Network Science
Robustness

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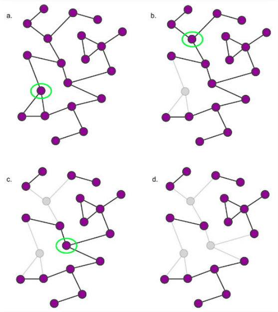
Introduction



[1]

- The removal of nodes can break a network into several components.
- The more nodes we remove, the higher are the chances to break (damage) a network.

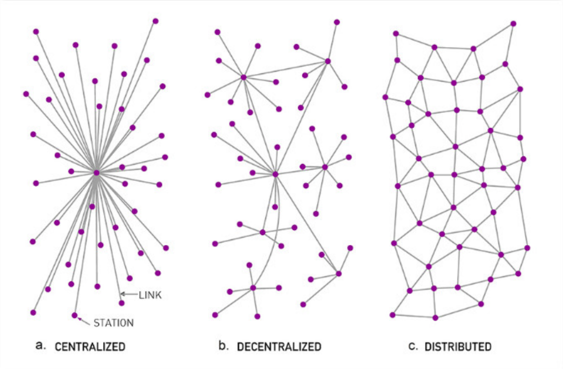
[1] *Images are taken from Barabási book*

Introduction

- How to predict the failure of an ecosystem when faced with the disruptive effects of human activity?
- Why some mutations lead to diseases and others do not?
- Natural and social systems have a remarkably ability to sustain their basic functions even when some of their components fail.

Understanding the origins of this robustness is important for many disciplines and networks play a key role in the robustness of biological, social and technological systems.

Introduction



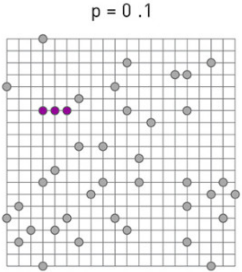
Possible configurations of communications networks, as envisioned by Paul Baran in 1959.

Introduction

- How many nodes do we have to delete to fragment a network into isolated components?
- How the structure of the underlying network plays an essential role in a system's ability to survive random failures or deliberate attacks?
- Are there universal laws governing the error and attack tolerance of complex networks?

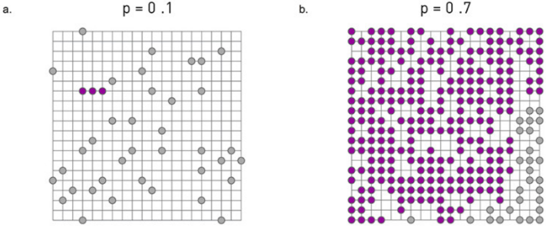
To answer these questions, we must familiarize ourselves with the mathematical fundamentals of network robustness, offered by percolation theory.

Percolation theory

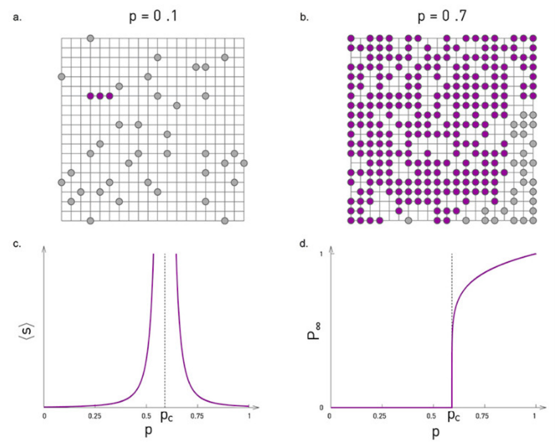


- Let us suppose a square lattice, where we place pebbles with probability p at each intersection.
- neighboring pebbles are considered connected forming clusters of size two or more.
- Given that the position of the pebble is decided by chance, we ask:
  - What is the expected size of the large cluster?
  - What is the average cluster size?

Percolation theory



- The higher is p, the larger are the clusters.
- Percolation theory predicts is that the cluster size does not change gradually with p.
- For a wide range of p, the lattice is populated with numerous tiny clusters.
- If p approaches a critical value p_c, the small clusters grow and coalesce, giving rise to a large cluster (percolation cluster) at p_c.



(c) The average cluster size $\langle s \rangle$, as a function pf p, (d) The probability P_∞ that a pebble belongs to the largest cluster.

Percolation theory

- **Average cluster size:** $\langle s \rangle$. It diverges as we approach p_c

$$\langle s \rangle = |p - p_c|^{-\gamma_p}$$

- **Order parameter:** P_∞. As p decreases towards p_c, the probability that a pebble belongs to the largest cluster drops to zero.

$$P_\infty \sim (p - p_c)^{\beta_p}$$

- **Correlation length:** ξ. The mean distance between two pebbles that belong to the same cluster follows:

$$\xi \sim |p - p_c|^{-v}$$

at p_c the size of the largest cluster becomes infinite, percolating the whole lattice.

Percolation theory

- The exponents γ_p,β_p and v are called the *critical exponents*, as they characterize the system's behavior near the critical point p_c.
- If we place the pebbles on a triangular or a hexagonal lattice, the behavior of $\langle s\rangle$, P_∞ and ξ is characterized by the same γ_p,β_p and v exponents.

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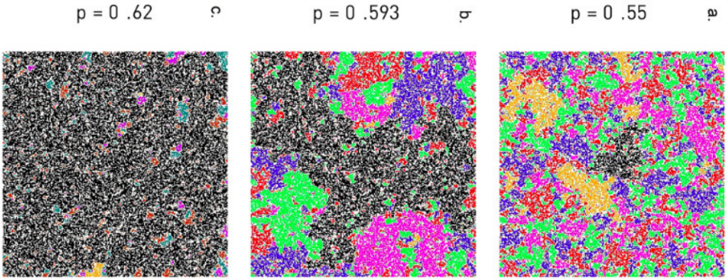
Percolation theory predicts that these exponents are *universal*, meaning that they are independent of the nature of the lattice or the precise value of p_c.

Percolation theory

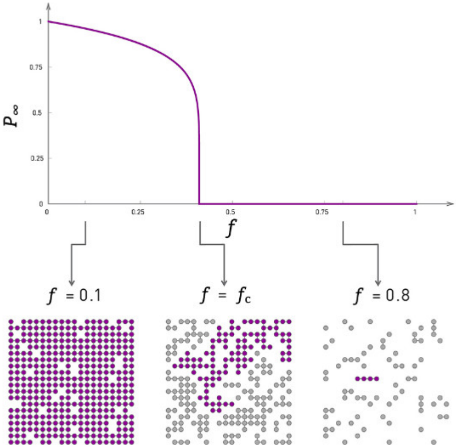


- Each pebble is a tree and the lattices describe a forest.
- If a tree catches the fire, it ignites their neighbors and these, in turn, ignite their neighbors.
- If we randomly ignites a tree, what fraction of the forest burns down?

Inverse percolation transition and robustness

- We are interested on how the node failures affects the integrity of the network.
- We can view a square lattice as a network whose nodes are the intersections.
- We randomly remove a fraction f of nodes and evaluate how their absence impacts the integrity of the lattice.

Inverse percolation transition and robustness



$0 < f < f_c$:

There is a giant component.

$P_\infty \sim |f - f_c|^\beta$

$f = f_c$:

The giant component vanishes.

$f > f_c$:

The lattice breaks into many tiny components.

Inverse percolation transition and robustness

- If f is small, the missing nodes do little damage to the network.
- Increasing f can isolate groups of nodes from the giant component.
- Once f exceeds f_c the giant component vanishes.
- This process can be mapped to classical percolation using $f = 1 - p$.
- The critical exponents γ_p,β_p,v are the same as those encountered in regular lattices.

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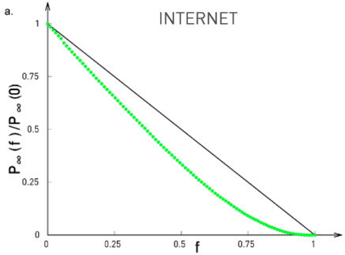
Random networks under random node failures share the same scaling components as infinite-dimensional percolation

Robustness of scale-free networks

- In regular lattices, the nodes have identical degrees.
- In random networks, the nodes have comparable degree.
- What appends, if the network is scale-free?

Robustness of scale-free networks



- Nodes are randomly selected and removed one by one.
- According to the percolation theory, once the number of removed nodes reaches a critical value f_c, the internet should fragmented into many isolated subgraphs.
- However, the simulation shows that the internet has a unusual robustness to random node failures. Its robustness is rooted in its scale-free topology.

Robustness of scale-free networks

- Random node failures are blinded to degree.
- In a scale-free network we have far more small-degree nodes than hubs.
- Hubs are responsible for this remarkable robustness.

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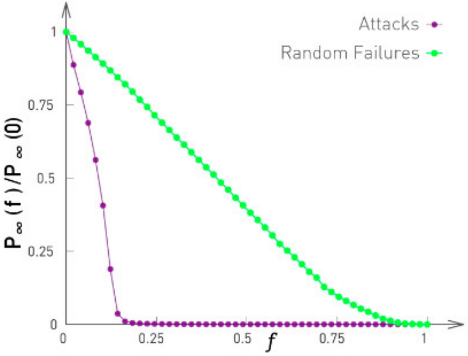
What if we do not remove the nodes randomly but go after the hubs?

Robustness of scale-free networks



For an attack the nodes are removed in decreasing order of their degree

From percolation to robustness