Explanation about BSM and Heston model derivation

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0.1 Explanation about BSM and Heston model derivation

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0.1.1 Geometric Brownian Motion

GBM is the easiest stochastic process, thus this part of the notebook would thoroughly explain the derivation process of call option value under GBM. Under GBM asset value change follows constant drift and constant diffusion, with risk-neutral dynamic given by stochastic differential equation:

$$dS_t/S_t = rdt + \sigma dW_t$$

Here the notebook replace asset (A) to more common used spot price (S) in derivative pricing

By definition, the value of call option equals to 'discounted all possible positive residual value at end of period', or like the formula below:

$$C(s,K,t,T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \middle| S_t = s\right]$$

Separate end price and strike price into two formula part:

$$C(s,K,t,T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{S_T>K} \middle| S_t = s\right] - e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[K \mathbb{1}_{S_T>K} \middle| S_t = s\right]$$

Please be advised, the prior part should be treated under spot price numeraire, thus change of measure is required.

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{S_T}{\mathbb{E}^{\mathbb{Q}}[S_T|S_t]} = \frac{S_T}{S_t e^{r(T-t)}}$$

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{S_T>K} \middle| S_t = s\right] &= e^{-r(T-t)}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} S_T \mathbb{1}_{S_T>K} \middle| S_t = s\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\frac{S_t e^{r(T-t)}}{S_T} S_T \mathbb{1}_{S_T>K} \middle| S_t = s\right] \end{aligned}$$

Remove the duplicated S_T and extract the $S_t e^{r(T-t)}$ out of $\mathbb{E}^{\tilde{\mathbb{Q}}}$, then the discount factor $e^{-r(T-t)}$ is neutralized. Last but least, S_t equals to s, a constant. The prior part can be expressed as:

$$s\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathbb{1}_{S_T>K} \middle| S_t = s\right] = s\tilde{\mathbb{Q}}[S_T > K | S_t = s]$$

The prior part equals to initial spot price times risk neutrarl probability of $S_T > K$. The latter part is much more straightforward:

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[K\mathbb{1}_{S_T>K}\middle|S_t = s\right] &= e^{-r(T-t)}K\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{S_T>K}\middle|S_t = s\right] \\ &= e^{-r(T-t)}K\mathbb{Q}[S_T > K|S_t = s\Big] \end{aligned}$$

The latter part equals to strike price discounted by risk free rate and times the physical probability of $S_T > K$.

As the notebook assumed before, the S_T follows GBM. Write the formula again:

$$dS_t/S_t = rdt + \sigma dW_t$$

Obviously, the dS_t/S can be rewrited as $dlog(S_t)$. Further, based on Ito's lemma, below formula exists:

$$df = f^{'}(S_t)dS_t + \frac{1}{2}f^{''}(S_t)d^2S_t$$

Substitute f with log, then:

$$dlog(S_t) = \frac{1}{S_t}dS_t + \frac{1}{2}(-1)(S_t)^{-2}d^2S_t$$

Bring in the GBM process assumption.

$$dlog(S_t) = (rdt + \sigma_t dW_t) + \frac{1}{2}(-1)(rdt + \sigma_t dW_t)^2$$

Here the notebook needs to make an assumption, that dt^2 equals zero, dW_t^2 equals dt, and $dW_t dt$ equals zero. Find explicit derivation process here: https://www.youtube.com/watch?v=I0OiWYZRXoA.

$$dlog(S_t) = (rdt + \sigma dW_t) + \frac{1}{2}(-1)(\sigma^2 dt) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

Integral from t to T:

$$log(\frac{S_T}{S_t}) = (r - \frac{1}{2}\sigma^2)(T - t) + \sigma W_{T-t}$$

Or:

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}}$$

By Girsanov theorem:

$$d\tilde{W}_t = dW_t - \sigma t$$

The dynamic of GBM under risk neutral measure is like:

$$\begin{aligned} dlog(S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t &= (r - \frac{1}{2}\sigma^2)dt + \sigma(d\tilde{W}_t + \sigma t) \\ &= (r + \frac{1}{2}\sigma^2)dt + \sigma(d\tilde{W}_t) \end{aligned}$$

Integral the differential equation, then get:

$$S_T = S_t e^{(r+\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}}$$

$\mathbb{Q}$ means the conditional probability, like the probability of spot price at time T will be greater than strike price K if the starting spot price at t is s. The prior part is:

$$
\begin{aligned}
s\tilde{\mathbb{Q}}[S_T > K|S_t = s] &= sP[S_t e^{(r+\frac{1}{2}\sigma^2)(T-t)+\sigma W_(T-t)} > K|S_t = s] \\
&= sP[W_{T-t} > \frac{log(\frac{K}{S_t}) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma}|S_t = s]
\end{aligned}
$$

Based on normal distribution, probability of right tail equals to left tail under same deviation. To make the derivation more clear, the notebook chooses to calculate the left tail.

$$
sP[W_{T-t} > \frac{log(\frac{K}{S_t}) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma}|S_t = s] = sP[W_{T-t} < \frac{log(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma}|S_t = s]
$$

Based on the probability distribution definition:

$$
P[W_{T-t} < x|S_t = s] = \phi(\frac{x}{\sqrt{T-t}})
$$

The prior part:

$$
s\tilde{\mathbb{Q}}[S_T > K|S_t = s] = s\phi(\frac{log(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}})
$$

Similarly the latter part:

$$
e^{-r(T-t)}K\mathbb{Q}[S_T > K|S_t = s] = e^{-r(T-t)}K\phi(\frac{log(\frac{S_t}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}})
$$

Combine the prior and latter part, then the closed form solution of call price should be:

$$
C(s, K, t, T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \middle| S_t = s\right] = s\phi(\frac{log(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}) - e^{-r(T-t)}K\phi(\frac{log(\frac{S_t}{K}) + (r - \frac{1}{2}\sigma^2)(}{\sigma\sqrt{T-t}}
$$

0.1.2 Heston Process and its directed integral

The asset price follows GBM process while the volatility itself follows CIR process, which gives mean-reverting drift. The risk neutral dynamic of Heston process is a little bit complicated:

$$
dS_t/S_t = udt + \sqrt{v_t}dW_{1,t}
$$

$$
dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t}
$$

The risk neutral process for dS_t is :

$$
dS_t = rS_t dt + \sqrt{v_t}S_t d\tilde{W}_{1,t}
$$

Where

$$
d\tilde{W}_{1,t} = dW_{1,t} + \frac{\mu - r}{\sqrt{v_t}}dt
$$

Similarly, for risk neutral dynamic of volatility:

$$
dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \sigma\sqrt{v_t}d\tilde{W}_{2,t}
$$

$$
d\tilde{W}_{2,t} = dW_{2,t} + \frac{\lambda(S_t, v_t, t)}{\sigma\sqrt{v_t}}dt
$$

The below derivation is based on risk neutral measure.

The notebook would start with partial differential equation, or PDE of heston process. Apply Ito's lemma, the below formula holds:

$$df = f^{'}(x)dx + \frac{1}{2}f^{''}(x)d^2x$$

Define the value of option as $V(S, v, t)$, considering Taylor Expansion, the the below formula holds:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}d^2S + \frac{1}{2}\frac{\partial^2 V}{\partial^2 v}d^2v + \frac{1}{2}\frac{\partial^2 V}{\partial^2 t}d^2t + \frac{\partial^2 V}{\partial S \partial v}dSdv + \frac{\partial^2 V}{\partial S \partial t}dSdt + \frac{\partial^2 V}{\partial v \partial t}dvdt$$

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}vS^2dt + \frac{1}{2}\frac{\partial^2 V}{\partial^2 v}\sigma^2 vdt + \frac{\partial^2 V}{\partial S \partial v}\sigma vS\rho dt$$

Given that:

$$d^2S = (rSdt + \sqrt{v}SdW_1)^2 = vS^2dt$$

$$d^2v = (\kappa(\theta - v)dt + \sigma\sqrt{v}dW_{2,t})^2 = \sigma^2 vdt$$

$$dSdv = (rSdt + \sqrt{v}SdW_1)(\kappa(\theta - v)dt + \sigma\sqrt{v}dW_2) = \sqrt{v}SdW_1\sigma\sqrt{v}dW_2 = \sigma vS\rho dt$$

$$d^2t = 0; dtdW = 0$$

For further analysis, create a risk free synthetic portfolio Π consists of one unit of option V, Φ units of option U, and Δ units of underlying asset S.

$$d\Pi = dV + \Delta dS + \Phi dU$$

The PDE of U is indentical to V, substitute the PDE and rewrite above formula:

$$d\Pi = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}vS^2dt + \frac{1}{2}\frac{\partial^2 V}{\partial^2 v}\sigma^2 vdt + \frac{\partial^2 V}{\partial S \partial v}\sigma vS\rho dt + \Delta dS + \Phi(\frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial t}dt + \frac{1}{2}\frac{\partial^2 U}{\partial^2 S}vS$$

$$d\Pi = (\frac{\partial V}{\partial S} + \Delta + \Phi\frac{\partial U}{\partial S})dS + (\frac{\partial V}{\partial v} + \Phi\frac{\partial U}{\partial v})dv + (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}vS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial^2 v}\sigma^2 v + \frac{\partial^2 V}{\partial S \partial v}\sigma vS\rho)dt + \Phi(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial^2 S}vS^2 + \frac{1}{2}$$

Since the synthetic portfolio is risk free, by definition, the portfolio differential $d\Pi$ is zero in terms of asset price differential dS and volatility differential dv. To achieve that, Δ and Φ should satisfy below formula:

$$\Phi = -\frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}}$$

$$\Delta = -\Phi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}$$

Risk free portfolio has risk free return, besides, the notebook use A and B as proxy.

$$d\Pi = r\Pi dt = r(V + \Delta S + \Phi U)dt = (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}vS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial^2 v}\sigma^2 v + \frac{\partial^2 V}{\partial S \partial v}\sigma vS\rho)dt + \Phi(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial^2 S}vS^2 + \frac{1}{2}\frac{\partial^2 U}{\partial^2 v}\sigma^2 v +$$

Substitue Φ & Δ and rewrite above formula:

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}$$

The left-hand side is equivalent to the right-hand side, thus both sides can be written as a function specified by Heston as:

$$f(S,v,t) = -\kappa(\theta - v) + \lambda(S,v,t)$$

Then the balanced equation becomes:

$$-\kappa(\theta - v) + \lambda(S,v,t) = \frac{(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial^2 S}vS^2 + \frac{1}{2}\frac{\partial^2 U}{\partial^2 v}\sigma^2 v + \frac{\partial^2 U}{\partial S \partial v}\sigma v S \rho) - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}$$

Rewrite

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial^2 S}vS^2 + \frac{1}{2}\frac{\partial^2 U}{\partial^2 v}\sigma^2 v + \frac{\partial^2 U}{\partial S \partial v}\sigma v S \rho - rU + rS\frac{\partial U}{\partial S} - (-\kappa(\theta - v) + \lambda(S,v,t))\frac{\partial U}{\partial v} = 0$$

Use $x = lnS$ rewrite the above equation:

$$\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial^2 x} + \frac{1}{2}\frac{\partial^2 U}{\partial^2 v}\sigma^2 v + \sigma v \rho \frac{\partial^2 U}{\partial x \partial v} - rU + (r - \frac{1}{2}v)\frac{\partial U}{\partial x} + (\kappa(\theta - v) - \lambda v)\frac{\partial U}{\partial v} = 0$$

Use $C_T(K)$ repalce U, and since τ equals $(T-t)$, $d\tau = -dt$

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2}v\frac{\partial^2 C}{\partial^2 x} + \frac{1}{2}\frac{\partial^2 C}{\partial^2 v}\sigma^2 v + \sigma v \rho \frac{\partial^2 C}{\partial x \partial v} - rC + (r - \frac{1}{2}v)\frac{\partial C}{\partial x} + (\kappa(\theta - v) - \lambda v)\frac{\partial C}{\partial v} = 0$$

The call option can be defined as followed:

$$C_T(K) = e^{-\tau T}E[(S_T - K)^+] = e^{x_t}P_1(x,v,\tau) - e^{-r\tau}KP_2(x,v,\tau)$$

Use P_j to rewrite the formula:

$$-\frac{\partial P_j}{\partial \tau} + \frac{1}{2}v\frac{\partial^2 P_j}{\partial^2 x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial^2 v} + \sigma v \rho \frac{\partial^2 P_j}{\partial x \partial v} + (r + u_j v)\frac{\partial P_j}{\partial x} + (a - b_j v)\frac{\partial P_j}{\partial v} = 0$$

Where:

$$u_1 = \frac{1}{2}; u_2 = -\frac{1}{2}; a = \kappa\theta; b_1 = \kappa + \lambda - \rho\sigma; b_2 = \kappa + \lambda$$

Based on Gil-Pelaez inversion theorem:

$$P_j = P(\ln S_T > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big[\frac{e^{-i\phi \ln K}f_j(\phi; x, v)}{i\phi}\Big]d\phi$$

The characteristic functions are in the form of such, defined by Heston:

$$f_j(\phi; x, v) = e^{C_j(\tau,\phi) + D_j(\tau,\phi)v0 + i\phi x}$$

f_j also follows the P_j PDE, substitue f_j with C_j and D_j and rearrange, then get two differential equation:

$$\frac{\partial D_j}{\partial \tau} = \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j$$

$$\frac{\partial C_j}{\partial \tau} = ri\phi + aD_j$$

Solving the Heston Riccati Equation

$$\frac{\partial D_j}{\partial \tau} = P_j - Q_j D_j + R D_j^2$$

The corresponding ordinary differential equation is:

$$w^{''} + Q_j w^{'} + P_j R = 0$$

The solution to Heston Riccati Equation is:

$$D_j = -\frac{1}{R}\left(\frac{K\alpha e^{\alpha\tau} + \beta e^{\beta\tau}}{K e^{\alpha\tau} + e^{\beta\tau}}\right)$$

Since $D_j(0, \phi) = 0$:

$$D_j = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2} \frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}}$$

Where

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}$$

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}$$

C_j is found by integral the differential equation

$$C_j = \int_0^\tau ri\phi dy + a(\frac{Q_j + d_j}{2R}) \int_0^\tau (\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}}) dy + K_1$$

Since $C_j(0, \phi) = 0$:

$$C_j = ri\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln(\frac{1 - g_j e^{d_j\tau}}{1 - g_j})\right]$$

Where $a = \kappa\theta$

0.1.3 Carr and Madan(1999) approach

The above Heston model derivation gives the closed form option price, while the integral process is extremely costy and low efficiency. To boost the efficiency of calculation, Carr and Madan(1999) develop fast Fourier transform(FFT) to value options. First, based on option definition:

$$\begin{aligned} C_T(k) &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \middle| S_t = s\right] \\ &= e^{-r(T-t)} \int_k^\infty (e^x - e^k) q(x) dx \end{aligned}$$

Where

$$x = log(S_T)$$
$$k = log(K)$$

If K is close to zero, then $\mathbb{E}^{\mathbb{Q}}[S_T - 0] = S_0 e^{rT}$. Thus $C_T(s, K, t, T)$ is close to S_0, which is not close zero, makes the C not a integrable function. Introduce damping factor $e^{\alpha k}$ makes $c_T(s, K, t, T)$ integrable.

$$c_T(k) = e^{\alpha k} C_T(k)$$

The Fourier transform is given by,

$$\begin{aligned}
\psi(\nu) &= \int_{-\infty}^{\infty} e^{i\nu k} c(k) dk \\
&= \int_{-\infty}^{\infty} e^{i\nu k} e^{\alpha k} e^{-r(T-t)} \int_{k}^{\infty} (e^x - e^k) q(x) dx dk \\
&= \int_{-\infty}^{\infty} e^{-r(T-t)} q(x) \int_{k}^{\infty} e^{i\nu k} e^{\alpha k} (e^x - e^k) dk dx \\
&= \int_{-\infty}^{\infty} e^{-r(T-t)} q(x) \int_{-\infty}^{x} e^{i\nu k} e^{\alpha k} (e^x - e^k) dk dx
\end{aligned}$$

The inner integral,

$$\begin{aligned}
\int_{-\infty}^{x} e^{i\nu k} e^{\alpha k} (e^x - e^k) dk &= \int_{-\infty}^{x} e^{i\nu k} e^{\alpha k} e^x dk - \int_{-\infty}^{x} e^{i\nu k} e^{\alpha k} e^k dk \\
&= e^x \int_{-\infty}^{x} e^{(i\nu+\alpha)k} dk - \int_{-\infty}^{x} e^{(i\nu+\alpha+1)k} dk \\
&= e^x \frac{1}{(i\nu + \alpha)} [e^{(i\nu+\alpha)k}]_{-\infty}^{x} - \frac{1}{(i\nu + \alpha + 1)} [e^{(i\nu+\alpha+1)k}]_{-\infty}^{x} \\
&= \frac{1}{(i\nu + \alpha)} [e^{(i\nu+\alpha+1)x}] - \frac{1}{(i\nu + \alpha + 1)} [e^{(i\nu+\alpha+1)x}]
\end{aligned}$$

Back to Fourier transform,

$$\begin{aligned}
\psi(\nu) &= \int_{-\infty}^{\infty} e^{-r(T-t)} q(x) \int_{-\infty}^{x} e^{i\nu k} e^{\alpha k} (e^x - e^k) dk dx \\
&= \int_{-\infty}^{\infty} e^{-r(T-t)} q(x) [\frac{1}{(i\nu + \alpha)} [e^{(i\nu+\alpha+1)x}] - \frac{1}{(i\nu + \alpha + 1)} [e^{(i\nu+\alpha+1)x}]] dx \\
&= \int_{-\infty}^{\infty} e^{-r(T-t)} q(x) \frac{e^{(i\nu+\alpha+1)x}}{(i\nu + \alpha)(i\nu + \alpha + 1)} dx
\end{aligned}$$

Further transform,

$$\begin{aligned}
\int_{-\infty}^{\infty} q(x) e^{(i\nu+\alpha+1)x} dx &= \int_{-\infty}^{\infty} q(x) e^{i[-i(i\nu+\alpha+1)]x} dx \\
&= \int_{-\infty}^{\infty} q(x) e^{i(\nu - i\alpha - i)x} dx \\
&= \int_{-\infty}^{\infty} q(x) e^{i[(\nu-(\alpha+1)i]x} dx \\
&= \phi(\nu - (\alpha + 1)i)
\end{aligned}$$

Thus,

$$\psi(\nu) = \int_{-\infty}^{\infty} e^{-r(T-t)} q(x) \frac{e^{(i\nu+\alpha+1)x}}{(i\nu+\alpha)(i\nu+\alpha+1)} dx$$
$$= e^{-r(T-t)} \frac{\phi(\nu-(\alpha+1)i)}{(i\nu+\alpha)(i\nu+\alpha+1)}$$
$$= e^{-r(T-t)} \frac{\phi(\nu-(\alpha+1)i)}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}$$

Finally, given that ψ is Fourier transform of $c(k)$, use the inverse Fourier,

$$c_T(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu k} \psi(\nu) d\nu$$
$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} Re[e^{-i\nu k} \psi(\nu)] d\nu$$

Similarly, for put option,

$$\psi_p(\nu) = e^{-r(T-t)} \frac{\phi(\nu-(-\alpha+1)i)}{\alpha^2-\alpha-\nu^2+i(-2\alpha+1)\nu}$$
$$P_T(k) = \frac{e^{\alpha k}}{\pi} \int_0^{\infty} Re[e^{-i\nu k} \psi_p(\nu)] d\nu$$

In the ZSAMC fixed income pricing context, the target is to price the out-of-money put option, for further calculation of spread. $z_T(k)$ is the price of out-of-money option. Carr and Madan focus on time value, or extrinsic value, of OTM option, because intrinsic value is zero for OTM option.

$$\xi_T(\nu) = \int_{-\infty}^{\infty} e^{i\nu k} z_T(k) dk$$

Inverting the transform,

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu k} \xi_T(\nu) d\nu$$

Define the $z_T(k)$ as OTM option. For OTM call, the payoff would be $e^x - e^k$ satisfied $k > 0$. For OTM put, the payoff would be $e^k - e^s$ satisfied $k < 0$. Assume, $S_0 = 1$

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} [(e^k - e^x)\mathbb{1}_{x<k,k<0} + (e^x - e^k)\mathbb{1}_{x>k,k>0}] q(x) dx$$
$$\xi_T(\nu) = \int_{-\infty}^{\infty} e^{i\nu k} e^{-rT} \int_{-\infty}^{\infty} [(e^k - e^x)\mathbb{1}_{x<k,k<0} + (e^x - e^k)\mathbb{1}_{x>k,k>0}] q(x) dx dk$$

Further rearrange,

$$
\begin{aligned}
\xi_T(\nu) &= \int_{-\infty}^{\infty} e^{i\nu k} e^{-rT} \int_{-\infty}^{\infty} [(e^k - e^x)\mathbb{1}_{x<k,k<0}]q(x)dxdk + \int_{-\infty}^{\infty} e^{i\nu k} e^{-rT} \int_{-\infty}^{\infty} [(e^x - e^k)\mathbb{1}_{x>k,k>0}]q(x)dxdk \\
&= \int_{-\infty}^{\infty} e^{i\nu k} e^{-rT} \int_{-\infty}^{k} (e^k - e^x)q(x)dxdk + \int_{-\infty}^{\infty} e^{i\nu k} e^{-rT} \int_{k}^{\infty} (e^x - e^k)q(x)dxdk \\
&= \int_{-\infty}^{\infty} e^{-rT} q(x) \int_{-\infty}^{k} (e^k - e^x)e^{i\nu k} dkdx + \int_{-\infty}^{\infty} e^{-rT} q(x) \int_{k}^{\infty} (e^x - e^k)e^{i\nu k} dkdx \\
&= \int_{-\infty}^{\infty} e^{-rT} q(x) \int_{x}^{\infty} (e^k - e^x)e^{i\nu k} dkdx + \int_{-\infty}^{\infty} e^{-rT} q(x) \int_{-\infty}^{x} (e^x - e^k)e^{i\nu k} dkdx \\
&= \int_{-\infty}^{\infty} e^{-rT} q(x) \int_{x}^{\infty} (e^{(i\nu+1)k} - e^{i\nu k + x})dkdx + \int_{-\infty}^{\infty} e^{-rT} q(x) \int_{-\infty}^{x} (e^{i\nu k + x} - e^{(i\nu+1)k})dkdx
\end{aligned}
$$

Inner integral and rewrite,

$$
\xi_T(\nu) = e^{-rT}\left[\frac{1}{1+i\nu} - \frac{e^{rT}}{i\nu} - \frac{\phi_T(\nu - i)}{\nu^2 - i\nu}\right]
$$

$$
\begin{aligned}
\gamma_T(\nu) &= \xi_T(\nu) sinh(\alpha k) \\
&= \int_{-\infty}^{\infty} e^{i\nu k} sinh(\alpha k) z_T(k)dk \\
&= \int_{-\infty}^{\infty} e^{i\nu k} \frac{e^{\alpha k} - e^{-\alpha k}}{2} z_T(k)dk \\
&= \int_{-\infty}^{\infty} e^{i\nu k} \frac{e^{\alpha k}}{2} z_T(k)dk - \int_{-\infty}^{\infty} e^{i\nu k} \frac{e^{-\alpha k}}{2} z_T(k)dk \\
&= \int_{-\infty}^{\infty} e^{(i\nu+\alpha)k} \frac{1}{2} z_T(k)dk - \int_{-\infty}^{\infty} e^{(i\nu-\alpha)k} \frac{1}{2} z_T(k)dk \\
&= \frac{1}{2}(\xi_T(\nu - i\alpha) - \xi_T(\nu + i\alpha))
\end{aligned}
$$

Inverse transform,

$$
z_T(k) = \frac{1}{2\pi sinh(\alpha k)} \int_{-\infty}^{\infty} e^{-i\nu k} \gamma_T(\nu) d\nu
$$

[ ]: