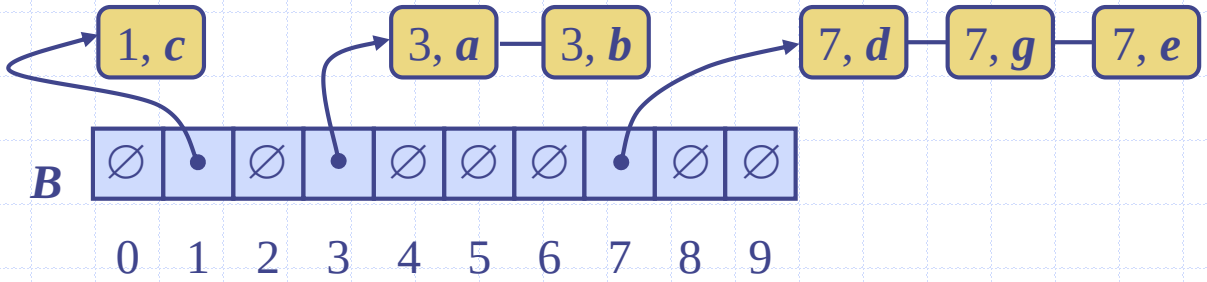
Bucket-Sort and Radix-Sort



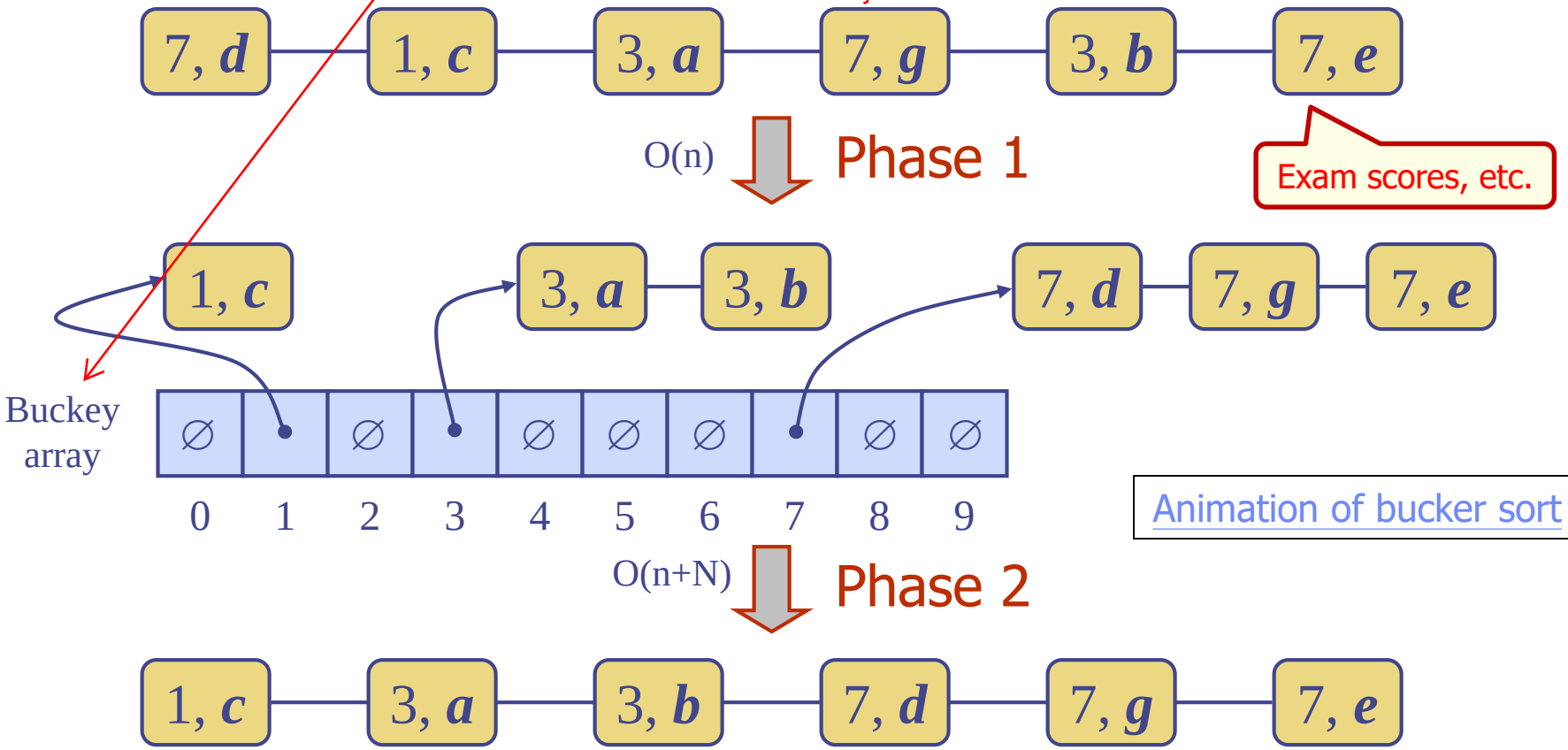
Introduction

- Comparison-based sorting algorithms ➔ $\Omega(n \log n)$
  - Quick-sort, heap-sort, merge-sort
- Can we do better?
  - Yes, if the keys have a restricted type
  - Examples: Bucket-sort and radix-sort
  - Complexity: Linear!

No comparison needed!

Bucket-sort Example

- Key range [0, 9]



Animation of bucker sort

Bucket-Sort

- Let be S be a sequence of n (key, element) entries with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$
  - Phase 2: For $i = 0, \ldots, N-1$, move the entries of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time

Bucket-sort takes $O(n + N)$ time

Linear time if N << n

Algorithm *bucketSort*(S, N)

Input sequence S of (key, element) items with keys in the range $[0, N-1]$

Output sequence S sorted by increasing keys

$B \leftarrow$ array of N empty sequences

while $\neg S.empty()$

$(k, o) \leftarrow S.front()$

$S.eraseFront()$

$B[k].insertBack((k, o))$

for $i \leftarrow 0$ **to** $N-1$

while $\neg B[i].empty()$

$(k, o) \leftarrow B[i].front()$

$B[i].eraseFront()$

$S.insertBack((k, o))$

Counting-sort

- Characteristics of counting-sort
  - A special case of bucket-sort
  - Avoid the use of linked list by counting the number of occurrence of each key

Counting-sort Example

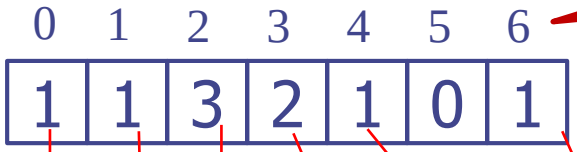
Quiz!

- Input array:

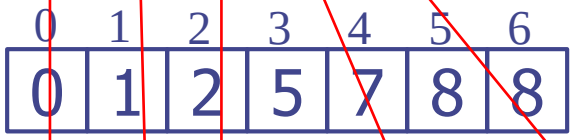
Keys in 0~6

- Count array:

Use keys as indices



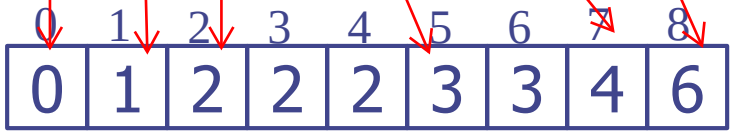
- Position array:



Cumulative sum of the count array

Starting positions

- Output array:



Animation of count-sort

Properties and Extensions

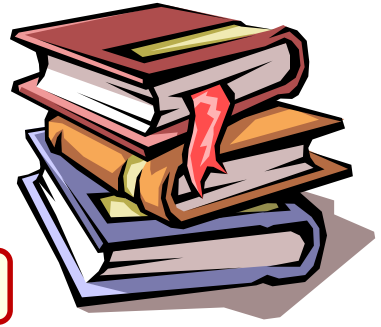
- Key-type Property
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- Stable Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions

- Integer keys in the range $[a, b]$
  - Put entry (k, o) into bucket $B[k - a]$
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put entry (k, o) into bucket $B[r(k)]$

You need to convert keys into buckey array indices.

Lexicographic Order



Dictionary order

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$

$$\Leftrightarrow$$

$$x_1 < y_1 \vee x_1 = y_1 \wedge (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let *stableSort*(S, C) be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm *stableSort*, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of *stableSort*

Algorithm *lexicographicSort*(S)

Input sequence S of d-tuples
Output sequence S sorted in lexicographic order

for $i \leftarrow d$ **downto** 1
  stableSort(S, C_i)

Another example: Sort dates

Example:

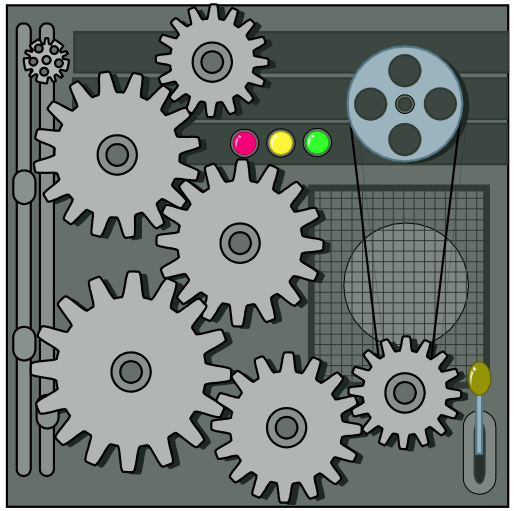
(7, 4, 6) (5, 1, 5) (2, 4, 6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5, 1, 5) (7, 4, 6) (2, 4, 6)

(2, 1, 4) (5, 1, 5) (3, 2, 4) (7, 4, 6) (2, 4, 6)

(2, 1, 4) (2, 4, 6) (3, 2, 4) (5, 1, 5) (7, 4, 6)

Radix-Sort



- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d*(n+N))$

Animation of radix sort

Algorithm *radixSort*(S, N)

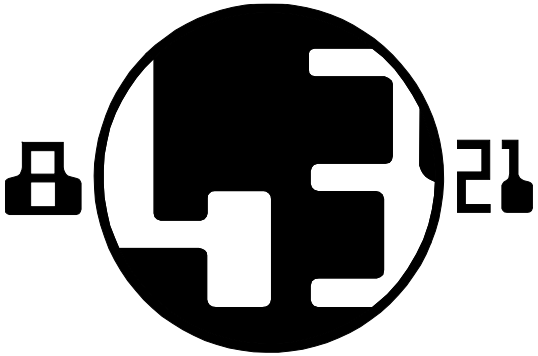
Input sequence S of d-tuples such that $(0, \ldots, 0) \leq (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \leq (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S

Output sequence S sorted in lexicographic order

for $i \leftarrow d$ **downto** 1

bucketSort(S, N)

Radix-Sort for Binary Numbers



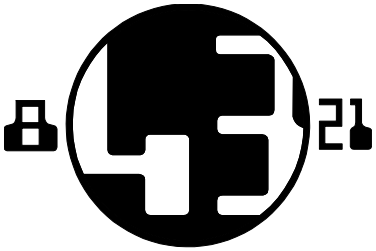
- Consider a sequence of n b-bit integers

$$x = x_{b-1} \ldots x_1 x_0$$

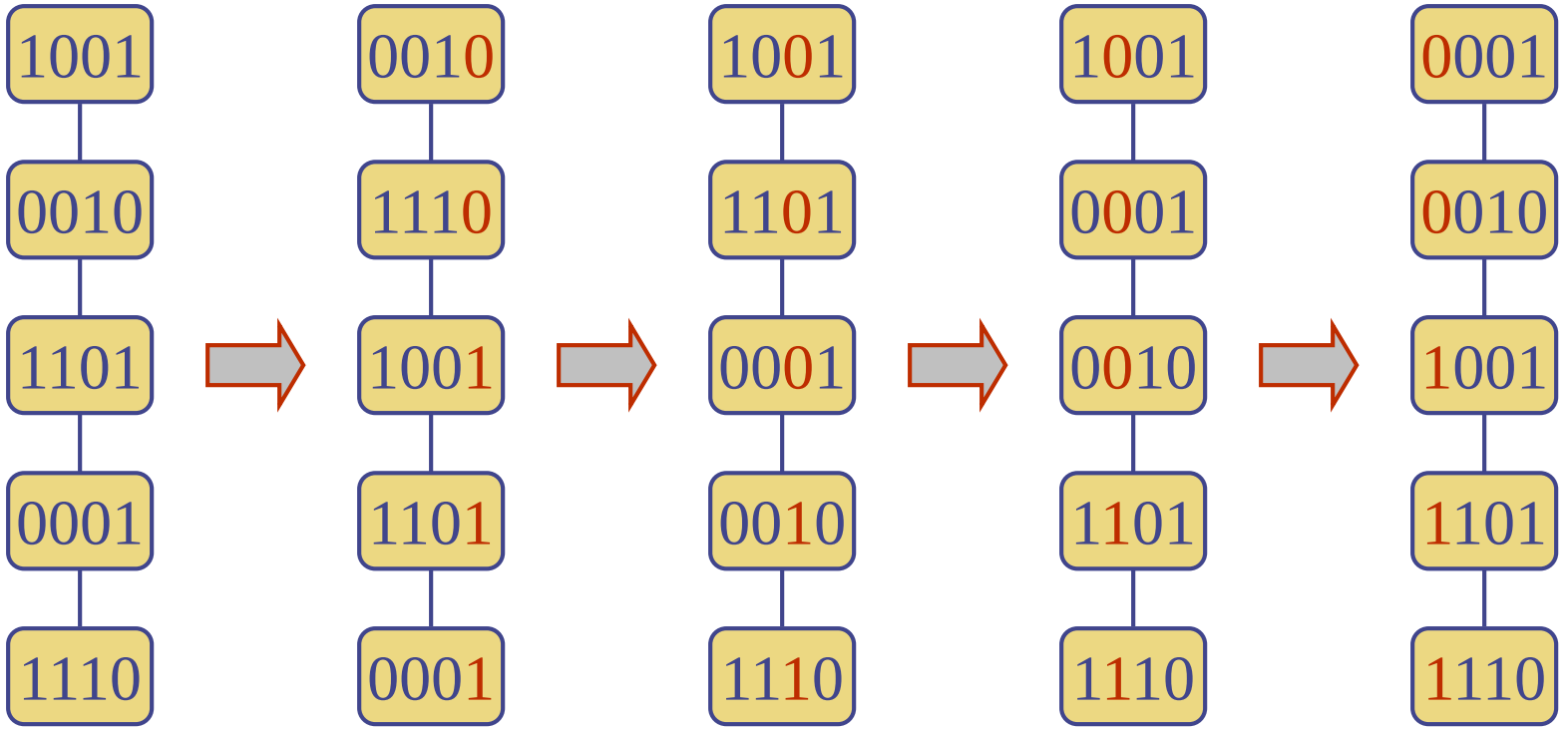
- We represent each element as a b-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm *binaryRadixSort*(S)
  Input sequence S of b-bit integers
  Output sequence S sorted
  replace each element x of S with the item $(0, x)$
  for $i \leftarrow 0$ **to** $b - 1$
    replace the key k of each item (k, x) of S with bit x_i of x
    bucketSort(S, 2)

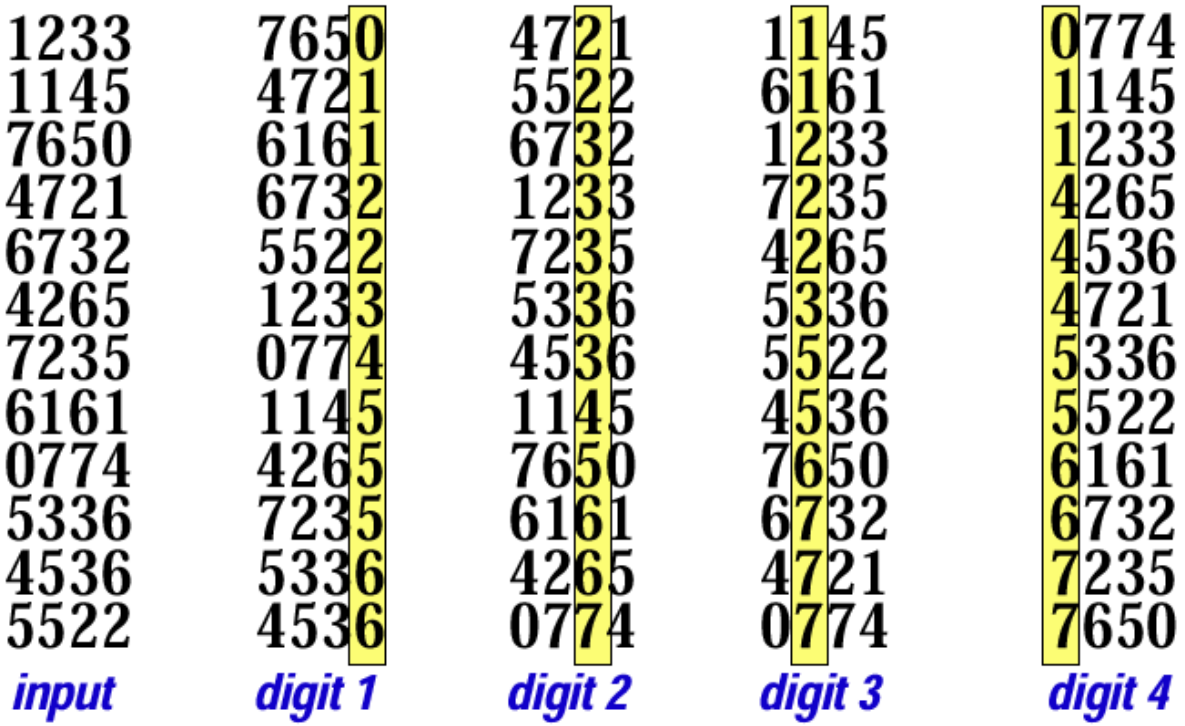
Radix-sort Example



- Sorting a sequence of 4-bit integers



Another Radix-sort Example



- Quicksort: O(n log n)
- Radix sort: O(d*(n+10))

Quiz!