Chapter 4

$T_2(2)$ $F_2=\{a+bi \mid a,b\in Q\}$ 其中 $i=\sqrt{-1}$, Q 为有理数.

和 $(a_1+b_1i)+(a_2+b_2i)=(a_1+a_2)+(b_1+b_2)i$

差 $(a_1+b_1i)-(a_2+b_2i)=(a_1-a_2)+(b_1-b_2)i$

积 $(a_1+b_1i)(a_2+b_2i)=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i$

商 $\frac{a_1+b_1i}{a_2+b_2i}=\frac{(a_1+b_1i)(a_2-b_2i)}{a_2^2+b_2^2}=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+\frac{-a_1b_2+a_2b_1}{a_2^2+b_2^2}i$; a,b 也是有理数 → 数域

$T_3(2)$ 线性空间

① $\alpha+\beta=\beta+\alpha$

② $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$

③ 0: $\alpha+0=\alpha$.

④ $\forall\alpha\in V$: $\alpha+\xi=0$, $\xi=-\alpha$

⑤ $1\alpha=\alpha$

⑥ $k(l\alpha)=kl\alpha$

⑦ $(k+l)\alpha=k\alpha+l\alpha$

⑧ $k(\alpha+\beta)=k\alpha+k\beta$

若(2)中为不平行零向量,则不是线性空间,因为没有零元。

其他情况中,由于向量可任意选取,两向量相加可以生成该"不平行向量"故不可形成线性空间

$T_3(4)$ 全体复数

① 交换律 √ ② 结合律 ③ 0 √ ④ $-\alpha \Rightarrow -a+(-b)i$ √

⑤ $1\alpha=\alpha$ √ ⑥ $k(l\alpha)=kl\alpha$ √ ⑦ $(k+l)\alpha=k\alpha+l\alpha$ √

⑧ $k(\alpha+\beta)=k\alpha+k\beta$ √

$T_4(1)$ 子空间 : 线性空间 + 加法数乘封闭

(1) $x+ky\in W_1$ 是子空间

(3) $x+ky\notin W_1$ 且 $-x\notin W_1$ 不是子空间

(5) $x=(1,0,1)\in W_5$, 而 $-x\notin W_5$, 不是子空间

T5 (2) $U_2=\{f\in V \mid f(1)-f(-1)=0\}$;

设 $f_2\in U_2$, $g_2\in U_2$

有 $f_2(1)-f_2(-1)=0$ $\Rightarrow$ $(f_2+kg_2)(1)-(f_2+kg_2)(-1)$

$g_2(1)-g_2(-1)=0$ $=[f_2(1)-f_2(-1)]+k[g_2(1)-g_2(-1)]=0$

是线性子空间

(4) $U_4=\{f\in V \mid f(1)-f^2(1)=0\}$

而 $-f(1)-[-f^2(1)]^2=-f(1)-f^2(1)$ 不一定为 0

不是线性子空间

(5) $U_6=\{f\in V \mid f(x)-f(x^2)=0\}$

$f(x)-f(x^2)=0$, $g(x)-g(x^2)=0$

$(f+kg)(x)-(f+kg)(x^2)=f(x)+kg(x)-f(x^2)-kg(x^2)$

$=(f(x)-f(x^2))+k(g(x)-g(x^2))=0$

是线性子空间

T7 $\begin{cases} 2x_1+x_2-x_3+x_4-3x_5=0 \\ x_1+x_2-x_3+x_5=0 \end{cases}$ 解空间维数和基

$A=\begin{pmatrix} 2 & 1 & -1 & 1 & -3 \\ 1 & 1 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 & 5 \end{pmatrix}=B$

基解系是 $\begin{pmatrix} 0\\1\\1\\0\\0 \end{pmatrix}$ $\begin{pmatrix} -1\\1\\0\\1\\0 \end{pmatrix}$ $\begin{pmatrix} 4\\-5\\0\\0\\1 \end{pmatrix}$ 解空间维数 3

T8 $\{(a_{ij})_{2\times2}\in M \mid a_{11}+a_{12}+a_{21}+a_{22}=0\}$ 是 $M^{2\times2}$ 的子空间。

① $(a_{ij})_{2\times2}$ 和 $(b_{ij})_{2\times2}$ 是集合中元素 $(a_{ij})_{2\times2}+k(b_{ij})_{2\times2}$ 是该集合, 子空间 ✓

② 设 $\varepsilon_1=\begin{pmatrix} 0 & 0\\ 1 & -1 \end{pmatrix}$ $\varepsilon_2=\begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix}$ $\varepsilon_3=\begin{pmatrix} 0 & 1\\ 0 & -1 \end{pmatrix}$ $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 独立, 线性无关

$\eta=a\varepsilon_1+b\varepsilon_2+c\varepsilon_3$, 子空间维数为 3, $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 是一组基.

T9

$\alpha_1=\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$ $\alpha_2=\begin{pmatrix}4\\5\\-2\\-7\end{pmatrix}$ $\alpha_3=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$ $\alpha_4=\begin{pmatrix}3\\2\\-1\\-4\end{pmatrix}$ $\alpha_5=\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$ 生成子空间维数及基

$$A=\begin{pmatrix}1&4&0&3&-1\\1&5&1&2&0\\-1&-2&0&-1&0\\-1&-7&-1&-4&1\end{pmatrix}\to\begin{pmatrix}1&4&0&3&-1\\0&1&1&-1&1\\0&0&-2&4&-3\\0&0&0&0&0\end{pmatrix}=B$$

$r(A)=r(B)=3$. 且 1,2,3列线性无关

故子空间维数为3, $\alpha_1\alpha_2\alpha_3$是一组基

T11 $\alpha_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\alpha_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\alpha_3=\begin{pmatrix}1\\2\\2\end{pmatrix}$ $\beta=\begin{pmatrix}1\\3\\0\end{pmatrix}$

显然 $\alpha_1,\alpha_2,\alpha_3$ 独立

$$A=\begin{pmatrix}1&0&1&1\\0&1&2&3\\1&0&2&0\end{pmatrix}\Rightarrow\begin{pmatrix}1&0&0&2\\0&1&0&5\\0&0&1&-1\end{pmatrix}=B.$$

β在这 $\alpha_1\alpha_2\alpha_3$ 的坐标为 $(2,5,-1)$

T12

$$A=\begin{pmatrix}1&1&1&1&1\\1&1&-1&-1&2\\1&-1&1&-1&1\\1&-1&-1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&\frac{5}{4}\\0&1&0&0&\frac{1}{4}\\0&0&1&0&-\frac{1}{4}\\0&0&0&1&-\frac{1}{4}\end{pmatrix}=B$$

是一组基且

β的坐标为 $(\frac{5}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4})^T$

T13 (1) 过渡

$\alpha_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\alpha_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\alpha_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $\alpha_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, $\beta_1=\begin{pmatrix}2\\1\\-1\\1\end{pmatrix}$ $\beta_2=\begin{pmatrix}0\\3\\1\\0\end{pmatrix}$ $\beta_3=\begin{pmatrix}5\\3\\2\\1\end{pmatrix}$ $\beta_4=\begin{pmatrix}6\\6\\1\\3\end{pmatrix}$ $\xi=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

① 过渡矩阵 $\alpha \xleftarrow{\text{到}} \beta$

$$\begin{pmatrix}2&0&5&6\\1&3&3&6\\-1&1&2&1\\1&0&1&3\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}C \quad 得C=\begin{pmatrix}2&0&5&6\\1&3&3&6\\-1&1&2&1\\1&0&1&3\end{pmatrix}$$

$$\xi=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4=(\alpha_1\alpha_2\alpha_3\alpha_4)\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\to$$

在β的坐标 $$x'=C^{-1}x=\begin{pmatrix}\frac{4x_1+3x_2-9x_3-11x_4}{9}\\\frac{x_1+12x_2-9x_3-23x_4}{27}\\\frac{x_1-2x_4}{3}\\\frac{-7x_1-3x_2+9x_3+26x_4}{27}\end{pmatrix}$$

T14 $X' = CX \Rightarrow \eta = C\eta$

令 $\begin{cases} \dfrac{4x_1+3x_2-9x_3-11x_4}{9} = x_1 \\ \dfrac{x_1+12x_2-9x_3-23x_4}{27} = x_2 \\ \dfrac{x_1-2x_4}{3} = x_3 \\ \dfrac{-7x_1-3x_2+9x_3+26x_4}{27} = x_4 \end{cases}$ 得 $\begin{cases} x_1 = -1 \\ x_2 = -1 \\ x_3 = -1 \\ x_4 = 1 \end{cases}$ $\eta = C\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

T15

$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \to \alpha$ 在 $\beta_1, \beta_2, \beta_3$ 坐标为 $(1,0,0)^T$

则 $\begin{cases} \overset{\beta}{(1,0,0)} = C^{-1}\overset{\alpha}{(a_1, a_2, a_3)} \\ (\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3)C \end{cases}$ 得:

$(1,0,0)\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (a_1\, a_2\, a_3)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ 得 $\begin{cases} \beta_1 = \alpha \\ \beta_2 = \alpha_2 \\ \beta_3 = \alpha_3 \end{cases}$ 取

T16 $(1, x, x^2, x^3) \to (1, 1+x, 1+x+x^2, 1+x+x^2+x^3)$

(1) C 为 $1, 1+x, 1+x+x^2, 1+x+x^2+x^3$ 在 $1, x, x^2, x^3$ 下坐标.

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(2) $1+2x+3x^2+3x^3$ 在前基坐标为 $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$

在后基坐标为 $C^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$ 得 $\begin{pmatrix} -1 \\ -1 \\ 0 \\ 3 \end{pmatrix}$

(3) $p(x)$ 在后基坐标为 $(1,2,3,4)$

在前基坐标 $C\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 7 \\ 4 \end{pmatrix}$

T17

(M_1, M_2, M_3, M_4) 在 (E_1, E_2, E_3, E_4) 坐标 ($C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;)

M 在 (E_1, E_2, E_3, E_4) 坐标为 $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

在 (M_1, M_2, M_3, M_4) 坐标为 $C^{-1}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

T19

(1) $|\alpha_1| = \sqrt{1^2+1^2+1^2+2^2} = \sqrt{7}$ $\quad |\beta_1| = \sqrt{3^2+1^2+(-1)^2+0} = \sqrt{11}$

(2) $\langle \alpha_1, \beta_1 \rangle = \mathrm{arc}\dfrac{\alpha_1 \beta_1}{|\alpha_1||\beta_1|} = \arccos\dfrac{3}{\sqrt{77}}$

$\langle \alpha_2, \beta_2 \rangle = \arccos\dfrac{\alpha_2 \beta_2}{|\alpha_2||\beta_2|} = \arccos 0 = \dfrac{\pi}{2}$

(3) $|\alpha_1 - \beta_1| = \sqrt{(1-3)^2+(1-1)^2+(1+1)^2+(2-0)^2} = 2\sqrt{3}$

$|\alpha_2 - \beta_2| = \sqrt{(2-1)^2+(1-2)^2+(3+2)^2+(2-1)^2} = 2\sqrt{7}$

T20. 设 $(a_1\, a_2\, a_3\, a_4)$

$\begin{cases} a_1 + a_2 + a_3 - a_4 = 0 \\ a_1 - a_2 + a_3 - a_4 = 0 \\ 2a_1 + a_2 + 3a_3 + a_4 = 0 \\ a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 \end{cases}$ 得: $\begin{cases} a_1 = \pm\frac{4}{\sqrt{26}} \\ a_2 = 0 \\ a_3 = \mp\frac{3}{\sqrt{26}} \\ a_4 = \pm\frac{1}{\sqrt{26}} \end{cases}$ 得 $\pm\left(\frac{4}{\sqrt{26}}, 0, -\frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)^T$

T22. $(\beta, \alpha_i) = 0$. 证 β 与 $L(\alpha_1 \cdots \alpha_m)$ 都正交

有 $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + \cdots + k_m\alpha_m$

$(\beta, \alpha_i) = 0$

$(\beta, \alpha) = k_1(\beta\, \alpha_i) + \cdots + k_m(\beta\, \alpha_m) = 0$ 都正交

T23

(1) $\beta_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}\frac{2}{3}\\-\frac{4}{3}\\\frac{2}{3}\end{pmatrix}\to\begin{pmatrix}1\\-2\\1\end{pmatrix}$

单位化 $\beta_1\to\eta_1=\begin{pmatrix}\frac{1}{\sqrt3}\\\frac{1}{\sqrt3}\\\frac{1}{\sqrt3}\end{pmatrix}$ $\beta_2\to\eta_2=\begin{pmatrix}\frac{1}{\sqrt6}\\\frac{-2}{\sqrt6}\\\frac{1}{\sqrt6}\end{pmatrix}$

(2) $\beta_1=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ $\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\\0\end{pmatrix}$

$\beta_3=\alpha_3-\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=\begin{pmatrix}1\\0\\1\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}-\frac{\frac{1}{2}}{\frac{3}{2}}\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\\0\end{pmatrix}=\begin{pmatrix}2/3\\-2/3\\2/3\\1\end{pmatrix}$

得 $\varepsilon_1=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ $\varepsilon_2=\frac{1}{\sqrt6}\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}$ $\varepsilon_3=\frac{1}{\sqrt{21}}\begin{pmatrix}2\\-2\\2\\3\end{pmatrix}$

(3) 过程略

$\varepsilon_1=\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $\varepsilon_2=\frac{1}{\sqrt{14}}\begin{pmatrix}3\\0\\-1\\-2\end{pmatrix}$ $\varepsilon_3=\frac{1}{\sqrt{42}}\begin{pmatrix}1\\0\\-5\\4\end{pmatrix}$ 标准正交基

T25. (1) $S=\{\alpha\in\mathbb{R}^4\mid\alpha\perp\alpha_1,\alpha\perp\alpha_2\}$.

① $\alpha,\beta\in S$ 则 $\alpha\perp\alpha_1$, $\alpha\perp\alpha_2$

$\beta\perp\alpha_1$ $\beta\perp\alpha_2$.

有 $(\alpha+k\beta)\perp\alpha_1$, $(\alpha+k\beta)\perp\alpha_2$ 子空间 ✓

② $\alpha=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$ $\begin{cases}x_1+x_2+x_3+x_4=0\\x_1-2x_2=0\end{cases}$ $A=\begin{pmatrix}1&1&1&1\\1&-2&0&0\end{pmatrix}=\begin{pmatrix}1&0&\frac{2}{3}&\frac{2}{3}\\0&1&\frac{1}{3}&\frac{1}{3}\end{pmatrix}$

得 $\eta_1=\begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\1\\0\end{pmatrix}$ $\eta_2=\begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\0\\1\end{pmatrix}$ 基

正交+单位化 $\varepsilon_1=\frac{1}{\sqrt{14}}\begin{pmatrix}-2\\-1\\3\\0\end{pmatrix}$ $\varepsilon_2=\frac{1}{\sqrt{266}}\begin{pmatrix}-6\\-3\\-5\\14\end{pmatrix}$

(3) 任取η_3, η_4，之后$\varepsilon_1, \varepsilon_2, \eta_1, \eta_2$为一组基(在$R^4$) 之后 正交+单位化即可

T33 $(x, y) = x^T A y \quad A = C^T C$

$= x^T C^T C y =$

(1) ① $(\alpha, \beta) = (\beta, \alpha) \longrightarrow (x, y) = x^T A y \quad (y, x) = y^T A x$

② $(k\alpha, \beta) = k(\alpha, \beta) \longrightarrow (kx, y) = k x^T A y$

③ $(\alpha+\beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma) \longrightarrow (x+y, z) = (x+y)^T A z = x^T A z + y^T A z$

④ $(\alpha, \alpha) \geq 0$ 而 $(\alpha, \alpha) = 0$. 当且仅当$\alpha = 0 \rightarrow (x, x) = x^T A x \geq 0$

(2) $|(\alpha, \beta)| \leq |\alpha||\beta|$

$x^T A y$ 的长度 小于等于 x的长度 · y的长度

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i y_j\right| \leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j} \cdot \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} y_i y_j}$$

(3) $(e_1, e_2) = (1, 0, 0)\, C^T C \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

$(e_2, e_3) = (0, 1, 0)\, C^T C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

$(e_1, e_3) = (1, 0, 0)\, C^T C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

T35 $(p, q) = \int_{-1}^{1} p(x) q(x) dx$.

$P[x]$的基 $1, x, x^2, x^3$. 求标准正交基.

$a_1 = 1, a_2 = x, a_3 = x^2, a_3 = x^3$

取 $b_1 = a_1 = 1$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = x - \frac{\int_{-1}^{1} x dx}{2} 1 = x$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = x^2 - \frac{\int_{-1}^{1} x^2 dx}{2} - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} = x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$$

$$b_4 = \cdots = x^3 - \frac{3}{5}x = \frac{1}{5}(5x^3 - 3x)$$

$$\begin{cases} C_1 = \frac{b_1}{\|b_1\|} = \frac{\sqrt{2}}{2} \\ C_2 = \frac{b_2}{\|b_2\|} = \frac{\sqrt{6}}{2} x \end{cases}$$

$$\begin{cases} C_3 = \frac{b_3}{\|b_3\|} = \frac{\sqrt{10}}{4}(3x^2 - 1) \\ C_4 = \frac{\sqrt{15}}{4}(5x^3 - 3x) \end{cases}$$