机率

P.1-14 EX18: $f(x) = e^{-\pi(x-1)^2}$; $-\infty < x < \infty$ express the cumulative distribution funtion (CDF) of X using Q-funtion.

<Sol> $Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ $\qquad f(x)$ v.s. $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\sqrt{2\pi}\sigma = 1 \quad \sigma = \frac{1}{\sqrt{2\pi}} \quad \mu = 1 \quad X \sim N(1, \frac{1}{2\pi})$

$F_X(x) = P(X \le x) = P\left(\frac{X-1}{1/\sqrt{2\pi}} \le \frac{x-1}{1/\sqrt{2\pi}}\right) = P(Z \le \frac{x-1}{1/\sqrt{2\pi}}) = \Phi(\sqrt{2\pi}(x-1)) = 1 - Q(\sqrt{2\pi}(x-1))$

P.1-22 EX26: $P(0,0,1) = \frac{1}{8}$ $\quad P(0,1,1) = \frac{1}{16}$ $\quad P(0,1,2) = \frac{1}{8}$ $\quad P(1,0,0) = \frac{1}{16}$

$P(1,0,1) = \frac{1}{8}$ $\quad P(1,1,1) = \frac{1}{16}$ $\quad P(1,2,0) = \frac{1}{8}$ $\quad P(1,2,2) = \frac{1}{32}$

$P(2,0,1) = \frac{1}{16}$ $\quad P(2,0,0) = \frac{1}{8}$ $\quad P(2,2,1) = \frac{1}{32}$ $\quad P(2,2,2) = \frac{1}{16}$, else zero.

(a) Are X、Z statistically independent ?

X \ Z	0	1	2	P(X)
0	0	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{5}{16}$
1	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{32}$	$\frac{13}{32}$
2	$\frac{1}{8}$	$\frac{3}{32}$	$\frac{1}{16}$	$\frac{9}{32}$
P(Z)	$\frac{5}{16}$	$\frac{15}{32}$	$\frac{7}{32}$	

$P(X=0, Z=0) \ne P(X=0)\,P(Z=0)$

Not independent

(b) what is the conditional expectation $E[X|Y=y]$

$X \backslash Y$	0	1	2	$P(X)$
0	$\frac{1}{8}$	$\frac{3}{16}$	0	$\frac{5}{16}$
1	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{5}{32}$	$\frac{13}{32}$
2	$\frac{3}{16}$	0	$\frac{3}{32}$	$\frac{9}{32}$
$P(Y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	

X	0	1	2	$E[X|y]$
$P(X|y=0)$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
$P(X|y=1)$	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$
$P(X|y=2)$	0	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{11}{8}$

P.1-45 EX49: Let N be a geometric random varible with its sample space

$S_N = \{1, 2, 3 \ldots\}$, Find the $P(N \text{ is odd} \mid N \le m)$

$$P(N=n) = (1-p)^{n-1} p \quad ; \quad n = 1, 2, 3 \ldots$$

$$P(N \le m) = \sum_{n=1}^{m} (1-p)^{n-1} p = \frac{p(1-(1-p)^m)}{1-(1-p)} = 1-(1-p)^m$$

$$P(N \text{ is odd}, N \le m) = \begin{cases} \dfrac{1-(1-p)^m}{2-p} & ; \ m \in N \text{ 且 even} \quad r = (1-p)^2 \quad r^n = (1-p)^m \\[2ex] \dfrac{1-(1-p)^{m+1}}{2-p} & ; \ m \in N \text{ 且 odd} \quad r = (1-p)^2 \quad r^n = (1-p)^{m+1} \end{cases}$$

$$P(N \text{ is odd} \mid N \le m) = \frac{P(N \text{ is odd}, N \le m)}{P(N \le m)}$$

P.1-55 EX59: $X \sim U(0,1)$, $Y|x \sim B(n, x)$, $n=3$

Find $f(x|Y)$

$f(x)=1$ $\quad f(Y|x) = C^3_y (x)^y (1-x)^{3-y}$; $y = 0.1.2.3\ldots$ $\quad 0 \le x \le 1$

$$f(Y|x) = \frac{f(x,y)}{f(x)} = f(x,y)$$

$$f(y) = \int_0^1 C^3_y (x)^y (1-x)^{3-y} dx = C^3_y \int_0^1 x^y (1-x)^{3-y} dx \qquad B(\alpha,\beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

$$= C^3_y \; B(y+1, 4-y) = \frac{3!}{(3-y)!\,y!} \frac{\Gamma(y+1)\Gamma(4-y)}{\Gamma(5)} = \frac{3!\; y!\; 3-y!}{(3-y)!\; y!\; 4!} = \frac{1}{4}$$

$$f(x|Y) = \frac{f(x,y)}{f(y)} = 4 f(x,y) = 4 C^3_y \, x^y (1-x)^{3-y} \; ; \; y = 0.1.2\ldots \quad 0 \le x \le 1$$

P.1-57 EX61: X: the number of earthquakes $\quad f_X(x) = 2^x \frac{e^{-2}}{x!}$; $x \ge 0$

When X occurs, Y also occurs $p = 0.5$, what's $f_Y(y)$?

$$\text{r.v. } Y|x \sim B(x, \tfrac{1}{2}) = C^x_y (\tfrac{1}{2})^x = \frac{f(x,y)}{f(x)}$$

$$f(x,y) = C^x_y (\tfrac{1}{2})^x \cdot 2^x \frac{e^{-2}}{x!} = C^x_y \frac{e^{-2}}{x!}$$

$$f(y) = \sum_{x=y}^{\infty} C^x_y \frac{e^{-2}}{x!} = e^{-2} \sum_{x=y}^{\infty} \frac{x!}{(x-y)!\,y!} \frac{1}{x!} = \frac{e^{-2}}{y!} \sum_{x=y}^{\infty} \frac{1}{(x-y)!}$$

$$= \frac{e^{-2}}{y!} \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \right) = \frac{e^{-2}}{y!} e^{1} = \frac{e^{-1}}{y!}$$

課本(3) EX41: $M_X(t) = \frac{1}{81}(e^t+2)^4$ Find $P(X>2)$

$M_X(t) = \frac{1}{81}(e^{4t} + 8e^{3t} + 24e^{2t} + 32e^{t} + 16) = \sum_x e^{tx} P(X=x)$

$x=0 \quad e^0 P(X=0) = \frac{16}{81}$

$x=1 \quad e^t P(X=1) = \frac{32}{81} e^t$

$x=2 \quad e^{2t} P(X=2) = \frac{24}{81} e^{2t}$

$x=3 \quad e^{3t} P(X=3) = \frac{8}{81} e^{3t}$

$x=4 \quad e^{4t} P(X=4) = \frac{1}{81} e^{4t}$ $\quad P(X>2) = 1 - P(X\le 2) = 1 - P(X=0) - P(X=1) - P(X=2)$

P.2-90 Let $X|Y=y \sim B(y,p)$, $Y|\Lambda = \lambda \sim \text{Poisson}(\lambda)$, and $\Lambda \sim \text{Exponential}(\beta)$, $Var(X) = ?$

$E[E[X|Y]] = E[pY] = pE[E[Y|\Lambda]] = pE[\lambda] = p\frac{1}{\beta}$

$E[E[x^2|Y]] = E[yp(1-p) + y^2p^2] = p(1-p)E[y] + p^2E[y^2]$

$= p(1-p)E[E[Y|\Lambda]] + p^2E[E[Y^2|\Lambda]] = p(1-p)E[\lambda] + p^2E[\lambda^2+\lambda]$

$= p(1-p)\frac{1}{\beta} + p^2(\frac{1}{\beta} + \frac{2}{\beta^2}) = p\frac{1}{\beta} - p^2\frac{1}{\beta} + p^2\frac{1}{\beta} + p^2\frac{2}{\beta^2} = p\frac{1}{\beta} + p^2\frac{2}{\beta^2}$

$Var(x) = E[x^2] - E[x]^2 = p\frac{1}{\beta} + p^2\frac{1}{\beta}$

P.2-102 EX109 : Let $P(Y=y \mid X=x) = \dfrac{e^{-x} x^{y}}{y!}$; $y = 0.1.2 \ldots$ and $X \sim N(0,1)$ $E[Y] = ?$

$Y|X \sim \text{Poisson}(x)$ $\quad E[Y] = E[E[Y|X]] = E[x \mid x>0]$

$$f(x \mid x>0) = \frac{f(x)}{P(x>0)} = 2f(x) = \sqrt{2}\,\frac{1}{\sqrt{\pi}}\,e^{-\frac{x^2}{2}}$$

$$E[x \mid x>0] = \int_0^{\infty} x \sqrt{\tfrac{2}{\pi}}\, e^{-\frac{x^2}{2}}\,dx = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}}\,dx$$ 令 $u = \frac{x^2}{2}$ $\quad du = x\,dx$ $\quad dx = \frac{1}{x}du$

$$\sqrt{\tfrac{2}{\pi}} \int_0^{\infty} e^{-u}\,du = \sqrt{\tfrac{2}{\pi}}\left(-e^{-u}\Big|_0^{\infty}\right) = \sqrt{\tfrac{2}{\pi}}$$

P.3-5 EX4: $X \sim G(p)$; $P(X=k) = (1-p)^{k-1}p$; $0<p<1$ for $k = 1.2.3 \ldots$

Let $Y = \min(X, 10)$, i.e. $Y(\omega)$ is the min. of $X(\omega)$ and 10 for each outcome ω.

Find $F_Y(y) = ?$

$S_Y = \{1.2.3.4 \ldots 10\}$ $\quad P(Y=1) = P(X=1) = (1-p)^0 p = p$

$P(Y=2) = P(X=2) = (1-p)p$

$P(Y=3) = P(X=3) = (1-p)^2 p$

$\vdots$

$P(Y=10) = P(X \geq 10) = (1-p)^9 p + (1-p)^{10} p + \ldots = (1-p)^9$

$P(Z) = F(Z) - F(Z-1)$ (離散) $\quad F(Z) = P(Z) + F(Z-1)$

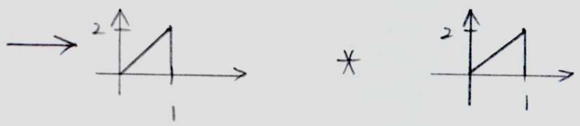
$$F(Y) = \begin{cases} P(Y=1) + 0 = p & ; \; 1 \le Y < 2 \\ P(Y=2) + F(Y-1) = p + (1-p)p & ; \; 2 \le Y < 3 \\ P(Y=3) + F(Y-1) = p + (1-p)p + (1-p)^2 p & ; \; 3 \le Y < 4 \\ \vdots \\ P(Y=9) + F(Y-1) = p + (1-p)p + \cdots + (1-p)^8 p & ; \; 9 \le Y < 10 \\ 1 & ; \; 10 \le Y \end{cases}$$

P.3-45 EX45: $f(x,y) = \begin{cases} 4xy & ; \; 0<x, y<1 \\ 0 & ; \; \text{else} \end{cases}$, $Z = X+Y$, Find $f_Z(z)$

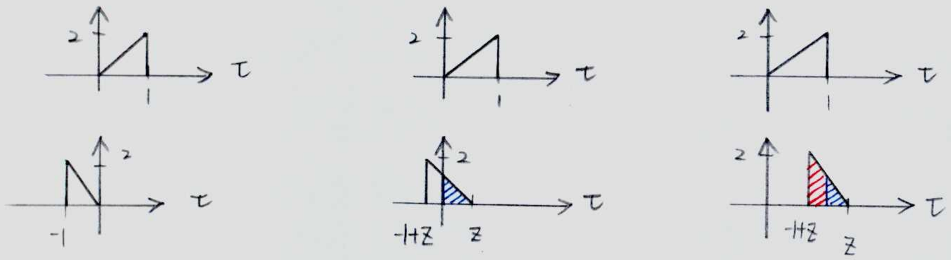
$f(x) = \int_0^1 4xy \, dy = \left(\frac{4}{2} x y^2 \Big|_0^1 \right) = 2x \quad ; \; 0<x<1$

$f(Y) = \int_0^1 4xy \, dx = \left(\frac{4}{2} x^2 y \Big|_0^1 \right) = 2y \qquad f_{X,Y}(x,y) = f(x) f(y) \longrightarrow X, Y$ indep.

$Z = X+Y \quad , \quad f(z) = f_X(z) * f_Y(z)$



$\longrightarrow \int_0^1 2\tau \cdot 2(z-\tau) \, d\tau = \int_0^1 2\tau(2z - 2\tau) \, d\tau = \int_0^1 4\tau z - 4\tau^2 \, d\tau$



$z = 0$ | $0 < z < 1$ | $1 < z < 2$

$f(z) = 0$

$$f(z) = \int_0^z 4\tau z - 4\tau^2 \, d\tau$$

$$= \left(\frac{4}{2} z\tau^2 - \frac{4}{3}\tau^3 \Big|_0^z \right)$$

$$= \left(2z^3 - \frac{4}{3}z^3\right) - (0)$$

$$= \frac{2}{3}z^3$$

$$f(z) = \int_{-1+z}^{1} 4\tau z - 4\tau^2 \, d\tau$$

$$= \left(\frac{4}{2} z\tau^2 - \frac{4}{3}\tau^3 \Big|_{-1+z}^{1}\right)$$

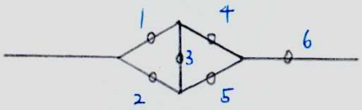
$$= \left(\frac{4}{2}z - \frac{4}{3}\right) - \left(2z(z-1)^2 - \frac{4}{3}(z-1)^3\right)$$

$$= 2z - \frac{4}{3} - 2z^3 + 4z^2 - 2z + \frac{4}{3}z^3 - 4z^2 + 4z - \frac{4}{3}$$

$$= -\frac{2}{3}z^3 + 4z - \frac{8}{3}$$

$$f(z) = \begin{cases} 0 & ; \quad z \le 0 \\ \frac{2}{3}z^3 & ; \quad 0 < z < 1 \\ -\frac{2}{3}z^3 + 4z - \frac{8}{3} & ; \quad 1 < z < 2 \\ 0 & ; \quad 2 \le z \end{cases}$$

P.4-52 EX65



(導通机率)

$$P = (1-P_3)\left[(P_1 \cap P_4) \cup (P_2 \cap P_5)\right] P_6 + P_3\left[(P_1 \cup P_2) \cap (P_4 \cup P_5)\right] P_6$$

$$= (1-P_3)\left[P_1P_4 + P_2P_5 - P_1P_2P_4P_5\right] P_6 + P_3\left[(P_1+P_2-P_1P_2)(P_4+P_5-P_4P_5)\right] P_6$$

P.4-124 EX141 : X, Y are indep. $\sim E(\lambda)$, $\lambda = a$, Find the Pdf of $Z = \frac{X}{X+Y}$

(i) $\begin{cases} V = X \\ W = X+Y \end{cases}$ $\begin{cases} X = V \\ Y = W-V \end{cases}$

$$f(v,w) = f_{X,Y}(x=v,\ y=w-v)\,|J| \quad , \quad f(x,y) = a e^{-ax} a e^{-ay} \ ; \ x, y \geq 0$$

$$f(v,w) = a^2 e^{-av} e^{-a(w-v)} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = a^2 e^{-aw} \ ; \quad w \geq 0$$

(ii) $Z = \frac{X}{X+Y} = \frac{V}{W}$ $\qquad f(z) = \int_{\infty}^{\infty} |w|\, f_{V,W}(v = zw, w)\, dw$

$$= \int_0^\infty w \cdot a^2 e^{-aw}\, dw = a^2 \int_0^\infty w e^{-aw}\, dw$$

$$\begin{array}{ll} +\ w & e^{-aw} \\ -\ 1 & -\frac{1}{a} e^{-aw} \\ \ \ 0 & \frac{1}{a^2} e^{-aw} \end{array}$$

$$= a^2\left(-\frac{w}{a} e^{-aw} - \frac{1}{a^2} e^{-aw}\Big|_0^\infty\right) = 1 \ ; \ 0 < Z < 1$$

失敗率 : $\frac{f_T(t)}{1-F_T(t)}$ (hazard rate)

P. 4-165 EX 180 : X, Y indep. $\sim U(0,1)$ $\begin{cases} W = \sqrt{-2\log x}\ \sin(2\pi Y) \\ V = \sqrt{-2\log x}\ \cos(2\pi Y) \end{cases}$

Show that W and V are indep. and each has the standard normal distribution.

$$\begin{cases} W = \sqrt{-2\log x}\ \sin(2\pi Y) \\ V = \sqrt{-2\log x}\ \cos(2\pi Y) \end{cases} \qquad \begin{aligned} W^2 + V^2 &= (-2\log x)\sin^2(2\pi Y) + (-2\log x)\cos^2(2\pi Y) \\ &= -2\log x \end{aligned}$$

$$\begin{cases} X = e^{-\frac{W^2+V^2}{2}} \\ Y = \frac{1}{2\pi}\tan^{-1}\left(\frac{W}{V}\right) \end{cases} \qquad f(W,V) = f_{x,Y}\left(x = e^{-\frac{W^2+V^2}{2}},\ Y = \frac{1}{2\pi}\tan^{-1}\left(\frac{W}{V}\right)\right)|J| = 1 \cdot |J|$$

$$|J| = \begin{vmatrix} \frac{\partial e^{-\frac{W^2+V^2}{2}}}{\partial W} = -We^{-\frac{W^2+V^2}{2}} & \frac{\partial \frac{1}{2\pi}\tan^{-1}\left(\frac{W}{V}\right)}{\partial W} = \frac{1}{2\pi}\frac{1}{1+\frac{W^2}{V^2}} \cdot \frac{1}{V} \\ \frac{\partial e^{-\frac{W^2+V^2}{2}}}{\partial V} = -Ve^{-\frac{W^2+V^2}{2}} & \frac{\partial \frac{1}{2\pi}\tan^{-1}\left(\frac{W}{V}\right)}{\partial V} = \frac{1}{2\pi}\frac{1}{1+\frac{W^2}{V^2}} \cdot -WV^{-2} \end{vmatrix}$$

$$= \left(W^2 \cdot \frac{1}{2\pi}\frac{1}{W^2+V^2} + \frac{V^2}{2\pi}\frac{1}{W^2+V^2}\right) e^{-\frac{1}{2}(W^2+V^2)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(W^2+V^2)} = f_{W,V}(W,V)$$

$$\longrightarrow f_{W,V}(W,V) = f_W(W) \cdot f_V(V) = \frac{1}{\sqrt{2\pi}} e^{-\frac{W^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{V^2}{2}}$$