the Matsubara formalism. Since in that case all vertices are identical, the simplified combinatoric rule applies, which then gives back the bubble diagram expansion.

For illustration we shall compute the interaction pressure for a  $SU(N)$  gauge theory up to order  $g^2$ . In the high-temperature regime, we may neglect vacuum diagrams and subdiagrams which are finite and  $T$ -independent after renormalization. We are then left with the diagrams of fig. 4.5. The combinatorial factors may be determined by rules (ii) and (iii) above, but also by the simplified rule of treating all vertices as being identical. For example, the second diagram is invariant under permutation of the two thermal propagators giving a factor one-half. Another factor one-half comes from rule (iii) or from the permutation symmetry of the vertices. The sum of these diagrams can be shown to be gauge invariant, so we can work in the Feynman gauge.

We note that in this special case no type-2 vertices appear. This would require diagrams like the second one but with three thermal propagators. However, the corresponding Feynman integrals contain the factor  $\delta(k^2)\delta(p^2)\delta((k+p)^2)$  and vanish identically in virtue of the explicit expression for the  $SU(N)$  vertices. In a massless  $\phi^3$  theory, though, such diagrams would contribute.

Let us compute the second diagram as an example. The  $SU(N)$  vertices in combination with the propagator expression (3.3.28) through (3.3.31) give rise to the integral expression

$$\text{II} = \frac{g}{2} N(N^2 - 1) g^2 \int \frac{d^4 k}{(2\pi)^3} \frac{d^4 p}{(2\pi)^3} N(|k_0|) N(|p_0|) \delta(k^2) \delta(p^2) \frac{k^2 + p^2 + k \cdot p}{(k+p)^2 + i\epsilon}. \quad (4.2.29)$$

With the aid of the formulae (A.14) and (A.21) we get

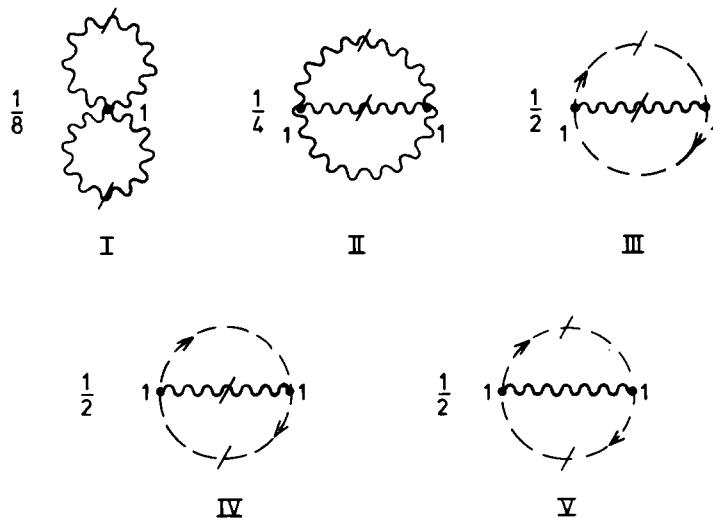


Fig. 4.5. Lowest-order diagrams contribution to  $P_1$  in the high-temperature limit.

$$\text{II} = \frac{1}{64} N(N^2 - 1) g^2 T^4 . \quad (4.2.30)$$

In a similar manner we determine the other contributions. Listed consecutively, they are:

$$\begin{aligned} \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} &= N(N^2 - 1) g^2 T^4 \left( -\frac{1}{48} + \frac{1}{64} - \frac{1}{576} - \frac{1}{576} + \frac{1}{576} \right) \\ &= -\frac{1}{144} N(N^2 - 1) g^2 T^4 . \end{aligned} \quad (4.2.31)$$

The final result is identical to that found by Kapusta [6.2] in the Matsubara formalism.

### 4.3. Hydrostatic pressure

In this section we discuss an alternative real-time method for determining both the free and interaction pressures, which is based on the evaluation of the energy-momentum tensor [3.16]. Namely, instead of calculating the thermodynamic pressure via the partition function, we shall direct our attention to the hydrostatic pressure (sometimes called the kinetic pressure) [11.16], defined by the space-like trace of the energy-momentum tensor. The virial theorem [11.16, 11.17] states that in equilibrium the hydrostatic and thermodynamic pressures are equal. A field-theoretic proof of this statement will be given after the notion of a dilatation has been introduced.

#### 4.3.1. Dilatations

A (rigid) spatial dilatation [9.2, 9.3, 9.5–9.7, 9.30] is a transformation of space-time coordinates and fields of the form

$$x \rightarrow e^\lambda x , \quad t \rightarrow t , \quad (4.3.1)$$

$$\phi(x) \rightarrow e^{d\lambda} \phi(t, e^\lambda x) , \quad (4.3.2)$$

or, infinitesimally,

$$\delta\phi(x) = \lambda(d + x \cdot \nabla) \phi(x) . \quad (4.3.3)$$

The number  $d$  is called the scale dimension of the field. In some cases  $d$  may be adapted so as to make the dilatation a symmetry transformation. In general  $d$  is only defined for elementary fields and often chosen to be equal to the dimension of the field in mass units. An elementary analysis [9.5, 9.6] shows that the spatial dilatation is generated by the charge

$$\hat{D}(t) = \int d^3x [x_i \hat{\theta}_B^{0i}(x) + \hat{V}^0(x)] . \quad (4.3.4)$$

Here  $\hat{\theta}_B^{\mu\nu}$  is the Belinfante energy-momentum tensor, and  $V^0$  is the time component of the so-called field virial

$$V^\mu = \sum_a (g^{\mu\nu} d_a + \Sigma_a^{\mu\nu}) \phi_a \cdot \frac{\partial \mathcal{L}}{\partial (\partial^\nu \phi_a)} . \quad (4.3.5)$$

The sum is over all fields in the Lagrangian, and  $d_a$  and  $\Sigma_a^{\mu\nu}$  are the relevant scale dimension and spin tensor pertaining to  $\phi_a$ , respectively.

For all renormalizable theories except gauge theories, the dilatation generator (4.3.4) can be rewritten as

$$\hat{D}(t) = \int d^3x x_i \hat{\theta}^{0i}(x) \quad (4.3.6)$$

in terms of the “improved” energy-momentum tensor [9.5, 9.6]

$$\theta^{\mu\nu} = \theta_B^{\mu\nu} - \frac{1}{6} \sum_a' (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi_a^2, \quad (4.3.7)$$

where now the sum is over scalar fields only; cf. section 4.1.2. The additional term is of the form (4.1.19) and does not contribute to the total energy-momentum of the system. Note, that the singular term with  $f(d, \lambda)$  in (4.1.20), needed to give  $\theta^{\mu\nu}$  finite matrix elements, plays no role in this context. Equation (4.3.6) is a consequence of the fact that the field virial equals  $\frac{1}{2} \partial^\mu \phi^2$  for a scalar field, and can be absorbed in a redefined energy-momentum tensor, while the virial vanishes for spin  $\frac{1}{2}$  fields and renormalizable massive spin 1 fields. Gauge fields form an exception. For example, the  $SU(N)$  Lagrangian (2.2.39) yields

$$V^\mu = 2\lambda A^{\mu a} \partial \cdot A^a - \partial^\mu (\bar{\omega}^a \omega^a) - gf^{abc} \bar{\omega}^a A^{\mu b} \omega^c. \quad (4.3.8)$$

We shall shortly see how to deal with this term.

Let us now derive some equal-time commutation relations. The time component of a current which arises from an internal symmetry transformation can generally be written as

$$j^0(x) = \pi(x) T \phi(x). \quad (4.3.9)$$

Here  $\pi(x)$  is the canonical momentum conjugate to  $\phi(x)$ , and  $T$  some symmetry generator. (Internal indices are suppressed.) Canonical quantization and the use of eqs. (4.3.6), (4.3.7) and (4.3.9) then leads to the equal-time commutator

$$i[\hat{D}(t), \hat{j}^0(t, \mathbf{x})] = (3 + \mathbf{x} \cdot \nabla) \hat{j}^0(t, \mathbf{x}), \quad (4.3.10)$$

showing that the charge density has scale dimension three. It should be noted that the extra term in (4.3.7) does not contribute to the commutator. Equation (4.3.10) holds true for gauge theories as well; if  $j^\mu$  is the current corresponding to global gauge transformations, then  $\hat{j}^0$  can be shown to commute with  $\hat{V}^0$  taken from (4.3.8), while the same is trivially true if  $j^\mu$  is a matter current.

We shall need the commutator with the energy density  $\hat{\theta}^{00}(x)$  as well. This quantity follows from the general commutator expression [9.1, 9.4]

$$i[\hat{\theta}^{00}(x), \hat{\theta}^{0i}(y)] = [\hat{\theta}^{ji}(x) - g^{ij} \hat{\theta}^{00}(y)] \partial_j \delta(\mathbf{x} - \mathbf{y}) + \hat{\tau}^{00,0i}(x, y), \quad (4.3.11)$$

where  $x_0 = y_0$ . The first two terms at the right-hand side follow from the demand that  $\hat{P}^\mu = \int d^3x \hat{\theta}^{0\mu}(x)$  and  $\hat{J}^{\mu\nu} = \int d^3x [x^\mu \hat{\theta}^{0\nu}(x) - x^\nu \hat{\theta}^{0\mu}(x)]$  generate the Poincaré algebra. The last term is a Schwinger term

involving higher derivatives of spatial delta functions, but such that it does not contribute to the Poincaré charges, e.g. [9.1, 9.4]

$$\int d^3x \mathbf{x} \hat{\tau}^{00,0i}(x, y) = 0. \quad (4.3.12)$$

Equation (4.3.11) holds for any energy-momentum tensor, thus, in particular for the Belinfante tensor featuring in (4.3.4) and for the improved tensor in (4.3.6). Combining eqs. (4.3.6), (4.3.11) and (4.3.12), we deduce, for non-gauge theories:

$$i[\hat{D}(t), \hat{\theta}^{00}(t, \mathbf{x})] = (3 + \mathbf{x} \cdot \nabla) \hat{\theta}^{00}(t, \mathbf{x}) - \hat{\theta}_i^i(t, \mathbf{x}), \quad (4.3.13)$$

where  $\theta^{00}$  is understood to be the “improved” energy density. This shows that the energy density has no definite scale dimension.

For  $SU(N)$  gauge theory we must use the original definition (4.3.4) with the Belinfante tensor (4.1.24). Canonically quantizing the Lagrangian (2.2.39) [12.17], one may check the validity of (4.3.11) for  $\theta_B^{\mu\nu}$  by explicit calculation (apart from the Schwinger term, which can never be found from the naive canonical equal-time commutators). As an aside, we remark that (4.3.11) also holds identically for the non-observable canonical tensor (4.1.23), while the gauge-invariant tensor (4.1.22) satisfies (4.3.11) if the Gauss-law constraint is (weakly) imposed. We can now use (4.3.4), (4.3.8) and (4.3.11) to arrive at the remarkable result

$$i[\hat{D}(t), \hat{\theta}_B^{00}(t, \mathbf{x})] = (3 + \mathbf{x} \cdot \nabla) \hat{\theta}_B^{00}(t, \mathbf{x}) - \hat{\theta}_{Bi}^i(t, \mathbf{x}) - \xi^{-1} \delta_{BRS}(\hat{\omega}^a \partial \cdot \hat{A}^a + 2\hat{A}^{0a} \partial_0 \hat{\omega}^a + 2\hat{A}^{ia} \partial_i \hat{\omega}^a). \quad (4.3.14)$$

Thus we see that the  $V^0$  term in (4.3.4) modifies (4.3.13) by a BRS-transformed operator only; this will be of significance in the next subsection.

### 4.3.2. Virial theorem

We are now ready to prove the following equality for the thermodynamic pressure:

$$P = -\frac{1}{3} \langle \hat{\theta}_i^i(x) \rangle \quad (4.3.15)$$

in the rest frame, provided the energy-momentum tensor is chosen suitably. We refer to eq. (4.3.15) as the virial theorem because the classical virial theorem of Clausius, and its various generalizations, are implied by it. (For an extensive review of the non-relativistic virial theorem see ref. [11.16, 11.17].) Let us first remark that eq. (4.3.15) is compatible with thermodynamics. Namely,  $P$  is a function of  $\mu$  and  $T$ , and as such a characteristic function [11.1, 11.9]. On the other hand, the hydrostatic pressure, i.e. the right-hand side of eq. (4.3.15), is found by evaluating the expectation value of the energy-momentum tensor. This yields the hydrostatic pressure as a functional of bare propagators depending on  $\mu$  and  $T$ . Hence, eq. (4.3.15) is consistent in this respect.

As in the classical case [11.16], the proof involves essentially a scaling argument. Possibly it could be modified or even invalidated if conformal (trace) anomalies [9.5, 9.6, 9.14, 9.15, 9.21, 9.24, 9.25, 9.29] occur. This we exclude explicitly here, although their influence would certainly be a topic for further investigation.

We shall leave renormalization out of consideration for the moment and start from the identity

$$\text{Tr } \hat{\rho}[\hat{D}(t), \beta \hat{H} + \alpha \hat{Q}] = 0, \quad (4.3.16)$$

where operators and trace are defined for a finite volume. For simplicity's sake we take only one conserved charge into account; the extension to several charges would only involve notational modifications. The identity is a consequence of the form (2.1.1) of the grand-canonical operator, and the cyclic invariance of the trace. Writing

$$\beta \hat{H} + \alpha \hat{Q} = \int_V d^3x [\beta \hat{\theta}^{00}(x) + \alpha \hat{j}^0(x)], \quad (4.3.17)$$

we may use (4.3.10) and (4.3.13) to calculate the commutator. For non-gauge theories this yields

$$i[\hat{D}(t), \beta \hat{H} + \alpha \hat{Q}] = -\beta \int_V d^3x \hat{\theta}_i^i(x) + \frac{\partial}{\partial \lambda} \int_{e^{3\lambda}V} d^3x [\beta \hat{\theta}^{00}(x) + \alpha \hat{j}^0(x)]|_{\lambda=0}. \quad (4.3.18)$$

Taking the grand-canonical average and recalling the definition of the grand-canonical partition function, we obtain

$$\beta \langle \hat{\theta}_i^i(x) \rangle + \frac{1}{V} \frac{\partial}{\partial \lambda} \log Z(\alpha, \beta, e^{3\lambda}V)|_{\lambda=0} = 0. \quad (4.3.19)$$

We may now use the fact that  $\log Z$  is an extensive quantity (see section 4.2.1). Hence, we conclude that the virial theorem (4.3.15) holds true for the unrenormalized pressure [11.13]. For gauge theories the right-hand side of (4.3.18) contains a contribution coming from the extra term in (4.3.14). However, as we mentioned in subsection 4.1.3, the BRS transformation of any operator has vanishing matrix elements between physical states. Now the grand-canonical density operator contains a projector on physical states [2.13, 4.6]. Hence, the extra term disappears on taking the grand-canonical average in going from (4.3.18) to (4.3.19). Therefore, our proof goes through equally for gauge theories.

The proof is formal to the extent that it employs the unrenormalized commutator expressions (4.3.10) and (4.3.11). However, our discussion in section 4.1.4 has shown that the renormalized expectation value (4.1.26) of  $\theta^{\mu\nu}$  differs from the unrenormalized one by a c-number only, namely its vacuum expectation value. Notice, that both the left-hand side (trivial) and the right-hand side of (4.3.11) are unaltered by the replacement  $\hat{\theta}^{\mu\nu} \rightarrow \hat{\theta}^{\mu\nu} - \langle 0|\hat{\theta}^{\mu\nu}|0\rangle$ , provided that the vacuum term has the form  $\lambda g^{\mu\nu}$  dictated by Lorentz invariance [9.1]. Since eq. (4.3.19) is valid in the vacuum limit as well, the equality of the renormalized hydrostatic and thermodynamic pressures follows as a matter of course. This final step looks rather trivial, but one should be well aware of the fact that it is the result of the entire analysis of section 4.1, and of section 4.1.4 in particular.

#### 4.3.3. Examples: $\phi^4$ and $SU(N)$

In this section the formalism developed above will be illustrated with some explicit calculations [3.16]. We begin with a  $\phi^4$  theory, for which the canonical energy-momentum tensor (4.1.21) reads

$$t^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \left( \frac{1}{2} \partial\phi \cdot \partial\phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right). \quad (4.3.20)$$

It leads to the composite vertices depicted in fig. 4.6. With the help of the real-time Feynman rules given in section 2.4.3, we may compute the free contribution to the renormalized energy density as

$$\begin{aligned}
 \langle t^{00} \rangle_{\text{ren}}^0 &= \frac{1}{2} \quad \text{(Diagram 1)} \\
 &= \frac{1}{2} \int \frac{d^4 k}{(2\pi)^3} \frac{\delta(k^2 - m^2)}{\exp(\beta|k_0|) - 1} (2k_0^2 - k^2 + m^2). \quad (4.3.21) \\
 \text{(Diagram 2)} \quad p=0 &= 2k^\mu k^\nu - g^{\mu\nu}(k^2 - m^2) \\
 \text{(Diagram 3)} \quad p=0 &= g^{\mu\nu}\lambda
 \end{aligned}$$

Fig. 4.6. Composite vertices generated by (4.3.20).

Using the expansions (A.21), (A.22), we obtain

$$\langle t^{00} \rangle_{\text{ren}}^0 = \frac{\pi^2 T^4}{30} - \frac{m^2 T^2}{24} + \mathcal{O}(m^3), \quad (4.3.22)$$

which for  $m = 0$  is nothing but the Stefan–Boltzmann law. Similarly, for the hydrostatic pressure we have

$$-\frac{1}{3} \langle t_i^i \rangle_{\text{ren}}^0 = \frac{1}{3} \int \frac{d^4 k}{(2\pi)^3} \frac{k^2 \delta(k^2 - m^2)}{\exp(\beta|k_0|) - 1}. \quad (4.3.23)$$

By a simple integration by parts we obtain the well-known formula

$$P_0 = \frac{1}{\beta} \int \frac{d^3 k}{(2\pi)^3} \log[1 + N(\omega_k)] \quad (4.3.24)$$

for the thermodynamic pressure, thus verifying the virial theorem.

We proceed with the first-order correction. After mass renormalization the diagrams of fig. 4.7 contribute. The first three diagrams are each “pathological” in the sense of section 3.2.2, but their sum is regular, as it must be, and even vanishes [3.16]. For the next diagram we have (see (A.14))

$$\text{IV} = -\frac{\lambda}{128\pi^4\beta^4} (L_2^+(\beta m, 0))^2. \quad (4.3.25)$$

The last two diagrams are each  $\beta$ -dependent and UV-divergent. According to the analysis of section

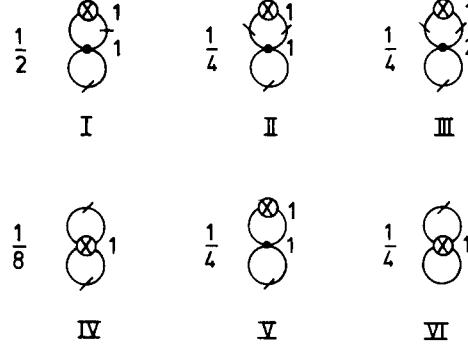


Fig. 4.7. First-order diagrams.

4.1.4 these divergencies have to cancel out. We use dimensional regularization [9.22], and setting  $\frac{1}{3}k^2 = k_i^2$ , we find

$$\begin{aligned} \text{V} &= \frac{i\lambda L_2^+(\beta m, 0)}{16\pi^2\beta^2} \int \frac{d^d k}{(2\pi)^d} \frac{k_0^2 - k_i^2 - m^2}{(k^2 - m^2 + i\epsilon)^2} \\ &= \frac{\lambda L_2^+(\beta m, 0)}{16\pi^2\beta^2} \frac{\Gamma(1-d/2)}{(4\pi)^{d/2} m^{2-d}}, \end{aligned} \quad (4.3.26)$$

$$\begin{aligned} \text{VI} &= -\frac{i\lambda L_2^+(\beta m, 0)}{16\pi^2\beta^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\epsilon} \\ &= -\frac{\lambda L_2^+(\beta m, 0)}{16\pi^2\beta^2} \frac{\Gamma(1-d/2)}{(4\pi)^{d/2} m^{2-d}}. \end{aligned} \quad (4.3.27)$$

Hence, these two contributions exactly cancel. Thus, the complete  $\mathcal{O}(\lambda)$  pressure is given solely by (4.3.25). It equals the thermodynamic pressure as evaluated in the Matsubara formalism [3.16]. This again vindicates the virial theorem.

As a second example we calculate the free pressure and the energy density of a  $SU(N)$  gauge theory. We have already explained in section 4.1.3 that we may use the gauge invariant energy-momentum tensor (4.1.22). It leads to the two-point vertex

$$\begin{array}{c} \mu \nu \quad | \quad k=0 \\ \text{a,}\alpha \quad \text{b,}\beta \\ \text{p} \quad -\text{p} \end{array} = \delta^{ab} [2\delta_{(\nu}^\alpha p_{\mu)} p^\beta + 2\delta_{(\nu}^\beta p_{\mu)} p^\alpha - 2g^{\alpha\beta} p_\mu p_\nu - 2\delta_{(\mu}^\alpha \delta_{\nu)}^\beta p^2 + g^{\mu\nu} (g^{\alpha\beta} p^2 - p^\alpha p^\beta)]. \quad (4.3.28)$$

The parentheses denote symmetrization with strength one. There are also three-point and four-point vertices [3.16], but these are not needed here. The relevant propagators are (3.3.28) through (3.3.31).

The steps leading to (4.3.21) and (4.3.23) may now be repeated for the present gluon case. We find

$$\langle t^{\mu\nu} \rangle_{\text{ren}}^0 = \frac{1}{2}(N^2 - 1) \int \frac{d^4 k}{(2\pi)^3} \frac{\delta(k^2)}{\exp(\beta|k_0|) - 1} (4k^\mu k^\nu - g^{\mu\nu} k^2). \quad (4.3.29)$$

With the help of (A.14) and (A.22) it is then a straightforward exercise to obtain the energy density and pressure

$$E_0 = 3P_0 = \frac{1}{15} (N^2 - 1) \pi^2 T^4 , \quad (4.3.30)$$

which is exactly what one expects [6.2, 6.4].

It would be possible, in principle, to go on to the next order, but the number of diagrams and the amount of algebra rapidly increase so as to be almost prohibitive [3.16]. Fortunately, we can fall back on the method of section 4.2.3 which is much less involved.

#### 4.4. Plasmon effect

At very high temperature and density hadronic matter as described by QCD is thought to undergo a phase transition to a state composed of quarks and gluons [6.1–6.9]. Renormalization-group arguments indicate that the equation of state of this quark-gluon plasma may be calculated in perturbation theory as an asymptotic expansion in an effective coupling which decreases as temperature and/or density are raised. Explicit low-order calculations have been carried out [6.2–6.4, 6.9, 6.20] and they yield IR-finite results for the thermodynamic potential of the quark-gluon plasma. Nevertheless, one expects that high-temperature perturbation theory actually breaks down beyond a certain order [7.3, 6.5, 6.24, 6.25]. This conclusion is based on an analysis of the perturbative IR behaviour of the space-like (transverse) gluon propagator which, unlike the time-like (longitudinal) one, does not develop a static mass in the one-loop approximation.

There may be, however, a contribution to the thermodynamic potential generated by the dynamical mass of the transverse gluon which has been ignored in the past. This transverse plasmon effect was calculated recently in refs. [10.11, 10.13, 10.15] where it was pointed out that the standard treatment in the Matsubara formalism [6.2, 6.3, 6.5, 6.20] is at fault. Subsequently, it has been shown [10.16] that this sort of difficulty is avoided if the plasmon effect is calculated by way of the real-time method. This suggests that the study of the plasmon effect may have some bearing on the infrared problem of QCD [10.16]. Be that as it may, the plasmon effect is a case in point where a naive application of the Matsubara method gives an ambiguous result, in contrast to the real-time method.

##### 4.4.1. Ring approximation

To illustrate the treachery of the Matsubara method in the case of an IR-singular theory, we take a look at the plasmon contribution to the thermodynamic potential of a pure  $SU(N)$  gluon plasma. The plasmon effect arises because a subset of the bubble diagrams is increasingly IR-divergent. The most divergent diagrams consist of a single gluon loop with an arbitrary number of (lowest-order) self-energy insertions. This selected class of ring diagrams gives the following contribution to the pressure:

$$P^{\text{ring}} = \frac{1}{2} \sum_{m=2}^{\infty} \text{Tr} \frac{1}{m} [-\tilde{\Pi}(k)\tilde{D}(k)]^m , \quad (4.4.1)$$

where  $\tilde{\Pi}_{\mu\nu}^{ab}(k) = \delta^{ab}\tilde{\Pi}_{\mu\nu}(k)$  is the lowest-order gluon self-energy [6.2, 6.5, 6.9, 6.20]. Since this lowest-order polarization tensor is transverse, the bare gluon propagator (2.3.20) may be taken in the Feynman gauge. The trace is over colour and Minkowski indices, and includes the loop integration and summation.

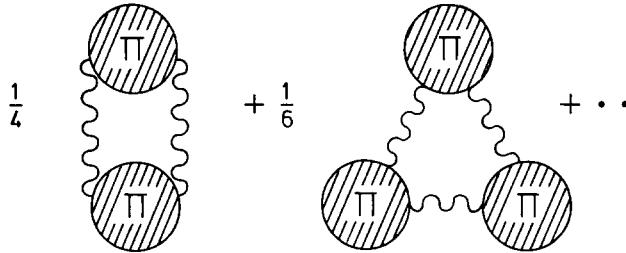


Fig. 4.8. Gluon ring diagrams.

The standard calculation of the plasmon pressure at high temperatures goes back to the early work of Akhiezer and Peletminskií [7.1]. First one sums (4.4.1) to a logarithm and one then argues that solely the zero-frequency term need be retained:

$$P^{\text{ring}} = -\frac{N^2 - 1}{2\beta} \int \frac{d^3 k}{(2\pi)^3} \text{tr} \left\{ \log[1 + \tilde{\Pi}(0, \mathbf{k})/|\mathbf{k}|^2] - \tilde{\Pi}(0, \mathbf{k})/|\mathbf{k}|^2 \right\}. \quad (4.4.2)$$

The trace is now over Minkowski indices only. Subsequently, one replaces the static polarization tensor by its IR limit [6.2, 6.5, 6.9, 6.20]

$$\tilde{\Pi}_{\mu\nu}(0, \mathbf{k}) = g_{\mu 0} g_{\nu 0} \kappa_L^2 + \mathcal{O}(|\mathbf{k}|), \quad (4.4.3)$$

with  $\kappa_L^2 = \tilde{\Pi}_L(0, \mathbf{k} \rightarrow 0)$  as given in (3.3.38), and performs the integration. This immediately yields the  $\mathcal{O}(g^3)$  longitudinal plasmon contribution

$$P_L^{\text{ring}} = \frac{N^2 - 1}{12\pi\beta} \kappa_L^3 + \mathcal{O}(g^4 \log g). \quad (4.4.4)$$

The transverse part of the polarization tensor gives a null result because it does not develop a static magnetic mass  $\kappa_T$  of order  $gT$ ; see section 3.3.3, also cf. [6.5, 10.9, 10.12].

Physically, the absence of a transverse plasmon effect is rather puzzling. Indeed it is well known that at temperatures above the plasma frequency a chromoplasma, like a QED plasma [2.2, 7.4], can support propagating plasma modes [7.11, 6.9]. By this mechanism transverse gluons (or rather the one-particle states in the transverse excitation spectrum [6.9]) acquire a dynamically generated mass, whereas longitudinal gluons do not. This situation is the exact opposite to that for the static case where the longitudinal propagator is screened but the transverse propagator is not [6.16]. Hence, one would expect the screening mass and the dynamic mass, both being of order  $gT$ , to appear in the thermodynamic potential on equal footing.

By taking this argument seriously, one is led to the tentative conclusion that perhaps some important region of phase space has been overlooked. What is wrong is the perturbative expression for the static transverse gluon propagator. In the temporal axial gauge, for example, one finds for small  $|\mathbf{k}|$  [6.15]

$$|\mathbf{k}|^2 + \tilde{\Pi}_T(0, \mathbf{k}) \simeq |\mathbf{k}|^2 - \frac{3}{16} N g^2 |\mathbf{k}| T. \quad (4.4.5)$$

This expression becomes negative for  $|\mathbf{k}| \leq g^2 T$ . The same behaviour is found in covariant gauges [6.9].

Therefore, the transverse gluon propagator violates the positivity requirement (3.1.13) which should ensure that the propagator has no pole on the imaginary axis. It also implies that, as far as the second-order approximation to the transverse propagator is concerned, the logarithm in (4.4.2) is ill defined on the imaginary axis, so that the pressure (4.4.2) has an imaginary part (cf. related observations in [10.12]). Because (4.4.5) is formally of order  $g^4$  where it is negative, it can in principle not be excluded that higher-order contributions restore the correct analyticity properties [10.18]. If so, this would lead to a wicked mixing of the various orders of perturbation theory.

A way around this problem is to go back to the sum of ring diagrams (4.4.1), and to extend separately each ring diagram away from the imaginary axis [10.13]. To be able to do this we need an analytic continuation of  $\tilde{\Pi}(k)$  from the discrete values  $k_0 = i\omega_n$  to the complex plane. As is conventional, we impose that this continuation  $\Pi(z, k)$  be bounded as  $|z|$  goes to infinity, and be analytic off the real axis. We have already explained in section 3.1.3 that this particular analytic continuation may be constructed by first performing the internal frequency sums implicit in  $\tilde{\Pi}(k)$ , followed by a replacement of  $i\omega_n$  by the continuous variable  $z$ .

By using this analytic extension we accomplish that products of  $\Pi(z, k) D(z, k)$  are analytic everywhere with the exception of the real axis. Moreover, it can be shown that  $D^{-1}(z, k) + \Pi(z, k)$  can have neither a zero nor a pole in the whole region outside both the imaginary and real axes. Consequently, we can formally sum to a logarithm in this region. Employing now formula (2.3.22), we get for the ring contribution

$$P^{\text{ring}} = -(N^2 - 1) \int \frac{d^4 k}{(2\pi)^4 i} \theta(k_0) [N(k_0) + \tfrac{1}{2}] \text{tr Disc}[\log(1 + D\Pi) - D\Pi]. \quad (4.4.6)$$

Since we already know that the polarization tensor is transverse, we may decompose it into a longitudinal and spatially transverse part, as in (3.3.19). If we confine ourselves now to the contribution of the transverse gluon in the high-temperature limit, we are left with [10.13]:

$$P_T^{\text{ring}} = -4(N^2 - 1) \int \frac{d^4 k}{(2\pi)^4} \frac{\theta(k_0)}{\beta k_0} \text{Im} \left\{ \log \left[ 1 - \frac{\Pi_T(k)}{k^2} \right] + \frac{\Pi_T(k)}{k^2} \right\}, \quad (4.4.7)$$

where the distribution function has been approximated by  $1/\beta k_0 - \tfrac{1}{2}$ , and

$$\Pi_T(k) = -\tfrac{1}{2}(\Pi_\mu^\mu + k^2 \Pi_{00}/|\mathbf{k}|^2) \quad (4.4.8)$$

is the transverse polarization function extended to the real axis. The imaginary part is defined by the prescription  $k_0 \rightarrow k_0 + i\varepsilon$ , that is, by approaching the real axis from above.

Before presenting the real-time derivation of expression (4.4.7) in the next subsection, we like to make two remarks. The first one is that the zero-frequency term (4.4.2) and the integral (4.4.7) over the discontinuity across the real axis, are *not* just related to each other by analytic continuation via the high-temperature limit of the contour integral identity (2.3.22). This would require the logarithm to be analytic on the imaginary axis, which it is not in this case. Hence, the summation of ring diagrams in the case of an IR-singular interaction is not a unique operation.

The second remark is that the difficulty of producing the correct IR limit of the transverse gauge-boson propagator in the Matsubara formalism, is not typical of QCD, because the same happens in QED [10.13, 10.15]. In fact, the occurrence of non-physical poles in the photon propagator was

already discussed in the context of vacuum QED by Bogoliubov, Logunov and Shirkov [12.3]. These authors argued that the correct analytic continuation of the photon propagator is obtained by a summation of ring contributions in the integrand of the Källen–Lehmann spectral representation. The propagator obtained by this prescription satisfies the analyticity requirements by construction. In other words, in singular theories a direct perturbative computation of some quantity may lead to an incorrect result, as opposed to the procedure of perturbatively calculating the relevant imaginary part and then reconstructing the original quantity by its dispersion integral. We like to suggest here that a similar procedure might also work in finite-temperature QCD.

#### 4.4.2. Chromoplasmon pressure

Perhaps the non-trivial procedure of analytic continuation employed above has thrown some doubt on the validity of the final expression (4.4.7) for the transverse plasmon pressure. Therefore, we will first ascertain its correctness by an alternative calculation with the help of the real-time method. The diagrams taken into account are the same as before, but now we apply the Feynman rules of section 4.2.3. We recall that, in general, one of the vertices has to be treated as distinct, but in the case of ring diagrams its position is immaterial. Hence we can immediately write down the following alternative expression for the ring pressure [10.16]:

$$P^{\text{ring}} = (N^2 - 1) \sum_{m=2}^{\infty} \frac{1}{2m} \int \frac{d^4 k}{(2\pi)^4 i} \text{tr}[(\tilde{H}\tilde{D})^m]^{(11)}. \quad (4.4.9)$$

The trace is over Minkowski indices; the colour trace is trivial and has already been performed.

Using the general matrix representations (2.4.30) and (3.2.17) for the thermo field matrix propagator and self-energy, respectively, we may write the matrix product appearing in (4.4.9) as

$$\tilde{H}(k)\tilde{D}(k) = M_+^{-1} \begin{pmatrix} -\tilde{\Pi}_{\mu\nu}(k) & \\ \frac{k^2 + i\varepsilon}{k^2 + i\varepsilon} & -\tilde{\Pi}_{\mu\nu}^*(k) \\ & \frac{k^2 - i\varepsilon}{k^2 - i\varepsilon} \end{pmatrix} M_+. \quad (4.4.10)$$

Since we know from the discussion in section 3.3.2 that the polarization tensor is transverse, the longitudinal part of the propagator drops out. Moreover, because the matrices  $A_{\mu\nu}$  and  $B_{\mu\nu}$  ion (3.3.19) are idempotent and orthogonal, the trace of the  $m$ -fold matrix product is trivial to take. Keeping only the thermal (11) component, we obtain

$$P^{\text{ring}} = (N^2 - 1) \sum_{m=2}^{\infty} \frac{1}{2m} \int \frac{d^4 k}{(2\pi)^4 i} \left\{ \cosh^2 \Theta_k \left[ \frac{\tilde{\Pi}_L(k)}{k^2 + i\varepsilon} \right]^m + 2 \left[ \frac{\tilde{\Pi}_T(k)}{k^2 + i\varepsilon} \right]^m \right\} - \sinh^2 \Theta_k (\text{c.c.}), \quad (4.4.11)$$

Here  $\tilde{\Pi}_L$  and  $\tilde{\Pi}_T$  are the polarization scalars as defined in (3.3.21) and (3.3.22), respectively.

In the high-temperature limit, the factor  $\cosh^2 \Theta_k$ , which comes from (2.4.31), may be replaced by  $\sinh^2 \Theta_k = N(|k_0|)$ . We then sum the contributions of all ring diagrams to arrive at  $P^{\text{ring}}$  as the sum of the longitudinal contribution

$$P_L^{\text{ring}} = -(N^2 - 1) \int \frac{d^4 k}{(2\pi)^4} N(|k_0|) \text{Im} \left\{ \log \left[ 1 - \frac{\tilde{\Pi}_L(k)}{k^2 + i\varepsilon} \right] + \frac{\tilde{\Pi}_L(k)}{k^2 + i\varepsilon} \right\}, \quad (4.4.12)$$

and the transverse contribution

$$P_T^{\text{ring}} = -2(N^2 - 1) \int \frac{d^4 k}{(2\pi)^4} N(|k_0|) \text{Im} \left\{ \log \left[ 1 - \frac{\tilde{\Pi}_T(k)}{k^2 + i\epsilon} \right] + \frac{\tilde{\Pi}_T(k)}{k^2 + i\epsilon} \right\}. \quad (4.4.13)$$

The logarithm is defined to have a branch cut on the negative real axis. Comparing the transverse expression with (4.4.7), and taking the connection (3.1.25) between the Feynman and analytic self-energies into account, we conclude that they are entirely equivalent. We refer to the literature for a detailed analysis of (4.4.12) [10.13, 6.2] and (4.4.13) [10.16]. As could be expected from the physical arguments of the preceding subsection, it turns out that (4.4.12) receives its dominant  $\mathcal{O}(g^3)$  contribution from the static infrared domain  $k_0 = 0, k = 0$ , while the non-perturbative content of (4.4.13) comes from the  $k^2 \geq 0$  region which supports the propagating plasma modes. In these respective regions (and only there!) we approximate  $\tilde{\Pi}_L(k)$  and  $\tilde{\Pi}_T(k)$  by their limits (3.3.23) and (3.3.26), respectively. (The latter approximation is disputed in [10.18].) We see that the situation for the transverse term is complementary to that of the longitudinal one, and this is reflected by the limiting values (3.3.25,38,39).

Eventually, we obtain the total plasmon contribution in the ring approximation [10.13, 10.16]

$$P^{\text{ring}} = \frac{(N^2 - 1)}{12\pi\beta} (\kappa_L^3 + 2\omega_T^3). \quad (4.4.14)$$

This answer is easy to understand. The first term is nothing but the well-known Debye–Hückel correction to the equation of state of an ionized gas. The second term is the  $\omega_T^3$  term in the expansion of the pressure of an ideal gas of bosons with a mass  $\omega_T$ , which, in this case, has been dynamically generated.

Physically appealing as these lowest-order results may be, they should not be understood as an attempt to validate perturbation theory. In view of the many conundra (magnetic mass [6.5, 10.9, 10.12], Landau ghost [10.12], breakdown of gauge invariance [6.25], plasmon puzzle [6.11]) that plague perturbative QCD at finite temperature, it may well be that perturbative results are of little relevance, if any.

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## Appendix

### A.1. Singular functions

In the main text the following regularizations are used [1.27]. For the step function ( $\epsilon > 0$ ):

$$\begin{aligned}
2\pi i \theta(x) &= - \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega x}}{\omega + i\varepsilon} \\
&= \log(-x + i\varepsilon) - \log(-x - i\varepsilon) \\
&=: \text{Disc } \log -x .
\end{aligned} \tag{A.1}$$

The logarithm has a cut along the negative real axis. The corresponding regularization of the Dirac function and its derivatives

$$\delta^{(n)}(x) := \frac{1}{n!} \frac{\partial^n}{\partial x^n} \delta(x) \tag{A.2}$$

is given by

$$\begin{aligned}
2\pi i \delta^{(n)}(x) &= \left( \frac{-1}{x + i\varepsilon} \right)^{n+1} - \left( \frac{-1}{x - i\varepsilon} \right)^{n+1} \\
&=: \text{Disc} \left( \frac{-1}{x} \right)^{n+1} .
\end{aligned} \tag{A.3}$$

Principal parts are regularized by

$$\text{PP } \frac{1}{x} = \frac{x}{x^2 + \varepsilon^2} \tag{A.4}$$

When these formulae are applied to functions of the energy, we use the prescription  $k_0 \rightarrow k_0 \pm i\varepsilon$  in the definition of the discontinuity. Hence

$$\text{Disc } \log(m^2 - k^2) = -2\pi i \varepsilon(k_0) \theta(k^2 - m^2), \tag{A.5}$$

$$\text{Disc}(m^2 - k^2)^{-n-1} = 2\pi i \varepsilon(k_0) \delta^{(n)}(k^2 - m^2). \tag{A.6}$$

Another important formula is [3.12]:

$$\frac{\delta(k^2 - m^2)}{k^2 - m^2 + i\varepsilon} = -\frac{1}{2} \delta^{(1)}(k^2 - m^2) - i\pi [\delta(k^2 - m^2)]^2, \tag{A.7}$$

where the  $\delta$ -functions are understood to be regularized according to (A.3). Finally we quote a generalization of the useful ‘‘mass-derivative formula’’ [4.12]:

$$\{[\tilde{D}_{\alpha\beta}(p)\tau_3]^{n+1}\}^{(11)} = [d_{\alpha\beta}(p)]^{n+1} \frac{1}{n!} \left( \frac{\partial}{\partial m^2} \right)^n \left[ \frac{1}{p^2 - m^2 + i\varepsilon} - 2\pi i \eta n(p_0) \delta(p^2 - m^2) \right], \tag{A.8}$$

with

$$n(p_0) = \theta(p_0) \{\exp[\beta(p_0 - \mu)] - \eta\}^{-1} + \theta(-p_0) \{\exp[\beta(\mu - p_0)] - \eta\}^{-1}, \tag{A.9}$$

for the thermal (11) component of a direct product of free propagators (2.4.30) in the single-mass case.

### A.2. Mellin summation technique

Integrals of the type (4.3.21) encountered in the main text will be evaluated here by a summation method based on the Mellin integral transform [13.1, 13.6, 13.8]. This technique can be used to sum series, and as such it sometimes provides an alternative to the summation formulae (2.3.22), (2.3.23). It also often allows a slowly convergent series to be resummed into a rapidly convergent one, to which purpose we shall apply it below.

For integrable functions the Mellin integral transformation and its inverse are given by

$$f^*(s) = \int_0^\infty dx x^{s-1} f(x) , \quad (\text{A.10})$$

$$f(x) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} x^{-s} f^*(s) . \quad (\text{A.11})$$

The Mellin transform  $f^*(s)$ , also denoted as  $\mathcal{M}[f(x); s]$ , is analytic in a strip  $a < \operatorname{Re} s < b$  for some real numbers  $a, b$  with the real constant  $c$  lying in this interval. Using

$$\sum_{n=1}^{\infty} \eta^{n-1} n^{-s} = [1 - (1 - \eta)^{1-s}] \zeta(s) , \quad \operatorname{Re} s > 1 , \quad (\text{A.12})$$

where  $\zeta(s)$  is the Riemann zeta function and  $\eta = \pm 1$ , as well as some elementary properties of the Mellin transform, one easily derives the summation formula

$$\sum_{n=1}^{\infty} \eta^{n-1} n^{-\nu} f(ny) = y^\nu \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} [1 - (1 - \eta)^{1-s}] \zeta(s) y^{-s} f^*(s - \nu) . \quad (\text{A.13})$$

The sum at the left-hand side is assumed to converge. If  $f^*(s)$  is analytic on  $a < s < b$ ,  $c$  must satisfy  $\max(1, a + \nu) < c < b + \nu$ , if  $\eta = 1$ , and  $a + \nu < c < b + \nu$ , if  $\eta = -1$ . By closing the contour the right-hand side can be evaluated with the aid of the residue theorem.

We are going to use this formula to evaluate the integral

$$\frac{L_{2\nu}^\eta(x, y)}{4\pi^2 \beta^{2\nu}} = \frac{1}{(2\nu - 1)!} \int \frac{d^4 p}{(2\pi)^3} n(p_0) |p|^{2\nu-2} \delta(p^2 - m^2) , \quad (\text{A.14})$$

where  $n(p_0)$  is the distribution function (A.9), and  $\nu \geq 1$  an integer. The normalization has been chosen in such a way that

$$L_{2\nu}^\eta(x=0, y) = G_{2\nu}(y) + G_{2\nu}(-y) \quad (\text{A.15})$$

is the sum of two ordinary Bose–Einstein or Fermi–Dirac integrals  $G_{2\nu}$  for  $\beta m = x = 0$  with  $y = \beta\mu$  [11.8, 13.5]. For bosons ( $\eta = 1$ ) we have the restriction  $|y| < x$  in order that (A.14) exists. This constraint arises from the demand that the occupation numbers be positive, cf. subsection 2.3.1. Fermionic chemical potentials are unconstrained, but for the moment we shall assume  $|\mu/m| < 1$  for both statistics, in order that some of the manipulations to follow be valid. As we shall see, the final result can immediately be continued to any value of  $\mu$  for fermions, whereas a natural barrier will arise for bosons.

The function  $L_{2\nu}^\eta$  as defined in (A.14) has many applications. We have encountered it in the computation of real-time loop integrals. It also plays a role in relativistic kinetic theory [11.8]. For example, the free pressure of an ideal quantum gas in  $d = 2\nu$  dimensions is given by [13.11]  $\beta^{2\nu} P_0 = \pi^{-\nu} (\nu - 1)! L_{2\nu}^\eta$ .

In the following we need the representation [13.3; formula 7.3(15)] of the modified Bessel function of the second kind  $K_\nu(z)$ . Performing the  $p$  integration in (A.14), and writing (A.9) as a geometric series, we can write

$$L_{2\nu}^\eta(x, y) = \frac{4}{\Gamma(\nu)} \left(\frac{x}{2}\right)^\nu I_{2\nu}^\eta(x, y) \quad (\text{A.16})$$

in terms of the function

$$I_{2\nu}^\eta(x, y) := \sum_{n=1}^{\infty} \eta^{n-1} n^{-\nu} (\cosh ny) K_\nu(nx) \quad (\text{A.17})$$

introduced by Braden [13.9] for  $\mu = 0$ . (It is sometimes denoted as  $B(F)$  [3.12].) Due to the asymptotic behaviour  $K_\nu(z) \sim e^{-z}$ , the right-hand side of (A.17) clearly converges. However, expanding  $K_\nu$  in (A.17) gives a double series which is very slowly convergent. To obtain a convergent (not asymptotic) high-temperature expansion, it is better to resum (A.17) by the Mellin summation technique. Another method, which will not be pursued here, is the so-called  $\zeta$ -function regularization [13.11–13.13].

The following calculations are an extension of those of Haber and Weldon [13.10]. In view of (A.13) we obviously need the Mellin transform of  $(\cosh bx)K_\nu(x)$  with  $b = \mu/m$ . Using the tabulated result [13.7; formula I.11.6], as well as [13.2; formula 3.4(11)], we find

$$\mathcal{M}[(\cosh bx)K_\nu(x); s] = \frac{1}{4} 2^s \Gamma(\frac{1}{2}s + \frac{1}{2}\nu) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu) {}_2F_1(\frac{1}{2}s - \frac{1}{2}\nu, \frac{1}{2}s + \frac{1}{2}\nu, \frac{1}{2}; b^2), \quad (\text{A.18})$$

where  ${}_2F_1$  is the hypergeometric function, valid for  $|b| < 1$  and  $\operatorname{Re} s > |\nu|$ . From (A.13), (A.17) and (A.18), we then have

$$I_{2\nu}^\eta = \frac{1}{4} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} [1 - (1-\eta)^{1-s}] \zeta(s) (\frac{1}{2}x)^{\nu-s} \Gamma(\frac{1}{2}s) \Gamma(\frac{1}{2}s - \nu) {}_2F_1(\frac{1}{2}s - \nu, \frac{1}{2}s, \frac{1}{2}; y^2/x^2), \quad (\text{A.19})$$

with  $c > 2\nu$ . The contour can be closed to the left. The integrand has simple poles in  $s = 2\nu - 2l$  ( $l = 0, 1, \dots, \nu - 1$ ),  $s = -2l$  ( $l = 1, 2, \dots$ ),  $s = 1$  (only for bosons), and a double pole in  $s = 0$ .

Using a number of properties of the special functions in (A.19), the residues can be summed, and after some rearrangements we finally find the following expression for  $L_{2\nu}^\eta$  as defined in (A.14):

$$\begin{aligned}
(\nu - 1)! L_{2\nu}^\eta(x, y) = & \sum_{l=0}^{\nu} \sum_{n=\delta(l, \nu)}^l (-1)^{\nu+n+1} (4^{n-l} - (1-\eta)4^{n-\nu}) (2\pi)^{2\nu-2l} \frac{B_{2\nu-2l}(\nu-l+n-1)!}{(2\nu-2l)!(l-n)!(2n)!} \\
& \times x^{2l-2n} y^{2n} + (-1)^\nu (1+\eta) \pi \frac{\nu!}{(2\nu)!} (x^2 - y^2)^{\nu-1/2} \\
& + (-1)^\nu 2\eta \frac{4^{-\nu}}{\nu!} x^{2\nu} \left[ \log \frac{x}{4\pi} + (1-\eta) \log 2 + \gamma - \frac{1}{2} \sum_{k=1}^{\nu} \frac{1}{k} \right] \\
& + 2 \sum_{l=1}^{\infty} \sum_{n=0}^l (-1)^{l+\nu} (4^{n-l-\nu} - (1-\eta)4^{n-\nu}) \\
& \times \frac{(2\pi)^{-2l} (2l)! \zeta(2l+1)}{(2n)!(l-n)!(l+\nu-n)!} x^{2\nu+2l-2n} y^{2n}. \tag{A.20}
\end{aligned}$$

Here the  $B_n$ 's are the Bernoulli numbers ( $B_0 = 1$ ,  $B_2 = \frac{1}{6}$ ,  $B_4 = -\frac{1}{30}$ ) [13.4]. The zeta functions  $\zeta(2l+1)$  in the last term are only known numerically. Formula (A.20) summarizes results found previously [13.5, 13.9–13.13]. Its feature is that, although valid for any  $\beta$ , it rapidly converges for high temperatures. Furthermore, we see that the second term, which is non-analytic in  $\mu/m$  enforces the bosonic constraint  $|\mu| < m$ . For fermions ( $\eta = -1$ ) this very term is absent.

In the main text the following special cases occur:

$$L_2^+(x, 0) = \frac{1}{3}\pi^2 - \pi\beta m + \mathcal{O}(m^2), \tag{A.21}$$

$$L_4^+(x, 0) = \frac{1}{45}\pi^4 - \frac{1}{12}\pi^2\beta^2m^2 + \frac{1}{6}\pi\beta^3m^3 + \mathcal{O}(m^4). \tag{A.22}$$

For massless fermions one recovers the remarkable result that  $L_{2\nu}^-(0, y)$  is a finite polynomial in  $y$  [13.11].

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