Math 494: Mathematical Statistics
Solutions to Midterm2

Suppose the lifetime of a certain brand of bulbs follows an exponential distribution with parameter θ ($\theta > 0$) for which the pdf is

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, x > 0.$$

(**Remark**: This exponential distribution has mean θ and variance θ^2.) Let $X_1, \ldots, X_n$ be the lifetime of a random sample of n bulbs of this brand.

1. (15 points) Write down the likelihood function $L(\theta)$.

Solution: [15pts]

$$\begin{aligned} L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) &= \prod_{i=0}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \\ &= \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}} \end{aligned}$$

2. (15 points) Show that the sample mean $\bar{X}_n$ is a sufficient statistic for θ using the factorization theorem.

Solution: The joint pdf can be rewritten as [10pts]

$$\frac{1}{\theta^n} e^{-\frac{n\bar{X}_n}{\theta}}$$

[5pts] From the factorization theorem, we see that $\bar{X}_n$ is a sufficient statistic for θ.

3. (15 points) Find the Fisher information $I(\theta)$.

Solution: [5pts] $\log f = -\log\theta - \frac{x}{\theta}$; $\frac{\partial \log f}{\partial\theta} = -\frac{1}{\theta} + \frac{x}{\theta^2}$; $\frac{\partial^2 \log f}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$.
[5pts] $I(\theta) = -E\left[\frac{\partial^2 \log f}{\partial\theta^2}\right]$
[5pts] $I(\theta) = -\frac{1}{\theta^2} + \frac{2E[X]}{\theta^3} = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$.

4. (10 points) Find the Rao-Cramer lower bound for the variance of unbiased estimators of θ.

Solution: [5pts] $\because$ unbiased $\quad\quad \therefore\ CRLB = \frac{1}{nI(\theta)}$
[5pts] So, $CRLB = \frac{\theta^2}{n}$.

5. (15 point) Find the MLE of θ. Is it a MVUE of θ? Explain by comparing its variance to the Rao-Cramer lower bound in (d).

Solution: [8pts] $l(\theta) = \log L(\theta) = -n\log\theta - \frac{n\overline{X}}{\theta}$
$l'(\theta) = -\frac{n}{\theta} + \frac{n\overline{X}}{\theta^2} = 0 \Rightarrow \hat{\theta}_{MLE} = \overline{X}$
[2pts] $E(\hat{\theta}_{MLE}) = E(\overline{X}) = \theta \ \Rightarrow$ unbiased
[5pts] $Var(\hat{\theta}_{MLE}) = Var(\overline{X}) = \frac{Var(X)}{n} = \frac{\theta^2}{n}$, which attains the RCLB.
Therefore, $\hat{\theta}_{MLE}$ is a MVUE.

6. (30 points) Consider the following hypothesis testing problem, $H_0 : \theta = 100$ versus $H_1 : \theta > 100$.

   (a) (15 points) Show that the maximum likelihood ratio test statistic is a function of $\bar{X}_n$.

   (b) (15 points) Give a test based on $\bar{X}_n$ with significance level α based on the asymptotic normal distribution of $\bar{X}_n$ from central limiting theorem. (**Note**: The alternative hypotheis is one-sided.)

Solution:

(a)

$$\begin{aligned}\Lambda &= \frac{L(\theta_0)}{L(\hat{\theta}_{MLE})} \quad [5pts] \\ &= \frac{\frac{1}{100^n}\exp\left(-\frac{n\bar{X}_n}{100}\right)}{\frac{1}{\bar{X}_n^n}\exp\left(-\frac{n\bar{X}_n}{\bar{X}_n}\right)} \quad [3pts] \\ &= \left(\frac{\bar{X}_n}{100}\right)^{100}\exp\left(n - \frac{n\bar{X}_n}{100}\right), \quad [3pts]\end{aligned}$$

which is based on $\bar{X}_n$. So the maximum likelihood ratio test statistic $-2\log\Lambda$ also based on $\bar{X}_n$. [4pts]

(b) By CLT,

$$\sqrt{n}\frac{(\bar{X}_n - \theta)}{\theta} \to^d N(0,1). \quad [5pts]$$

Then under $H_0: \ \theta = 100$,

$$\sqrt{n}\left(\frac{\bar{X}_n}{100} - 1\right) \to^d N(0,1). \quad [3pts] .$$

So at level α, we reject H_0 when

$$\sqrt{n}\left(\frac{\bar{X}_n}{100} - 1\right) > z_{1-\alpha},$$

or when

$$\bar{X}_n > \frac{100 z_{1-\alpha}}{\sqrt{n}} + 100. \quad \text{[7pts]}$$