Illustrative Example

Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets

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References

[1] Ashkan Jasour, Necdet Serhat Aybat, and Constantino Lagoa "Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets", *SIAM Journal on Optimization*, 25(3), 1411 - 1440, 2015.

[2] Ashkan Jasour, Necdet Serhat Aybat, and Constantino Lagoa "Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets", arXiv:1402.6382v3, 2014.

[3] Ashkan Jasour, Constantino Lagoa, "Semidefinite Relaxations of Chance Constrained Algebraic Problems", *51st IEEE Conference on Decision and Control*, Maui, Hawaii, 2012.

1 Chance Optimization Problem

Let $q \in \mathbb{R}$ be the uncertain parameter with a uniform distribution on [-1,1]. Then, consider the following chance optimization problem:

$$\sup_{x\in\mathbb{R}} \left\{ \text{Probobality}_{\mu_q}\left(\{q \in \mathbb{R}: \ \mathcal{P}_1(x,q) \geq 0, \mathcal{P}_2(x,q) \geq 0 \ \}\right) \right\}, \tag{1}$$

where

$$\mathcal{P}_1\left(x,q\right) = 0.5\ q\left(q^2 + (x-0.5)^2\right) - \left(q^4 + q^2(x-0.5)^2 + (x-0.5)^4\right) \tag{2}$$

$$\mathcal{P}_2\left(x,q\right) = 0.3^2 - (x-0.5)^2 - (q-0.4)^2 \tag{3}$$

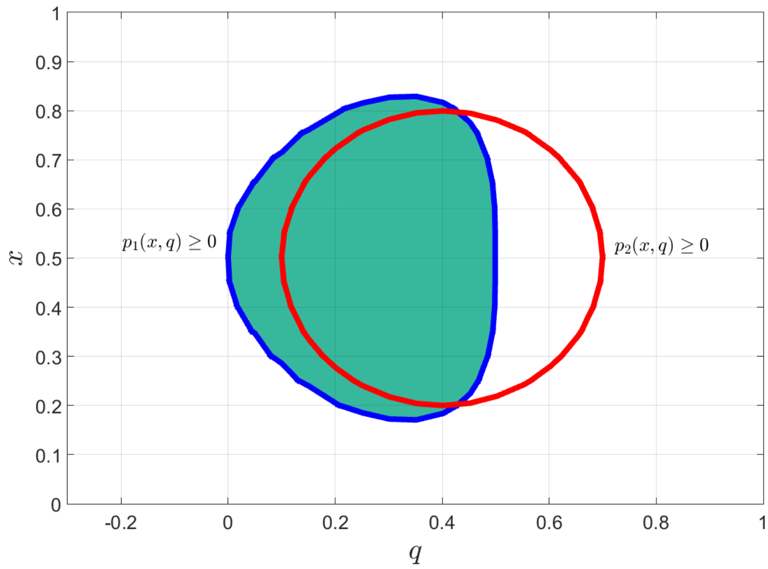

Figure 1: Semialgebraic set $(\{(x,q) \in \mathbb{R}\times\mathbb{R}: \ \mathcal{P}_1(x,q) \geq 0, \mathcal{P}_2(x,q) \geq 0 \ \})$

In problem 1 we aim at finding decision parameter x such that the probability of given semialgebraic set becomes maximum. In other words, decision parameter x should be chosen such that the probability of the random point (x,q) belonging to the set $\{(x,q) \in \mathbb{R}\times\mathbb{R}: \ \mathcal{P}_1(x,q) \geq 0, \mathcal{P}_2(x,q) \geq 0 \ \}$ becomes maximum.

2 Semidefinite Programming

To obtain an approximate solution of chance optimization problem in 1, we solve the following SDP ([1], Theorem 3.3):

$$
\begin{aligned}
\mathbf{P}_d := & \sup_{\mathbf{y}\in\mathbb{R}^{S_{2,2d}},\ \mathbf{y_x}\in\mathbb{R}^{S_{1,2d}}} (\mathbf{y})_{\mathbf{0}}, && (4)\\
\text{s.t.}\quad & M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-r_1}(\mathbf{y};\mathcal{P}_1) \succcurlyeq 0,\ M_{d-r_2}(\mathbf{y};\mathcal{P}_2) \succcurlyeq 0 && (5)\\
& M_d(\mathbf{y_x}) \succcurlyeq 0,\ ||\mathbf{y_x}||_\infty \le 1,\ (\mathbf{y_x})_{\mathbf{0}} = 1, && (6)\\
& M_d(A_d\mathbf{y_x} - \mathbf{y}) \succcurlyeq 0, && (7)
\end{aligned}
$$

where $(\mathbf{y})_\mathbf{0}$ and $(\mathbf{y_x})_\mathbf{0}$ are first elements of vectors $\mathbf{y}$, $\mathbf{y_x}$, respectively. $\delta_1 = 4$ and $\delta_2 = 2$ are the degrees of polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$, $r_1 := \lceil \frac{\delta_1}{2} \rceil = 2$ and $r_2 := \lceil \frac{\delta_2}{2} \rceil = 1$. Given n and d in $\mathbb{N}$, we define $S_{n,d} := \binom{d+n}{n}$ and $\mathbb{N}^{\mathrm{n}}_d := \{\alpha \in \mathbb{N}^n : ||\alpha||_1 \le d\}$. $A_d : \mathbb{R}^{S_{1,2d}} \to \mathbb{R}^{S_{2,2d}}$ is a linear map depending only on moments of given probability measure μ_q. Indeed, let $\mathbf{y_q} := \{y_{q_\beta}\}_{\beta\in\mathbb{N}^1_{2d}}$ be the truncated moment sequence of μ_q. Then for any given $\mathbf{y_x} = \{y_{x_\alpha}\}_{\alpha\in\mathbb{N}^1_{2d}}$, $A_d\mathbf{y_x} = \bar{\mathbf{y}}$ such that $(\bar{\mathbf{y}})_\theta = (\mathbf{y_q})_\beta(\mathbf{y_x})_\alpha$ for all $\theta = (\beta,\alpha) \in \mathbb{N}^2_{2d}$. The k-th moment of uniform distribution U[a,b] is $(\mathbf{y_q})_k = \frac{b^{k+1}-a^{k+1}}{(b-a)(k+1)}$. $M_d(\mathbf{y})$, $M_d(\mathbf{y_x})$, and $M_d(A_d\mathbf{y_x} - \mathbf{y})$ are d-th order moment matrices constructed by vectors $\mathbf{y}$, $\mathbf{y_x}$, and $(A_d\mathbf{y_x} - \mathbf{y})$. $M_{d-r_1}(\mathbf{y};\mathcal{P}_1)$ and $M_{d-r_2}(\mathbf{y};\mathcal{P}_2)$ are localization matrices constructed by vector $\mathbf{y}$ and polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$, ([1], Section 2.1).

For example for $d = 2$, the vectors and matrices of SDP in (4) are as follow.

◊ The vectors $\mathbf{y} \in \mathbb{R}^{S_{2,4}}$, $\mathbf{y_x} \in \mathbb{R}^{S_{1,4}}$ and $\mathbf{y_q}$ are

$$\mathbf{y}_\mathbf{q}^T = \left[1,\ 0,\ \tfrac{1}{3},\ 0,\ \tfrac{1}{5}\right], \quad \mathbf{y}_\mathbf{x}^T = [\,y_{x_0},\ y_{x_1},\ y_{x_2},\ y_{x_3},\ y_{x_4}],$$

$$\mathbf{y}^T = [y_{00} \mid y_{10},\ y_{01} \mid y_{20},\ y_{11},\ y_{02} \mid y_{30},\ y_{21},\ y_{12},\ y_{03} \mid y_{40},\ y_{31},\ y_{22},\ y_{13},\ y_{04}].$$

◊ Given the vectors $\mathbf{y_q}$, the vector $A_2\mathbf{y_x}$ has the form

$$
\begin{aligned}
(A_2\mathbf{y_x})^T &= [\,y_{x_0} \mid y_{x_1}y_{q_0},\ y_{x_0}y_{q_1} \mid y_{x_2}y_{q_0},\ y_{x_1}y_{q_1},\ y_{x_0}y_{q_2} \mid y_{x_3}y_{q_0},\ y_{x_2}y_{q_1},\ y_{x_1}y_{q_2},\ y_{x_0}y_{q_3} \mid y_{x_4}y_{q_0},\ y_{x_3}y_{q_1},\ y_{x_2}y_{q_2},\ y_{x_1}y_{q_3},\ y_{x_0}y_{q_4}],\\
&= [\,y_{x_0} \mid y_{x_1},\ 0 \mid y_{x_2},\ 0,\ \tfrac{1}{3}\ y_{x_0} \mid y_{x_3},\ 0,\ \tfrac{1}{3}y_{x_1},\ 0 \mid y_{x_4},\ 0,\ \tfrac{1}{3}y_{x_2},\ 0,\ \tfrac{1}{5}\ y_{x_0}]
\end{aligned}
$$

◊ Moment matrices:

$$
M_2(\mathbf{y}) = \left[\begin{array}{c|cc|ccc}
y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02}\\
\hline
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12}\\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03}\\
\hline
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22}\\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13}\\
y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{array}\right] \tag{8}
$$

$$
M_2(\mathbf{y}_x) = \begin{bmatrix}
y_{x_0} & y_{x_1} & y_{x_2}\\
y_{x_1} & y_{x_2} & y_{x_3}\\
y_{x_2} & y_{x_3} & y_{x_4}
\end{bmatrix} \tag{9}
$$

$$
M_2(A_2\mathbf{y}_x - \mathbf{y}) = \left[\begin{array}{c|cc|ccc}
y_{x_0} & y_{x_1} & 0 & y_{x_2} & 0 & \frac{1}{3}y_{x_0}\\
\hline
y_{x_1} & y_{x_2} & 0 & y_{x_3} & 0 & \frac{1}{3}y_{x_1}\\
0 & 0 & \frac{1}{3}y_{x_0} & 0 & \frac{1}{3}y_{x_1} & 0\\
\hline
y_{x_2} & y_{x_3} & 0 & y_{x_4} & 0 & \frac{1}{3}y_{x_2}\\
0 & 0 & \frac{1}{3}y_{x_1} & 0 & \frac{1}{3}y_{x_2} & 0\\
\frac{1}{3}y_{x_0} & \frac{1}{3}y_{x_1} & 0 & \frac{1}{3}y_{x_2} & 0 & \frac{1}{5}y_{x_0}
\end{array}\right] - \left[\begin{array}{c|cc|ccc}
y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02}\\
\hline
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12}\\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03}\\
\hline
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22}\\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13}\\
y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{array}\right] \tag{10}
$$

◊ Localization matrices:

$$
M_0(\mathbf{y};\mathcal{P}_1) = \left[\ -y_{04} + \tfrac{1}{2}y_{03} - y_{22} + y_{12} - \tfrac{1}{4}y_{02} + \tfrac{1}{2}y_{21} - \tfrac{1}{2}y_{11} + \tfrac{1}{8}y_{01} - y_{40} + 2y_{30} - \tfrac{3}{2}3y_{20} + \tfrac{1}{2}y_{10} - \tfrac{1}{16}y_{00}\ \right] \tag{11}
$$

$$
M_2(\mathbf{y};\mathcal{P}_2) = \left[\begin{array}{c|c|c}
-y_{02} + \frac{4}{5}y_{01} - y_{20} + y_{10} - \frac{8}{25}y_{00} & -y_{12} + \frac{4}{5}y_{11} - y_{30} + y_{20} - \frac{8}{25}y_{10} & -y_{03} + \frac{4}{5}y_{02} - y_{21} + y_{11} - \frac{8}{25}y_{01}\\
-y_{12} + \frac{4}{5}y_{11} - y_{30} + y_{20} - \frac{8}{25}y_{10} & -y_{22} + \frac{4}{5}y_{21} - y_{40} + y_{30} - \frac{8}{25}y_{20} & -y_{13} + \frac{4}{5}y_{12} - y_{31} + y_{21} - \frac{8}{25}y_{11}\\
-y_{03} + \frac{4}{5}y_{02} - y_{21} + y_{11} - \frac{8}{25}y_{01} & -y_{13} + \frac{4}{5}y_{12} - y_{31} + y_{21} - \frac{8}{25}y_{11} & -y_{04} + \frac{4}{5}y_{03} - y_{22} + y_{12} - \frac{8}{25}y_{02}
\end{array}\right] \tag{12}
$$

Problem in (4) is semidefinite program (SDP) where a linear function is minimized subject to linear matrix inequality (LMI) constraints and can be solved by Matlab-based optimization toolboxes like GloptiPoly, CVX, and Yalmip where use interior-point solvers like SeDuMi and Mosek.

The obtained results by solving SDP in (4) for d=2 using GloptiPoly are as follow:

$$\mathbf{y}^{*T} = [0.5876, 0.2938, 0.2150, 0.1469, 0.1075, 0.0999, 0.0735, 0.0538, 0.0499, 0.0493, 0.0367, 0.0269, 0.0250, 0.0247, 0.0247],$$
$$\mathbf{y}_{\mathbf{x}}^{*T} = [1.0000, 0.5000, 0.2503, 0.1256, 0.8132].$$

We approximate the solution to the chance optimization in (1) with $y^*_{x_1} = 0.5$ and estimate the optimal probability $\mathbf{P}^*$ with $\mathbf{P}_2 = y^*_{00} = 0.5876$, ([1], See section 3.3 where we make a case for this approximation under some simplifying assumptions). To test the accuracy of the obtained results, we used Monte Carlo simulation to estimate $\mathbf{P}^*$ and an optimal solution to (1), ([1], Section 5.3.1). This computationally intensive method estimated that $x^* = 0.5$ with optimal probability of 0.2. To obtain better estimates of the optimum probability, one needs to increase the relaxation order d.

3 Improving Estimates of Probability

To improve the estimated optimal probability $\mathbf{P}^*$, we can solve following SDP for obtained solution x^*, ([1], Problem 3.12).

$$\mathbf{P}_d := \sup_{\tilde{\mathbf{y}} \in \mathbb{R}^{S_{1,2d}}} \; L_{\tilde{\mathbf{y}}} \left(\mathcal{P}_1(x^*) \mathcal{P}_2(x^*) \right), \tag{13}$$
$$\text{s.t.} \quad M_d(\tilde{\mathbf{y}}) \succcurlyeq 0, \;\; M_{d-r_1}(\tilde{\mathbf{y}}; \mathcal{P}_1(x^*)) \succcurlyeq 0, \;\; M_{d-r_2}(\tilde{\mathbf{y}}; \mathcal{P}_2(x^*)) \succcurlyeq 0 \tag{14}$$
$$M_d(\mathbf{y_q} - \tilde{\mathbf{y}}) \succcurlyeq 0, \tag{15}$$

where $\mathcal{P}_1(x^*) = \mathcal{P}_1(x^*, q) = 0.5q^3 - q^4$ and $\mathcal{P}_2(x^*) = \mathcal{P}_2(x^*, q) = -0.07 + 0.8q - q^2$. $L_{\tilde{\mathbf{y}}}(.)$ is a linear map that gives the moment representation of given polynomial, e.g., $L_{\tilde{\mathbf{y}}} \left(\mathcal{P}_1(x^*) \mathcal{P}_2(x^*) \right) = L_{\tilde{\mathbf{y}}} \left(-0.035q^3 + 0.47q^4 - 1.3q^5 + q^6 \right) = -0.035\tilde{y}_3 + 0.47\tilde{y}_4 - 1.3\tilde{y}_5 + \tilde{y}_6$, ([1], Section 2.1). $\delta_1 = 4$ and $\delta_2 = 2$ are the degrees of $\mathcal{P}_1(x^*)$ and $\mathcal{P}_2(x^*)$, $r_1 := \lceil \frac{\delta_1}{2} \rceil = 2$ and $r_2 := \lceil \frac{\delta_2}{2} \rceil = 1$. $\mathbf{y_q}$ is truncated moment vector of given measure μ_q. $M_d(\tilde{\mathbf{y}})$ and $M_d(\mathbf{y_q} - \tilde{\mathbf{y}})$ are d-th order moment matrices constructed by vectors $\tilde{\mathbf{y}}$ and $(\mathbf{y_q} - \tilde{\mathbf{y}})$. $M_{d-r_1}(\tilde{\mathbf{y}}; \mathcal{P}_1(x^*))$ and $M_{d-r_2}(\tilde{\mathbf{y}}; \mathcal{P}_2(x^*))$ are localization matrices constructed by vector $\tilde{\mathbf{y}}$ and polynomials $\mathcal{P}_1(x^*)$ and $\mathcal{P}_2(x^*)$. We approximate the optimal probability $\mathbf{P}^*$ for a given point x^* with $\tilde{\mathbf{P}} = (\tilde{\mathbf{y}})_{\mathbf{0}}$, the first elements of vector $\tilde{\mathbf{y}}$.

For example for $d = 5$, the vectors and matrices of SDP in (13) are as follow.

◊ The vectors $\tilde{\mathbf{y}} \in \mathbb{R}^{S_{1,10}}$ and $\mathbf{y_q}$ are

$$\mathbf{y_q}^T = \left[1,\ 0,\ \tfrac{1}{3},\ 0,\ \tfrac{1}{5},\ 0,\ \tfrac{1}{7},\ 0,\ \tfrac{1}{9},\ 0,\ \tfrac{1}{11}\right]$$

$$\tilde{\mathbf{y}}^T = [\tilde{y}_0,\ \tilde{y}_1,\ \tilde{y}_2,\ \tilde{y}_3,\ \tilde{y}_4,\ \tilde{y}_5,\ \tilde{y}_6,\ \tilde{y}_7,\ \tilde{y}_8,\ \tilde{y}_9,\ \tilde{y}_{10}]$$

◊ Moment matrices:

$$M_5(\tilde{\mathbf{y}}) = \begin{bmatrix} \tilde{y}_0 & \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 \\ \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 \\ \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 & \tilde{y}_8 \\ \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 & \tilde{y}_8 & \tilde{y}_9 \\ \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 & \tilde{y}_8 & \tilde{y}_9 & \tilde{y}_{10} \end{bmatrix} \tag{16}$$

$$M_5(\mathbf{y}_q - \tilde{\mathbf{y}}) = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 & \frac{1}{9} \\ \frac{1}{5} & 0 & \frac{1}{7} & 0 & \frac{1}{9} & 0 \\ 0 & \frac{1}{7} & 0 & \frac{1}{9} & 0 & \frac{1}{11} \end{bmatrix} - \begin{bmatrix} \tilde{y}_0 & \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 \\ \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 \\ \tilde{y}_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 & \tilde{y}_8 \\ \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 & \tilde{y}_8 & \tilde{y}_9 \\ \tilde{y}_5 & \tilde{y}_6 & \tilde{y}_7 & \tilde{y}_8 & \tilde{y}_9 & \tilde{y}_{10} \end{bmatrix} \tag{17}$$

◊ Localization matrices:

$$M_3(\tilde{\mathbf{y}}; \mathcal{P}_1(x^*)) = \begin{bmatrix} \frac{1}{2}\tilde{y}_3 - \tilde{y}_4 & \frac{1}{2}\tilde{y}_4 - \tilde{y}_5 & \frac{1}{2}\tilde{y}_5 - \tilde{y}_6 & \frac{1}{2}\tilde{y}_6 - \tilde{y}_7 \\ \frac{1}{2}\tilde{y}_4 - \tilde{y}_5 & \frac{1}{2}\tilde{y}_5 - \tilde{y}_6 & \frac{1}{2}\tilde{y}_6 - \tilde{y}_7 & \frac{1}{2}\tilde{y}_7 - \tilde{y}_8 \\ \frac{1}{2}\tilde{y}_5 - \tilde{y}_6 & \frac{1}{2}\tilde{y}_6 - \tilde{y}_7 & \frac{1}{2}\tilde{y}_7 - \tilde{y}_8 & \frac{1}{2}\tilde{y}_8 - \tilde{y}_9 \\ \frac{1}{2}\tilde{y}_6 - \tilde{y}_7 & \frac{1}{2}\tilde{y}_7 - \tilde{y}_8 & \frac{1}{2}\tilde{y}_8 - \tilde{y}_9 & \frac{1}{2}\tilde{y}_9 - \tilde{y}_{10} \end{bmatrix} \tag{18}$$

$$M_4(\tilde{\mathbf{y}}; \mathcal{P}_2(x^*)) = \begin{bmatrix} -\frac{7}{100}\tilde{y}_0 + \frac{4}{5}\tilde{y}_1 - \tilde{y}_2 & -\frac{7}{100}\tilde{y}_1 + \frac{4}{5}\tilde{y}_2 - \tilde{y}_3 & -\frac{7}{100}\tilde{y}_2 + \frac{4}{5}\tilde{y}_3 - \tilde{y}_4 & -\frac{7}{100}\tilde{y}_3 + \frac{4}{5}\tilde{y}_4 - \tilde{y}_5 & -\frac{7}{100}\tilde{y}_4 + \frac{4}{5}\tilde{y}_5 - \tilde{y}_6 \\ -\frac{7}{100}\tilde{y}_1 + \frac{4}{5}\tilde{y}_2 - \tilde{y}_3 & -\frac{7}{100}\tilde{y}_2 + \frac{4}{5}\tilde{y}_3 - \tilde{y}_4 & -\frac{7}{100}\tilde{y}_3 + \frac{4}{5}\tilde{y}_4 - \tilde{y}_5 & -\frac{7}{100}\tilde{y}_4 + \frac{4}{5}\tilde{y}_5 - \tilde{y}_6 & -\frac{7}{100}\tilde{y}_5 + \frac{4}{5}\tilde{y}_6 - \tilde{y}_7 \\ -\frac{7}{100}\tilde{y}_2 + \frac{4}{5}\tilde{y}_3 - \tilde{y}_4 & -\frac{7}{100}\tilde{y}_3 + \frac{4}{5}\tilde{y}_4 - \tilde{y}_5 & -\frac{7}{100}\tilde{y}_4 + \frac{4}{5}\tilde{y}_5 - \tilde{y}_6 & -\frac{7}{100}\tilde{y}_5 + \frac{4}{5}\tilde{y}_6 - \tilde{y}_7 & -\frac{7}{100}\tilde{y}_6 + \frac{4}{5}\tilde{y}_7 - \tilde{y}_8 \\ -\frac{7}{100}\tilde{y}_3 + \frac{4}{5}\tilde{y}_4 - \tilde{y}_5 & -\frac{7}{100}\tilde{y}_4 + \frac{4}{5}\tilde{y}_5 - \tilde{y}_6 & -\frac{7}{100}\tilde{y}_5 + \frac{4}{5}\tilde{y}_6 - \tilde{y}_7 & -\frac{7}{100}\tilde{y}_6 + \frac{4}{5}\tilde{y}_7 - \tilde{y}_8 & -\frac{7}{100}\tilde{y}_7 + \frac{4}{5}\tilde{y}_8 - \tilde{y}_9 \\ -\frac{7}{100}\tilde{y}_4 + \frac{4}{5}\tilde{y}_5 - \tilde{y}_6 & -\frac{7}{100}\tilde{y}_5 + \frac{4}{5}\tilde{y}_6 - \tilde{y}_7 & -\frac{7}{100}\tilde{y}_6 + \frac{4}{5}\tilde{y}_7 - \tilde{y}_8 & -\frac{7}{100}\tilde{y}_7 + \frac{4}{5}\tilde{y}_8 - \tilde{y}_9 & -\frac{7}{100}\tilde{y}_8 + \frac{4}{5}\tilde{y}_9 - \tilde{y}_{10} \end{bmatrix} \tag{19}$$

Figure 2 depicts the estimated probability $\tilde{\mathbf{P}}_\mathbf{d}$ for point $x^* = 0.5$ obtained by solving SDP in (13) for d=3:50 using GloptiPoly. Figure 3 shows the run time in *seconds* required for computing $\tilde{\mathbf{P}}_\mathbf{d}$.

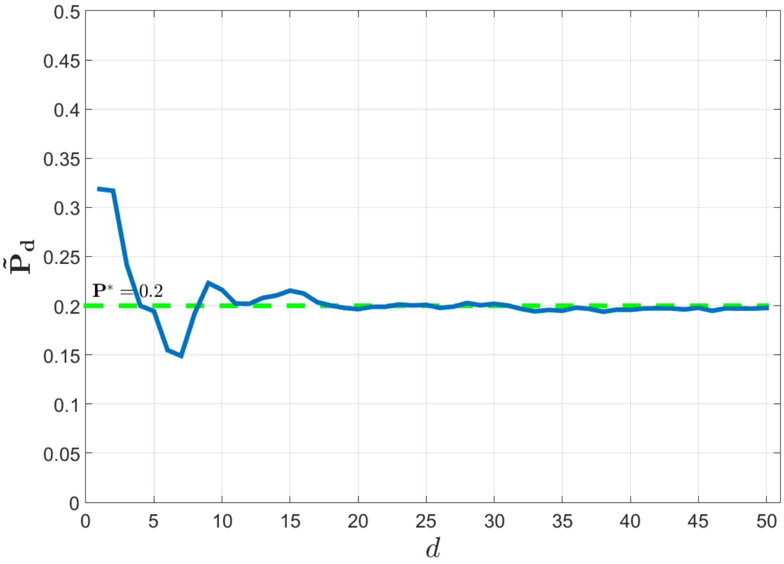

Figure 2: Probability estimates $\tilde{\mathbf{P}}_\mathbf{d} = (\tilde{\mathbf{y}})_\mathbf{0}$ obtained by solving SDP in (13)

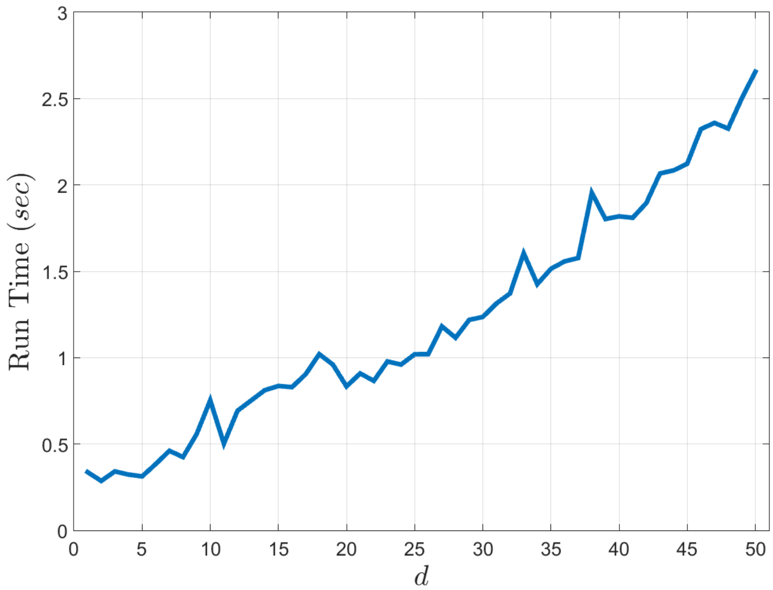

Figure 3: Run Time required for computing $\tilde{\mathbf{P}}_\mathbf{d}$