Solving Laplace's Equation in two dimensions

We'll solve Laplace's Equation in polar coordinates for the scalar field $\Phi(r, \theta)$:

$$(r\Phi_r)_r + \frac{1}{r}(\Phi_\theta)_\theta = 0$$

Assuming we can separate variables such that $\Phi(r, \theta) = R(r)\Theta(\theta)$:

$$\Phi_r = R'\Theta, \Phi_\theta = R\Theta' \Rightarrow$$

$$(rR'\Theta)_r + \frac{1}{r}(R\Theta')_\theta = 0 \Leftrightarrow$$

$$R'\Theta + rR''\Theta + \frac{1}{r}R\Theta'' = 0 \Leftrightarrow$$

$$\frac{rR' + r^2R''}{R} = \frac{-\Theta''}{\Theta} \Leftrightarrow$$

$$\frac{r(rR')'}{R} = \frac{-\Theta''}{\Theta}$$

Set both sides equal to λ^2:

$$(1) \quad \frac{r(rR')'}{R} = \lambda^2, \qquad (2) \quad \frac{-\Theta''}{\Theta} = \lambda^2$$

Solve the two differential equations separately, assuming at first that $\lambda \neq 0$. For (1), use a solution of the form $R(r) = \gamma r^\alpha$:

$$R' = \gamma \alpha r^{\alpha-1} \Rightarrow$$

$$\frac{r(r\gamma \alpha r^{\alpha-1})'}{\gamma r^\alpha} = \lambda^2 \Leftrightarrow$$

$$\alpha^2 \frac{\gamma r r^{\alpha-1}}{\gamma r^\alpha} = \lambda^2 \Leftrightarrow$$

$$\alpha^2 = \lambda^2 \Rightarrow \alpha = \pm\lambda$$

For (2), we have $\Theta'' + \lambda^2\Theta = 0$, so the general form of the solution is:

$$\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$$

Since $\Theta(0) = \Theta(2\pi)$ we know that $\lambda \in \mathbb{N}$ (or, more generally, that $\lambda \in \mathbb{Z}$ but due to the symmetry in cos and sin that information can be encoded into A and B). When $\lambda = 0$ we can in both cases integrate to find the solutions:

$$(1) \quad R(r) = c_1 \ln r + c_2, \qquad (2) \quad \Theta(\theta) = c_3\theta + c_4$$

Since Laplace's Equation is linear, we can add the solutions together to get the general solution set:

$$\Phi(r, \theta) = R(r)\Theta(\theta)_{\lambda=0} + R(r)\Theta(\theta)_{\lambda\in\mathbb{N}} =$$

$$(c_1 \ln r + c_2)(c_3\theta + c_4) + \sum_{n\in\mathbb{N}}(\gamma_{1n}r^n + \gamma_{2n}r^{-n})\big[A\cos(n\theta) + B\sin(n\theta)\big]$$

Particular solution for infinite cylinder in $\underline{E}$ field

We will add two different solutions together to find the total potential. Let the $\underline{E}$ field be uniform in the x direction such that $\underline{E} = E_0 \hat{x}$. Since $\underline{E} = -\nabla\Phi$, $\Phi = -E_0 x = -E_0 r \cos\theta$, which is in the general solution set we defined above. Our first boundary condition is then:

$$\lim_{r\to\infty} \Phi(r,\theta)_{total} = -E_0 r \cos\theta$$

Next we want to find the field from the cylinder. Let the radius of the cylinder be R. The second boundary condition is that the potential on the surface of the cylinder must be constant, say V_0, so:

$$(3) \quad \Phi(R,\theta)_{total} = -E_0 R \cos\theta + \Phi(R,\theta)_{cylinder} = V_0$$

$\Phi(r,\theta)_{cylinder}$ must also be in the general solution set defined above. We need a constant so take $c_2 c_4$ from the first part of the general solution, but discard the $\ln r$ and θ terms to keep within the first boundary condition. From the second part we need to get an expression which lets us fulfill (3), so it is logical to take the $n = 1$ term which has $\cos\theta$:

$$\Phi(R,\theta)_{total} = -E_0 R \cos\theta + c_2 c_4 + (\gamma_{11} R + \gamma_{21} R^{-1})(A\cos\theta + B\sin\theta) = V_0$$

Now B must be zero since the sin term would violate the symmetry we require about the x axis, and all that is left is to solve the following equation:

$$\Phi(R,\theta)_{total} = -E_0 R \cos\theta + c_2 c_4 + (\gamma_{11} R + \gamma_{21} R^{-1})A\cos\theta = V_0 \Leftrightarrow$$

$$c_2 c_4 - V_0 + R\cos\theta \left(A\gamma_{11} - E_0 + \frac{A\gamma_{21}}{R^2} \right) = 0$$

Hence set $c_2 c_4 = V_0$, $\gamma_{11} = 0$, and $A\gamma_{21} = E_0 R^2$ to fulfill (3) and get the final solution:

$$\Phi(r,\theta)_{total} = V_0 + E_0 cos\theta \left(\frac{R^2}{r} - r \right)$$

Neutral conductors, local gauge invariance and conservation of charge

One powerful tool for investigating the validity of our numerical solutions is conservation of charge. Since we are adding neutral conductors to the field, the total integrated field should not change as a result. To understand how it is possible for some numerical solutions to break this fundamental rule we will first consider the electric potential created by a conductor in the field, and then use the power of gauge theory and Noether's Theorem to qualitatively explain the effect.

As the redistribution of charges in the conductor cancels out the electric field inside it, and it has a total charge of 0, we can approximate the charge distribution in the stable state to be similar to an electric dipole where the charges are separated by the length of the conductor in the direction of the field, l. This allows us to derive a mathematical expression for the electric potential caused by it:

Consider two charges, q_+ and q_-, separated by l. The total potential at a point $\bar{r}$, at an angle θ from the normal to l, separated from the charges by r_+ and r_- can then be written by

$$V(\bar{r}) = V(r_+) + V(r_-)$$

$$= \frac{q_+}{4\pi\epsilon_0 r_+} + \frac{q_-}{4\pi\epsilon_0 r_-}$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_+} - \frac{1}{r_-} \right)$$

$$\approx \frac{q}{4\pi\epsilon_0} \left(\frac{l\sin\theta}{r^2} \right)$$

Here r is the distance from the midpoint between the two charges to $\bar{r}$. Note that this potential is proportional to $1/r^2$ unlike the electric potential from a point charge, which limits its effective range considerably.

Gauge theory is a powerful tool for analysing fields where the observable quantities are invariant under certain transformations of the field itself. By Noether's theorem, these symmetries all have corresponding conservation laws: the conserved Noether currents. The symmetry group of electromagnetism is $U(1)$, which requires (for local gauge invariance) that its gauge boson is massless and hence gives the force infinite range. Gauge invariance is here the symmetry that results in conservation of charge. The same symmetry is actually required for the weak force as well, which is why the Higgs mechanism was invented to introduce a way to consistently give mass to the Z and $W^\pm$ bosons and hence limit the range of the weak force without breaking gauge invariance. In our case we are interested in gauge theory because it allows us to predict that meaningfully limiting the range of the electromagnetic force by using a finite grid size with the assumption that the conductor won't change the field at the boundaries will break conservation of charge. 'Meaningfully' here refers to the fact that the potential from the conductor will tend to 0 quickly as we move further away from it due to the inverse square law, so we can estimate the field to be effectively zero at some finite distance away. This allows us to justify the use of the finite grid sizes we require to make sensible numerical simulations,

and gives us a good variable to use for determining how small we can make the grid in comparison to the conductor without creating unphysical solutions: conservation of charge. The dipole potential calculation also tells us that the required size of the grid varies roughly linearly with the length l of the conductor in the direction of the field, whereas the size in the orthogonal direction to the field is less important.

Multiple conductors

The problem of adding multiple conductors to the same field is that in order to obey charge conservation, they must all take the fields generated by the others into account when deciding their equilibrium potential. In other words, adding one and solving before before adding a new one will give an unphysical result. The simplest correct way of dealing with this problem is to find solutions for each individual conductor first, averaging them all, and then adding all the conductors into the averaged field at once. This ensures that charge conservation is obeyed and gives the correct physical solution.