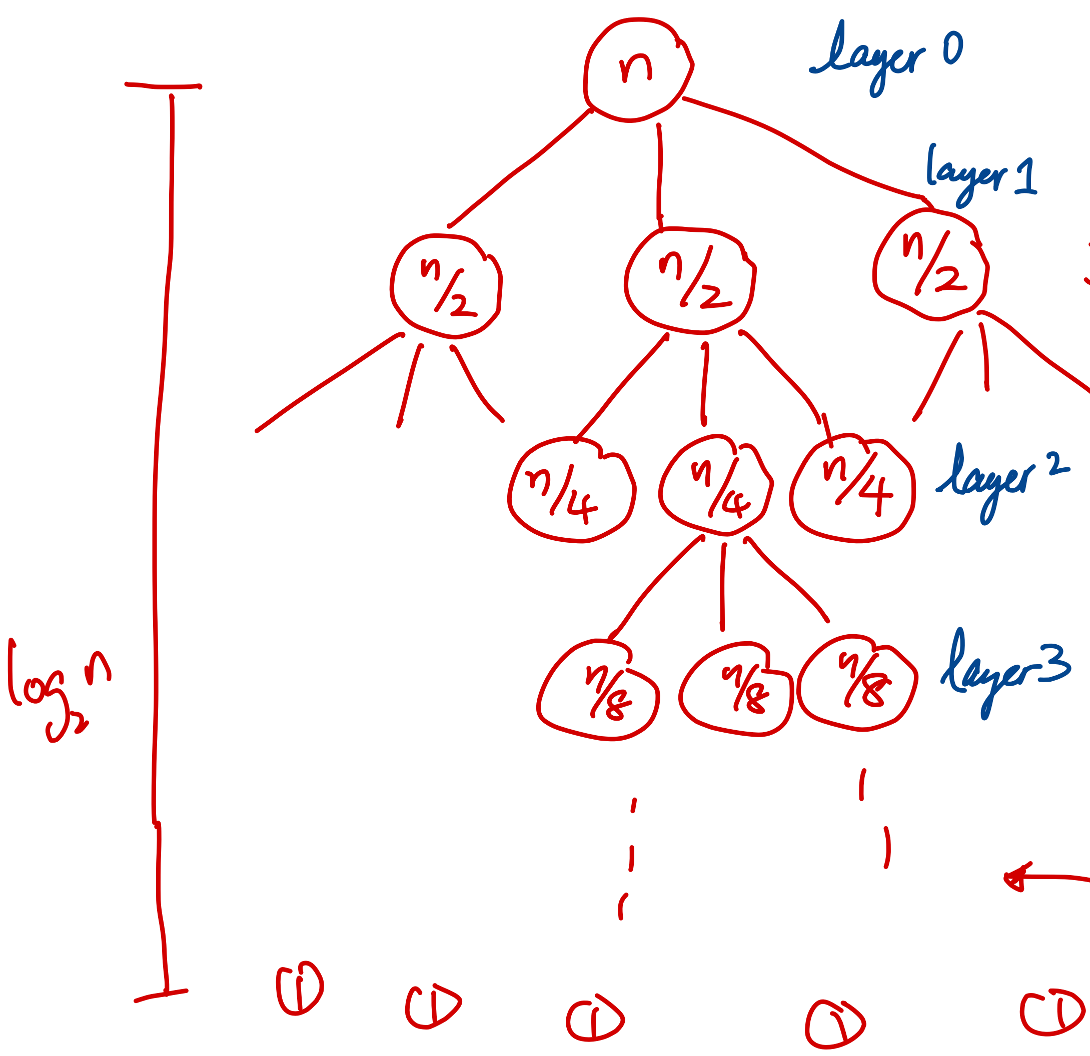
Divide and Conquer

- Recurrence Relations

$$T(n) = 3T\left(\frac{n}{2}\right) + n$$



layer $\log_2 n - 1$

$\boxed{n}$

$3 * \frac{n}{2} = \boxed{\frac{3n}{2}}$ $\qquad T\left(\frac{n}{2}\right) = 3T\left(\frac{n}{4}\right) + \frac{n}{2}$

$9 * \frac{n}{4} = \boxed{\frac{9n}{4}}$ $\qquad T\left(\frac{n}{4}\right) = 3T\left(\frac{n}{8}\right) + \frac{n}{4}$

$\boxed{\frac{27n}{8}}$

i^{th} layer = # of node * (time taken for each)

$3^i * \frac{n}{2^i}$

$\boxed{\frac{3^i n}{2^i}}$

$$T(n) = n + \frac{3n}{2} + \frac{9n}{4} + \frac{27n}{8} + \cdots +$$

$$= \sum_{i=0}^{\log_2 n - 1} \left(\frac{3}{2}\right)^i \cdot n = n \sum_{i=0}^{\log_2 n - 1} \left(\frac{3}{2}\right)^i$$

$$\boxed{1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}}$$

$$= n \underbrace{\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^{\log_2 n - 1}\right)}$$

$$= n \left(\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1}\right) \quad \frac{1}{2}$$

$$= 2n\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right) \leq 2n\left(\frac{3}{2}\right)^{\log_2 n}$$

$$= 2\cancel{n} \frac{3^{\log_2 n}}{\cancel{2^{\log_2 n}}} = 2 \cdot 3^{\log_2 n}$$

$$= 2 \cdot n^{\log_2 3} = 2 \cdot n^{1.585}$$

$$= O(n^{1.59})$$

Logarithmic Rule:

For $a > 1$ and $b > 1$.

$$a^{\log b} = b^{\log a}$$

Integer Multiplication

$$T(n) = 3\underbrace{T(n/2)}_{\substack{\text{time taken to} \\ \text{multiply two} \\ \frac{n}{2}\text{-digit numbers.}}} + n$$

$T(n)$: time taken to multiply two n-digit numbers

Example: $n = 4$

$$\begin{array}{r} 2311 \\ \times\ 3234 \end{array}$$

Input: 2311, 3234

11×34 (A) $\quad 23 \times 32$ (B) $\quad 34 \times 66$ (C)

Output: $23 \times 32 \times 10000 + (34 \times 66 - 11 \times 34 - 23 \times 32) \times 100 + 11 \times 34$

Observe. $2311 \times 3234 = (23 \times 100 + 11) \times (32 \times 100 + 34)$

$= (23 \times 32 \times 10000) + (23 \times 34 \times \underline{100}) + (11 \times 32 \times \underline{100}) + (11 \times 34)$

$= 23 \times 32 \times 10000 + (23 \times 34 + 11 \times 32) \times 100 + 11 \times 34$

if we did

$11 \times 34 \quad 23 \times 32 \quad 23 \times 34 \quad 11 \times 32$

$T(n) = 4T(n/2) + n$

Ex: $O(n^2)$

$23 \times 34 + 11 \times 32 \quad\quad 23 \times 32$ (A) $\quad\quad 11 \times 34$ (B)

$= C - A - B$

$34 \times 66 = (23 + 11) \times (34 + 32)$ (C)

$= 23 \times 34 + 23 \times 32 + 11 \times 34 + 11 \times 32$ (A) (B)

multiply (X, Y)

$X = X_\ell \cdot 10^{n/2} + X_r$

$Y = Y_\ell \cdot 10^{n/2} + Y_r$

Compute $X_\ell + X_r$

$Y_\ell + Y_r$

$X = x_1 x_2 x_3 \cdots x_n$

$Y = y_1 y_2 y_3 \cdots y_n$

$X_\ell = x_1 x_2 \cdots x_{n/2}$

$X_r = x_{n/2+1} x_{n/2+2} \cdots x_n$

$X = 2311$

$X_\ell = 23 \quad X_r = 11$

$2311 = X_\ell \cdot 10^2 + X_r$

$P_1 = \text{multiply}(X_l, Y_l) \qquad T(n/2)$

$P_2 = \text{multiply}(X_r, Y_r) \qquad T(n/2)$

$P_3 = \text{multiply}(X_l + X_r, Y_l + Y_r) \qquad T(n/2)$

Output : $P_1 \cdot 10^n + (P_3 - P_1 - P_2) \cdot 10^{n/2} + P_2$

$$T(n) = 3T(n/2) + O(n)$$
$$= O(n^{1.59})$$

Naive : $P_4 = \text{multiply}(X_l, Y_r) \quad P_5 = \text{multiply}(X_r, Y_l)$

Output : $P_1 \cdot 10^n + (P_4 + P_5) \cdot 10^{n/2} + P_2$

$$T(n) = 4T(n/2) + n = O(n^2)$$

Matrix Multiplication

A, B be $n \times n$ matrices . $C = A \cdot B$

$$\begin{bmatrix} \bigcirc & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} = C = \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 \\ 4 & 2 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & 3 & 1 & 1 \\ 3 & 2 & 5 & 6 \\ 4 & 1 & 2 & 2 \\ 2 & 3 & 6 & 0 \end{bmatrix}}_{B}$$

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 2$

Naive : 64 single-digit multiplication $\quad O(n^3)$

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

$T(n) = 8\,T(n/2) + n \quad \leftarrow \quad$ Ex: $O(n^3)$

Strassen's Algorithm : reduce to 7

Maximum Subarry Problem

Input : Array $A = [-2, 4, 6, -5, -4, 2, -7, 8]$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$

$S(3,5) = -3$, $S(2,3) = 10$ which is largest.

Output : Possitions i and j and $S(i,j)$ which is largest possible.

Naive : nested for loop check all subarry. $O(n^2)$

Use D & C. in

Recurrence Relation : $T(n) = 2T\left(\frac{n}{2}\right) +$

$A_l \quad\quad\quad A_r \quad\quad A$

maxsub(A): $\leftarrow$ BC

$A_l = A[1, \frac{n}{2}]$, $A_r = [\frac{n}{2}+1, n]$ $\quad\quad \frac{n}{2} + \frac{n}{2} = n$

$(i_l, j_l, s(i_l, j_l)) = \text{maxsub}(A_l)$ $\quad T(n/2)$

$(i_r, j_r, s(i_r, j_r)) = \text{maxsub}(A_r)$ $\quad T(n/2)$

$(i_m, j_m, s(i_m, j_m)) = \text{mid best}$ $\quad \leftarrow n$

return maximum of the 3 above.

$$T(n) = 2T(n/2) + O(n)$$
$$= O(n \log n)$$

$\boxed{(2, 3, 10)}$ (i, j, $s(i,j)$)

$\max(10, 8, 4) = 10$

Example :

$A = [\underbrace{-2, \boxed{4, 6,} -5}_{10}, \underbrace{-4, 2, -7, 8}_{8}]$ $\quad m = (5) + (-1) = 4$

$\underbrace{-2, 4}_{4}, \underbrace{6, -5}_{6}$ $\quad m = 10$ $\quad\quad \underbrace{-4, 2}_{2}, \underbrace{-7, 8}_{8}$ $\quad m = 3$

$\underbrace{-2}_{-2}, \underbrace{4}_{4}$ $\quad m = 2$ $\quad\quad \underbrace{6}_{6}, \underbrace{-5}_{-5}$ $\quad m = 1$ $\quad\quad -4, 2 \quad\quad -7, 8$

$-2 \quad 4 \quad\quad 6 \quad -5$

$$T(n) = AT\left(\frac{n}{B}\right) + f(n)$$