Practice problems for midterm 1

1. Suppose $X_1, \ldots, X_n$ are a random sample of size n from $Unif(0, \theta)$.

   (a) Show that $2\bar{X}_n$ is an unbiased estimator of θ.

   (b) Let Y_n be the sample maximum, i.e. $\max\{X_1, \ldots, X_n\}$. Find its pdf.

   (c) Find the bias of Y_n as an estimator of θ. And construct an unbiased estimator based on Y_n by correcting its bias.

2. Suppose $X_1, \ldots, X_n$ are a random sample of size n from $Bernoulli(\theta)$, for which the pdf is

   $$f(x; \theta) = \theta^x (1 - \theta)^{1-x}, x = 0, 1, 0 < \theta < 1.$$

   Consider testing $H_0 : \theta = 0.5$ versus $H_1 : \theta > 0.5$.

   (a) For a test with critical region $\{(x_1, \ldots, x_n) : \bar{x}_n > c\}$, find the value of c for the test to have an asymptotic significance level $\alpha = 0.05$.

   (b) Suppose $\theta = 0.7$. Derive the power of test in (a).

   (c) If it is observed that $\bar{x}_{25} = 0.7$, should H_0 be rejected at the significance level $\alpha = 0.05$? Make your conclusion based on the p-value from the test in (a).

3. Let $x_1, \ldots, x_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are unknown. Outline how you construct a 95% confidence interval for σ^2. And what is the expected length of your confidence interval?

4. Suppose $X_1, \ldots, X_n$ are a random sample of size n from $N(0, \theta)$. Then the MLE of θ is $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i^2$.

   (a) Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, 2\theta^2)$ using the Central Limit Theorem.

   (b) Let $g(\theta) = \log \theta$. And show that $\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} N(0, c)$, where c is a constant. And give the constant c.

   (c) Find an asymptotic 95% confidence interval for $g(\theta)$.

   (d) Transform the above confidence interval to an asymptotic 95% confidence interval for θ.