Sample solution

1.(a) $E(X_i)=\frac{\theta}{2}$, so, $E(\bar{X}_n)=\frac{\theta}{2}$ and $E(2\bar{X}_n)=\theta$

(b) $P(Y_n\le t)=P(X_1\le t \cdots X_n\le t)=\left(\frac{t}{\theta}\right)^n$, for $0<t<\theta$

pdf is then $f_{Y_n}(t)=\frac{nt^{n-1}}{\theta^n}$, $0<t<\theta$.

(c) $E(Y_n)=\int_0^\theta t\cdot\frac{nt^{n-1}}{\theta^n}\,dt=\frac{n}{n+1}\theta$

$\mathrm{Bias}(Y_n)=E(Y_n)-\theta=-\frac{\theta}{n+1}$

$E\left(\frac{n+1}{n}Y_n\right)=\theta$, so it is an unbiased estimator.

2.(a) Under H_0, $\bar{X}_n$ is approximately $N\left(0.5,\frac{0.5\times0.5}{n}\right)$

Let $P(\bar{X}_n>c)=\alpha$. then $P\left(\frac{\bar{X}_n-0.5}{\sqrt{\frac{0.5\times0.5}{n}}}>\frac{c-0.5}{0.5\sqrt{\frac{1}{n}}}\right)=\alpha$.

Therefore, $\frac{c-0.5}{0.5\sqrt{\frac{1}{n}}}=z_\alpha$. So, $c=0.5+z_\alpha\frac{0.5}{\sqrt{n}}$, where $z_\alpha=1.96$.

(b) When $\theta=0.7$, $\bar{X}_n$ is approximately $N\left(0.7,\frac{0.7\times0.3}{n}\right)$

$$\text{power}=P(\bar{X}_n>c\mid H_A)$$

$$=P\left(\frac{\bar{X}_n-0.7}{\sqrt{\frac{0.7\times0.3}{n}}}>\frac{0.5+z_\alpha\frac{0.5}{\sqrt{n}}-0.7}{\sqrt{\frac{0.7\times0.3}{n}}}\right)$$

$$=1-\Phi\left(\frac{-0.2+z_\alpha\frac{0.5}{\sqrt{n}}}{\sqrt{\frac{0.21}{n}}}\right),$$ where Φ is the cdf of $N(0,1)$

(c) $\text{pvalue}=P(\bar{X}_n>0.7\mid H_0)=1-\Phi\left(\frac{0.7-0.5}{0.5\sqrt{\frac{1}{25}}}\right)=1-\Phi(2)<0.05$.

So, reject H_0.

3. $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

From Student's theorem, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

So, $P\left(\chi^2_{n-1,1-\frac{\alpha}{2}} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,\frac{\alpha}{2}}\right) = 1-\alpha$.

Then, a $(1-\alpha)$ CI of σ^2 is $\left(\frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}, \frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}\right)$

4. (a) $\cancel{E(\hat{\theta}_n)} = E(X_i^2) = Var(X_i) + [E(X_i)]^2 = \theta$

As $\frac{X_i^2}{\theta} \sim \chi^2_1$, $Var(X_i^2) = \theta^2 \cdot Var\left(\frac{X_i^2}{\theta}\right) = \theta^2 \cdot 2 = 2\theta^2$

Then the result follows from the CLT as X_i^2's are iid.

(b) By the Δ-method, $c = [g'(\theta)]^2 \cdot 2\theta^2 = \frac{1}{\theta^2} \cdot 2\theta^2 = 2$.

(c) $g(\hat{\theta}_n) \pm z_{\alpha/2}\sqrt{\frac{2}{n}} = (L, U)$, $\alpha = 0.05$

(d) (e^L, e^U).