Capturing Bisimulation-Invariant Complexity Classes by Polyadic Higher-Order Fixpoint Logic

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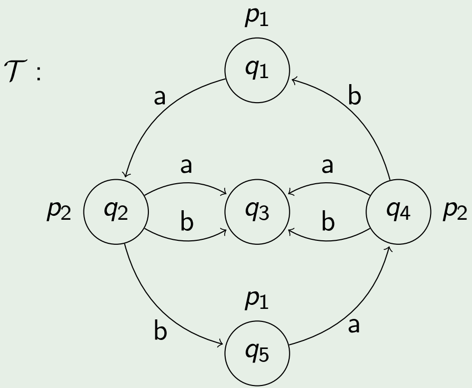
Outline

1. k-EXPTIME/$\sim$
2. PHFL
3. Upper Bounds
4. HO + LFP

Example 1.1



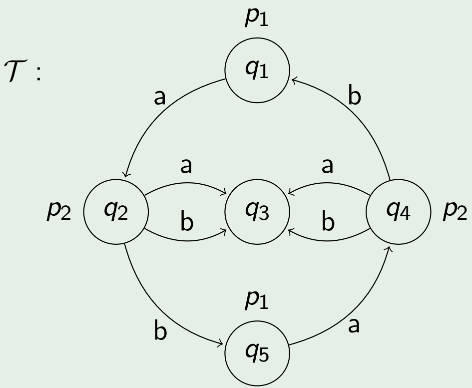
Definition 1.2

Let be $\mathcal{T}_1 = (Q_1, \Sigma, P, \Delta_1, v_1)$ and $\mathcal{T}_2 = (Q_2, \Sigma, P, \Delta_2, v_2)$ two LTS. A **bisimulation** is a binary relation $R \subseteq Q_1 \times Q_2$ that fulfills for all $(q_1, q_2) \in R$

- $v_1(q_1) = v_2(q_2)$,
- for all $a \in \Sigma$ and all $q_1' \in Q_1$, if $q_1 \xrightarrow{a} q_1'$, then there is a state $q_2' \in Q_2$, $q_2 \xrightarrow{a} q_2'$ and $(q_1', q_2') \in R$ and
- for all $a \in \Sigma$ and all $q_2' \in Q_2$, if $q_2 \xrightarrow{a} q_2'$, then there is a state $q_1' \in Q_1$, $q_1 \xrightarrow{a} q_1'$ and $(q_1', q_2') \in R$.

We call two states $q_1 \in Q_1$, $q_2 \in Q_2$ **bisimilar**, noted as $(\mathcal{T}_1, q_1) \sim (\mathcal{T}_2, q_2)$, if there is a bisimulation R such that $(q_1, q_2) \in R$.

Example 1.3



Definition 2.1

PHFL types are given by the grammar

$$\sigma, \tau ::= \bullet \mid \sigma^{v} \to \tau,$$

where v is called *variance*. The **variances** of PHFL are defined by the grammar

$$v ::= + \mid - \mid 0.$$

Definition 2.2

$$[\![\tau]\!]_{\mathcal{T}} = \begin{cases} (\mathcal{P}(Q^d), \subseteq), & \text{if } \tau = \bullet \\ (([\![\sigma_1]\!]_{\mathcal{T}})^{v} \to [\![\sigma_2]\!]_{\mathcal{T}}, \leq_{([\![\sigma_1]\!]_{\mathcal{T}})^{v} \to [\![\sigma_2]\!]_{\mathcal{T}}}), & \text{if } \tau = \sigma_1^{v} \to \sigma_2. \end{cases}$$

Definition 2.3

$$\Phi, \Psi ::= \top \mid p_i \mid \Phi \vee \Psi \mid \neg\Phi \mid \langle a \rangle_i \Phi \mid \{\mathbf{i} \leftarrow \mathbf{j}\}\Phi \mid X \mid \lambda(X^v : \tau).\Phi \mid \Phi\Psi \mid \mu(X : \tau).\Phi$$

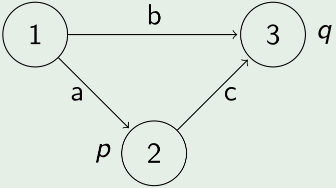
Example 2.4

The following 2-adic PHFL^0 formula Φ describes bisimilarity i.e. it denotes those pairs (q_1, q_2) such that $q_1 \sim q_2$ and vice versa.

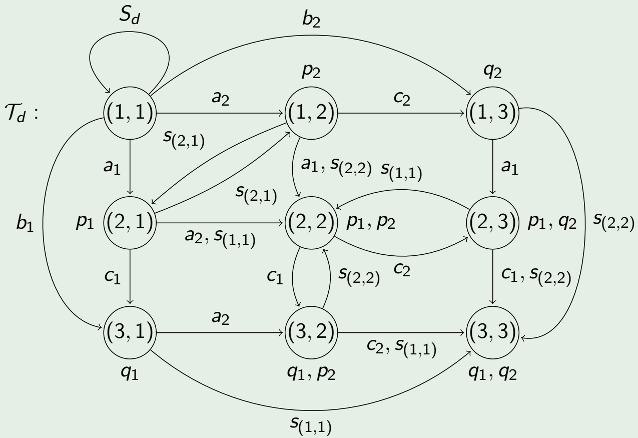
$$\Phi = \nu(X : \bullet). \bigwedge_{a \in \Sigma} [a]_1 \langle a \rangle_2 X \wedge [a]_2 \langle a \rangle_1 X \wedge \bigwedge_{p \in P} p_1 \Leftrightarrow p_2$$

Example 3.1

$\mathcal{T}$:



Example 3.2


$\mathcal{T}_d$:
S_d
b_2
p_2
q_2
$(1,1)$
a_2
$(1,2)$
c_2
$(1,3)$
$s_{(2,1)}$
a_1
$a_1, s_{(2,2)}$
$s_{(1,1)}$
a_1
$s_{(2,1)}$
b_1
p_1
$(2,1)$
$(2,2)$
p_1, p_2
$(2,3)$
p_1, q_2
$s_{(2,2)}$
$a_2, s_{(1,1)}$
c_1
c_1
$s_{(2,2)}$
c_2
$c_1, s_{(2,2)}$
a_2
$(3,1)$
$(3,2)$
$(3,3)$
$c_2, s_{(1,1)}$
q_1
q_1, p_2
q_1, q_2
$s_{(1,1)}$

Definition 3.3

$$
\begin{aligned}
F(\top) &= \top \\
F(X) &= X \\
F(p_i) &= p_i \\
F(\langle a\rangle_i \psi) &= \langle a_i\rangle F(\psi) \\
F(\psi \vee \psi') &= F(\psi) \vee F(\psi') \\
F(\neg\psi) &= \neg F(\psi) \\
F(\{\mathbf{i} \leftarrow \mathbf{j}\}\psi) &= \langle s_{(e(1),\ldots,e(d))}\rangle F(\psi) \\
F(\mu(X : \tau).\,\psi) &= \mu(X :\ T(\tau)).\,F(\psi) \\
F(\lambda(X^v : \tau).\,\psi) &= \lambda(X^v :\ T(\tau)).\,F(\psi) \\
F(\psi\psi') &= F(\psi)F(\psi')
\end{aligned}
$$

Definition 4.1

HO types are defined inductive as follows:

$\tau = \odot$ is a HO type,

$\tau = (\tau_1, \ldots, \tau_n)$ is a HO type, if $\tau_1, \ldots, \tau_n$ are HO types.

Definition 4.2

Let $\mathcal{A}$ be a σ-structure over universe $\mathcal{U}$ then the universes of the HO types are defined inductively as follows:

$D_{\odot}(\mathcal{U}) = \mathcal{U}$,

$D_{(\tau_1,\ldots,\tau_n)}(\mathcal{U}) = \mathcal{P}(D_{\tau_1}(\mathcal{U}) \times \cdots \times D_{\tau_n}(\mathcal{U}))$

Definition 4.3

The set of **HO formulas** over σ is defined inductively as follows:

- $R(x_1, \ldots, x_n)$ is a HO formula over σ if $R \in \sigma$ is a relation with arity n and $x_1, \ldots, x_n$ are variables of type $\odot$,
- $X(x_1, \ldots, x_n)$ is a HO formula over σ if X is a variable of type $(\tau_1, \ldots, \tau_n)$ and x_i is a variable of type τ_i, with $i \in \{1, \ldots, n\}$,
- if φ and ψ are two HO formulas over σ, then $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ are also HO formulas over σ,
- if φ is a HO formula over σ and X a variable of arbitrary type τ, then $\exists X : \tau. \varphi$ and $\forall X : \tau. \varphi$ are also HO formulas over σ.

Definition 4.4

Let σ an arbitrary signature, X a relation variable of HO type $\tau = (\tau_1, \ldots, \tau_k)$, $\tau_1, \ldots \tau_k$ arbitrary HO types, $x_1, \ldots, x_k$ variables of HO type $\tau_1, \ldots, \tau_k$ respectively and $\varphi(X, x_1, \ldots, x_k)$ a formula over σ with free variables $X, x_1, \ldots, x_k$. For each σ-structure $\mathcal{A}$ with universe $\mathcal{U}$, $\varphi(X, x_1, \ldots, x_k)$ induces the operator

$$\begin{aligned} F_\varphi^{\mathcal{A}} \colon \mathscr{P}(D_\tau(\mathcal{U})) &\longrightarrow \mathscr{P}(D_\tau(\mathcal{U})) \\ A &\longmapsto F_\varphi^{\mathcal{A}}(A) := \{(a_1, \ldots, a_k) \mid \mathcal{A} \models \varphi(A, a_1, \ldots, a_k)\} \end{aligned}$$

where $a_1 \in D_{\tau_1}(\mathcal{U})$, $\ldots$, $a_k \in D_{\tau_k}(\mathcal{U})$.

Definition 4.5

Let $\mathcal{A}$ be a σ-structure and α a variable assignment over universe $\mathcal{U}$. The semantics of a HO(LFP) formula extends that of HO formulas with the following definition:

$\mathcal{A}, \alpha \models [LFP_{X,x_1,\ldots,x_k}\varphi(X, x_1, \ldots, x_k)](v_1, \ldots, v_k)$ iff $(\alpha(v_1), \ldots, \alpha(v_k)) \in LFP(F_\varphi^{\mathcal{A}})$.

Definition 4.6

Let $F\colon \mathscr{P}(A) \to \mathscr{P}(A)$ be an operator on a finite set A, then the **partial fixpoint** of F, abbreviated as $PFP(F)$, is defined as follows:

$$PFP(F) := \begin{cases} F^{i+1}(\emptyset) = F^{i}(\emptyset), & \text{if such } i \in \{0, \dots, |A|\} \text{ exists} \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $F^0(\emptyset) = \emptyset$, $F^1(\emptyset) = F(\emptyset)$, $F^2(\emptyset) = F(F(\emptyset))$, and so on.

End

Thank you for your attention!