Catalan Numbers

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Intro to Catalan numbers

- Background
  - Named after the Belgian mathematician Eugène Charles Catalan (1814–1894).
  - 清代數學家明安圖(1692年－1763年)在其《割圜密率捷法》中最先發明這種計數方式，遠遠早於Catalan
- Appear in more than 100 counting problems
  - Stack-sortable permutations
  - Dyck words
  - Full binary trees
  - Convex polygons
  - Mountain range
  - …

Stack-sortable Permutations

- Numbers of all possible ways of sending a sequence of 1 to n to a stack and pop them out.
- $C_1=1$, $C_2=2$, $C_3=5$, $C_4=14$, …
- Recursion

123O
132O
213O
231O
312X
321O

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0.$$

- Analytic formula

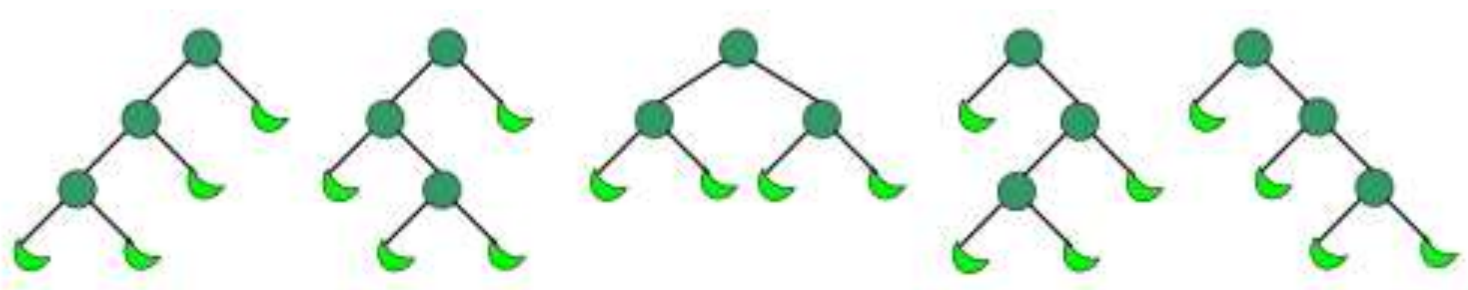
$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} \quad \text{for } n \geq 0.$$

Can be proved by generating functions

Paring of Binary Operators

- The number of ways of associating n applications of a binary operator (with n+1 operands)

((ab)c)d (a(bc))d (ab)(cd) a((bc)d) a(b(cd))



- Same recursion as the Catalan number

Full binary tree with n+1 leaves

Dyck Words

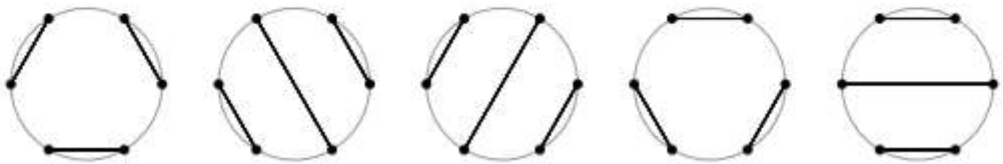
- Number of Dyck words of length 2n, which consists of *n* X's and *n* Y's such that no initial segment of the string has more Y's than X's
- Example
  - n=3

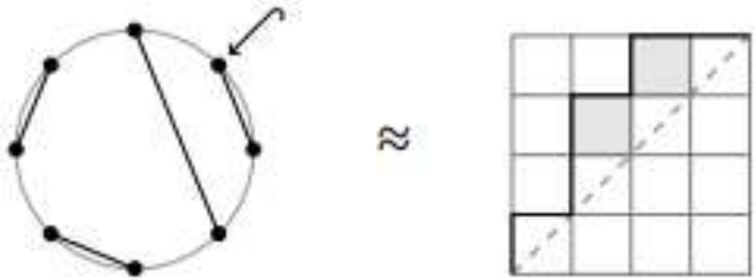
    XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY.
  - n=3

    ((())) ()(()) ()()() (())() (()())

Non-crossing Handshake Patterns

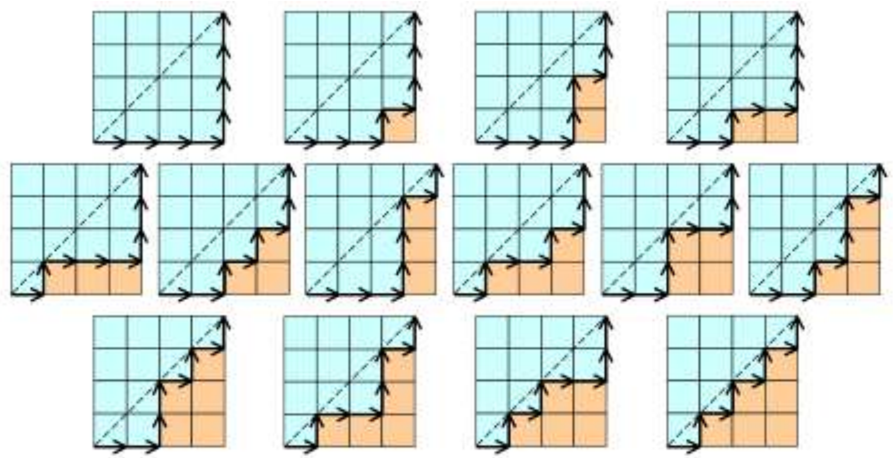
- 2n nodes located at the boundary of a circle
- How many ways to pair the 2n nodes with edges that do not intersect

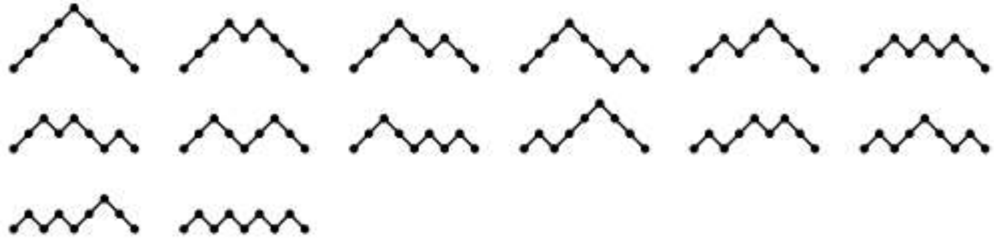




Constrained Lattice Paths

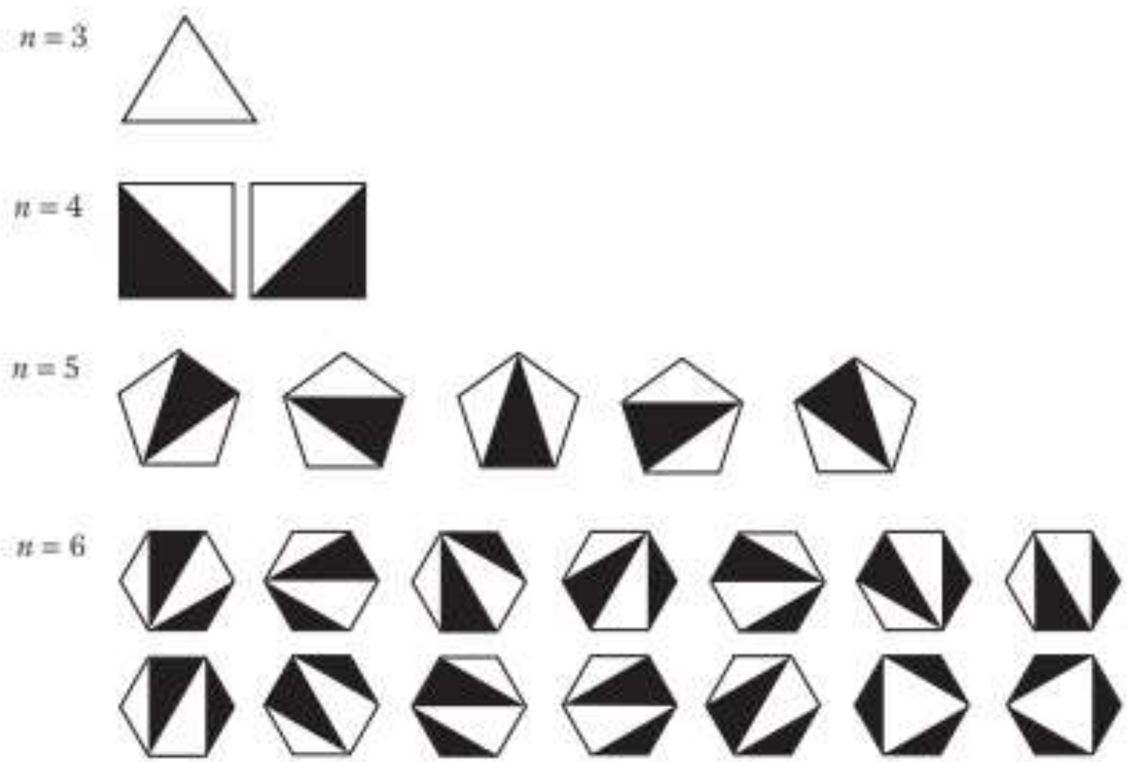
- Number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal.





Triangulations of N-gons

- Number of ways a convex n-gon can be partitioned into triangles by drawing non-intersecting diagonals.



Proof 1: By Generating Function

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0.$$

$$c(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$c(x) = 1 + xc(x)^2$$

$$c(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1-4x}}$$

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} y^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \cdots$$

$$c(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}$$

Proof 2: By Bijection

- 0➔push, 1➔pop
- When n=5
  - Legal sequence: 0 1 0 0 1 0 1 1 0 1
  - Illegal sequence: $\underbrace{0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0}_{\text{5 0's, 5 1's}}$ ➔ $\underbrace{0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1}_{\text{4 0's, 6 1's}}$

Transform is bijective!

  - So we have:

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} \quad \text{for } n \geq 0.$$

References

- Wikipedia
  - https://en.wikipedia.org/wiki/Catalan_number
  - https://zh.wikipedia.org/wiki/卡塔兰数
- Dyck Paths and The Symmetry Problem
- The Ubiquitous Catalan Number