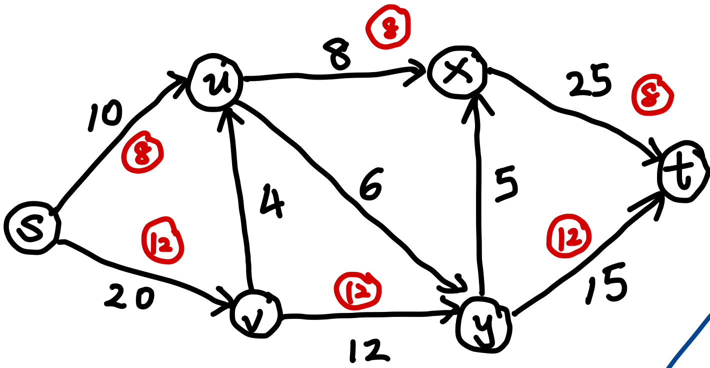
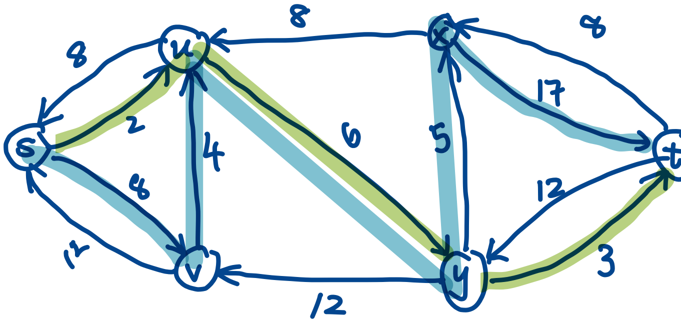
Network Flow [Chapter 7]



1) What is the current flow value?

20

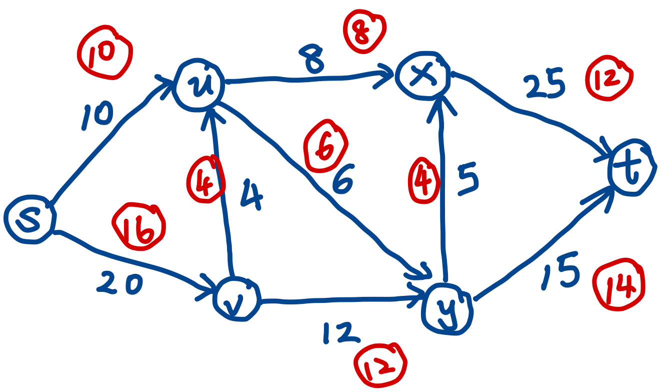
2) Draw the Residual (G, f).

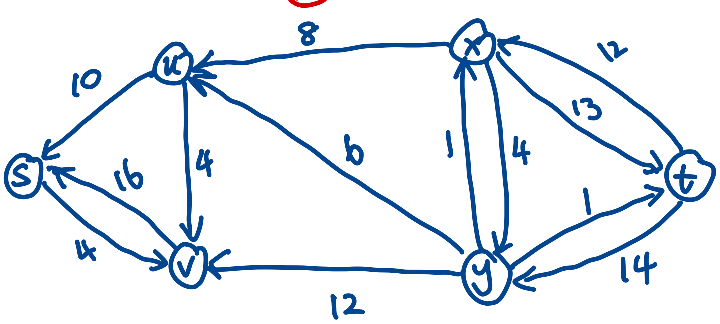


3) Does the Residual (G, f) have any more s-t path? If so, find it and update the flow, until you get the max flow.

s, u, y, t → push 2 on flow.

s, v, u, y, x, t → push 4 on flow.





Max flow : 26.

Ford - Fulkerson Algorithm

Initially $f(e) = 0$ for edges.

BFS : $O(m+n) = O(m)$

While there is an s-t path in Residual (G, f)

Let p be such a simple (no cycle) path.

$f' = \text{augment}(f, p)$ ["push" flow along p] $O(n)$

Update f to f'

Update $R(G, f)$ to $R(G, f')$ $\Big\} \leftarrow O(m+n) = O(m)$

endwhile.

augment (f, p):

$O(n)$

$b = \text{bottleneck}(p)$ [find the min capacity of the edge on p]

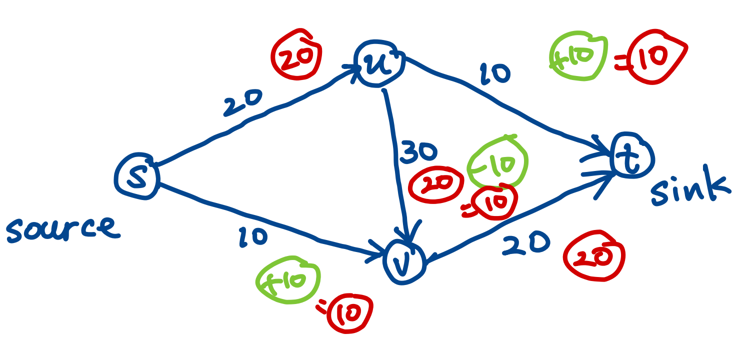
for all $e \in P$. (at most $n-1$) edges

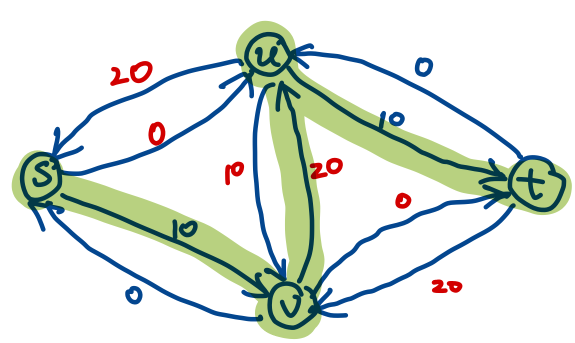
$O(n)$

if e is a forward edge $f'(e) = f(e) + b$.

if e is a backward edge $f'(e) = f(e) - b$.

endfor.





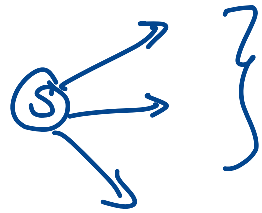
Each iteration of FF algorithm takes $O(m+n) = O(m)$

$m \geq n$.

How many iterations does FF take?

1) Every iteration increase the value of flow by at least 1. $(b \geq 1)$.

2) Define $C = \sum_{e \text{ out of } s} c(e)$



$t = C$.

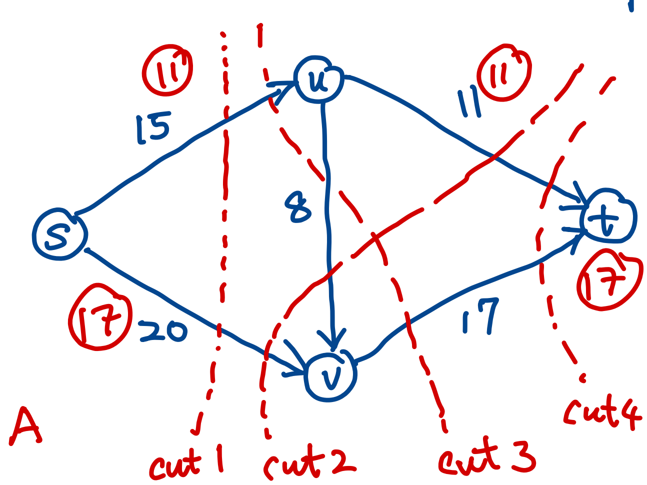
FF takes at most C iterations. Max Flow $\leq C$.

Total Runtime : $O(mC)$.

Minimum Cut Problem:

Given a flow network, dividing the nodes of the graph into two set A and B such that $s \in A$ and $t \in B$.

A cut of a flow is an upper bound on the max flow value for the flow across from A to B. i.e. $cut(A,B) = \sum_{e \text{ out of } A} c(e)$



$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = f^{out}(A) - f^{in}(A)$

cut $\geq$ flow

cut1 $(A,B) = 15 + 20 = 35$

cut2 $(A,B) = 11 + 8 + 20 = 39$

cut3 $(A,B) = 15 + 17 = 32$

cut4 $(A,B) = 11 + 17 = 28$ ← = max-flow.

minimum cut.

Max-Flow Min-Cut Theorem

In every flow network, the maximum value of a flow is equal to the minimum capacity of a cut.

1) If f is a max-flow in G, then there is no s-t path in $R(G,f)$.

Pf. Assume f is a max-flow, and there is a s-t path P.

Then $f' = f + f(P)$ is a flow and

$$val(f') = val(f) + \underbrace{val(P)}_{\text{at least } 1} > \underbrace{val(f)}_{\text{max flow.}} \quad \text{contradiction.}$$

2) If there is no s-t path in $R(G,f)$, then there exists a cut (A,B) where $cut(A,B) = val(f)$.

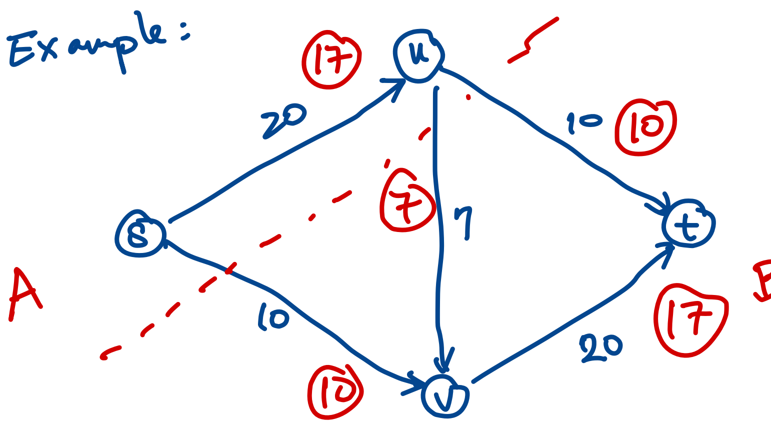
Let A be the set of vertices reachable in $R(G,f)$ (MR) from s.

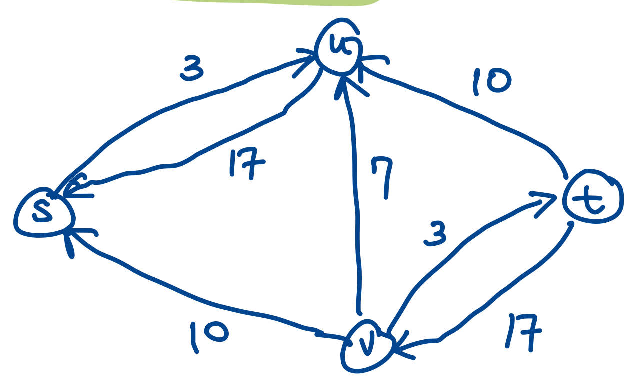
and $B = V - A$.

$(u,v) \in A \times B \rightarrow f(u,v) = c(u,v)$

$(u,v) \in B \times A \rightarrow f(u,v) = 0$

Example:





$val(f) = 27$

$cut(A,B) = 10 + 7 + 10 = 27$

$A = \{s, u\}$

$B = \{v, t\}$

Recall: $val(f) = f^{out}(A) - f^{in}(A) = \sum_{e \text{ out of } A} f(e) = \sum_{e \text{ out of } A} c(e)$

$= cut(A,B)$

3) If there exists a cut (A,B) s.t $cut(A,B) = val(f)$, then f is max-flow.

Assume (A,B) is a cut where $cut(A,B) = val(f)$.
and f is not max-flow. $\rightarrow \exists f' : val(f') > val(f)$

$$\underbrace{val(f') > val(f) = cut(A,B)}_{} \quad ↯$$

$\uparrow$ upper bound of any flow.

Thus, min-cut = max-flow.