Reductions & NP-completeness [chapter 8]

Problem $Y \leq_p$ Problem X : [polynomial time reduction]

if X can be solved in poly. time, then after solving X and doing some poly. work, we can solve Y.

$\Rightarrow$ "Y is polynomial-time reducible to X".

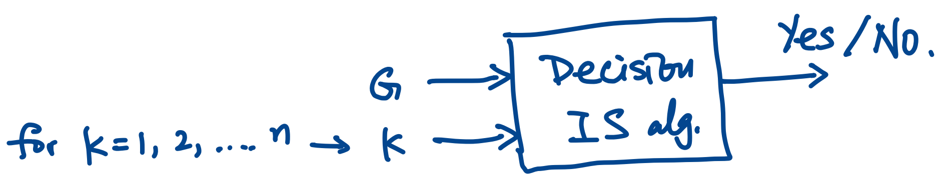
$\Rightarrow$ "X is at least as hard as Y (with respect to poly. time)".

Decision Problems : Problems with answer : Yes or No.

Max IS $\rightarrow$ Decision Version : Given $k \geq 1$ and $G=(V,E)$.

Is there an independent set of size at least k ?

Let say we have $O(n^2)$ algorithm decision version IS.



At some k, its answer switches from Yes to No.

The last k for which it answers Yes is the size of MIS.

Then we got an $O(n^3)$ algorithm for max-version.

$\Rightarrow$ MIS $\leq_p$ Decision-IS.

Is Decision-IS $\leq_p$ MIS ?

if $k \leq$ MIS, then answer to Decision IS is Yes.

otherwise, answer No.

Set Cover

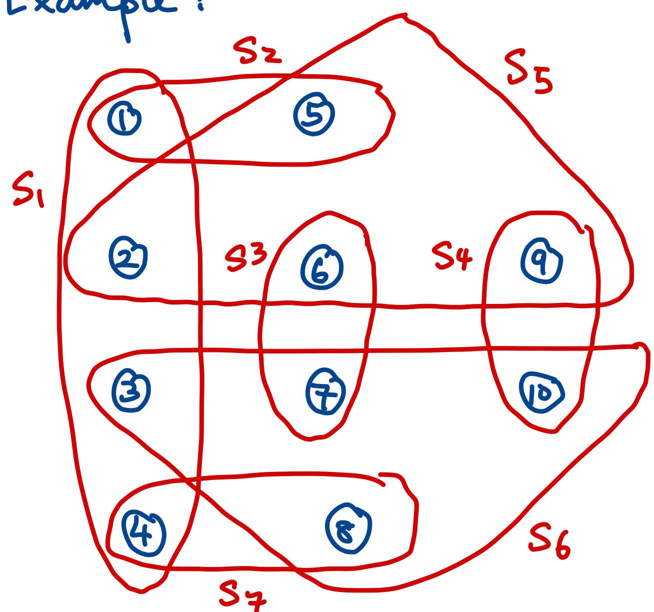
Input : 1) A set $\mathcal{U}$ of n elements

2) A collection $S_1, \ldots, S_m$ of subsets of $\mathcal{U}$, where $\bigcup_{i=1}^{m} S_i = \mathcal{U}$, union of these subsets $= \mathcal{U}$.

3) $k \geq 1$.

Question/Output : Does there exist a collection of at most k of these sets whose union equals $\mathcal{U}$?

Example :



$\mathcal{U} = \{1, 2, 3, \ldots, 10\}$

$S_1 = \{1, 2, 3, 4\}$

$S_2 = \{1, 5\}$

$S_3 = \{6, 7\}$

$S_4 = \{9, 10\}$

$S_5 = \{2, 5, 6, 9\}$

$S_6 = \{3, 7, 8, 10\}$

$S_7 = \{4, 8\}$

$\{S_1, S_5, S_6\}$ is a collection of 3 sets that cover $\mathcal{U}$. ← a set cover.

We want to show Vertex Cover $\leq_P$ Set Cover.

Given 1) An instance I of vertex Cover and

2) An algorithm A of Set Cover,

use A to solve I. (maybe some extra poly. work).

Step 1 : Convert the input I of Vertex Cover to an input I' of Set Cover.

Step 2: Solve I' using A.

Step 3: Use the solution of I' to solve I.

Step 1: $I : (G = (V, E), k)$ - Is there a vertex cover of size at most k ?

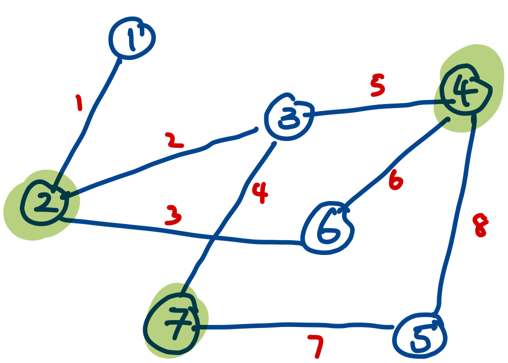
Decision version VC.

I' : $\mathcal{U} = \{e_1, e_2, \ldots, e_m\}$, i.e., $e_i \in E$, $\mathcal{U} = E$.

$\left.\begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{matrix}\right\}$ For every vertex $v \in V$, we make a set

$S_v = \{$all edges incident to $v\}$

We have n sets, one for each vertex $v \in V$ and Universal set $\mathcal{U}$ has m elements, one for each edge.



$\mathcal{U} = \{1, 2, \ldots\ 8\}$

$S_1 = \{1\}$

$S_2 = \{1, 2, 3\}$

$S_3 = \{2, 4, 5\}$

$S_4 = \{5, 6, 8\}$

$\vdots$

$S_7 = \{4, 7\}$

$I : G = (V, E) \longrightarrow I'$

$\{2, 4, 7\}$ is a VC. $\rightarrow$ is $\{S_2, S_4, S_7\}$ a S.C. ?

<u>Claim</u>: $\mathcal{U}$ can be covered by k of the sets $\{S_v : v \in V\}$, if and only if, there is a vertex cover of size k in G.

If claim is true : Given I, we look up I', feed I' to the set cover Algorithm.

if it returns "Yes" to I',

we return "Yes" to I.

else "No".

3-SAT (satisfiability).

$x_1, x_2, \ldots, x_n$ variables, $x_i \in \{0,1\}$

$$F = (x_5 \vee x_7 \vee \overline{x_8}) \wedge (x_1 \vee \overline{x_2} \vee x_4) \wedge \cdots \cdots \wedge (x_6 \vee x_7 \vee \overline{x_4})$$

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_m, \quad C = \underbrace{(x_i \vee \overline{x_j} \vee x_k)}_{\text{3 variables only.}}$$

F: formula, C_1: class, $\wedge$: and.

Question : Given F, can we set the x_i to 0 or 1 such that the formula is satisfied, i.e. the formula F evaluate to 1?

Example: $F = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_2}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_1})$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$ is a satisfying assignment.

Brute Force Algorithm : check all 2^n assignments.

$O(2^n * m)$ time.

Show 3-SAT $\leq_P$ Max IS.

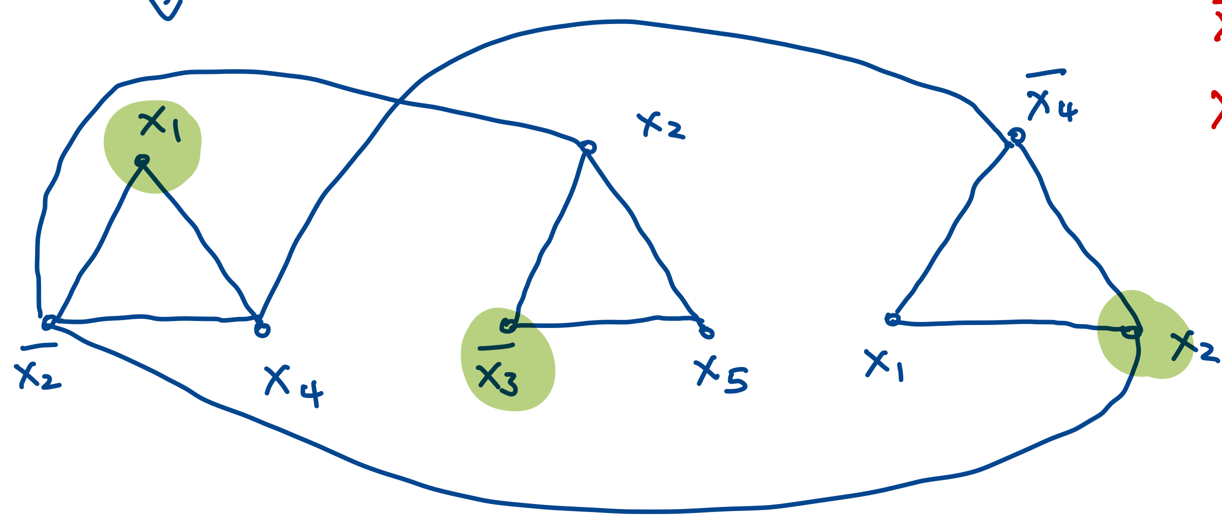
Given 1) An instance I of 3-SAT (a formula) and
2) An algorithm A of MIS

use A to solve I. (maybe some extra poly. work).

$\Rightarrow$ Given F, we need to build a graph, solve MIS. and somehow use the solution to solve 3-SAT on F.

Given $F = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee x_5) \wedge (\overline{x_4} \vee x_1 \vee x_2)$

$\downarrow$ Build G.



$x_1 = 1$

$\overline{x_3} = 1, \; x_3 = 0$

$x_2 = 1$

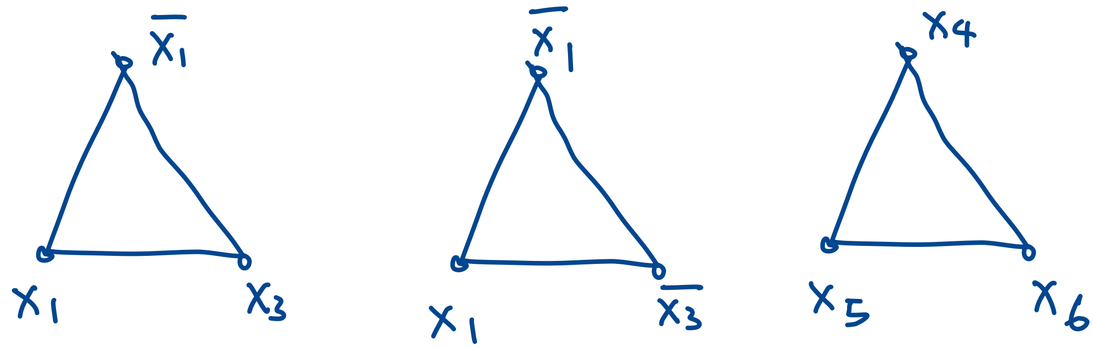
m triangles = m classes

$|MIS| \leq m$

$F = (\overline{x_1} \vee x_3 \vee x_1) \wedge (\overline{x_2} \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$

$\downarrow$

G:



If $|MIS| = m$, Answer "Yes" to 3-SAT.

If $|MIS| < m$, Answer "No" to 3-SAT.