Case Study 2: Conceptual Rocket Design

Linearity II

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Figure 1: Rocket Racoon has virtually nothing to do with this case study. But it was a fun movie.

SpaceY Inc.

SpaceY is a new startup that is exploring getting into the space business. They have contracted with you to propose a conceptual rocket design for inserting a 1000 kg mass into low earth orbit (LEO). They have framed the question pretty simply: *How many stages should the rocket have, and how big should each stage be?*

Why are they interested in these two questions? Well, your initial research on rocks yielded *some* useful general information. First, it turns out that making a rocket with multiple stages can be quite helpful with respect to rocket performance, but can also increase the complexity and cost of the rocket. According to `wikipedia`,

> A multistage (or multi-stage) rocket is a rocket that uses two or more stages, each of which contains its own engines and propellant. A tandem or serial stage is mounted on top of another stage; a parallel stage is attached alongside another stage. The result is effectively two or more rockets stacked on top of or attached next to each other. Taken together these are sometimes called a launch vehicle. Two-stage rockets are quite common, but rockets with as many as five separate stages have been successfully launched. By jettisoning stages when they run out of propellant, the mass of the remaining rocket is decreased. This staging allows the thrust of the remaining stages to more easily accelerate the rocket to its final speed and height.
>
> In serial or tandem staging schemes, the first stage is at the bottom and is usually the largest, the second stage and subsequent upper stages are above it, usually decreasing in size. In parallel staging schemes solid or liquid rocket boosters are used to assist with lift-off. These are sometimes referred to as 'stage 0'. In the typical case, the first-stage and booster engines fire to propel the entire rocket upwards. When the boosters run out of fuel, they are detached from the rest of the rocket (usually with some kind of small explosive charge) and fall away. The first stage then burns to completion and falls off. This leaves a smaller rocket, with the second stage on the bottom, which then fires. Known in rocketry circles as staging, this process is repeated until the final stage's motor burns to completion.
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> The main reason for multi-stage rockets and boosters is that once the fuel is exhausted, the space and structure which contained it and the motors themselves are useless and only add weight to the vehicle which slows down its future acceleration. By dropping the stages which are no longer useful, the rocket lightens itself. The thrust of

future stages is able to provide more acceleration than if the earlier stage were still attached, or a single, large rocket would be capable of. When a stage drops off, the rest of the rocket is still traveling near the speed that the whole assembly reached at burn-out time. This means that it needs less total fuel to reach a given velocity and/or altitude.

You've also found that when engineers discuss rocket mass in a conceptual design, they typically consider three categories: structural mass, m_s; propellant or fuel mass, m_f, and payload mass, m_p.

The structural mass represents the stuff that is used to hold the rocket together and confine the propellant mass (so that it's a rocket, not a bomb). These two masses, together with the payload, constitute the rocket's fully loaded mass:

$$m_{total} = m_s + m_p + m_f$$

Note that the payload mass is defined by the customer, while the fuel mass and structural mass are both defined by the people designing the rocket.

SpaceY has informed you that they have a number of hard engineering constraints that your conceptual design must take into account: First, SpaceY's structural technology requires that the fuel mass of any given stage cannot be greater than about $4\times$ the structural mass of that stage. In addition, SpaceY's proprietary propellant technology produces a consistent exhaust velocity is $v_e = 3000$ m/sec.

Background: Rocket Models

There are a variety of approaches for modeling the behavior of a given rocket. A common model for rocket performance is the Tsiolkovsky rocket equation, which calculates the change in a rocket's velocity (Δv) as a function of the rocket's exhaust velocity v_e (i.e., the speed at which propellant is expelled from the rocket's nozzle), and the rocket's masses:

$$\Delta v = v_e \ln \frac{m_s + m_f + m_p}{m_s + m_p}$$

As can be seen, the more of your mass that is fuel, the better the performance.

For a multi-stage rocket, one could extend the model to calculate the total change in velocity:

$$\Delta v_{total} = \Delta v_2 + \Delta v_2 + ...$$

where Δv_n is the change in the rocket's velocity associated with a given stage.

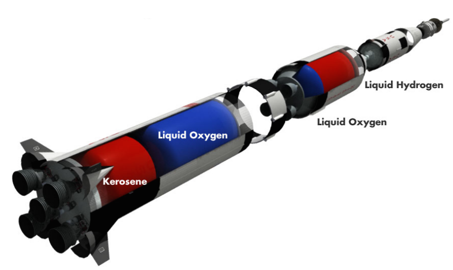

Figure 2: An example of a multi-stage rocket. Note the large fraction of volume devoted to propellant.

This model ignores the effects of gravitational impulse and drag, and as a consequence somewhat overestimates the change in velocity.

The Tsiokolvsky equation can be easily extended to include the effect of gravity if the gravitational acceleration is constant:

$$\Delta v = v_e \ln \frac{m_s + m_f + m_p}{m_s + m_p} - g t_{tot}$$

where t_{tot} is the total time of the burn.

Note that neither Tsiokolvsky equation nor that modified Tsiokolvsky equation indicates how *far* the rocket goes; rather they only specify the change in velocity. Furthermore neither includes the effect of drag, or the effect of changing gravitational conditions.

If one wants to include drag and gravity in modeling the rocket, it becomes necessary to solve the differential equation:

$$\frac{d\vec{v}}{dt} = -v_e \frac{dm}{dt}\hat{v} - \frac{\vec{F}_g}{m(t)} - -\frac{\vec{F}_D}{m(t)}$$

where $m(t)$ is the mass of the rocket as a function of time, v_e is the exhaust speed of the propellant measured relative to the rocket, F_g is the gravitational force, and F_D is the drag force. Note that the drag force depends both on where the rocket is (through the atmospheric density) and on how fast the rocket is moving.

This differential equation can only be solved numerically (e.g., using ODE45).

Deliverable

You should do the following:

1. Formulate the problem as one or more a rigorously-framed optimization problem(s). Be sure to define your notation clearly, using appropriate mathematical notation.

2. Situate the problem (or problems) in the optimization taxonomy: what kind of problem is this?

3. Identify an appropriate computational tool to solve the problem. Some places to start: MATLAB has an optimization toolbox (that we have a license for). There are also good things on `http://www.neos-guide.org/NEOS-Resources` if you are so inclined, as well as various things in the open source world.

4. Write up your recommendations regarding the conceptual rocket design in a brief memo to the head of engineering at SpaceY. Note that the head of engineering is probably more mathematically trained than the director of the water authority, so it might be

appropriate to include one or two graphs or equations in your memo. The memo should be clear about (1) what the question is that you are addressing; (2) how you went about addressing the question (your method), (3) what the results of your calculations are, and (4) what your recommendations are based on these results. Given the nature of the question you're addressing, a memo that is more than 2 or 3 pages long (not counting appendices) is too long. Also include your mathematical work and code as appendices.