第二章作业

T1 $A+B=\begin{pmatrix}3&3&1\\3&8&2\end{pmatrix}$ $A-C=\begin{pmatrix}1&-3&0\\3&6&-1\end{pmatrix}$ $\Rightarrow A-3C=\begin{pmatrix}4&2&2\\0&9&1\end{pmatrix}$ //

T2 $A=\begin{pmatrix}3&1&0\\-1&2&1\\3&4&2\end{pmatrix}$ $B=\begin{pmatrix}1&0&2\\-1&1&1\\2&1&1\end{pmatrix}$ $3A-2X=B$ 求 X

加法性质

$$X=\frac{1}{2}(3A-B)=\begin{pmatrix}4&\frac{3}{2}&-1\\-1&\frac{5}{2}&1\\\frac{7}{2}&\frac{11}{2}&\frac{5}{2}\end{pmatrix}$$ //

T3 $\begin{pmatrix}a+2b&2a-b\\2c+d&c-2d\end{pmatrix}=\begin{pmatrix}4&-2\\4&-3\end{pmatrix}$ 求 a,b,c,d

对应元素相等

$$\begin{cases}a+2b=4\\2a-b=-2\\2c+d=4\\c-2d=-3\end{cases}\Rightarrow\begin{cases}a=0\\b=2\\c=1\\d=2\end{cases}$$ //

T4 A − $m\times n$, $kA=0$ 证明 $k=0$ 或 $A=0$

考察0矩阵性质

设 $A=(a_{ij})_{m\times n}$, $kA=0\Rightarrow ka_{ij}=0$

① $k=2$ ② a_{ij} 都为0 $\Rightarrow A=0$ //

T5

(1) $\begin{bmatrix}3&2\\-1&4\\5&1\end{bmatrix}\cdot\begin{bmatrix}1&8&-1\\2&0&3\end{bmatrix}=\begin{bmatrix}7&24&3\\7&-8&13\\7&40&-2\end{bmatrix}$ (4) $\begin{pmatrix}1&3&2\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}5\end{pmatrix}$

注意1×1矩阵不为数

(2) $\begin{pmatrix}1&3&-1\\0&4&2\\7&0&1\end{pmatrix}\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}-8\\-2\\10\end{pmatrix}$

(5) $\begin{pmatrix}1&-1&2\end{pmatrix}\begin{pmatrix}2&-1&0\\1&1&3\\1&0&-1\end{pmatrix}=\begin{pmatrix}3&-2&-5\end{pmatrix}$

(3) $\begin{pmatrix}1\\-1\\2\\3\end{pmatrix}\begin{pmatrix}3&2&-1&0\end{pmatrix}=\begin{pmatrix}3&2&-1&0\\-3&-2&1&0\\6&4&-2&0\\9&6&-3&0\end{pmatrix}$

(6) $(x,y,z)\begin{pmatrix}a_{11}&a_{12}&b_1\\a_{12}&a_{22}&b_2\\b_1&b_2&c\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}$

$$=\left(a_{11}x^2+2a_{12}xy+a_{22}y^2+2b_1x+2b_2y+c\right)$$ //

T6. $A=\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}$

$A^2=\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}$

$A^3=\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 3\lambda & 1\end{pmatrix}$

A^n 数学归纳：设 $A^{n-1}=\begin{pmatrix}1 & 0\\ (n-1)\lambda & 1\end{pmatrix}$，$A^n=A^{n-1}\cdot A=\begin{pmatrix}1 & 0\\ n\lambda & 1\end{pmatrix}$ //

T7 (1) $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}^n \Rightarrow \begin{pmatrix}\cos n\theta, & -\sin n\theta\\ \sin n\theta, & \cos n\theta\end{pmatrix}$ 非作业题，过程略

(2) $\begin{pmatrix}1 & 4 & 2\\ 0 & -3 & -2\\ 0 & 4 & 3\end{pmatrix}^n \Rightarrow \begin{pmatrix}1 & 4 & 2\\ 0 & -3 & -2\\ 0 & 4 & 3\end{pmatrix}^2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

则 原式 $=\begin{cases}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} & \text{当 } n=2k \quad k\geq 1 \text{ 整数}\\ \begin{pmatrix}1 & 4 & 2\\ 0 & -3 & -2\\ 0 & 4 & 3\end{pmatrix} & \text{当 } n=2k+1 \quad k \text{ 为非负整数}\end{cases}$ //

(3) $\begin{pmatrix}1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{pmatrix}^n \Rightarrow A^2=4E \quad A^n \begin{cases}=\begin{pmatrix}4^k & 0 & 0 & 0\\ 0 & 4^k & 0 & 0\\ 0 & 0 & 4^k & 0\\ 0 & 0 & 0 & 4^k\end{pmatrix} & n=2k,\ k \text{正整}\\ =4^k\begin{pmatrix}1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{pmatrix} & n=2k+1 \quad k \text{ 为非负整数}\end{cases}$ /

(4) $(\alpha\beta^T)^n \Rightarrow \alpha\beta=\begin{pmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & a_3b_3\end{pmatrix}$

$(\alpha\beta^T)^n=\underbrace{\alpha\beta^T\cdot\alpha\beta^T\cdots\alpha\beta^T}_{n\text{对}}=\alpha\underbrace{\beta^T\alpha\beta^T\alpha\beta^T\alpha\beta^T}_{n-1\text{对}}\alpha\beta^T$

$=\alpha(a_1b_1+a_2b_2+a_3b_3)^{n-1}\cdot\beta^T$

$=(a_1b_1+a_2b_2+a_3b_3)^{n-1}\begin{pmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & a_3b_3\end{pmatrix}$ //

T8 可交换

T9 $A=\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$ 证明:与A可交换矩阵必为对角矩阵. $(a\neq b\neq c)$

设 $B=\begin{pmatrix} x_{11}, & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ 可交换.

$$AB=\begin{pmatrix} ax_{11} & ax_{12} & ax_{13} \\ bx_{21} & bx_{22} & bx_{23} \\ cx_{31} & cx_{32} & cx_{33} \end{pmatrix} \quad BA=\begin{pmatrix} ax_{11} & bx_{12} & cx_{13} \\ ax_{21} & bx_{22} & cx_{23} \\ ax_{31} & bx_{32} & cx_{33} \end{pmatrix}$$

令 $AB=BA$, 则 $\begin{cases} x_{12}(a-b)=0 \\ x_{13}(c-a)=0 \\ x_{21}(a-b)=0 \\ x_{23}(b-c)=0 \\ x_{31}(a-c)=0 \\ x_{32}(b-c)=0 \end{cases} \xrightarrow{a\neq b\neq c} \begin{cases} x_{12}=0 \\ x_{13}=0 \\ x_{21}=0 \\ x_{23}=0 \\ x_{31}=0 \\ x_{32}=0 \end{cases}$

$B=\begin{pmatrix} x_{11} & & \\ & x_{22} & \\ & & x_{33} \end{pmatrix} \longrightarrow$ 对角阵 //

T10 A. P n阶. A为对称, 证明 P^TAP 也对称

$$(P^TAP)^T=P^TA^T(P^T)^T=P^TA^TP=P^TAP \quad //$$

T11 AB对称充要是 $AB=BA$,(A.B为对称阵)

(1) AB对称 $\rightarrow$

$$AB=(AB)^T=B^TA^T=BA \;,$$

(2) $AB=BA$ $\rightarrow$

$$AB=BA=B^TA^T=(AB)^T \;,$$

综上, 充要 //

T12 $(AB=BA)$

(1) $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$ /

(2) $A^2 - B^2 \Leftarrow$ [从 $(A+B)(A-B)$ 开始]

$$= A^2 + BA - AB - B^2 = A^2 - B^2 \ /$$

(3) $(AB)^p = \underbrace{(AB)(AB)\cdots(AB)}_{p} = A\underbrace{(BA)(BA)\cdots(BA)}_{p-1}B$

$$= A\underbrace{(AB)(AB)\cdots(AB)}_{p-1}B$$

$$= A(AB)^{p-1}B$$

$$= \cdots\cdots$$

$$= A^p B^p \ //$$

T13 $A^2 = E$ 则 $A \neq E$ 的 A

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

T14 $A = \frac{1}{2}(B+E)$ 证明 $A^2 = A$ 的充要 $B^2 = E$

(1) $A^2 = A \rightarrow$

$$A^2 = \frac{1}{2}(B+E)\cdot\frac{1}{2}(B+E) = \frac{1}{2}\cdot\frac{1}{2}\cdot(B^2+2B+E)$$

$$= \frac{1}{2}(B+E)$$

即 $B^2 + 2B + E = 2B + 2E$

即 $B^2 = E$

(2) $B^2 = E \rightarrow$

$$A^2 = \frac{1}{2}\cdot\frac{1}{2}(B^2 + 2B + E)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot 2(B+E)$$

$$= \frac{1}{2}(B+E)$$

$$= A \qquad //$$

T15 A 反对称. B 对称：$A' = -A$，$B' = B$

证明题(对称)
直接取 T

(1) A^2 对称：

$$(A^2)^T = (AA)^T = A^T A^T = (-A)(-A) = A^2$$ /

(2) $(AB-BA)$ 对称

$$\begin{aligned}(AB-BA)^T &= (AB)^T - (BA)^T \\ &= B^T A^T - A^T B^T \\ &= B(-A) - (-A)B \\ &= AB - BA\end{aligned}$$ /

(3) AB. 反对称的充要 $AB = BA$

$$\xrightarrow{\quad} AB \xrightarrow{=AB}$$

$$(AB)^T = B^T A^T = -BA$$

有 $(AB)^T = -BA = -AB$ 则 $AB = BA$ /

$$\xrightarrow{AB=BA} (AB)^T = B^T A^T = -BA = -AB$$ //

T16. 实对称 $A^2 = O$，证明 $A = O$

设 $A = (a_{ij})_n$ 主对角元素的 $A^2 \rightarrow$

$a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2$，

$a_{12}^2 + a_{22}^2 + \cdots + a_{2n}^2$，都为 0

$\vdots$

$a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2$，

则 $a_{ij} = 0$ 则 $A = O$ //

T17 A 为 $m\times n$，证明 $AA^T=0$ 则 $A=0$.

$A=(a_{ij})_{m\times n}$ $AA^T\Rightarrow$ 主对角 $a_{11}^2+a_{12}^2+a_{13}^2+\cdots+a_{1n}^2$

$\vdots$

$a_{m1}^2+a_{m2}^2+a_{m3}^2+\cdots+a_{mn}^2$

都为 $\Rightarrow 0$

则 $a_{ij}=0$；则 $A=0$ //

T18 $A=\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$

(1) $A^2-2A=\begin{pmatrix}-3 & -2\\ 4 & 1\end{pmatrix}$ (2) $3A^3-2A^2+5A-4E=\begin{pmatrix}-24 & -30\\ 60 & 36\end{pmatrix}$

T19 $f(x)=\lambda^2-\lambda+1$ $A=\begin{pmatrix}2 & 2 & 3\\ 1 & -1 & 0\\ 3 & 1 & 2\end{pmatrix}$ 求 $f(A)$

转E

$$f(A)=A^2-A+E$$

$$=\begin{pmatrix}15 & 5 & 12\\ 1 & 3 & 13\\ 13 & 7 & 13\end{pmatrix}-\begin{pmatrix}2 & 2 & 3\\ 1 & -1 & 0\\ 3 & 1 & 2\end{pmatrix}+\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}14 & 3 & 9\\ 0 & 5 & 3\\ 10 & 6 & 12\end{pmatrix}$$ /