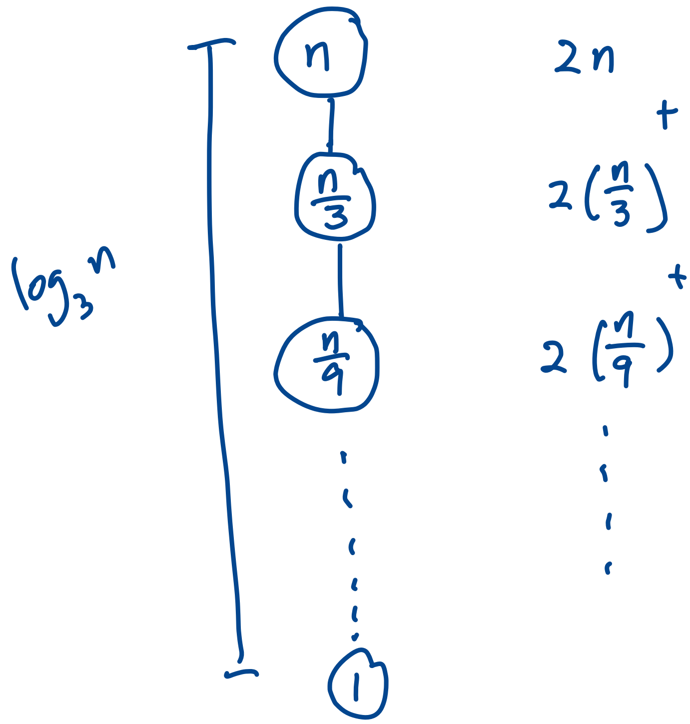
Divide and Conquer

Review: $T(n) = T\left(\frac{n}{3}\right) + 2n$



$$T(n) = 2n + 2\left(\frac{n}{3}\right) + 2\left(\frac{n}{9}\right) + \cdots\cdots$$

$$= 2n\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\cdots\right)$$

$$= 2n\left(\frac{1}{1-\frac{1}{3}}\right)$$

$$= 2n\left(\frac{1}{\frac{2}{3}}\right)$$

$$= 2n\left(\frac{3}{2}\right)$$

$$T(n) = 3n = O(n)$$

$a < 1$.

$$1 + a + a^2 + a^3 + \cdots$$

$$= \frac{1}{1-a}$$

$a > 1$

$$1 + a + a^2 + \cdots + a^n$$

$$\frac{a^{n+1} - 1}{a - 1}$$

$O(n)$

Linear-Time Median Finding

Question: Given n unsorted numbers, find the median.

Naive: Sort the numbers, select $\frac{n}{2}$th entry. $O(n \log n)$

What recurrence relation solves to $O(n)$?

$$T(n) = T(an) + T(bn) + O(n)$$

where $a + b < 1$

Example: $T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{n}{2}\right) + O(n)$

$a = \frac{1}{3}$, $b = \frac{1}{2}$, $a+b = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < 1$.

$T(n) = O(n)$ ← $\boxed{\text{Exercise}}$

Our algorithm will satisfy this relation.

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n). \Rightarrow O(n).$$

$a = \frac{1}{5}$, $b = \frac{7}{10}$, $a + b = \frac{1}{5} + \frac{7}{10} = \frac{9}{10} < 1$.

Rank of a element e in a list/array is the number of keys (value) smaller (or equal to) than e.

median has rank $\lceil \frac{n}{2} \rceil$.

Input n unsorted numbers. $T(n)$

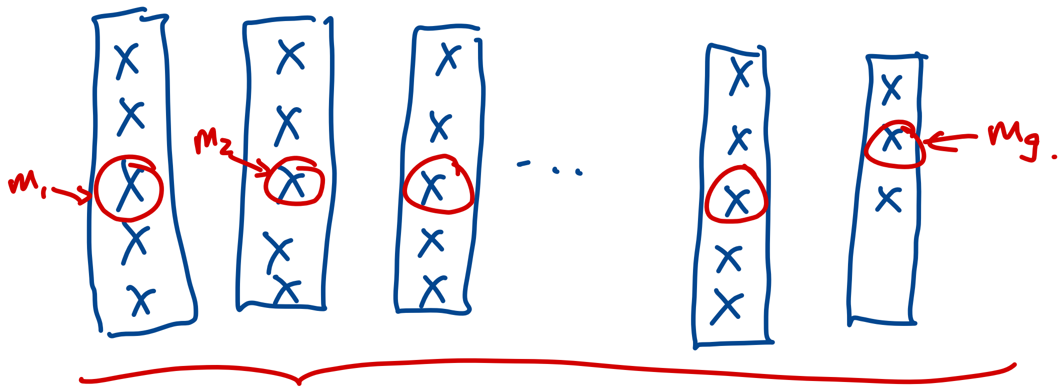
1) Partition input into groups of size 5. $O(n)$

(last list/group may have less than 5 element).

2) Sort each group and find its median. $O(\frac{n}{5})$

$5 \log 5 = O(1)$ $\quad\quad O(1)$

For all groups, $g = \frac{n}{5}$, $\quad \frac{n}{5} O(1) = O(\frac{n}{5})$



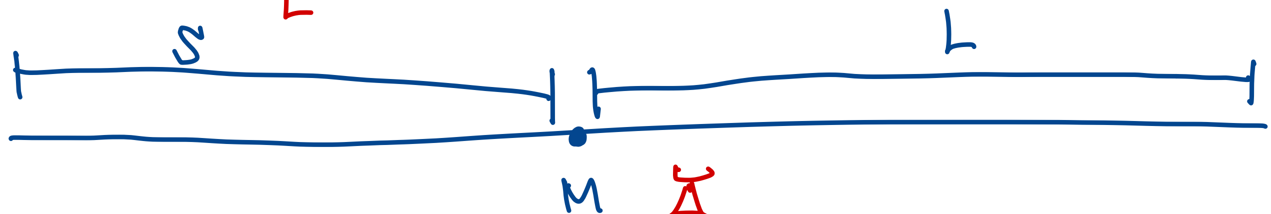
$g = \lceil \frac{n}{5} \rceil$ groups.

3) Recursively find median $(m_1, m_2, \dots, m_g)$

Say $M = \text{median}(m_1, m_2, \dots, m_g)$ $\quad \frac{n}{5}$. $\quad T(\frac{n}{5})$

(Quick Sort)

4) Partition the input into $\underbrace{\text{keys smaller than M}}_{S}$ and $\underbrace{\text{keys greater than M}}_{L}$.



rank K

$\text{Rank}(M) = r = |S|$ $\quad\quad O(n)$

$K = \lceil \frac{n}{2} \rceil$

5) if rank $(M) > k$, recursively find rank k^{th} element in S. $T(|S|)$

if rank $(M) = k$, output M. $\leftarrow O(1)$

if rank $(m) < k$, recursively find rank $(k-r)^{th}$ element in L. $T(|L|)$

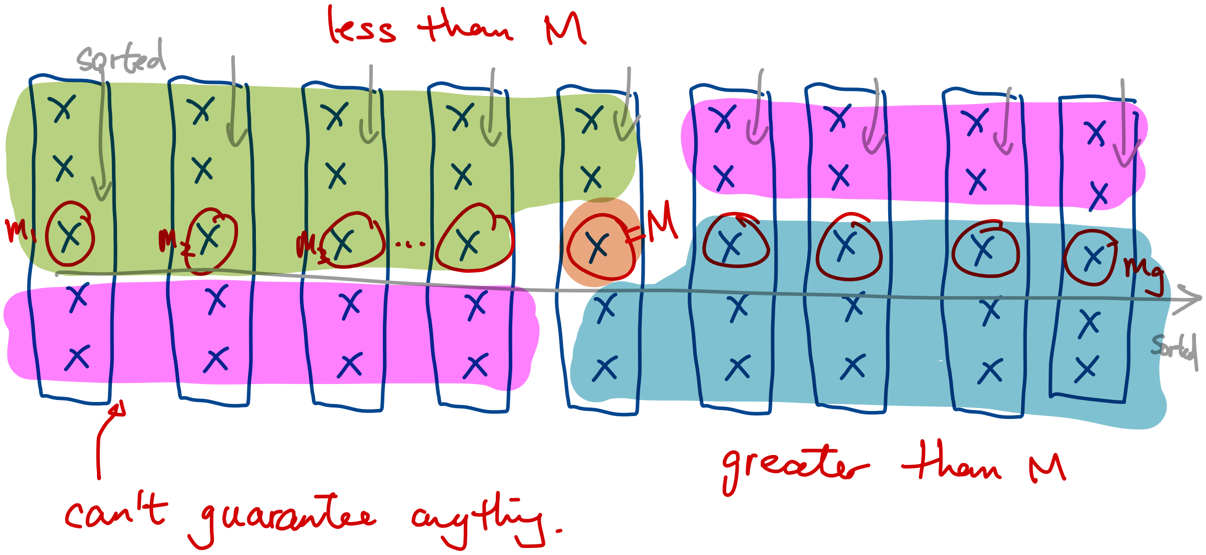
Recurrence : $T(n) = T(n/5) + T(|S|) + O(n)$

or

$T(|L|)$

$|S| \leq \frac{7}{10} n$ and $|L| \leq \frac{7}{10} n$.

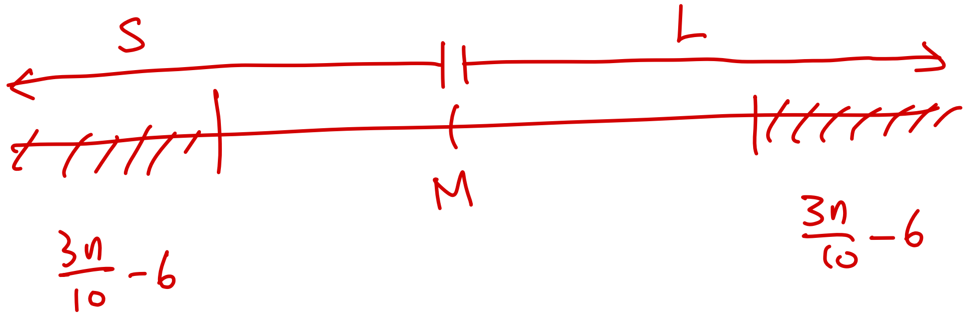
$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7}{10} n\right) + O(n)$$



$\frac{n}{5} \cdot \frac{1}{2} \cdot 3 - 6$ element smaller than M.

$\frac{n}{5} \cdot \frac{1}{2} \cdot 3 - 6$ element greater than M.

$$3 \left\lceil \frac{\lfloor \frac{n}{5} \rfloor + 1}{2} \right\rceil - 1 \geq \frac{3n}{10} - 6$$



if $|S| \geq \frac{3n}{10} - 6$, then $|L| \leq \frac{7n}{10} + 6$

if $|L| \geq \frac{3n}{10} - 6$, then $|S| \leq \frac{7n}{10} + 6$

Example of Median Find (Median of Medians). $k = \lceil \frac{17}{2} \rceil = 9$

$$\begin{array}{ccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array}$$

$A = [3, 7, 2, 10, 9, 8, 5, 13, 20, 17, 11, 6, 19, 1, 21, 15, 12]$

MF(A, k=9)

①
3 7 2 10 9 → 2 3 7 9 10 → 7
8 5 13 20 17 → 5 8 13 17 20 → 13
11 6 19 1 21 → 1 6 11 19 21 → 11
15 12 → 12 15 → 15

→ 7 11 13 15

M = 13.

② QuickSort(A, M) partition into S, L. M = 13

$$\begin{array}{ccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array}$$

$A = [3, 7, 2, 10, 9, 8, 5, \cancel{13}, \cancel{20}, \cancel{17}, \cancel{11}, \cancel{6}, 19, \cancel{1}, 21, 15, \cancel{12}]$

12 ~~12~~ ~~1~~ ~~13~~ 13 17 ~~9~~ ~~9~~ 20
~~13~~ ~~13~~ 6
1 11

rank(13) = 12 > 9

S = A[1, rank(M−1)] = {3, 7, 2, 10, 9, 8, 5, 12, 1, 11, 6}

L = A[rank(M)+1, n] = {19, 17, 21, 15, 20}

MF(S, k=9)

S = {3, 7, 2, 10, 9, 8, 5, 12, 1, 11, 6}

3 7 2 10 9 → 2 3 7 9 10 → 7
8 5 12 1 11 → 1 5 8 11 12 → 8
6 → 6 → 6

→ 6, 7, 8

M = 7.

Quick Sort(S, 7)

S = {3, 7, ~~2~~, ~~10~~, ~~9~~, ~~8~~, ~~5~~, 12, ~~1~~, 11, ~~6~~}

2 ~~7~~ 6 ~~1~~ ~~8~~ 8 9 10
6 ~~7~~ ~~7~~ 7
1 5

rank(7) = 6 < 9

S = {3, 2, 6, 1, 5}

L = {8, 12, 9, 11, 10}

$\Downarrow$

$MF(L,\ k = 9-6 = 3)$

$8, 12, 9, 11, 10 \rightarrow 8, 9, 10, 11, 12 \rightarrow 10$

$M = 10$

$\text{Quick Sort}(L,\ 10)$

$8,\ \cancel{12},\ \cancel{9},\ 11,\ \cancel{10}$

$\cancel{10}\quad 10\quad 12$

9

$\text{rank}(10) = 3 = k$

Median $= 10$ ✓

$k=9$
$\downarrow$

$$
\begin{array}{cccccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
A = & [3, & 7, & 2, & 10, & 9, & 8, & 5, & 13, & 20, & 17, & 11, & 6, & 19, & 1, & 21, & 15, & 12] \\
 & 1 & 2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 17 & 19 & 20 & 21
\end{array}
$$

$\uparrow$
median