# **Primal and Dual Problem**

## Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets

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## References

[1] Ashkan Jasour, Necdet Serhat Aybat, and Constantino Lagoa "Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets", *SIAM Journal on Optimization*, 25(3), 1411 - 1440, 2015.

[2] Ashkan Jasour, Necdet Serhat Aybat, and Constantino Lagoa "Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets", arXiv:1402.6382v3, 2014.

[3] Ashkan Jasour, Constantino Lagoa, "Semidefinite Relaxations of Chance Constrained Algebraic Problems", *51st IEEE Conference on Decision and Control*, Maui, Hawaii, 2012.

## 1 Chance Optimization Problem

Let $q \in \mathbb{R}$ be the uncertain parameter with a uniform distribution on [-1,1]. Then, consider the following chance optimization problem:

$$\sup_{x \in \mathbb{R}} \left\{ \text{Probobality}_{\mu_q} \left( \{ q \in \mathbb{R} : \ \mathcal{P}(x,q) \geq 0 \ \} \right) \right\}, \tag{1}$$

where

$$\mathcal{P}\,(x,q) = 0.5\ q \left( q^2 + (x-0.5)^2 \right) - \left( q^4 + q^2 (x-0.5)^2 + (x-0.5)^4 \right) \tag{2}$$

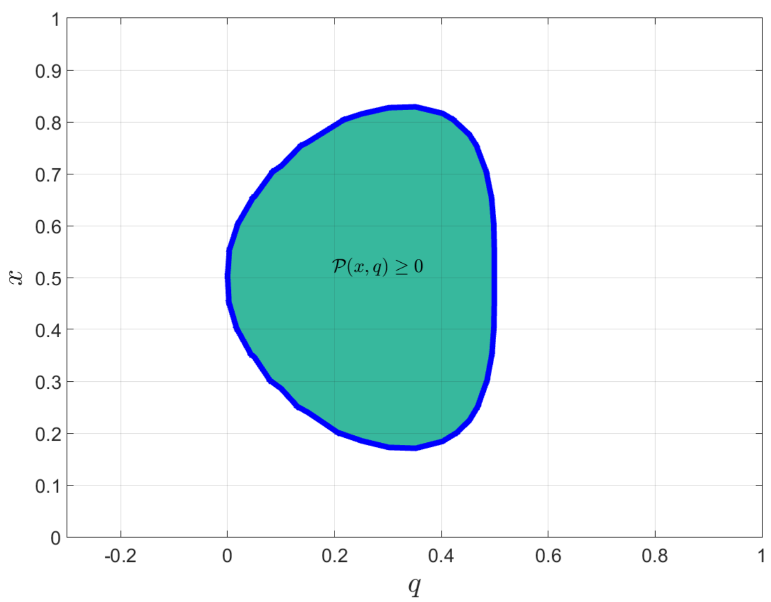


Figure 1: Semialgebraic set $\mathcal{K} = (\{(x,q) \in \mathbb{R} \times \mathbb{R} : \ \mathcal{P}(x,q) \geq 0 \ \})$

In problem 1 we aim at finding decision parameter $x$ such that the probability of given semialgebraic set becomes maximum. In other words, decision parameter $x$ should be chosen such that the probability of the random point $(x,q)$ belonging to the set $\mathcal{K} = \{(x,q) \in \mathbb{R} \times \mathbb{R} : \ \mathcal{P}(x,q) \geq 0\}$ becomes maximum.

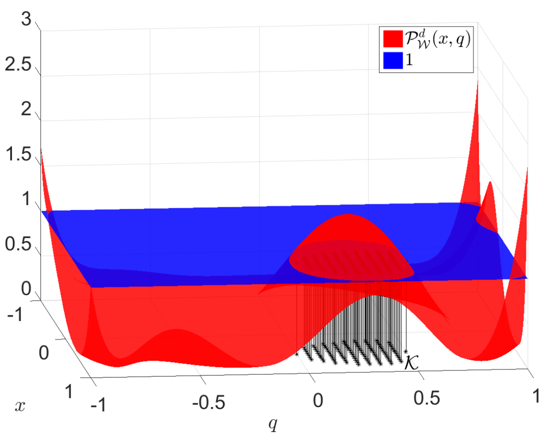

Figure 2: Obtained polynomial $\mathcal{P}^d_{\mathcal{W}}(x,q)$

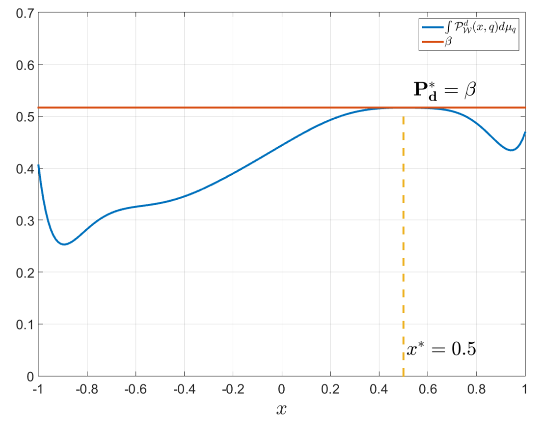

Figure 3: $\int \mathcal{P}^d_{\mathcal{W}}(x,q)d\mu_q$

## 2 Semidefinite Programming

To obtain an approximate solution of chance optimization problem in 1, we solve the following SDP ([1], Theorem 3.3):

$$
\begin{aligned}
\mathbf{P}_d := & \sup_{\mathbf{y}\in\mathbb{R}^{S_{2,2d}},\ \mathbf{y_x}\in\mathbb{R}^{S_{1,2d}}} (\mathbf{y})_\mathbf{0}, && (3)\\
\text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-r_1}(\mathbf{y};\mathcal{P}) \succcurlyeq 0,, && (4)\\
& M_d(\mathbf{y_x}) \succcurlyeq 0,\ ||\mathbf{y_x}||_\infty \leq 1,\ (\mathbf{y_x})_\mathbf{0} = 1, && (5)\\
& M_d(A_d\mathbf{y_x} - \mathbf{y}) \succcurlyeq 0, && (6)
\end{aligned}
$$

where $(\mathbf{y})_\mathbf{0}$ and $(\mathbf{y_x})_\mathbf{0}$ are first elements of vectors $\mathbf{y}$, $\mathbf{y_x}$, respectively. Given $n$ and $d$ in $\mathbb{N}$, we define $S_{n,d} := \binom{d+n}{n}$ and $\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n : ||\alpha||_1 \leq d\}$. $A_d : \mathbb{R}^{S_{1,2d}} \to \mathbb{R}^{S_{2,2d}}$ is a linear map depending only on moments of given probability measure $\mu_q$. Indeed, let $\mathbf{y_q} := \{y_{q_\beta}\}_{\beta\in\mathbb{N}^1_{2d}}$ be the truncated moment sequence of $\mu_q$. Then for any given $\mathbf{y_x} = \{y_{x_\alpha}\}_{\alpha\in\mathbb{N}^1_{2d}}$, $A_d\mathbf{y_x} = \bar{\mathbf{y}}$ such that $(\bar{\mathbf{y}})_\theta = (\mathbf{y_q})_\beta(\mathbf{y_x})_\alpha$ for all $\theta = (\beta,\alpha) \in \mathbb{N}^2_{2d}$. The $k$-th moment of uniform distribution U[a,b] is $(\mathbf{y_q})_k = \frac{b^{k+1}-a^{k+1}}{(b-a)(k+1)}$. $M_d(\mathbf{y})$, $M_d(\mathbf{y_x})$, and $M_d(A_d\mathbf{y_x} - \mathbf{y})$ are $d$-th order moment matrices constructed by vectors $\mathbf{y}$, $\mathbf{y_x}$, and $(A_d\mathbf{y_x} - \mathbf{y})$. $M_{d-r}(\mathbf{y};\mathcal{P})$ is localization matric constructed by vector $\mathbf{y}$ and polynomial $\mathcal{P}$, ([1], Section 2.1).

Problem in (3) is semidefinite program (SDP) where a linear function is minimized subject to linear matrix inequality (LMI) constraints and can be solved by Matlab-based optimization toolboxes like GloptiPoly, CVX, and Yalmip where use interior-point solvers like SeDuMi and Mosek. We solve SDP in (3) for $d$=5 (it needs information of moments up to order $2d$) using GloptiPoly and approximate the solution to the chance optimization in (1) with $y^*_x = 0.5$ and estimate the optimal probability with $y^*_{00} = 0.51$, ([1], See section 3.3 where we make a case for this approximation under some simplifying assumptions).

## 3 Dual on Polynomials

Consider following finite SDP on polynomials:

$$
\begin{aligned}
\mathbf{P^*_d} := & \min_{\beta\in\mathbb{R},\mathcal{P}^d_{\mathcal{W}}\in\mathbb{R}_d[x,q]} \beta, && (7)\\
\text{s.t.} \quad & \mathcal{P}^d_{\mathcal{W}}(x,q) - 1 \geq 0 \quad \text{on} \quad \mathcal{K} = \{(x,q) : \mathcal{P}(x,q) \geq 0\} && (8)\\
& \beta - \int_\chi \mathcal{P}^d_{\mathcal{W}}(x,q)d\mu_q \geq 0 \quad \text{on} \quad \chi = [-1,1], && (9)\\
& \mathcal{P}^d_{\mathcal{W}}(x,a) \geq 0,\ \beta \geq 0 && (10)
\end{aligned}
$$

where, $\mathcal{P}^d_{\mathcal{W}}(x,a)$ is finite order polynomial of order at most $d$. The following theorem establish the equivalence of problems on moments and polynomials.

**Theorem 1** *There is no duality gap between the finite SDP on moments and finite SDP on polynomials in the sense that the optimal values are the same.*

We solve SDP in (7) for $d$=10 using Yalmip and approximate the optimal probability with $\beta = 0.51$ and optimal solution with $x^* = 0.5$ for witch $\int \mathcal{P}^d_{\mathcal{W}}(x,q)d\mu_q$ is maximized. The obtained polynomials are as Fig. 2 and 3.