Analysis Tools

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Several Basic Functions

- Polynomial
  - Constant function 1.C
  - Linear function N: problem size
  - Quadratic function N^2
  - Cubic function N^3
- Logarithm function log(N)
- N-Log-N function NlogN
- Exponential function e^N

Family of Polynomials

- Constant function
  - $f(n)=1$
- Linear function
  - $f(n)=n$
- Quadratic function
  - $f(n)=n^2$
- Cubic function
  - $f(n)=n^3$
- A general polynomials
  - $f(n)=a_0+a_1n+a_2n^2+a_3n^3+\ldots+a_dn^d$

n: problem size

The Logarithm Function

- $f(n)=\log_2(n)=\log(n)$

The default base is 2.

- Definition of logarithm

$$x = \log_b n \quad \text{if and only if} \quad b^x = n.$$

- Some identities

$$\begin{aligned}
&1.\ \log_b ac = \log_b a + \log_b c \\
&2.\ \log_b a/c = \log_b a - \log_b c \\
&3.\ \log_b a^c = c \log_b a \\
&4.\ \log_b a = (\log_d a)/\log_d b \\
&5.\ b^{\log_d a} = a^{\log_d b}
\end{aligned}$$

- More…

$$y = \ln x \Rightarrow y' = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \ln x$$

The base is e=2.718281828….

The N-Log-N Function

- $f(n)=n*\log(n)$

The Exponential Function

- $f(n)=a^n$
- Some identities (for positive a, b, and c)

$a^{(b+c)} = a^b a^c$

$a^{bc} = (a^b)^c$

$a^b / a^c = a^{(b-c)}$

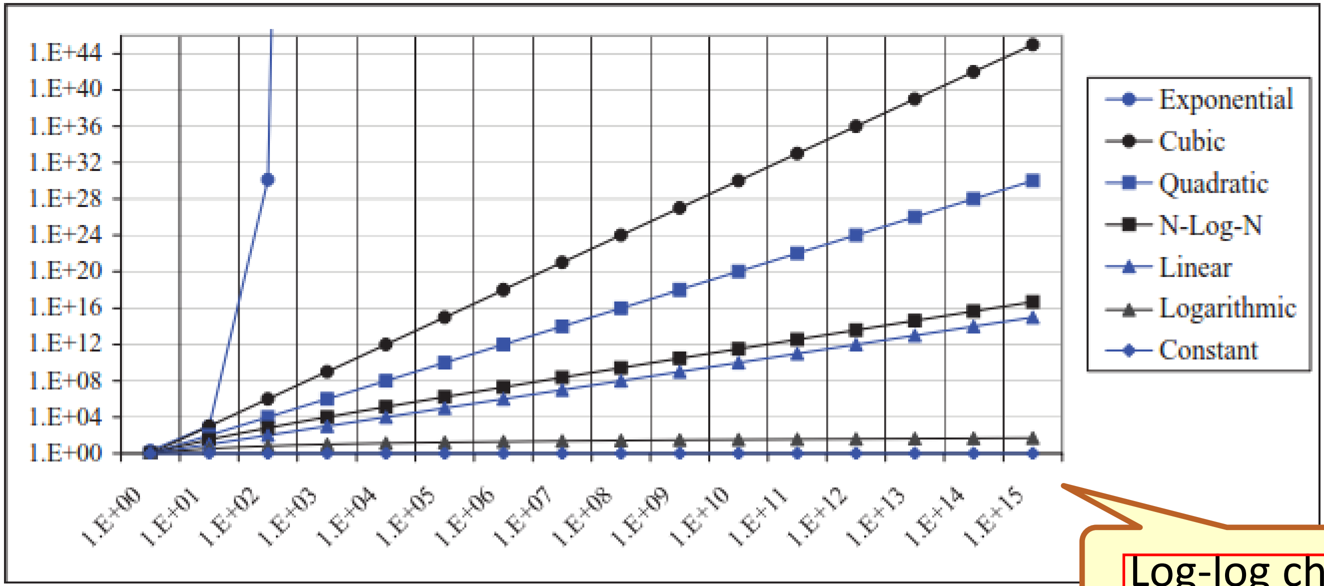
$b = a^{\log_a b}$

$b^c = a^{c*\log_a b}$

Growth Rate Comparisons

constant	*logarithm*	*linear*	*n-log-n*	*quadratic*	*cubic*	*exponential*
1	$\log n$	n	$n \log n$	n^2	n^3	a^n

Table 4.1: Classes of functions. Here we assume that $a > 1$ is a constant.



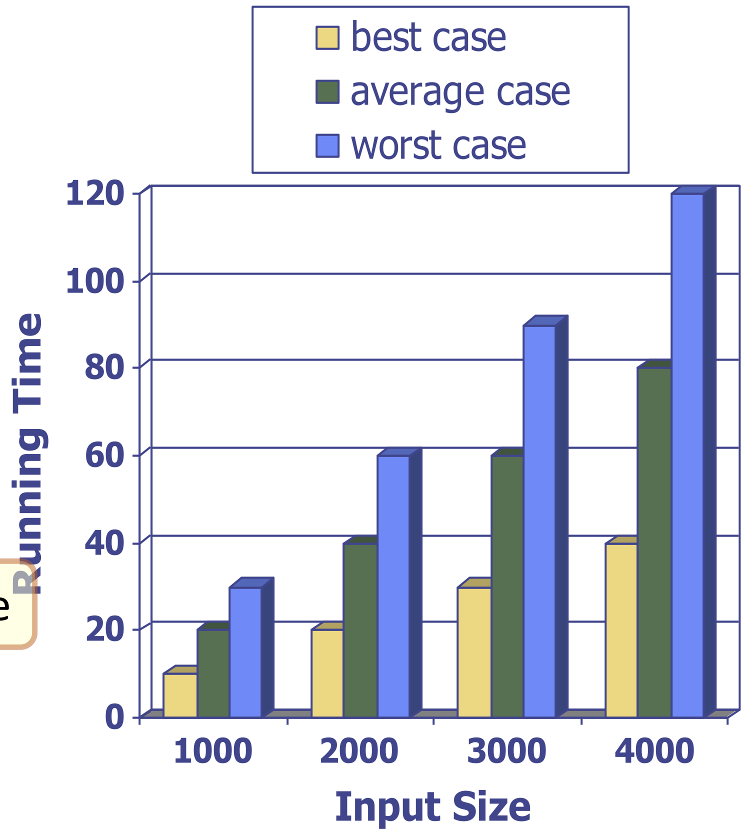
Log-log chart

$$y = n^k \Rightarrow \log y = k \log n \Rightarrow Y = kX$$

Running Time

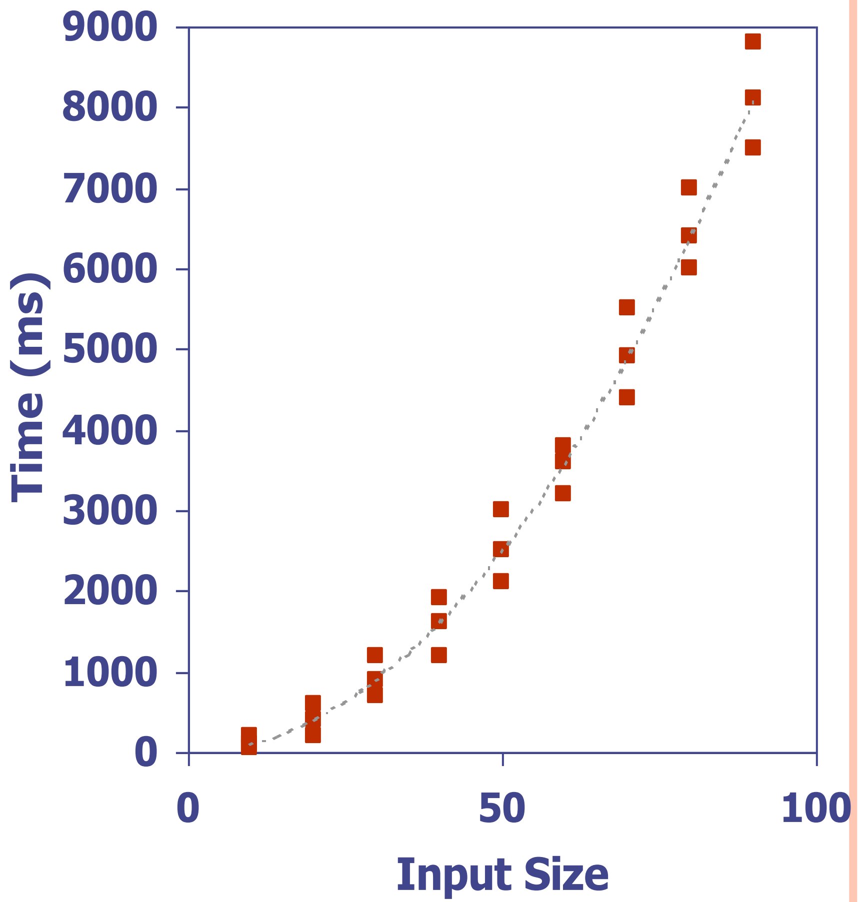
- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
    - Easier to analyze
    - Crucial to applications such as games, finance and robotics

Sorting, for example



Performance Measurement via Experiments

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a function like clock() to get an accurate measure of the actual running time
- Plot the results



Limitations of Experimental Studies

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Performance Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

About Primitive Operations

- Primitive operations
  - Basic computations performed by an algorithm
  - Identifiable in pseudocode
  - Largely independent from the programming language
  - Exact definition not important (we will see why later)
  - Assumed to take a constant amount of

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

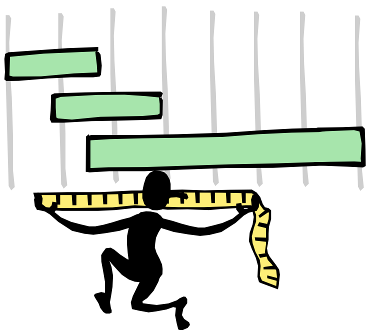
Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm *arrayMax*(*A*, *n*)	# operations	
$currentMax \leftarrow A[0]$	2	A[0], assign
for $i \leftarrow 1$ **to** $n-1$ **do**	$2n$	一路asign過去
if $A[i] > currentMax$ **then**	$2(n-1)$	
$currentMax \leftarrow A[i]$	$2(n-1)$	
{ increment counter i }	$2(n-1)$	
return $currentMax$	1	
Total	$8n-2$	

Estimating Running Time

- Algorithm ***arrayMax*** executes $8n - 2$ primitive operations in the worst case. Define:
  - a = Time taken by the fastest primitive operation
  - b = Time taken by the slowest primitive operation



- Let $T(n)$ be worst-case time of ***arrayMax*.** Then

$$a\,(8n - 2) \leq T(n) \leq b(8n - 2)$$

- Hence, the running time $T(n)$ is bounded by two linear functions
- Change the hardware environment only affects a and b.
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***

Big-Oh Notation

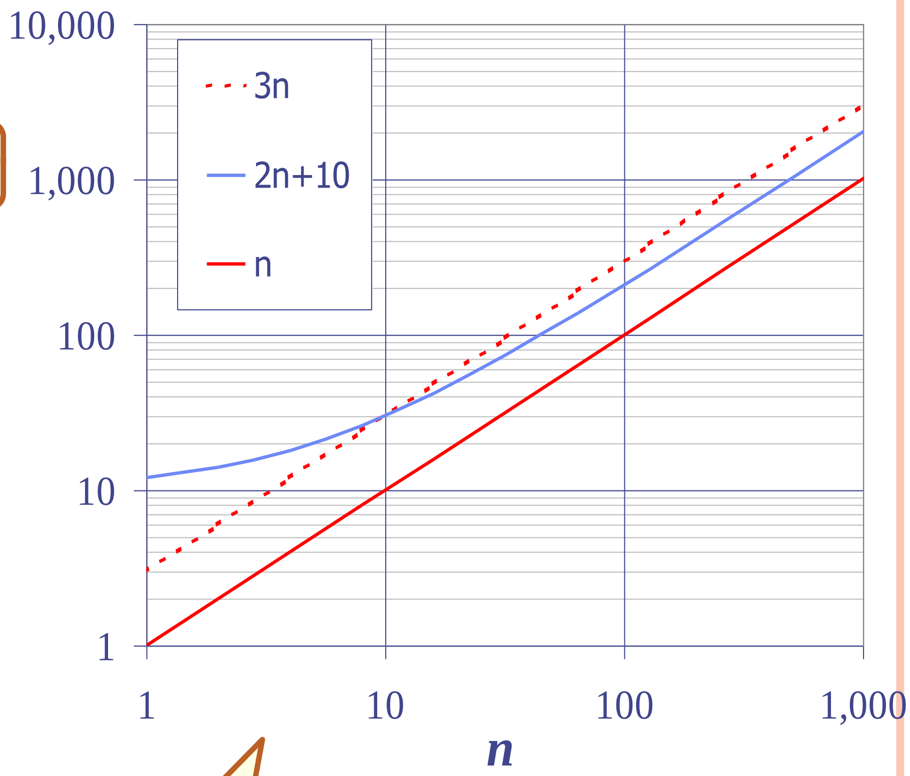
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

g(n): basic functions

$$f(n) \le cg(n) \quad \text{for } n \ge n_0$$

Quiz!

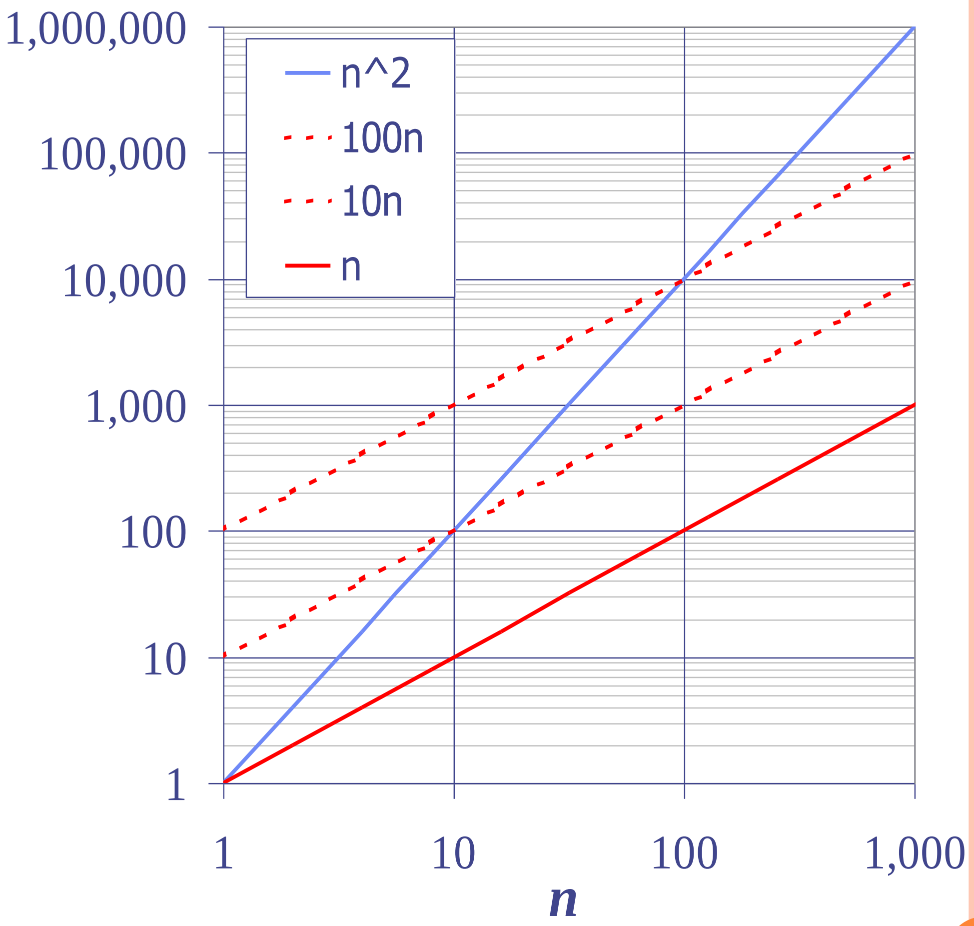
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \le cn$
  - $(c - 2)\, n \ge 10$
  - $n \ge 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Log scale

Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- 7n-2 is O(n)
  - Need $c > 0$ and $n_0 \geq 1$ such that $7n\text{-}2 \leq c \bullet n$ for $n \geq n_0$
  - This is true for $c = 7$ and $n_0 = 1$
- $3n^3 + 20n^2 + 5$ is $O(n^3)$
  - Need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$

    3n^3+20n^3+5n^3 = 28n^3
  - This is true for $c = 28$ and $n_0 = 1$
- $3 \log n + 5$ is $O(\log n)$

  3logn + 5logn = 8logn
  - Need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \bullet \log n$ for $n \geq n_0$
  - This is true for $c = 8$ and $n_0 = 2$

Quiz!

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation

That is, when n is big!

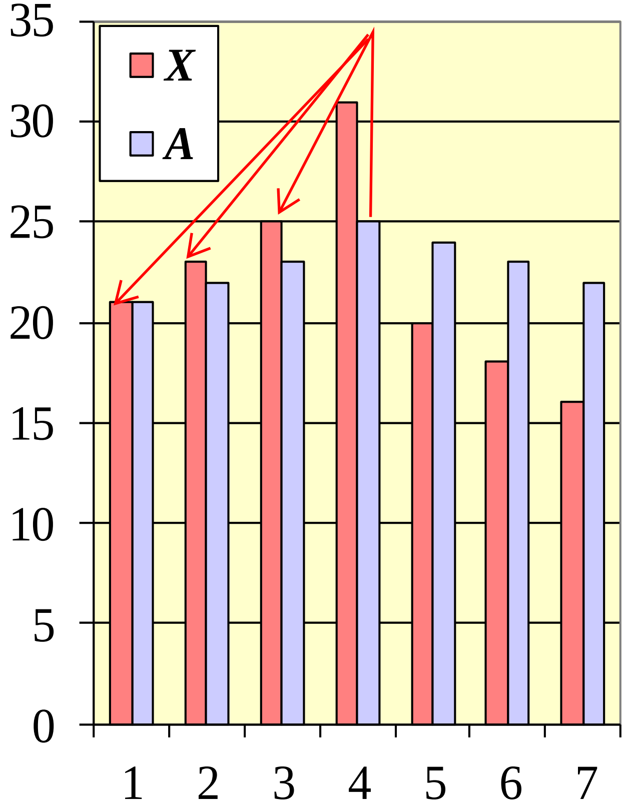
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $8n - 2$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The $\boldsymbol{i}$-th prefix average of an array $\boldsymbol{X}$ is average of the first $(\boldsymbol{i} + 1)$ elements of $\boldsymbol{X}$:

$$\boldsymbol{A}[\boldsymbol{i}] = (\boldsymbol{X}[0] + \boldsymbol{X}[1] + \ldots + \boldsymbol{X}[\boldsymbol{i}])/(i+1)$$

- Computing the array $\boldsymbol{A}$ of prefix averages of another array $\boldsymbol{X}$ has applications to financial analysis



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition directly.

```
Algorithm prefixAverages1(X, n)
    Input array X of n integers
    Output array A of prefix averages of X      #operations
    A ← new array of n integers                 n
    for i ← 0 to n − 1 do                       n
        s ← X[0]                                n
        for j ← 1 to i do                       1 + 2 + …+ (n − 1)
            s ← s + X[j]                        1 + 2 + …+ (n − 1)
        A[i] ← s / (i + 1)                      n
    return A                                    1
```

n(n-1)/2

Prefix Averages (Linear)

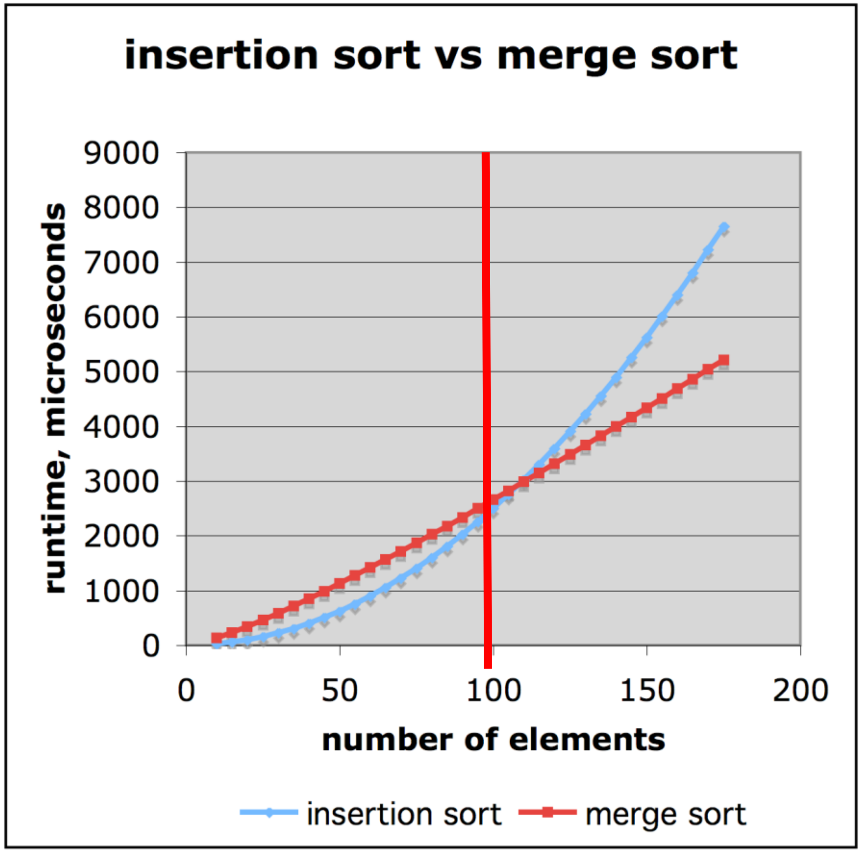
- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow s + X[i]$	n
$A[i] \leftarrow s / (i + 1)$ n	n
return A	1

Comparison of Two Algorithms

- Two sorting algorithms
  - Merge sort is O(n log n)
  - Insertion sort is O(n^2)
- To sort 1M items
  - Insertion sort ➔ 70 hours
  - Merge sort ➔ 40 seconds
- For a faster machine
  - Insertion sort ➔ 40 minutes
  - Merge sort ➔ 0.5 seconds

Slide by Matt Stallmann



Relatives of Big-Oh

- Big-Omega
  - f(n) is Ω(g(n)) if there is a constant c > 0 and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- Big-Theta
  - f(n) is Θ(g(n)) if there are constants c' > 0 and c'' > 0 and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$

Intuition for Asymptotic Notation

- Big-Oh
  - f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)
- Big-Omega
  - f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)
- Big-Theta
  - f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$ ➔ let $c = 5$ and $n_0 = 1$
- **$5n^2$ is $O(n^2)$**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \leq c \bullet g(n)$ for $n \geq n_0$ ➔ let $c = 5$ and $n_0 = 1$
- **$5n^2$ is $\Theta(n^2)$**
  - $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$.

Computing Powers

- To compute the power function p(x, n) = x^n
  - Method 1
    - n, n-1, n-2, …, 2, 1 ➔ O(n)

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{otherwise} \end{cases}$$

  - Method 2
    - n, n/2, n/4, … , 2, 1 ➔ O(log n)

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, (n-1)/2)^2 & \text{if } n > 0 \text{ is odd} \\ p(x, n/2)^2 & \text{if } n > 0 \text{ is even} \end{cases}$$

Element Uniqueness Problem (1/2)

- To determine if the elements in a vector are all unique
  - $O(2^n)$ implementation by recursion

```
bool isUnique(const vector<int>& arr, int start, int end) {
  if (start >= end) return true;
  if (!isUnique(arr, start, end-1))
    return false;
  if (!isUnique(arr, start+1, end))
    return false;
  return (arr[start] != arr[end]);
}
```

  - $O(n^2)$ implementation by looping

```
bool isUniqueLoop(const vector<int>& arr, int start, int end) {
  if (start >= end) return true;
  for (int i = start; i < end; i++)
    for (int j = i+1; j <= end; j++)
      if (arr[i] == arr[j]) return false;
  return true;
}
```

11 12 13 14...
23 24 25

Element Uniqueness Problem (2/2)

- To determine if the elements in a vector are all unique
  - O(n log n) implementation ➔ Sort the vector first and check for neighboring duplicate elements

```
bool isUniqueSort(const vector<int>& arr, int start, int end) {
  if (start >= end) return true;
  vector<int> buf(arr);                         // duplicate copy of arr
  sort(buf.begin()+start, buf.begin()+end);     // sort the subarray
  for (int i = start; i < end; i++)             // check for duplicates
    if (buf[i] == buf[i+1]) return false;
  return true;
}
```

  - A faster average-case running time can be achieved by using the hash table data structure (Section 9.2).

How to Prove Statements

- By giving counter example
  - Roger claims that every number of the form 2^i-1 is a prime, where i is an integer greater than 1. Prove he is wrong.
- By contrapositive: Switching the hypothesis and conclusion of a conditional statement and negating both.
  - If p is true, then q is true ←→ If q is not true, then p is not true
  - If a*b is even, then a is even or b is even. ←→ If a is odd and b is odd, then a*b is odd.
  - Be aware of "DeMorgan's Law"
- By math induction
  - Example: Prove that $F(n)<2^n$, where F(n) is the Fibonacci function with F(n+2)=F(n+1)+F(n), and F(1)=1, F(2)=2.

Exercise on Big Oh

- Given functions f(n) and g(n), under what condition do we we say that f(n) is O(g(n))?
  - Note that g(n) is one of the basic functions, such as n^3.
  - What is the physical meaning of the definition?
- Use the definition of big oh to explain why $3n^2+n\log(n)+2n+5\log(n)$ is $O(n^2)$.

Exercise on Proof by Induction

- Proof by induction
  - Prove that $F(n)<2^n$, where F(n) is the Fibonacci function satisfying $F(n+2)=F(n+1)+F(n)$, with $F(1)=1$, $F(2)=2$.
- 3 steps in proof by induction
  - Base case
  - Inductive Hypothesis
  - Inductive Step