考虑以下问题

$$\min_{x_1,x_2,y} f_1(x_1,y)+f_2(x_2,y)$$

其中$f_1(x_1,y)$和$f_2(x_2,y)$有部分独立性，可用分解方法来解决这两个问题

Primal decomposition

Let $\varphi_1(y)=\min_{x_1} f_1(x_1,y)$　$\varphi_2(y)=\min_{x_2} f_2(x_2,y)$

$$\therefore \min_{x_1,x_2,y} f_1(x_1,y)+f_2(x_2,y)=\min_{y} \varphi_1(y)+\varphi_2(y)$$

我们使用梯度下降法来解：

$$y\leftarrow y-\alpha g,\ g\in\partial(\varphi_1(y)+\varphi_2(y))=g_1+g_2$$

Algorithm (Primal decomposition)

for $k=1,2,\dots$

　$x_1=\arg\min_{x} f_1(x,y_k)$

　$x_2=\arg\min_{x} f_2(x,y_k)$

　Compute $g_1\in\partial\varphi_1(y_k)$ $g_2\in\partial\varphi_2(y_k)$

　$y\leftarrow y-\alpha_k(g_1+g_2)$

end for

其中，步骤1和步骤2可以并行

Dual Decomposition

考虑问题 $\min_{x_1,x_2,y_1,y_2} f_1(x_1,y_1)+f_2(x_2,y_2)$

$$\text{s.t. } y_1=y_2$$

拉格朗日函数 $L(x_1,x_2,y_1,y_2,\mu)=f_1(x_1,y_1)+f_2(x_2,y_2)-\mu^T(y_1-y_2)$　…1

$$\therefore g(\mu)=\min_{x_1,y_1}(f_1(x_1,y_1)-\mu^Ty_1)+\min_{x_2,y_2}(f_2(x_2,y_2)+\mu^Ty_2)$$

其中$\partial g(\mu)=y_2-y_1$，y_1和y_2是公式中的最小化

Algo (dual decomposition)

for $k=1,2,\dots$

　$x_1,y_1=\arg\min_{x_1,y_1} f_1(x_1,y_1)-\mu_k^Ty_1$

　$x_2,y_2=\arg\min_{x_2,y_2} f_2(x_2,y_2)+\mu_k^Ty_2$

　$\mu_{k+1}\leftarrow\mu_k+\alpha_k(y_2-y_1)$

end for.

可将问题转化为可分解方法可应用的，如以下

$$\min f_1(x_1)+f_2(x_2)$$

$$s.t.\ x_1\in C_1,\ x_2\in C_2$$

$$h_1(x_1)+h_2(x_2)\le 0$$

尽管x_1、x_2是耦合的（通过约束），我们可以引入额外的变量来降低依赖性

考虑，$\min\limits_{x_1,x_2,t} f_1(x_1)+f_2(x_2)$ 变作问题

$$s.t.\ x_1\in C_1,\ x_2\in C_2$$

$$h_1(x_1)\le t\quad h_2(x_2)\le -t$$

将问题分解为

$$\begin{cases}\varphi_1(t)=\min\limits_{x_1,t} f_1(x_1)\\ s.t.\ x_1\in C_1\ \ h(x_1)\le t\end{cases}\quad \cdots 1$$

$$\begin{cases}\varphi_2(t)=\min\limits_{x_2,t} f_2(x_2)\\ s.t.\ x_2\in C_2\ \ h(x_2)\le -t\end{cases}\quad \cdots 2$$

$$\min_t \varphi_1(t)+\varphi_2(t)\quad \cdots 3$$

寻找$\varphi_1(t)$和$\varphi_2(t)$的梯度：

我们声明：如果$\lambda(t)$是1的最优对耦变量，则$-\lambda(t)$是φ_1的梯度
（试）

$$L(x_1,t)=f_1(x_1)+\lambda^T(h_1(x_1)-t)\quad \text{当最优时，}\lambda^T(h_1(x_1)-t)=0$$

令另x_2是$\tilde{t}$的最小

$$\varphi_1(\tilde{t})=\sup_{\lambda\ge 0}\inf_{x\in X} f_1(x)+\lambda^T(h_1(x)-\tilde{t})\ \ge \inf_{x\in X} f_1(x)+\lambda(t)^T(h_1(x)-\tilde{t})$$

$$=\inf_{x\in X} f_1(x)+\lambda(t)^T(h_1(x)-t)+\lambda(t)^T(t-\tilde{t})\ =\varphi_1(t)+\lambda(t)^T(t-\tilde{t})$$

$$\therefore\ \varphi_1(\tilde{t})-\varphi_1(t)\ge(-\lambda(t))^T(\tilde{t}-t)\ \Rightarrow -\lambda(t)\in\partial\varphi_1(t)$$

Algo. for $k=1,2,\ldots$

  solve (1) with $t=t_k$ and 获得 x_1,λ_1

  solve (2) with $t=t_k$ and 获得 x_2,λ_2

  $t\leftarrow t+\alpha(\lambda_1+\lambda_2)$

end for

Dual decomposition 在下面例子中更直接.

$$L(x,t) = f_1(x_1)+f_2(x_2)+t(h_1(x_1)+h_2(x_2)) = f_1(x_1)+th_1(x_1)+f_2(x_2)+th_2(x_2)$$

Algo. for $k=1,2,3,\dots$

$$x_1 = \underset{x_1}{\operatorname{argmin}}\, f_1(x_1)+t_k h_1(x_1)$$

$$x_2 = \underset{x_2}{\operatorname{argmin}}\, f_2(x_2)+t_k h_2(x_2)$$

$$t_{k+1} \leftarrow (t_k+\alpha_k(h_1(x_1)+h_2(x_2)))_+$$

end for.

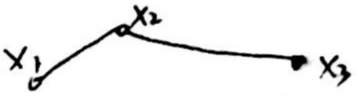
我们通过图形化表示decomposition.

1. 每一个本地变量代表一个node. 2. 每一个交互变量代表一个link.

比如，在上个问题中我们有

$\dots, 1$

问题 $\min f_1(x_1,y_1)+f_2(x_2,y_1,z)+f_3(z,x_3)$



一般我们通过 $y_i=E_i z$ 来表示交互变量，$card(z)$是自由度.

node的数量决定了并行的可能性.

比如，$y_1=y_2$, $y_3=y_4$ 可写为 $\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\end{pmatrix} = \begin{pmatrix}1&0\\1&0\\0&1\\0&1\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$

我们有 $\min \sum_{i=1}^{k} f_i(x_i,y_i)$

s.t. $(x_i,y_i)\in C_i$, $i=1,2,\dots,k$

$y_i = E_i z$, $i=1,2,\dots,k$

对于例1，$y_1=y$, $y_2=\begin{pmatrix}y\\ z\end{pmatrix}$ $y_3=z$

我们有 $\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \begin{pmatrix}1&0\\1&0\\0&1\\0&1\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix}$

(4)

对于第四次最后一个问题 primal decomposition 是非常直接的

$$\min_{x_i, z} \sum_{i=1}^{k} f_i(x_i, E_i z) = \min_{z} \sum_{i=1}^{k} \phi_i(z)$$

The dual method

$$L = \sum_{i=1}^{k} f_i(x_i, y_i) + \sum_{i=1}^{k} \mu_i^T (y_i - E_i z) = \sum_{i=1}^{k} (f_i(x_i, y_i) + \mu_i^T y_i) - \sum_{i=1}^{k} \mu_i^T E_i z$$

其中 $\mu_i^T E_i = 0$ 以避免 $L = -\infty$

dual 梯度为 $y - Ez$ 但需确保 $E^T \mu = 0$

$\therefore$ 投影梯度为

$$= (I - E(E^T E)^{-1} E^T)(y - Ez)$$

$$= y - Ez - E(E^T E)^{-1} E^T y + Ez$$

$$= (I - E(E^T E)^{-1} E^T) y$$

Algo.

for $k = 1, 2, \ldots$

  for $i = 1, 2, \ldots K$

    $x_i, y_i = \underset{x_i, y_i}{\arg\min} f_i(x_i, y_i) + \mu_i^T y_i$

  end for

  $g \leftarrow (I - E(E^T E)^{-1} E^T) y$

  $\mu \leftarrow y + \alpha_k g$

end for

⑤

交通问题

$\min \sum_{a\in A} x_a c_a$

s.t. $\sum_k f_k^{rs} = q_{rs}$

$f_k^{rs} \geq 0$

$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs}$

$x_a \leq k_a$

其中 $\delta_{a,k}^{rs} = \begin{cases} 1 & a \in f_k^{rs} \\ 0 & \text{otherwise} \end{cases}$

类似于

$\min \sum_a \frac{|x_a|}{c_a - |x_a|}$

s.t. $Ax = s$

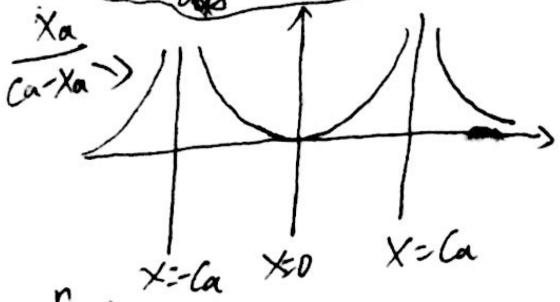
$\sum 0 \leq x_a \leq c_a$ for all $a \in A$

$\min \sum_a \frac{x_a}{c_a - x_a}$

s.t. $Ax = s$

$0 \leq x_a \leq c_a$, for $a \in A$

$$A = \begin{array}{c|c} & 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad \text{link} \\ \hline \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \\ \text{node} \end{array} & \begin{bmatrix} 1 & 1 & & & \\ -1 & & \ddots & & \\ & & & \ddots & \\ & -1 & & & \end{bmatrix} \end{array}$$



二、求解 $\min \sum_{j=1}^{n} f_j(x_j) = \sum_{j=1}^{n} \frac{x_j}{c_j - x_j}$

s.t. $Ax = s \Rightarrow \sum_j A_j x_j = s \quad A = (A_1, A_2, \cdots, A_n)$

$L(x,\mu) = \sum_{j=1} \frac{x_j}{c_j - x_j} + \mu^T \left(\sum_j A_j x_j - s\right) = \sum_{j=1}^{n} \frac{x_j}{c_j - x_j} + \sum_j \mu^T A_j x_j - \mu^T s$

$= \sum_{j=1}^{n} \underbrace{\left(\frac{x_j}{c_j - x_j} + \mu^T A_j x_j \right)}_{\varphi(x) = \frac{x}{c-x} + bx} - \mu^T s$

$\varphi'(x) = b + \frac{c - x + x}{(c-x)^2} = b + \frac{c}{(c-x)^2} \lessgtr 0$

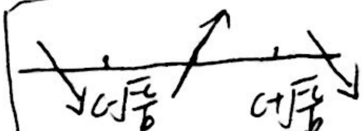
$\begin{cases} b > 0 \Rightarrow x = 0 \\ b < 0 \Rightarrow b + \frac{c}{(c-x)^2} = 0 \Rightarrow (c-x)^2 = -\frac{c}{b} \end{cases}$

$\Downarrow$

$c - x = \pm\sqrt{-\frac{c}{b}}$

$\Downarrow$

$x = c \pm \sqrt{-\frac{c}{b}}$



$c - \sqrt{-\frac{c}{b}} < 0 \Rightarrow x = 0$

$c - \sqrt{-\frac{c}{b}} > 0 \Rightarrow x = c - \sqrt{-\frac{c}{b}} \qquad c^2 < -\frac{c}{b} \Leftrightarrow b > -\frac{1}{c}$

$\arg\min_x \varphi(x) = \begin{cases} 0 & -\frac{1}{c} < b \\ c - \sqrt{-\frac{c}{b}} & b \leq -\frac{1}{c} \end{cases}$

$\therefore g = \sum_j A_j x_j - s$

Algo for $k = 1,2,3,\cdots$

for $j = 1,2,3,\cdots,n$

if $\mu_j^T A_j > -\frac{1}{c_j} \Rightarrow x_j = 0$

else $\Rightarrow x_j = c_j - \sqrt{\frac{-c_j}{\mu_j^T A_j}}$

end

$g = Ax - s$

$\mu \leftarrow \mu + \alpha g$

end for

我们需要一个初始可行点，可通过什么方法获得？

$\min 0$

s.t. $\begin{cases} x \leq c \\ -x \leq 0 \\ Ax = s \end{cases} \Leftrightarrow \begin{pmatrix} I \\ -I \end{pmatrix} x \leq \begin{pmatrix} c \\ 0 \end{pmatrix}$

（即解一个线性规划问题）