Momentum Operator Worked Example

1. Suggested study steps, based on Feynman study model [(a)]

1. Read the statement of the problem
2. Read slowly through the solution to get the big picture.
3. With pencil, paper and calculator work through the self-explanation prompts.
4. Write a summary of the problem and the solution as you would explain it to a friend.

2. Statement of the problem

The momentum is an important quantity in physics. In a classical framework the momentum is a variable but in quantum, is an operator. An operator is a mathematical structure that transforms a function to other function, e.g., let the function $f(x) = x^2$ and the operator $D = d/dx$, then $Df(x) = 2x$. In classical physics the momentum $\vec{p}$ of a particle with mass m is related with its velocity $\vec{v}$ and position $\vec{r}$ by $\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}$. The correspondence principle states that the behavior of a quantum system is equal to the predicted by classical physics at large quantum numbers; this principle can be applied to determinate the mathematical form of the momentum operator in quantum mechanics.

Suppose a one-dimensional quantum system in the limit of large quantum numbers, what is the mathematical definition of momentum operator?

3. Strategy

To resolve this problem, use the correspondence principle and the Schrödinger equation dependent on time. The correspondence principle gives you the mathematical structure of momentum, and the Schrödinger equation let to calculate the first time derivate.

4. Implementation

The correspondence principle implies the relation between momentum and position of classical physics, $p = m\frac{dx}{dt}$ have the similar mathematical structure than in quantum physics. However, in quantum physics the position and momentum are operator not variables, then, to compare obtain the expression of momentum in quantum physics it is necessary to calculate the expected values,

$$\langle p \rangle = m\frac{d}{dt}\langle x \rangle \tag{1}$$

The position expected value is given by,

$$\langle x\rangle = \int_{-\infty}^{\infty} \Psi^* \, x \, \Psi \, dx \tag{2}$$

Moreover, the Schrödinger equation dependent on time is,

$$\frac{\partial}{\partial t}\Psi(x) = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) - \frac{i}{\hbar}U(x)\Psi(x) \tag{3}$$

As $\langle p\rangle = m\frac{d}{dt}\langle x\rangle$, then,

$$\langle p\rangle = m\frac{d}{dt}\int_{-\infty}^{\infty} \Psi^* \, x \, \Psi \, dx = m\int_{-\infty}^{\infty}\left[\left(\frac{\partial}{\partial t}\Psi^*\right)(x\,\Psi) + \Psi^*\left(\frac{\partial}{\partial t}\Psi\right)\right]dx \tag{4}$$

Taken the conjugate to equation (3) we obtain,

$$\frac{\partial}{\partial t}\Psi^*(x) = -\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi^*(x) + \frac{i}{\hbar}U(x)\Psi^*(x) \tag{5}$$

Replacing equations (3) and (5) in (4) and integrating by parts,

$$\langle p\rangle = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\Psi dx \tag{6}$$

Do the comparison between equation (2) and equation (6) it is found that the one-dimensional momentum operator is given by,

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x} \tag{7}$$

5. Self-explanation prompts

5.1. Supply all the missing steps in the implementation.
5.2. Is it valid the equation (7) to any one-dimensional quantum system? If true, explain why.
5.3. What is the general expression of momentum operator (3D)?

6. References and further reading

(a) https://mattyford.com/blog/2014/1/23/the-feynman-technique-model

Prof. David A. Miranda proposed this Worked Example at April 1, 2018