Subject: 證明集　　No.: 1　Date: ..../..../....

(I) 夾擠法 (考最多)

(II) 數學歸納法

(III) 計算證明法

(IV) 邏輯推理法

(V) 反證法

(I) 夾擠法　<證> 左式 = 右式

↓(I) 左式 = …

(III) 為何會相等

↑(II) 右式 = …

EX: $A_{m\times n}$ , $B_{m\times n}$　證 $(AB)^T = B^TA^T$　<思考> 令 $C = (AB)^T$　$D = B^TA^T$

證 $C = D \leftrightarrow c_{ij} = d_{ij}$ ; $\forall i, j$

<策略> 夾擠法　令 $(AB)^T = C$ , $B^TA^T = D$

$$c_{ij} = [(AB)^T]_{ij} = [AB]_{ji} \quad \downarrow(I) \quad = \sum_{k=1}^{n} [A]_{jk}[B]_{ki}$$

$$(III) \quad = \sum_{k=1}^{n} [B]_{ki}[A]_{jk}$$

$A^T = B \leftrightarrow a_{ij} = b_{ji}$

$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}$

$$\uparrow(II) \quad = \sum_{k=1}^{n} [B^T]_{ik}[A^T]_{kj}$$

$$= [B^TA^T]_{ij} = d_{ij} \ ; \ \forall i, j$$　得證

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<習作> $A_{m\times n}$ , $B_{m\times n}$ 則 $AB = C_{m\times m}$ ; $BA = D_{n\times n}$ <證> $Tr(C) = Tr(D)$

(I) $c_{11} + c_{22} + \dots + c_{mm}$        求 $c_{11}\ c_{22}\ c_{mm}$

(III) $= d_{11} + d_{22} + \dots + d_{nn}$        $d_{11}\ d_{22}\ d_{nn}$

(II) $= Tr(D) = Tr(BA)$

$$
\begin{aligned}
tr(AB) = tr(C) &= \text{(I)}\quad c_{11} + c_{22} + \dots + c_{mm} \\
&= \text{(III)}\quad (a_{11}b_{11} \dots + a_{1n}b_{n1}) + (a_{21}b_{12} + \dots + a_{2n}b_{n2}) + \dots (a_{m1}b_{1m} + \dots + a_{mn}b_{nm}) \\
&= (b_{11}a_{11} + \dots + b_{1m}a_{m1}) + (b_{21}a_{12} + \dots + b_{2m}a_{m2}) + \dots + (b_{n1}a_{1n} + \dots + b_{nm}a_{mn}) \\
&= \text{(II)}\quad d_{11} + d_{22} + \dots + d_{nn} \\
&= tr(D) = tr(BA)
\end{aligned}
$$

$C = AB$    $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$    $d_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$    $D = BA$

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(Ⅱ) 數學歸納法　證公式用(有 $n=1,2,3\ldots$)

(Ⅰ) 證 $n=1$，公式成立

(Ⅱ) 假設 $n=k$，公式成立　拿來用

(Ⅲ) 證 $n=k+1$，公式成立

EX4: $A=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$，證 $A^n=\begin{bmatrix}\cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta)\end{bmatrix}$

<思考> 公式中有 $n=1,2,3\ldots\in N \longrightarrow$ 數學歸納法

(Ⅰ) $n=1$　$A^1=\begin{bmatrix}\cos(1\theta) & -\sin(1\theta) \\ \sin(1\theta) & \cos(1\theta)\end{bmatrix}$，公式成立

(Ⅱ) 假設 $n=k$，$A^k=\begin{bmatrix}\cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta)\end{bmatrix}$ 公式成立

(Ⅲ) $n=k+1$，$A^{k+1}=A^k\cdot A=\begin{bmatrix}\cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta)\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

$$=\begin{bmatrix}\cos(k\theta)\cos\theta-\sin(k\theta)\sin\theta & -\cos(k\theta)\sin(\theta)-\sin(k\theta)\cos(\theta) \\ \sin(k\theta)\cos\theta+\cos(k\theta)\sin\theta & -\sin(k\theta)\sin(\theta)+\cos(k\theta)\cos(\theta)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta\end{bmatrix}$$ 公式成立，得證

EX 33: show by induction that

$$\det\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{j>i} (x_j - x_i)$$

<Sol> (I) $n=2$ ; $\det\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = x_2 - x_1$ ，公式成立

(II) 假設 $n=k$ $\det\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{j>i} (x_j - x_i)$

(III) 當 $n=k+1$ $\det\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} & x_1^k \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^{k-1} & x_{k+1}^k \end{bmatrix}$

$\times(-x_1)$ $\times(-x_1)$ $\times(-x_1)$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{k+1}-x_1 & x_{k+1}(x_{k+1}-x_1) & \cdots & x_{k+1}^{k-2}(x_{k+1}-x_1) & x_{k+1}^{k-1}(x_{k+1}-x_1) \end{bmatrix}$$

$$= 1\times(-1)^{1+1}(x_2-x_1)(x_3-x_1)\ldots(x_{k+1}-x_1)\begin{bmatrix} 1 & x_2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & \cdots & x_{k+1}^{k-1} \end{bmatrix}$$

$$= (x_2-1)(x_3-x_1)\ldots(x_{k+1}-x_1)\prod_{i=2}^{k}(x_j-x_i) = \prod_{\substack{j>i \\ i=1}}^{k}(x_j-x_i)$$

公式亦成立，得證

〈定理〉 E為基本矩陣，A為n階方陣，則 $\det(EA) = \det(E)\det(A)$

$\det(AE) = \det(A)\det(E)$

〈證〉 (I) E：列基本矩陣

(i) $E = R_{ij} \rightarrow R_{ij}A$ = 將A之第i列、第j列對調

且 $\det(R_{ij}) = -1 \rightarrow \det(R_{ij}(A)) = -\det(A) = \det(R_{ij})\det(A)$

(ii) $E = R_i(k) \rightarrow R_i(k)A$ : 將A之第i列 ×k 且 $\det(R_i(k)) = k$

$\rightarrow \det(R_i(k)A) = k\det(A) = \det(R_i(k))\det(A)$

(iii) $E = R_{ij}(k) \rightarrow R_{ij}(k)A$ = 將A之第i列 ×k 加至第j列

且 $\det(R_{ij}(k)) = 1 \rightarrow \det(R_{ij}(k)A) = \det(A) = \det(R_{ij}(k))\det(A)$

(II) E：行基本矩陣

(i) $E = C_{ij}$ (ii) $E = C_i(k)$ (iii) $E = C_{ij}(k)$

對稱 反對稱

$V = F^{n\times n}$ $W_1 = \{A \in V \mid A^T = A\}$ $W_2 = \{A \in V \mid A^T = -A\}$

(a) 證 $W_1$ 為 $V$ 之子空間 (b) 證 $W_2$ 為 $V$ 之子空間

(I) 找出 $V$ 之零向量 $\vec{0} = ?$ 將 $\vec{0}$ 代入 $W$ 之控制方程若滿足 $\rightarrow \vec{0} \in W$

(II) $\forall \vec{x}, \vec{y} \in W$, $\alpha \in F$ (1) $\vec{x}$ 滿足控制方程 (2) $\vec{y}$ 滿足控制方程

將 $\alpha\vec{x} + \vec{y}$ 代入 $W$ 之控制方程若滿足：$\alpha\vec{x} + \vec{y} \in W$ 得證 $W$ 為 $V$ 之子空間

(a) (I) $V = F^{n\times n}$，零向量 $\vec{0} = \begin{bmatrix} 0 \end{bmatrix}_{n\times n}$ $\because 0^T = 0$ $\therefore 0 \in W_1$

(II) $\forall A, B \in W_1 \longrightarrow A^T = A$ 且 $B^T = B$

$\because (\alpha A + B)^T = \alpha A^T + B^T = \alpha A + B$ $\therefore \alpha A + B \in W_1$ 得證 $W$ 為 $V$ 之子空間

(b) (I) $V = F^{n\times n}$，零向量 $\vec{0} = \begin{bmatrix} 0 \end{bmatrix}_{n\times n}$ $\because 0^T = -0$ $\therefore 0 \in W_2$

(II) $\forall A, B \in W_2 \longrightarrow A^T = -A$ 且 $B^T = -B$

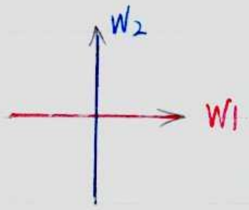
$\because (\alpha A + B)^T = \alpha A^T - B = -(\alpha A + B)$ $\therefore \alpha A + B \in W_2$ 得證 $W$ 為 $V$ 之子空間

<定理> $W_1$、$W_2$ 皆為 $V$ 之子空間，則 $W_1 \cap W_2$ 亦為 $V$ 之子空間

<觀念> $W_1$、$W_2$ 為 $V$ 之子空間，則 $W_1 \cup W_2$ 不一定為 $V$ 之子空間

<反例> $W_1 = \{ \langle a,0 \rangle ; a \in R \}$ $W_2 = \{ \langle 0,b \rangle ; b \in R \}$

$W_1 \cup W_2$ 不為 $R^2$ 子空間




<定理> $W_1$、$W_2$ 為 $V$ 之子空間，$W_1 \cup W_2$ 為 $V$ 之子空間 $\longleftrightarrow$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

<觀念> $p \leftrightarrow q$ (i) 證 $p \rightarrow q$ (ii) 證 $q \rightarrow p$

<證> (I) $W_1 \subseteq W_2$ 或 $W_2 \subseteq W_1$ $\longrightarrow$ $W_1 \cup W_2$ 為 $V$ 之子空間

若 (i) $W_1 \subseteq W_2$ $\longrightarrow$ $W_1 \cup W_2 = W_2$ 為 $V$ 之子空間

或 (ii) $W_2 \subseteq W_1$ $\longrightarrow$ $W_1 \cup W_2 = W_1$ 為 $V$ 之子空間

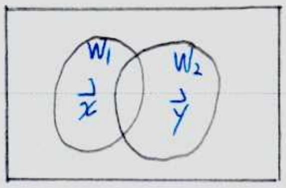
(II) $W_1 \cup W_2$ 為 $V$ 之子空間 $\longrightarrow$ $W_1 \subseteq W_2$ 或 $W_2 \subseteq W_1$

(V) 反證法 (I) 先假設與證明結論相反的論述

(II) 推導出矛盾的結果

(III) 故假設錯誤，應為正確的結論

(i) 假設 $W_1 \nsubseteq W_2$ 且 $W_2 \nsubseteq W_1$




$\exists\, \vec{x} \in W_1$ 但 $\vec{x} \notin W_2$ 且 $\exists\, \vec{y} \in W_2$ 但 $\vec{y} \notin W_1$

(ii) $\vec{x}, \vec{y} \in W_1 \cup W_2$ 且 $W_1 \cup W_2$ 為子空間，故 $\vec{x}+\vec{y} \in W_1 \cup W_2$

$\longrightarrow$ $\vec{x}+\vec{y} \in W_1$ 且 $\vec{x} \in W_1$

$\longrightarrow$ $(\vec{x}+\vec{y})+(-\vec{x}) = \vec{y} \in W_1$ : 矛盾

或 $\longrightarrow$ $\vec{x}+\vec{y} \in W_2$ 且 $\vec{y} \in W_2$

$\longrightarrow$ $(\vec{x}+\vec{y})+(-\vec{y}) = \vec{x} \in W_2$ : 矛盾

(iii) 故假設錯誤，應為 $W_1 \subseteq W_2$ 或 $W_2 \subseteq W_1$ 得證 <背>

EX: Show that (a) Row(AB) is the subspace of Row(B)

(b) if A is invertible, then Row(AB) = Row(B)

<觀念> 由於 Row(AB), Row(B) 皆已知為向量空間，故僅須證明：

(I) $\mathrm{Row}(AB) \subseteq \mathrm{Row}(B)$ $\qquad W = V \;;\; \begin{cases} (1)\ W \subseteq V \\ (2)\ V \subseteq W \end{cases}$

(II) $\mathrm{Row}(AB) = \mathrm{Row}(B)$

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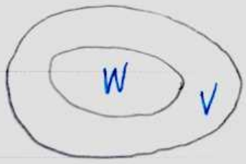
〈考型〉證 $W \subseteq V$ $\quad \forall \vec{x} \in W$ (1) → $\vec{x}$ 滿足 $W$ 之控制方程

(3) → …

(2) → $\vec{x}$ 亦滿足 $V$ 之控制方程

→ $\vec{x} \in V$




(a) $\forall \vec{x} \in Row(AB)$ (I) → $\exists \vec{w} \in F^{1\times m}$ s.t. $\vec{x} = \vec{w}AB$

(III) → 令 $\vec{w}A = \vec{v}$ , $\vec{x} = \vec{v}B$ , $\vec{v} \in F^{1\times m}$

(II) → $\vec{x} \in Row(B)$ → $Row(AB) \subseteq Row(B)$

(b) $\forall \vec{x} \in Row(B)$ → $\forall \vec{v} \in F^{1\times m}$ s.t. $\vec{x} = \vec{v}B$

$I = C^{-1}C = CC^{-1}$

$AB = AC^{-1}CB = ACC^{-1}B$

→ $\vec{x} = \vec{v}A^{-1}AB$

→ 令 $\vec{v}A^{-1} = \vec{w}$

→ $\vec{x} = \vec{w}AB$ , $\vec{w} \in F^{1\times m}$

→ $\vec{x} \in Row(AB)$

故 $Row(B) \subseteq Row(AB)$ $\quad \because Row(AB) \subseteq Row(B)$

且 $Row(B) \subseteq Row(A)$ $\quad \therefore Row(AB) = Row(B)$

EX 24: Prove if B and C are $n \times n$ matrices, then $\det(BC) = \det(B)\det(C)$

<Sol> 若 $\det(B) = 0 \longrightarrow \operatorname{rank}(B) < n$ 且 $\operatorname{rank}(BC) \leq \operatorname{rank}(B)$

故 $\operatorname{rank}(BC) < n \longrightarrow \det(BC) = 0 = 0 \cdot \det(C) = \det(B) \cdot \det(C)$

若 $\det(B) \neq 0 \longrightarrow$ B可逆，則 $B = E_1 E_2 \ldots E_k$，其中 $E_i$ 為基本矩陣

(I) $\det(B) = \det(E_1 E_2 \ldots E_k)$

$\det(E_i A) = \det(E_i)\det(A)$

$= \det(E_1)\det(E_2 \ldots E_k)$

$= \det(E_1)\det(E_2) \ldots \det(E_k)$

(II) $\det(BC) = \det(E_1 E_2 \ldots E_k C)$

$= \det(E_1)\det(E_2 \ldots E_k C)$

$= \ldots$

$= \det(B) \cdot \det(C)$

EX 86: Let $A$ be an $m \times n$ matrices show that (I) $rank(A^TA) = rank(A)$

(II) $rank(AA^T) = rank(A)$

<策略>

(I) 證 $Null(A^TA) = Null(A) \rightarrow nullity(A^TA) = nullity(A)$

(II) $rank(A) + nullity(A) = n \rightarrow rank(A^TA) = rank(A)$

<Sol> (I) 證 $Null(A^TA) = Null(A)$

(i) $Null(A) \subseteq Null(A^TA)$ , $\forall \vec{x} \in Null(A) \rightarrow A\vec{x} = \vec{0}$

$\rightarrow A^TA\vec{x} = A^T\vec{0} = \vec{0}$

$\rightarrow \vec{x} \in Null(A^TA)$ ,故 $Null(A) \subseteq Null(A^TA)$

(ii) $Null(A^TA) \subseteq Null(A)$ , $\forall \vec{x} \in Null(A^TA)$

$\rightarrow A^TA\vec{x} = \vec{0}$ $\qquad \langle \vec{x}, \vec{y} \rangle = \vec{y}^T\vec{x}$

$\rightarrow \vec{x}^TA^TA\vec{x} = \vec{x}^T\vec{0} = \vec{0}$ $\qquad (AB)^T = B^TA^T$

$\rightarrow (A\vec{x})^T(A\vec{x}) = 0$

$\rightarrow \|A\vec{x}\|^2 = 0$

$\rightarrow A\vec{x} = \vec{0}$ , $\vec{x} \in Null(A)$ ,故 $Null(A^TA) \subseteq Null(A)$

由 (i)、(ii) 得 $Null(A^TA) = Null(A) \rightarrow nullity(A^TA) = nullity(A)$

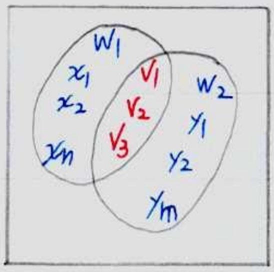
(Ⅱ) 證 $rank(A^TA) = rank(A)$

∵ $rank(A) + nullity(A) = n$，且 $rank(A^TA) + nullity(A^TA) = n$

∴ $rank(A) = n - nullity(A) = n - nullity(A^TA) = rank(A^TA)$

<定理> $W_1$、$W_2$ 為 $V$ 之子空間，$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1\cap W_2)$

<證>




(Ⅰ) 令 $\beta = \{\vec{v_1}, \vec{v_2} \ldots \vec{v_k}\}$ 為 $W_1\cap W_2$ 之基底，故 $\dim(W_1\cap W_2) = k$

(i) $\beta$ 為 $W_1$ 之獨立集，由擴增原理 $S_1 = \{\vec{v_1} \ldots \vec{v_k}\ \vec{x_1} \ldots \vec{x_n}\}$ 為 $W_1$ 之基底。

故 $\dim(W_1) = k+n$

(ii) $\beta$ 為 $W_2$ 之獨立集，由擴增原理 $S_2 = \{\vec{v_1} \ldots \vec{v_k}\ \vec{y_1} \ldots \vec{y_m}\}$ 為 $W_2$ 之基底。

故 $\dim(W_2) = k+m$

(Ⅱ) $W_1 + W_2 = span\{\vec{v_1} \ldots \vec{v_k}, \vec{x_1} \ldots \vec{x_n}, \vec{y_1} \ldots \vec{y_m}\}$

〈策略〉 證明 $\{\vec{v}_1 \dots \vec{v}_k, \vec{x}_1 \dots \vec{x}_n, \vec{y}_1 \dots \vec{y}_m\}$ 為 L.I $\longrightarrow$ 為 $W_1+W_2$ 基底

故 $\dim(W_1+W_2) = k+n+m$，得證。

(I) 令 $\sum_{i=1}^{k} \alpha_i \vec{v}_i + \sum_{j=1}^{n} \beta_j \vec{x}_j + \sum_{l=1}^{m} \gamma_l \vec{y}_l = \vec{0}$ 〈基本定義〉

$\longrightarrow$ 令 $\sum_i \alpha_i \vec{v}_i + \sum_j \beta_j \vec{x}_j = -\sum_l \gamma_l \vec{y}_l = \vec{z}$ 〈背〉

$\because$ $\vec{z} \in W_1$ 且 $\vec{z} \in W_2$ $\longrightarrow$ $\vec{z} \in W_1 \cap W_2$

$\longrightarrow$ $\vec{z} = -\sum_l \gamma_l \vec{y}_l = \sum_i c_i \vec{v}_i \longrightarrow \sum_i c_i \vec{v}_i + \sum_l \gamma_l \vec{y}_l = \vec{0}$

$\because$ $\{\vec{v}_i, \vec{y}_l\}$ 為 L.I， $\therefore$ $c_i = 0$ ， $\gamma_l = 0$

$c_i=0$

(II) $\vec{z} = \vec{0} = \sum_i \alpha_i \vec{v}_i + \sum_j \beta_j \vec{x}_j$ $\because$ $\{\vec{v}_i, \vec{x}_j\}$ 為 L.I， $\rightarrow \alpha_i = 0$ ， $\beta_j = 0$

$\therefore$ $\{\vec{v}_1 \dots \vec{v}_k, \vec{x}_1 \dots \vec{x}_n, \vec{y}_1 \dots \vec{y}_m\}$ 為 L.I， 為 $W_1+W_2$ 之基底

$\dim(W_1+W_2) = k+n+m = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ 得證

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〈定理〉 $T: V \to W$ 為線性 且 $\dim(V) < \infty$ 則 $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

〈證〉 (I) 令 $\{\vec{x}_1 \dots \vec{x}_k\}$ 為 $\text{Null}(T)$ 之一組基底 $\longrightarrow$ $\text{nullity}(T) = k$

$\because$ $\text{Null}(T) \subseteq V$，由擴增原理 $\{\vec{x}_1 \dots \vec{x}_k, \vec{x}_{k+1} \dots \vec{x}_n\}$ 為 $V$ 之基底 $\dim(V) = n$

(II) $\text{Range}(T) = \text{span}\{T(\vec{x}_1) \dots T(\vec{x}_k), T(\vec{x}_{k+1}) \dots T(\vec{x}_n)\}$

$= \text{span}\{\vec{0} \dots \vec{0}, T(\vec{x}_{k+1}) \dots T(\vec{x}_n)\}$

$= \text{span}\{T(\vec{x}_{k+1}) \dots T(\vec{x}_n)\}$

若 $\{T(\vec{x}_{k+1}) \dots T(\vec{x}_n)\}$ 為 L.I. $\longrightarrow$ 為 $\text{Range}(T)$ 之基底 $\longrightarrow$ $\text{rank}(T) = n-k$ 得證

令 $c_{k+1} T(\vec{x}_{k+1}) + c_{k+2} T(\vec{x}_{k+2}) + \dots + c_n T(\vec{x}_n) = \vec{0}$ $\quad$ $\because$ $T$ 為線性

$\longrightarrow$ $T(c_{k+1} \vec{x}_{k+1} + \dots + c_n \vec{x}_n) = \vec{0}$

$\longrightarrow$ $c_{k+1} \vec{x}_{k+1} + \dots + c_n \vec{x}_n \in \text{Null}(T)$

$\longrightarrow$ 故 $c_{k+1} \vec{x}_{k+1} + \dots + c_n \vec{x}_n = d_1 \vec{x}_1 + \dots + d_k \vec{x}_k$

$\longrightarrow$ $-d_1 \vec{x}_1 - \dots - d_k \vec{x}_k + c_{k+1} \vec{x}_{k+1} + \dots + c_n \vec{x}_n = \vec{0}$

由於 $\{\vec{x}_1 \dots \vec{x}_k, \vec{x}_{k+1} \dots \vec{x}_n\}$ 為 L.I. $\longrightarrow$ $\alpha_1 = \dots = \alpha_k = 0$ $\quad c_{k+1} = \dots = c_n = 0$

故 $\{T(\vec{x}_{k+1}) \dots T(\vec{x}_n)\}$ 為 L.I.，為 Range $(T)$ 之基底

$\longrightarrow$ $\text{rank}(T) = n-k$ $\quad \therefore$ $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

<定理> $A \in F^{n\times n}$ ; $\lambda_1, \lambda_2 \dots \lambda_r$ 為相異特徵值，且 $\vec{x}_1, \vec{x}_2 \dots \vec{x}_r$ 為相應特徵向量，則 $\{\vec{x}_1, \vec{x}_2 \dots \vec{x}_r\}$ 為線性獨立。

<證> 數學歸納法 (i) $r=1$ , $\{\vec{x}_1\}$ L.I.

(ii) $r=k$ , $\{\vec{x}_1, \vec{x}_2 \dots \vec{x}_k\}$ L.I.

(iii) $r=k+1$ , $\{\vec{x}_1, \vec{x}_2 \dots \vec{x}_k, \vec{x}_{k+1}\}$ L.I.?

令 $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_{k+1}\vec{x}_{k+1} = \vec{0}$ $\quad \dots (1)$

$\longrightarrow$ $A(c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_{k+1}\vec{x}_{k+1}) = A\vec{0}$

$\longrightarrow$ $c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \dots + c_{k+1}\lambda_{k+1}\vec{x}_{k+1} = \vec{0}$ $\quad \dots (2)$

$(2) - \lambda_{k+1}(1) \longrightarrow \overset{0}{\overset{\|}{c_1(\lambda_1 - \lambda_{k+1})}}\vec{x}_1 + \overset{0}{\overset{\|}{c_2(\lambda_2 - \lambda_{k+1})}}\vec{x}_2 \ldots + \overset{0}{\overset{\|}{c_k(\lambda_k - \lambda_{k+1})}}\vec{x}_k = \vec{0}$

$\because \{\vec{x}_1 \ldots \vec{x}_k\}$ 為 L.I. $\quad \therefore c_1\overset{\neq 0}{(\lambda_1 - \lambda_{k+1})} = 0 \quad \ldots \quad c_k\overset{\neq 0}{(\lambda_k - \lambda_{k+1})} = 0$ $\quad$ λ相異

$\longrightarrow \quad c_1 = c_2 = \ldots = c_k = 0$ 代回 $(1)$ : $c_{k+1}\vec{x}_{k+1} = \vec{0}$

$\because \vec{x}_{k+1} \neq \vec{0} \longrightarrow c_{k+1} = 0$ 故 $\{\vec{x}_1 \ldots \vec{x}_k\}$ 為 L.I.