Math 494: Mathematical Statistics
Solutions to HW10

Problem 1 (# 8.1.8)

$L(\theta_0;\mathbf{x}) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\prod_{i=1}^{n} x_i^0(1-x_i)^0 = 1.$
$L(\theta_1;\mathbf{x}) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)}\prod_{i=1}^{n} x_i^1(1-x_i)^1 = 6^n\prod_{i=1}^{n} x_i(1-x_i)$
$\frac{L(\theta_0;\mathbf{x})}{L(\theta_1;\mathbf{x})} = \frac{1}{6^n\prod_{i=1}^{n} x_i(1-x_i)} \le k \Longrightarrow c = \frac{1}{6^n k} \le \prod_{i=1}^{n} x_i(1-x_i)$
The best critical region is the set of all points $\mathbf{x} = (x_1,\cdots,x_n)$ such that $\prod_{i=1}^{n} x_i(1-x_i) \ge c$.

Problem 2 (# 8.1.9)

$\frac{L(p_0;\mathbf{x})}{L(p_1;\mathbf{x})} = \frac{(1/2)^{\sum_{i=1}^n x_i}(1/2)^{\sum_{i=1}^n(1-x_i)}}{(1/3)^{\sum_{i=1}^n x_i}(2/3)^{\sum_{i=1}^n(1-x_i)}} = (3/2)^{\sum_{i=1}^n x_i}(3/4)^{\sum_{i=1}^n(1-x_i)} = (\frac{3}{4})^n\cdot 2^{\sum_{i=1}^n x_i} \le k$
Equivalently, this is $\sum_{i=1}^n x_i \le c(n,k)$, where $c(n,k)$ is a constant dependent on n,k.
Therefore,the best critical region is $C = \{(x_1,\cdots,x_n) : \sum_{i=1}^n x_i \le c\}$.
By CLT, $\frac{\sum_{i=1}^n x_i - np}{\sqrt{np(1-p)}} \to^d N(0,1)$.
Hence, $P_{H_0}(\sum_{i=1}^n x_i \le c) = \Phi\left(\frac{c-\frac{1}{2}n}{\sqrt{n\frac{1}{2}(1-\frac{1}{2})}}\right)$ =0.1,
and $P_{H_1}(\sum_{i=1}^n x_i \le c) = \Phi\left(\frac{c-\frac{1}{3}n}{\sqrt{n\frac{1}{3}(1-\frac{1}{3})}}\right) = 0.8$.
Then $\frac{c-\frac{1}{2}n}{\sqrt{n\frac{1}{2}(1-\frac{1}{2})}} = -1.28$ and $\frac{c-\frac{1}{3}n}{\sqrt{n\frac{1}{3}(1-\frac{1}{3})}} = 0.84 \Rightarrow n = 39$ and $c = 15$.

Problem 3 (# 8.2.11)

Let $\theta' < \theta''$, consider the ratio of likelihoods

$$\frac{L(\theta_0'; x_1, x_2, ..., x_n)}{L(\theta''; x_1, x_2, ..., x_n)} = (\frac{\theta'}{\theta''})^n(\prod_{i=1}^{n} x_i)^{\theta'-\theta''}$$

Since $\theta' - \theta'' < 0$, the ratio is a monotone function of $y = \prod_{i=1}^n x_i$. Thus, the likelihood has mlr in the statistic $Y = \prod_{i=1}^n X_i$. Hence consider the hypotheses

$$H_0 : \theta = \theta' \text{ versus } H_1 : \theta < \theta', \text{for fixed } \theta' > 0$$

The UMP level α decision rule is,

$$\text{Reject } H_0 \text{ if } Y = \prod_{i=1}^{n} X_i \le c,$$

where c is such that $\alpha = P_{\theta'}[Y \le c]$.

Problem 4 (# 8.2.12)

(a)

$$\frac{L(\frac{1}{2};\mathbf{x})}{L(\theta;\mathbf{x})} = \frac{(\frac{1}{2})^5}{(\frac{\theta}{1-\theta})^{\sum_{i=1}^5 x_i}(1-\theta)^5} \leq k, \text{with } \theta < 1/2$$

Since $\theta < 1/2$, $\frac{\theta}{1-\theta} < 1$, the ratio is a monotone increasing function of $y = \sum_1^n x_i$. Thus, the likelihood has mlr in the statistic $Y = \sum_1^n X_i$. Hence consider the hypotheses

$$H_0 : \theta = \frac{1}{2} \text{ versus } H_1 : \theta < \frac{1}{2},$$

The decision rule is,

$$\text{Reject } H_0 \text{ if } Y = \sum_1^n X_i \leq c.$$

(b) When $c = 1$, the significant level

$$\alpha = P_{\theta=1/2}[\sum_{i=1}^5 x_i \leq 1] = \sum_{k=0}^1 \binom{5}{k}(\frac{1}{2})^5 = \frac{6}{32}$$

(c) When $c = 0$, the significant level

$$\alpha = P_{\theta=1/2}[\sum_{i=1}^5 x_i \leq 0] = \binom{5}{0}(\frac{1}{2})^5 = \frac{1}{32}$$

Problem 5 (# 8.3.12)

(a) Under $H_0 :\ \mu_0 = \mu_1 = 0$,

$$\hat{\sigma}_0^2 = \frac{1}{2n}\left[\sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2\right].$$

Under H_1, $\hat{\mu}_1 = \bar{X}$, $\hat{\mu}_2 = \bar{Y}$, and

$$\hat{\sigma}_1^2 = \frac{1}{2n}\left[\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2\right].$$

Then the test statistic

$$\Lambda = \frac{L(0,0,\hat{\sigma}_0^2)}{L(\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}_1^2)} = \frac{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}_0^2}}\right)^n \exp\left(\frac{\sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2}{-2\hat{\sigma}_0^2}\right)}{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}_1^2}}\right)^n \exp\left(\frac{\sum_{i=1}^n (X_i-\bar{X})^2 + \sum_{i=1}^n (Y_i-\bar{Y})^2}{-2\hat{\sigma}_1^2}\right)} = \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}\right)^n.$$

(b)

$$\begin{aligned}
\Lambda = \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}\right)^n &= \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{\sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2}\right]^n \\
&= \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2 + n\bar{X} + \bar{Y}^2}\right]^n \\
&= \left[\frac{1}{1 + \frac{n(\bar{X} + \bar{Y}^2)}{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}}\right]^n .
\end{aligned}$$

Note that under H_0,

$$\bar{X} \sim N(0, \frac{\sigma^2}{n}) \quad \perp\!\!\!\perp \quad \frac{\sum (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1},$$

$$\bar{Y} \sim N(0, \frac{\sigma^2}{n}) \quad \perp\!\!\!\perp \quad \frac{\sum (Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2_{n-1},$$

and $\{X_i,\ i = 1, \ldots, n\}$ and $\{Y_i,\ i = 1, \ldots, n\}$ are independent. So

$$\frac{\bar{X}^2 + \bar{Y}^2}{\sigma^2/n} \sim \chi^2_2,$$

and it is independent of

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2_{2(n-1)}.$$

So setting

$$Z = \frac{n(n-1)(\bar{X} + \bar{Y}^2)}{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2},$$

we have $Z \sim F_{2,2(n-1)}$ and

$$\Lambda = \left(1 + \frac{1}{n-1} z\right)^n .$$

(c) As discussed in (b), $Z \sim F_{2,2(n-1)}$ under H_0.

Math 494: Mathematical Statistics
Solutions to HW11

Problem 1 (# 10.2.3)

(a) The level of the test is

$$P_{H_0}[S \geq 16] = P[bin(25, 1/2) \geq 16] = 0.1148.$$

(b) The probability of success here is

$$p = P[X > 0] = P[Z > -0.5] = 0.6915.$$

Hence, the power of the sign test is

$$P_{0.6915}[S \geq 16] = P[bin(25, 0.6915) \geq 16] = 0.7836.$$

(c) To obtain the test, solve for k in the equation

$$0.1148 = P_{H_0}[\bar{X}/(1/\sqrt{25}) \geq k] = P[Z \geq k],$$

where Z has a standard normal distribution. The solution is $k = 1.20$. The power of this test to detect 0.5 is

$$P_{\mu=0.5}[\bar{X}/(1/\sqrt{25}) \geq 1.20] = P[Z \geq 1.20 - (0.5/(1/\sqrt{25}))] = 0.9032.$$

Problem 2 (# 11.2.1)

By the Bayes' rule (notice that $y = 9$, $p(\theta = 0.3) = 2/3$, and $p(\theta = 0.5) = 1/3$),

$$\begin{aligned} p(\theta|y) &= \frac{p(y|\theta)p(\theta)}{p(y|\theta = 0.3)p(\theta = 0.3) + p(y|\theta = 0.5)p(\theta = 0.5)} \\ &= \frac{\binom{20}{9}\theta^9(1-\theta)^{11}p(\theta)}{\binom{20}{9}0.3^9(1-0.3)^{11}(\frac{2}{3}) + \binom{20}{9}0.5^9(1-0.5)^{11}(\frac{1}{3})} \end{aligned}$$

Plugging $\theta = 0.3$ and $\theta = 0.5$ into the above equation, we have

$$p(\theta = 0.3|y) = \frac{\binom{20}{9}0.3^9(1-0.3)^{11}(\frac{2}{3})}{\binom{20}{9}0.3^9(1-0.3)^{11}(\frac{2}{3}) + \binom{20}{9}0.5^9(1-0.5)^{11}(\frac{1}{3})} \approx 0.4494$$

and

$$p(\theta = 0.5|y) = \frac{\binom{20}{9}0.5^9(1-0.5)^{11}(\frac{2}{3})}{\binom{20}{9}0.3^9(1-0.3)^{11}(\frac{2}{3}) + \binom{20}{9}0.5^9(1-0.5)^{11}(\frac{1}{3})} \approx 0.5506.$$

Problem 3 (# 11.2.5)

The pdf of Y_n is $f(y_n|\theta) = \frac{ny_n^{n-1}}{\theta^n}$, $0 < y_n < \theta$.
The prior $\pi(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}$, $\alpha < \theta < \infty$.
$f(y_n|\theta)\pi(\theta) = \frac{ny_n^{n-1}}{\theta^n}\frac{\beta\alpha^\beta}{\theta^{\beta+1}}I[\max(y_n,\alpha) < \theta < \infty] = \frac{ny_n^{n-1}\beta\alpha^\beta}{\theta^{n+\beta+1}}I[\max(y_n,\alpha) < \theta < \infty]$.
$\int f(y_n|\theta)\pi(\theta)d\theta = \int_{\max(y_n,\alpha)}^{\infty}\frac{ny_n^{n-1}\beta\alpha^\beta}{\theta^{n+\beta+1}}d\theta = \frac{ny_n^{n-1}\beta\alpha^\beta(n+\beta)}{[\max(y_n,\alpha)]^{n+\beta}}$.
So, the posterior $f(\theta|y_n) = \frac{f(y_n|\theta)\pi(\theta)}{\int f(y_n|\theta)\pi(\theta)d\theta} = \frac{1}{n+\beta}\frac{[\max(y_n,\alpha)]^{n+\beta}}{\theta^{n+\beta+1}}I[\max(y_n,\alpha) < \theta < \infty]$.
Since the squared-error loss function is used here, the Bayes solution $\delta(y_n)$ is the posterior mean

$$\delta(y_n) = \int \theta f(\theta|y_n)d\theta = \int_{\max(y_n,\alpha)}^{\infty}\frac{1}{n+\beta}\frac{[\max(y_n,\alpha)]^{n+\beta}}{\theta^{n+\beta}}d\theta = \frac{n+\beta-1}{n+\beta}\max(y_n,\alpha).$$

Problem 4 (# 11.2.8)

(a)

$$E[(\theta - \frac{10+Y}{45})^2] = (\theta - \frac{10+30\theta}{45})^2 + (\frac{1}{45})^2 30\theta(1-\theta)$$

(b)

$$E[(\theta - \frac{10+Y}{45})^2] < \frac{\theta(1-\theta)}{30}$$

requires that

$$k(\theta) = (\frac{\theta}{3} - \frac{2}{9})^2 - \frac{1}{54}\theta(1-\theta) < 0$$

Find the two zeros of $k(\theta)$, one of which is greater (less) than 2/3.

Sample solution

1.(a) $E(X_i)=\frac{\theta}{2}$, so, $E(\bar{X}_n)=\frac{\theta}{2}$ and $E(2\bar{X}_n)=\theta$

(b) $P(Y_n\le t)=P(X_1\le t\cdots X_n\le t)=\left(\frac{t}{\theta}\right)^n$, for $0<t<\theta$

pdf is then $f_{Y_n}(t)=\frac{nt^{n-1}}{\theta^n}$, $0<t<\theta$.

(c) $E(Y_n)=\int_0^\theta t\cdot\frac{nt^{n-1}}{\theta^n}\,dt=\frac{n}{n+1}\theta$

$\text{Bias}(Y_n)=E(Y_n)-\theta=-\frac{\theta}{n+1}$

$E\left(\frac{n+1}{n}Y_n\right)=\theta$, so it is an unbiased estimator.

2.(a) Under H_0, $\bar{X}_n$ is approximately $N\left(0.5,\frac{0.5\times0.5}{n}\right)$

Let $P(\bar{X}_n>c)=\alpha$. then $P\left(\frac{\bar{X}_n-0.5}{\sqrt{\frac{0.5\times0.5}{n}}}>\frac{c-0.5}{0.5\sqrt{\frac{1}{n}}}\right)=\alpha$.

Therefore, $\frac{c-0.5}{0.5\sqrt{\frac{1}{n}}}=z_\alpha$. So, $c=0.5+z_\alpha\frac{0.5}{\sqrt{n}}$, where $z_\alpha=1.96$.

(b) When $\theta=0.7$, $\bar{X}_n$ is approximately $N\left(0.7,\frac{0.7\times0.3}{n}\right)$

$$\text{power}=P(\bar{X}_n>c\mid H_A)$$

$$=P\left(\frac{\bar{X}_n-0.7}{\sqrt{\frac{0.7\times0.3}{n}}}>\frac{0.5+z_\alpha\frac{0.5}{\sqrt{n}}-0.7}{\sqrt{\frac{0.7\times0.3}{n}}}\right)$$

$$=1-\Phi\left(\frac{-0.2+z_\alpha\frac{0.5}{\sqrt{n}}}{\sqrt{\frac{0.21}{n}}}\right),$$ where Φ is the cdf of $N(0,1)$

(c) $\text{pvalue}=P(\bar{X}_n>0.7\mid H_0)=1-\Phi\left(\frac{0.7-0.5}{0.5\sqrt{\frac{1}{25}}}\right)=1-\Phi(2)<0.05$.

So, reject H_0.

3. $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

From Student's theorem, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

So, $P\left(\chi^2_{n-1,1-\frac{\alpha}{2}} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,\frac{\alpha}{2}}\right) = 1-\alpha$.

Then, a $(1-\alpha)$ CI of σ^2 is $\left(\frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}, \frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}\right)$

4. (a) $\cancel{E(\hat{\theta}_n)} = E(X_i^2) = Var(X_i) + [E(X_i)]^2 = \theta$

As $\frac{X_i^2}{\theta} \sim \chi^2_1$, $Var(X_i^2) = \theta^2 \cdot Var\left(\frac{X_i^2}{\theta}\right) = \theta^2 \cdot 2 = 2\theta^2$

Then the result follows from the CLT as X_i^2's are iid.

(b) By the Δ-method, $c = [g'(\theta)]^2 \cdot 2\theta^2 = \frac{1}{\theta^2} \cdot 2\theta^2 = 2$.

(c) $g(\hat{\theta}_n) \pm z_{\alpha/2}\sqrt{\frac{2}{n}} = (L, U)$, $\alpha = 0.05$

(d) (e^L, e^U).

Math 494: Mathematical Statistics
Solutions to Midterm2

Suppose the lifetime of a certain brand of bulbs follows an exponential distribution with parameter θ ($\theta > 0$) for which the pdf is

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, x > 0.$$

(**Remark**: This exponential distribution has mean θ and variance θ^2.) Let $X_1, \ldots, X_n$ be the lifetime of a random sample of n bulbs of this brand.

1. (15 points) Write down the likelihood function $L(\theta)$.

Solution: [15pts]

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=0}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}}$$
$$= \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta}}$$

2. (15 points) Show that the sample mean $\bar{X}_n$ is a sufficient statistic for θ using the factorization theorem.

Solution: The joint pdf can be rewritten as [10pts]

$$\frac{1}{\theta^n} e^{-\frac{n\bar{X}_n}{\theta}}$$

[5pts] From the factorization theorem, we see that $\bar{X}_n$ is a sufficient statistic for θ.

3. (15 points) Find the Fisher information $I(\theta)$.

Solution: [5pts] $\log f = -\log\theta - \frac{x}{\theta}$; $\frac{\partial \log f}{\partial\theta} = -\frac{1}{\theta} + \frac{x}{\theta^2}$; $\frac{\partial^2 \log f}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$.
[5pts] $I(\theta) = -E\left[\frac{\partial^2 \log f}{\partial\theta^2}\right]$
[5pts] $I(\theta) = -\frac{1}{\theta^2} + \frac{2E[X]}{\theta^3} = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$.

4. (10 points) Find the Rao-Cramer lower bound for the variance of unbiased estimators of θ.

Solution: [5pts] $\because$ unbiased $\quad\therefore\ CRLB = \frac{1}{nI(\theta)}$
[5pts] So, $CRLB = \frac{\theta^2}{n}$.

5. (15 point) Find the MLE of θ. Is it a MVUE of θ? Explain by comparing its variance to the Rao-Cramer lower bound in (d).

Solution: [8pts] $l(\theta) = \log L(\theta) = -n\log\theta - \frac{n\overline{X}}{\theta}$
$l'(\theta) = -\frac{n}{\theta} + \frac{n\overline{X}}{\theta^2} = 0 \Rightarrow \hat{\theta}_{MLE} = \overline{X}$
[2pts] $E(\hat{\theta}_{MLE}) = E(\overline{X}) = \theta \ \Rightarrow$ unbiased
[5pts] $Var(\hat{\theta}_{MLE}) = Var(\overline{X}) = \frac{Var(X)}{n} = \frac{\theta^2}{n}$, which attains the RCLB.
Therefore, $\hat{\theta}_{MLE}$ is a MVUE.

6. (30 points) Consider the following hypothesis testing problem, $H_0 : \theta = 100$ versus $H_1 : \theta > 100$.
   (a) (15 points) Show that the maximum likelihood ratio test statistic is a function of $\bar{X}_n$.
   (b) (15 points) Give a test based on $\bar{X}_n$ with significance level α based on the asymptotic normal distribution of $\bar{X}_n$ from central limiting theorem. (**Note**: The alternative hypotheis is one-sided.)

Solution:

(a)

$$\begin{aligned}\Lambda &= \frac{L(\theta_0)}{L(\hat{\theta}_{MLE})} \quad \text{[5pts]}\\ &= \frac{\frac{1}{100^n}\exp\left(-\frac{n\bar{X}_n}{100}\right)}{\frac{1}{\bar{X}_n^n}\exp\left(-\frac{n\bar{X}_n}{\bar{X}_n}\right)} \quad \text{[3pts]}\\ &= \left(\frac{\bar{X}_n}{100}\right)^{100}\exp\left(n - \frac{n\bar{X}_n}{100}\right), \quad \text{[3pts]}\end{aligned}$$

which is based on $\bar{X}_n$. So the maximum likelihood ratio test statistic $-2\log\Lambda$ also based on $\bar{X}_n$. [4pts]

(b) By CLT,

$$\sqrt{n}\frac{(\bar{X}_n - \theta)}{\theta} \to^d N(0,1). \quad \text{[5pts]}$$

Then under $H_0: \ \theta = 100$,

$$\sqrt{n}\left(\frac{\bar{X}_n}{100} - 1\right) \to^d N(0,1). \quad \text{[3pts]}\ .$$

So at level α, we reject H_0 when

$$\sqrt{n}\left(\frac{\bar{X}_n}{100} - 1\right) > z_{1-\alpha},$$

or when

$$\bar{X}_n > \frac{100 z_{1-\alpha}}{\sqrt{n}} + 100. \quad \text{[7pts]}$$

Math 494: Mathematical Statistics
Solutions to Extra practice exercises for the sections after midterm2

Problem 1 (# 7.5.1)

The pdf

$$f(x;\theta) = \exp\left\{\log\left(\frac{1}{6\theta^4}x^3\right) - \frac{x}{\theta}\right\} = \exp\left\{(-\frac{1}{\theta})x + 3\log x - \log(6\theta^4)\right\},$$

which follows the regular exponential form (7.5.1) on textbook with $K(x) = x$. Then form *Theorem 7.5.2*, a complete sufficient statistic for θ is given by $Y_1 = \sum_{i=1}^n X_i$. With some computation, we can verify that if $X \sim f(x;\theta)$, then

$$E(X) = \int_0^\infty (x)\frac{1}{6\theta^4}x^3 e^{-x\theta}dx = 4\theta,$$

and hence $E(Y_1/(4n)) = \theta$. So $\varphi(Y_1) = Y_1/(4n)$ is the unique function of Y_1 that is the MVUE of θ.

Problem 2 (# 7.5.11)

(a) The pdf

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} = \exp\left(x\log\theta + (1-x)\log(1-\theta)\right) = \exp\left(x\log\frac{\theta}{1-\theta} + \log(1-\theta)\right),$$

which in the form of a regular exponential family with $K(x) = x$, so by *Theorem 7.5.2*, a complete sufficient statistic for θ is given by $Y_1 = \sum_{i=1}^n X_i$.

(b) Since $E(X_i) = \theta$, we have $E(Y_1/n) = \theta$, so $\varphi(Y_1) = Y_1/n$.

(c) $E(Y_2) = \theta$.

(d) Since Y_1 is a complete sufficient statistic for θ, $T = E(U|Y_1)$ is a MVUE of θ for any unbiased estimator U of θ. Notice that T is also a function of Y_1. So by the uniqueness of MVUE in (b), we have $T = Y_1/n$. For the special case with $U = Y_2 = (X_1 + X_2)/2$ and $Y_1 = y_1$, we have

$$E(Y_2|Y_1 = y_1) = y_1/n.$$

Problem 3 (# 8.1.2)

$$\frac{L(\theta';x_1,x_2)}{L(\theta'';x_1,x_2)} = \frac{(1/2)^2 e^{-\frac{1}{2}(x_1+x_2)}}{(1/4)^2 e^{-\frac{1}{4}(x_1+x_2)}} = 4e^{-\frac{1}{4}(x_1+x_2)},$$

which is a function of $x_1 + x_2$. From *Theorem 8.1.1*, the conclusion immediately follows.

Problem 4 (# 8.1.3)

$$\frac{L(\theta'; x_1, x_2)}{L(\theta''; x_1, x_2)} = \frac{(1/2)^2 e^{-\frac{1}{2}(x_1+x_2)}}{(1/6)^2 e^{-\frac{1}{6}(x_1+x_2)}} = 9e^{-\frac{1}{3}(x_1+x_2)}$$

If $k > 0$, then $9e^{-\frac{1}{3}(x_1+x_2)} \leq k$ is a best critical region.
This inequality holds if and only if

$$x_1 + x_2 \geq -3\log(k/9) = c$$

We can see the best test of H_0 against H_1 can be carried out by use of the statistic $X_1 + X_2$, and the result holds for every $\theta'' > 2$.

Problem 5 (# 8.2.1)

$$\frac{L(1/4; x_1, x_2, ..., x_{10})}{L(\theta; x_1, x_2, ..., x_{10})} = \frac{(1/4)^{\sum x_i}(3/4)^{10-\sum x_i}}{\theta^{\sum x_i}(1-\theta)^{10-\sum x_i}} \geq k$$

The critical region is $C = \{(\sum_1^{10} x_i) : \sum_1^{10} x_i \leq 1\}$.
The power function $\gamma(\theta)$ of the test for $0 < \theta \leq 1/4$ is

$$\begin{aligned}
\gamma(\theta) &= \int_0^1 \theta^{\sum x_i}(1-\theta)^{10-\sum x_i} d\sum x_i \\
&= (1-\theta)^{10}\left[\frac{\theta/(1-\theta)}{\log\frac{\theta}{1-\theta}} - \frac{1}{\log\frac{\theta}{1-\theta}}\right] \\
&= \frac{(2\theta-1)(1-\theta)^9}{\log\frac{\theta}{1-\theta}}
\end{aligned}$$

Problem 6 (# 8.2.7)

$$\frac{L(\theta'; \mathbf{x})}{L(\theta''; \mathbf{x})} = \frac{(1/200\pi)^{n/2}\exp\left[-\sum(x_i-\theta')^2/200\right]}{(1/200\pi)^{n/2}\exp\left[-\sum(x_i-\theta'')^2/200\right]}$$

$$= \exp\left\{-(1/100)(\theta''-\theta')\sum_1^n x_i + \frac{n}{200}[(\theta'')^2 - (\theta')^2]\right\}$$

If $\theta' < \theta''$, the ratio is a monotone decreasing function of $y = \sum_1^n x_i$. Thus, the likelihood has mlr in the statistic $Y = \sum_1^n X_i$. Hence consider the hypotheses

$$H_0 : \theta = 75 \text{ versus } H_1 : \theta > 75,$$

The UMP level α decision rule is,

$$\text{Reject } H_0 \text{ if } Y = \sum_1^n X_i \geq c.$$

where c is such that $\alpha = P_{\theta=75}\left[\sum_{i=1}^{n} x_i \geq c\right]$.
Note that under H_0,

$$\frac{\sum_1^{25} x_i}{25} \sim N(75, 4)$$

Thus,

$$P_{\theta=75}\left[\sum_{i=1}^{25} \frac{x_i}{25} \geq \frac{c}{25}\right] = 0.10$$

We can get c from Table II in Appendix C

$$\frac{(c/25) - 75}{2} = z_\alpha = 1.28$$

Thus, $c = 1939$.

Problem 7 (# 8.3.4)

(a) $E(X_i) = \theta_1 + E(Z_i) = \theta_1$, $E(Y_i) = \theta_2 + E(Z_{n+i}) = \theta_2$.
$Var(X_i) = Var(Z_i) = \theta_3$, $Var(Y_i) = Var(Z_{n+i}) = \theta_3$.

(b) We know that under H_0, the test statistic T given in expression (8.3.4) has a t-distribution with $n + m - 2$ degrees of freedom. Then we only need to show when df $\to \infty$, a t-distribution is a standard normal.
The pdf of the t-distribution $t(r)$ is

$$f(x) = \frac{\Gamma[(r+1)/2])}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1 + x^2/r)^{(r+1)/2}}.$$

By the following two limit results,

$$\lim_{r\to\infty} \frac{\Gamma[r/2 + 1/2])}{\sqrt{\pi r}\Gamma(r/2)} = \frac{1}{\sqrt{2\pi}} \text{ and } \lim_{r\to\infty} \frac{1}{(1 + x^2/r)^{(r+1)/2}} = e^{-\frac{x^2}{2}},$$

the pdf of the t-distribution converges to the pdf of the standard normal distribution for any $x \in \mathbb{R}$.

(c) $T \to^D N(0, 1)$ as $m, n \to \infty$, so the critical region is $C = \{|T| \geq z_{\alpha/2}\}$.

Problem 8 (# 10.2.2)

Since the median of X_i is θ_0, $E[I(X_i > \theta_0)] = P(X_i > \theta_0) = 1/2$,
$Var[I(X_i > \theta_0)] = E[I(X_i > \theta_0)] - (E[I(X_i > \theta_0)])^2 = 1/4$.
By CLT, $\frac{S(\theta_0) - n\frac{1}{2}}{\sqrt{n}\sqrt{\frac{1}{4}}} = \frac{S(\theta_0) - (n/2)}{\sqrt{n}/2} \to Z$.

Problem 9 (# 11.2.2)

$$k(\theta|x_1, x_2, \cdots, x_n) \propto \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{\sum x_i+\alpha-1}(1-\theta)^{n-\sum x_i+\beta-1}.$$

This is the pdf of Beta($\sum x_i + \alpha$, $n - \sum x_i + \beta$), which is the same as that of Example 11.2.2.