Chapter 5

T1(1) $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ $|\lambda E-A|=0 \Rightarrow \begin{pmatrix}\lambda-1&-1\\1&\lambda-1\end{pmatrix}=(\lambda-1)^2-1=0$

$\lambda_1=2$ $\Rightarrow$ 基解系 $\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\to \eta_1=\begin{pmatrix}1\\1\end{pmatrix}$

$\lambda_2=0$ $\begin{pmatrix}1&1\\0&0\end{pmatrix}\to \eta_2=\begin{pmatrix}-1\\1\end{pmatrix}$

(3) $|\lambda E-A|=\begin{vmatrix}\lambda-2&+1&-2\\-5&\lambda+3&-3\\1&0&\lambda+2\end{vmatrix}=(\lambda+1)^3=0$

$\lambda_1=\lambda_2=\lambda_3=-1$

解 $(-E-A)X=0$ 得 基解系 $\begin{pmatrix}-1\\-1\\1\end{pmatrix}\to$ 特征向量 $c\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

(5) $|\lambda E-A|=\begin{vmatrix}\lambda-1&-3&-1&-2\\0&\lambda+1&-1&-3\\0&0&\lambda-2&-5\\0&0&0&\lambda-2\end{vmatrix}=0$ 得 $(\lambda-1)(\lambda+1)(\lambda-2)^2$

$\lambda_1=1,\ \lambda_2=-1,\ \lambda_3=\lambda_4=2$

$(\lambda_1=1)\to(A-\lambda_1 E)X=0$ $\begin{bmatrix}0&3&1&2\\0&-2&1&3\\0&0&1&5\\0&0&0&1\end{bmatrix}X=0\Rightarrow\eta_1=(1,0,0,0)^T$

$(\lambda_2)=-1\to\eta_2=(-\frac{3}{2},1,0,0)^T$

$\lambda_3=\lambda_4=2\to\eta_3=(2,\frac{1}{3},1,0)^T$

$\to\eta_1,\eta_2,\eta_3.$

T2: $\because A\alpha=\lambda\alpha$

$\therefore A^2x=AAx=A\lambda x=\lambda Ax=\lambda^2x$

$\therefore \lambda x=\lambda^2x$

$\lambda(\lambda-1)=0$ $\lambda_1=1$ 或 $\lambda=0$

T3 $Ax=\lambda x \Rightarrow A\lambda x = A^2x = \lambda^2 x$

$\Rightarrow A^2\lambda x = A^3 x = \lambda^3 x$

$A^k x = A^{k-1}\lambda x = \lambda^k x = 0$

$\lambda = 0.$

T5

设 $(P^{-1}AP)^T B = \lambda B$

$\rightarrow P^T A^T (P^{-1})^T B = \lambda B$

$A^T (P^{-1})^T B = (P^T)^{-1} \lambda B$

$A (P^T)^{-1} B = \lambda (P^T)^{-1} B$

$(P^T)^{-1} B = \alpha$

$B = P^T \alpha$,

T6. 设存在λ使得

$A\alpha = \lambda\alpha$

则 $A\alpha = \lambda(a\alpha_1 + b\alpha_2)$

$= \lambda_1 a\alpha_1 + \lambda_2 b\alpha_2 = \lambda\alpha = a\lambda\alpha_1 + b\lambda\alpha_2$

则 $\Rightarrow$

$a(\lambda-\lambda_1)\alpha_1 + b(\lambda-\lambda_2)\alpha_2 = 0$

$\because a\neq 0, b\neq 0, \alpha_1\alpha_2$不为0且线性无关

有$\lambda = \lambda_1$; $\lambda = \lambda_2$

与$\lambda_1 \neq \lambda_2$ 矛盾

$\therefore \alpha$不是特征向量

T7 $f_A(\lambda)=(\lambda+2)^4(\lambda-1)$

T8 $A\sim B \Rightarrow$ ∃可逆阵 $P^{-1}AP=B$

$\therefore P^TA^T(P^{-1})^T=B^T$

~~即~~ $P^TA^T(P^T)^{-1}=B^T \Rightarrow [(P^T)^{-1}]^{-1}A^T[(P^T)^{-1}]$

$(P^T)^{-1}$也是可逆阵

$\Rightarrow A^T\sim B^T$ ✓

T9 $B_1+B_2=P^{-1}A_1P+P^{-1}A_2P=P^{-1}(A_1+A_2)P$ ~~$\Rightarrow B_1+B_2\sim A_1+A_2$~~

$B_1B_2=P^{-1}A_1P\cdot P^{-1}A_2P=P^{-1}(A_1B_2)P$ ~~$\Rightarrow B_1B_2\sim A_1A_2$~~

$\Rightarrow A_1+A_2\sim B_1+B_2$; $A_1A_2\sim B_1B_2$

T10

$BA=A^{-1}ABA$

当$P=A$时有 $BA=P^{-1}ABA \rightarrow AB\sim BA$

T12 $A^2=A$, A特征值0或1

即$|\lambda E-A|=0$ 只有 $\lambda=0$或$\lambda=1$成立

$\therefore |3E-A|\neq 0$ 则 $(3E-A)$可逆

T13 $|A-A^2|=0$.

$|A(E-A)|=0$

$|A||E-A|=0$

即$|0E-A||1E-A|=0$ λ~~必为~~0,1中一个

T18

$$A=\begin{bmatrix}0 & 2 & -1\\ -2 & 5 & -2\\ -4 & 8 & -3\end{bmatrix}$$

$$|\lambda E-A|=\begin{vmatrix}\lambda & -2 & 1\\ 2 & \lambda-5 & 2\\ 4 & -8 & \lambda+3\end{vmatrix}=\lambda(\lambda-1)^2 \quad \lambda_1=0 \quad \lambda_2=\lambda_3=1.$$

$$\lambda_1=0 \Rightarrow AX=0 \Rightarrow \begin{pmatrix}0 & 2 & -1\\ -2 & 5 & -2\\ -4 & 8 & -3\end{pmatrix} \Rightarrow \begin{pmatrix}1 & -\frac{5}{2} & 1\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{pmatrix} \lambda_1\text{对应}\left(\tfrac{1}{4}\ \tfrac{1}{2}\ 1\right)^T$$

$$\lambda_2=\lambda_3=1 \Rightarrow \begin{pmatrix}1 & -2 & 1\\ 2 & -4 & 2\\ 4 & -8 & 4\end{pmatrix}X=0 \Rightarrow \eta_2=(2,1,0)^T \quad \eta_3=(-1,0,1)^T$$

$$Q=(4\eta_1,\eta_2,\eta_3)=\begin{pmatrix}1 & 2 & -1\\ 2 & 1 & 0\\ 4 & 0 & 1\end{pmatrix}$$

$$Q^{-1}AQ=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$(Q^{-1}AQ)^n=Q^{-1}A^nQ=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\text{得 } A^n=A=\begin{pmatrix}0 & 2 & -1\\ -2 & 5 & -2\\ -4 & 8 & -3\end{pmatrix}$$

T20 (1) $|\lambda E-A|=(\lambda+2)[(\lambda-x)(\lambda-1)-2]$ 由于 $A\sim B$，特征值相同

所以 $y=-2 \quad x=0$

(2) $\lambda_1=-1, \lambda_2=2, \lambda_3=-2$.

$$(\lambda_1E-A)=\begin{pmatrix}1 & 0 & 0\\ -2 & -1 & -2\\ -3 & -1 & -2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix} \rightarrow \eta_1=\begin{pmatrix}0\\ 2\\ -1\end{pmatrix}^{\text{T}}$$

$$(\lambda_2 E-A)=\begin{pmatrix}4&0&0\\-2&2&-2\\-3&-1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&-1\\0&0&0\end{pmatrix}\quad \eta_2=\begin{pmatrix}0\\1\\+1\end{pmatrix}$$

$$(\lambda_3 E-A)=\begin{pmatrix}0&0&0\\-2&-2&-2\\-3&-1&-3\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}\quad \eta_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$P=\begin{pmatrix}0&0&-1\\2&1&0\\-1&1&1\end{pmatrix}$$

T_{21}

$$Q=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\quad Q^{-1}AQ=\begin{pmatrix}\lambda_1&&\\&\lambda_2&\\&&\lambda_3\end{pmatrix}$$

$$(Q^{-1}AQ)^n=Q^{-1}A^nQ'=\begin{pmatrix}\lambda_1^n&&\\&\lambda_2^n&\\&&\lambda_3^n\end{pmatrix}$$

$$A^n=Q\begin{pmatrix}\lambda_1^n&&\\&\lambda_2^n&\\&&\lambda_3^n\end{pmatrix}Q^{-1}$$

$$=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}\lambda_1^n&&\\&\lambda_2^n&\\&&\lambda_3^n\end{pmatrix}Q^{-1}=\begin{pmatrix}\lambda_1^n&0&0\\\lambda_1^n-\lambda_2^n&\lambda_2^n&0\\\lambda_1^n-\lambda_2^n&(\lambda_2^n-\lambda_3^n)&\lambda_3^n\end{pmatrix}$$

$$(A^n)^T=\begin{pmatrix}\lambda_1^n&\lambda_1^n-\lambda_2^n&\lambda_1^n-\lambda_2^n\\0&\lambda_2^n&\lambda_2^n-\lambda_3^n\\0&0&\lambda_3^n\end{pmatrix}//$$

T_{22} $|\lambda E-A|=(\lambda-1)^2(\lambda-2)^2$ $\lambda_1=\lambda_2=1$; $\lambda_3=\lambda_4=2$.

$$(A-\lambda_1 E)=\begin{pmatrix}2&3&c&1\\2&b&1&0\\a&0&0&0\\0&0&0&0\end{pmatrix}$$ 若要A～对角阵

几何重数 =代数重数 $\therefore a=0$

$$(A-\lambda_3 E)=\begin{pmatrix}2&3&c&0\\2&b&0&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\quad c=0$$ b可取任意值

得: $\eta_1 = (-2, -2, 1, 0)^T$

$\eta_2 = (-3, -6, 0, 1)^T$ $\quad Q = (\eta_1 \eta_2 \eta_3 \eta_4)$

$\eta_3 = (0, 0, 1, 0)^T$ $\quad Q^{-1}AQ = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}$ //

$\eta_4 = (0, 0, 0, 1)^T$

T23 $\quad Q = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \quad Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$A = Q \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix} Q^{-1} = \frac{1}{9} \begin{bmatrix} -3 & 6 & 0 \\ 6 & 0 & 6 \\ 0 & 6 & 3 \end{bmatrix}$ //

T25

(1) $\begin{pmatrix} a & c \\ b & 2c \end{pmatrix}$

$\therefore \begin{cases} ac + 2bc = 0 \\ 2ac - bc = \pm 1 \\ a^2 + b^2 = 1 \\ c^2 + 4c^2 = 1 \end{cases} \Rightarrow \begin{cases} a = \pm \frac{2}{5}\sqrt{5} \\ b = \mp \frac{\sqrt{5}}{5} \\ c = \frac{\sqrt{5}}{5} \end{cases}$ 或 $\begin{cases} a = \mp \frac{2}{5}\sqrt{5} \\ b = \pm \frac{\sqrt{5}}{5} \\ c = -\frac{\sqrt{5}}{5} \end{cases}$

(2) $\begin{pmatrix} 0 & 1 & 0 \\ a & 0 & c \\ b & 0 & 1/2 \end{pmatrix} \quad \begin{cases} ac + \frac{1}{2}b = 0 \\ \frac{1}{2}a - bc = \pm 1 \\ a^2 + b^2 = 1 \\ c^2 + \frac{1}{4} = 1 \end{cases}$ 得

$\begin{cases} a = \pm \frac{1}{2} \\ b = \pm \frac{\sqrt{3}}{2} \\ c = -\frac{\sqrt{3}}{2} \end{cases}$ 或 $\begin{cases} a = \pm \frac{1}{2} \\ b = \mp \frac{\sqrt{3}}{2} \\ c = +\frac{\sqrt{3}}{2} \end{cases}$

T27 $A^2+6A+8E=0$

$\Rightarrow A^2+6A+9E=E$

$(A+3E)^2=E$

$(A+3E)(A+3E)=E$

$\Rightarrow (A+3E)^{-1}=A+3E$

且 $A=A^T$

$(A+3E)^{-1}=(A+3E)^T$

$\Rightarrow A+3E$ 正交阵

T29 $|E-A|=|A^TA-A|$

$=|A^T-E||A|$

$=|A-E||A|$

$=|A-E|$

$=(-1)^n|E-A|$

$n=2k+1$

即 $|E-A|=-|E-A|$

$|E-A|=0 \Rightarrow$ 不可逆

T31 (1)

$A=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix}$ $|\lambda E-A|=(\lambda-1)(\lambda-2)(\lambda-5)$

$\lambda_1=1$; $\lambda_2=2$; $\lambda_3=5$

$\lambda_1=1 \Rightarrow \eta_1=(0,-1,1)^T \Rightarrow \varepsilon_1=\frac{\sqrt{2}}{2}(0,-1,1)^T$

$\lambda_2=1 \Rightarrow \eta_2=(1,0,0)^T$

$\lambda_3=5 \Rightarrow \eta_3=(0\ 1\ 1)^T \Rightarrow \varepsilon_3=\frac{\sqrt{2}}{2}(0,1,1)^T$

$\Rightarrow Q=\begin{pmatrix} 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$

(3) $A=\begin{pmatrix} 3 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 1 \end{pmatrix} \Rightarrow |\lambda E-A|=(\lambda+1)(\lambda-2)(\lambda-5)$

$\lambda_1=-1$, $\lambda_2=2$; $\lambda_3=5$

$\eta_1=(1,2,2)^T$

$\varepsilon_1=\frac{1}{3}(1,2,2)^T$

$\eta_2=(-2,-1,2)^T$

$\varepsilon_2=\frac{1}{3}(-2,-1,2)^T$

$\eta_3=(2,-2,1)^T$

$\varepsilon_3=\frac{1}{3}(2,-2,1)^T$

$Q=\frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ //

T32

$$\begin{cases} x_1-2x_2+0x_3=0 \\ x_1+0x_2-x_3=0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \quad \eta_3=(2\ 1\ 2)^T$$

$$Q=\begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \quad Q^{-1}AQ=\begin{pmatrix} -1 & & \\ & -1 & \\ & & 8 \end{pmatrix}$$

$$A=Q\begin{pmatrix} -1 & & \\ & -1 & \\ & & 8 \end{pmatrix}Q^{-1}=\begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} diag(-1,-1,8)\begin{pmatrix} \frac{1}{9} & -\frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{2}{9} & -\frac{5}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

$$=\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

T33

A.B 实对称阵

$P^{-1}AP=diag(\lambda_1,\lambda_2,\lambda_3)$

$Q^{-1}BQ=diag(\lambda_4,\lambda_5,\lambda_6)$

A.B 有相同特征多项式 → 特征值

$P^{-1}AP=Q^{-1}BQ$

$QP^{-1}APQ^{-1}=B \rightarrow A\sim B$

T34 AB有n个不相等特征值

$\longrightarrow AB\neq 0 \rightarrow A\neq 0,\ B\neq 0$

设 $AB\alpha=\lambda\alpha$

若 $\lambda\neq 0$ 则

$AB\alpha=\lambda\alpha \Rightarrow BAB\alpha=\lambda B\alpha$

令 $B\alpha=\beta$

$BA\beta=\lambda\beta$

由于 $\lambda\neq 0$, $\alpha\neq 0$, $AB\alpha\neq 0$ 则 $B\alpha\neq 0$

$BA\beta=\lambda\beta$

若 $\lambda=0$ 则

$AB\alpha=\lambda\alpha=0$

$\alpha\neq 0$

$|AB|=|BA|=0$.

$BAX=\lambda X=0$

有非零解 β.

$\therefore \lambda$ 为特征值

综合$\lambda=0$, $\lambda\neq 0$都有

AB的特征值为BA的特征值

BA的特征值为AB的特征值

$\Rightarrow$ AB与BA有相同相似性

两者相似于同一个对角阵

T35

(1) $A=\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}$

$$\alpha^T\beta=(a_ib_j)=\begin{pmatrix} a_1b_1 & \cdots & a_1b_n \\ \vdots & & \vdots \\ a_nb_1 & \cdots & a_nb_n \end{pmatrix}=A$$

(2) $A^2=\alpha\beta^T\alpha\beta^T=\alpha(\beta^T\alpha)\beta^T=0$

(3) $A^*\Rightarrow$

对A进行变换 $A=\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ \frac{a_2}{a_2}b_1 & \frac{a_2}{a_2}b_2 & \cdots & \frac{a_i}{a_2}b_n \\ \vdots & \vdots & & \\ \frac{a_n}{a_n}b_1 & \frac{a_n}{a_n}b_2 & \cdots & \frac{a_n}{a_n}b_n \end{pmatrix}$ 设$a_1\neq 0$

$\begin{cases} \text{若}a_i=0\text{该行为}0 \\ \text{若}a_i\neq 0\text{该行进行行变换后为}0 \end{cases}$ $r(A)=1$

$A\Rightarrow\begin{pmatrix} a_1b_1 & \cdots & a_1b_n \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}$ 则超过1阶子式都为0

所以$A^*=0$.

(4) $AX=\lambda X$

$$AX=\begin{pmatrix} a_1\sum b_ix_i \\ a_2\sum b_ix_i \\ \vdots \\ a_n\sum b_ix_i \end{pmatrix}$$

$AX=\alpha\beta^TX$

$A^2X=\alpha\beta^T\alpha\beta^TX=\lambda^2X\Rightarrow\lambda=0$