Lecture 2 Note　　　　　　　　　　　　　　　　　　　　　　　Wednesday, 02/02/2022

Announcements:
- The Lecture Recordings will be available on the following YouTube Playlists Link: https://youtube.com/playlist?list=PLZaTmV9UMKlgYpo2cAiMaEWxqyvbiXDFd

Quick Review of what we did on Lecture 1

Continue with Growth of functions Exercises
- Show that $f(n) = \frac{n^3+2n}{2n+1}$ is $\Omega(n^2)$.
  We want to show that $\frac{n^3+2n}{2n+1} \geq c \cdot n^2$ whenever $n \geq n_0$ for some $c > 0$ and $n_0 \geq 0$.
  To underestimate the fraction, we can wither make the numerator smaller or the denominator larger.
  $\frac{n^3+2n}{2n+1} \geq \frac{n^3}{2n+1} \geq \frac{n^3}{2n+n} = \frac{n^3}{3n} = \frac{1}{3}n^2$ for $n \geq 1$. We have found $c = \frac{1}{3}$ and $n_0 = 1$.
  Since $f(n)$ is both $O(n^2)$ and $\Omega(n^2)$, therefore it's also $\Theta(n^{k+1})$.

- Let k be positive integer. Show that $1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$.
  We want to show that $1^k + 2^k + \cdots + n^k \leq c \cdot n^{k+1}$.
  $1^k + 2^k + \cdots + n^k \leq n^k + n^k + n^k + \cdots + n^k = n \cdot n^k = n^{k+1}$ for $n \geq 1$.
  We found that found $c = 1$ and $n_0 = 1$.

- Let k be positive integer. Show that $1^k + 2^k + \cdots + n^k$ is $\Omega(n^{k+1})$.
  We want to show that $1^k + 2^k + \cdots + n^k \geq c \cdot n^{k+1}$.
  $1^k + 2^k + \cdots + n^k \geq \left(\frac{n}{2}\right)^k + \left(\frac{n}{2}+1\right)^k + \cdots + n^k \geq \left(\frac{n}{2}\right)^k + \left(\frac{n}{2}\right)^k + \cdots + \left(\frac{n}{2}\right)^k = \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^k = \left(\frac{n}{2}\right)^{k+1} = \frac{n^{k+1}}{2^{k+1}} = \frac{1}{2^{k+1}}(n^{k+1})$
  for $n \geq 1$. $\frac{1}{2^{k+1}}$ is a constant. So, we found $c = \frac{1}{2^{k+1}}$ and $n_0 = 1$.
  We have shown that $1^k + 2^k + \cdots + n^k$ is both $O(n^{k+1})$ and $\Omega(n^{k+1})$, therefore it's $\Theta(n^{k+1})$ as well.

- Prove that 2^n is not $O(n^2)$.
  If 2^n is $O(n^2)$, then $2^n \leq c \cdot n^2$. We want to disprove it.
  If $2^n \leq c \cdot n^2$ is true, then $\frac{2^n}{n^2} \leq c$.
  We want to make 2^n and n^2 the same base, so we can check whether $\frac{2^n}{n^2} = \frac{2^n}{2^?} \leq c$.
  Well, $2^{\log_2 x} = x$, so we can have $n^2 = 2^{\log_2 n^2} = 2^{2\log_2 n}$.
  Then, $\frac{2^n}{n^2} = \frac{2^n}{2^{2\log_2 n}} = 2^{n-2\log_2 n}$, $2^{n-2\log_2 n}$ increases as n increases,
  therefore, $\frac{2^n}{n^2}$ cannot be bounded by a constant, which $\frac{2^n}{n^2} \nleq c$.
  Then $2^n \nleq c \cdot n^2$ and we have shown that 2^n is not $O(n^2)$.

Review of Logarithms & Exponentiations
- $\log_b n = x \iff b^x = n$
  - $\log_9 729 = 3$ since $9 \times 9 \times 9 = 9^3 = 729$
  - $\log_3 729 = 6$ since $(3^2)^3 = 729 \leftrightarrow 3^6 = 729$.
- $\log_b(n^k) = k\log_b n$
  - $b^x = n,\ n^k = (b^x)^k = b^{xk} \rightarrow \log_b(n^k) = \log_b(b^{xk}) = xk = k\log_b n$
- $\log_b(nm) = \log n + \log m$
  - $b^x = n,\ b^y = m, nm = b^x b^y = b^{x+y} \rightarrow \log_b(nm) = x + y = \log n + \log m$
- $\log_b\left(\frac{n}{m}\right) = \log n - \log m$
  - $b^x = n,\ b^y = m, \frac{n}{m} = \frac{b^x}{b^y} = b^{x-y} \rightarrow \log_b\left(\frac{n}{m}\right) = x - y = \log n - \log m$

- $\log_b n = \frac{\log_a n}{\log_a b}$
    - $b^x = n$, $a^p = n$, and $a^q = b$
      $\rightarrow\ (a^q)^x = a^{qx} = n,\ \log_a n = qx,\ q = \log_a b$
      $\rightarrow\ \log_b n = x = \frac{qx}{q} = \frac{\log_a n}{\log_a b}$.

- $\log_b n = \Theta(\log_a n)$
  To show $\log_b n$ is $\Theta(\log_a n)$, we want to have constant c_1 and c_2 where $c_1 \cdot \log_a n \le \log_b n \le c_2 \cdot \log_a n$ for $n \ge n_0$.
  We know that $\frac{1}{\log_a b} \cdot \log_a n = \log_b n$, then $\frac{1}{2 \cdot \log_a b} \cdot \log_a n \le \log_b n \le \frac{2}{\log_a b} \cdot \log_a n$.

- Show that $\log(n!)$ is $\Theta(n\log(n))$.
  In order to show $\log(n!)$ is $\Theta(n\log(n))$, we need to have both $\log(n!) = O(n\log(n))$ and $\log(n!) = \Omega(n\log(n))$.
  To show $\log(n!) = O(n\log(n))$, we want $\log(n!) \le c \cdot n\log(n) = c \cdot \log(n^n)$,
  $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \le n \cdot n \cdot n \cdot \ldots \cdot n = n^n$, since $n! \le n^n$, $\log(n!) \le \log(n^n) = n\log(n)$ for $c = 1$ and $n_0 = 1$. Therefore, $\log(n!) = O(n\log(n))$.
  To show $\log(n!) = \Omega(n\log(n))$, we want $\log(n!) \ge c \cdot n\log(n) = c \cdot \log(n^n)$,
  $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \ge \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) \ldots \cdot n \ge \left(\frac{n}{2}\right)\left(\frac{n}{2}\right) \ldots \left(\frac{n}{2}\right) = \left(\frac{n}{2}\right)^{\frac{n}{2}}$, then
  $\log(n!) \ge \log\left(\frac{n}{2}\right)^{\frac{n}{2}} = \frac{n}{2}\log\frac{n}{2} = \frac{n}{2}(\log n - \log 2) \ge \frac{n}{2}(\log n - 1)$, $(\log n - 1) \ge \frac{\log n}{2}$ when $n \ge 100$.
  Then $\log(n!) \ge \frac{n}{2}(\log n - 1) \ge \frac{n}{2} \cdot \frac{\log n}{2} = \frac{1}{4} \cdot n\log n$ for $c = \frac{1}{4}$ and $n_0 = 100$.
  Therefore, $\log(n!) = \Omega(n\log(n))$.
  Since $\log(n!)$ is both $O(n\log(n))$ and $\Omega(n\log(n))$, it's $\Theta(n\log(n))$.

- Solve for x and y in the following system of equations:
  $\begin{cases} 2^{\frac{1}{3}\log_2 y^3} + 4^{\log_2 \sqrt{x}} = 5 \\ 8^{\log_4 (x^{2/3})} \cdot 9^{\log_9 y} = 6 \end{cases}$
  $2^{\frac{1}{3}\log_2 y^3} = 2^{\frac{1}{3}\cdot 3 \cdot \log_2 y} = 2^{\log_2 y} = y$, $4^{\log_2 \sqrt{x}} = 4^{\log_2 x^{\frac{1}{2}}} = 4^{\frac{1}{2}\cdot \log_2 x} = \left(4^{\frac{1}{2}}\right)^{\log_2 x} = 2^{\log_2 x} = x$
  $8^{\log_4 (x^{2/3})} = 8^{\frac{2}{3}\cdot \log_4(x)} = \left(8^{\frac{2}{3}}\right)^{\log_4 x} = \left(\left(8^{\frac{1}{3}}\right)^2\right)^{\log_4 x} = ((2)^2)^{\log_4 x} = (4)^{\log_4 x} = x$, $9^{\log_9 y} = y$
  So, the system of equations can be simplified to $\begin{cases} x + y = 5 \\ x \cdot y = 6 \end{cases}$.
  The solution can be either $\begin{cases} x = 2 \\ y = 3 \end{cases}$ or $\begin{cases} x = 3 \\ y = 2 \end{cases}$.

What to expect or prepare for the next class:
- We will finish up the review of growth of function and look at some common running times.
- Expect an in-class quiz next class that will count toward the assignment/project credit.
    - You will only earn the credits if you get the questions correct. If you get the questions wrong, you won't get the credits but there is no penalty either.

Reading Assignment
Algorithm Design: 2.1, 2.2, 2.4

Suggested Problems
- Algorithm Design – Chapter 2 – 1,2,3,4,5,8
- Discrete Mathematics and its Application
    - 3.1 - 9, 13, 19, 23, 27, 33, 41
    - 3.2 - 5, 9, 18, 21, 22, 24, 27, 33, 41, 42, 48