Stationary Wave Equation and The Stationary Schrödinger Equation Worked Example

1. Suggested study steps (based on Feynman study model1)

1. Read the statement of the problem
2. Read slowly through the solution to get the big picture.
3. With pencil, paper and calculator work through the self-explanation prompts.
4. Write a summary of the problem and the solution as you would explain it to a friend.

2. Statement of the problem

Most of the modern physics and quantum mechanics textbooks introduce the Schrödinger equation as an Ad-hoc equation without no much deductions than the quantum mechanics principles enunciated to present its. To a better understanding of the birth of a new theory and its fundamentals principles, it is convenient to perform simple mental exercises orientated to show a way of knowledge creation related to the matter of study, in this case, the Scrhödinger equation as a material wave equation.

Suppose you are interested in studying ondulatory problems described by stationary waves in one dimension that interchanges momentum and energy. If the ondulatory problem involves the interchange of momentum in a way that each wave has a momentum $\vec{p}$ proportional to the wave vector $\vec{k}$, i.e. $\vec{p} = \hbar\,\vec{k}$, and the waves could interchange energy in a non-relativistic way, what is the stationary wave equation that represents this ondulatory problem?

3. Strategy

To resolve this problem, you need: first, to write the one-dimensional stationary wave equation to the wave function $\Psi(x)$ with wave vector $\vec{k}$, second, to write the energy associated to the wave function $\Psi(x)$ regarding the wave vector $\vec{k}$, finally, to write the one-dimensional differential equation that describe stationary ondulatory problems that involve energy and momentum interchange (one-dimensional Schödinger equation).

4. Implementation

4.1. The one-dimensional stationary wave equation

Let $\Psi(x)$ the wave function related to the stationary one-dimensional ondulatory problem to be described. If $\vec{k}$ is the one-dimensional wave vector, then, the stationary wave equation is given by,

$$\frac{d^2}{dx^2}\Psi(x) + k^2\Psi(x) = 0 \tag{1}$$

4.2. Total energy and the wave vector

The total energy E of a system with kinetic energy E_k and constrained in a potential U(x) is,

$$E = E_k + U(x) \tag{2}$$

The non-relativistic kinetic energy E_k is related with momentum by,

$$E_k = p^2/2m \tag{3}$$

If the wave vector $\vec{k}$ and momentum $\vec{p}$ are related by a linear relation, $\vec{p} = \hbar\,\vec{k}$, then, $E_k = \hbar k^2/2m$. Because $E = E_k + U(x)$, the wave vector can be expressed as,

$$k^2 = \frac{2m}{\hbar}[E - U(x)] \tag{4}$$

4.3. The one-dimensional stationary wave equation to waves with momentum

Replacing equation (4) into (1) and rearranging terms we obtain the differential equation that describe stationary ondulatory problems that imply energy and momentum interchange,

$$-\frac{1}{2m}\frac{d^2}{dx^2}\Psi(x) + U(x)\Psi(x) = E\Psi(x)$$

5. Self-explanation prompts

1) Where is the origin of the equation (1)?
2) What mean stationary in the context of this worked example?
3) If the wave vector and momentum are vectors, why in the equations (1), (3) and (4) appears as scalars?
4) In an ondulatory problem, is possible describe the wave vector of a wave packet by an only one (monochromatic) wave vector? Expose your arguments.
5) Is the a priori relation $\vec{p} = \hbar\,\vec{k}$ a description of a monochromatic wave or can this expression describe a wave packet? Expose your arguments.
6) What limitations have the description of the ondulatory mechanics proposed here?
7) Is the proposed situation a deduction of the Schödinger equation? Expose your arguments.

8) What is Schrödinger equation dependent of time and how obtain the independent of time Schrödinger equation from it?

6. References and further reading

- https://mattyford.com/blog/2014/1/23/the-feynman-technique-model
- http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr.html
- https://en.wikipedia.org/wiki/Schr%C3%B6dinger_equation
- http://home.tiscali.nl/physis/HistoricPaper/Schroedinger/Schroedinger1926c.pdf

This Worked Example was proposed by Prof. David A. Miranda at October 28, 2017