Reducibility of a C_{3v} regular representation

1. Suggested study steps, based on Feynman study model [a]

1. Read the statement of the problem
2. Read slowly through the solution to get the big picture.
3. With a pencil, paper, and calculator, work through the self-explanation prompts.
4. Write a summary of the problem and the solution as you would explain it to a friend.

2. Statement of the problem

To obtain the irreducible representations in the regular representation of the symmetry operations of the equilateral triangle.

3. Strategy

To solve this problem, first, it is necessary to review the definition of regular representation. Second, to get the regular representation of the equilateral triangle symmetry operations from its multiplication table. Third, to obtain the characters of the regular representation. Finally, with the character table of the C_{3v} group and the wonderful orthogonality theorem for characters, to get the number of irreducible representations in the reducible one.

4. Implementation

4.1. Definition of the regular representation

Regular representation of a group corresponds with matrixes got directly from the group multiplication table, ensuring the identity is a diagonal matrix, e.g., the matrix$D^{(\Gamma)}(g_1)$ for the element g_1 in the Γ representation of the group is a $h \times h$ matrix with ones in the positions where g_1 is, and zero in other positions, where h is the number of elements of the group[b].

4.2. Regular representation of the equilateral triangle symmetry operations

The equilateral triangle has six symmetry operations isomorphic with group C_{3v}: the identity (E), two rotations axis (C_3, C_3^2), and three reflection planes ($\sigma_A, \sigma_B, \sigma_C$). Table 1 is the multiplication table of the group of symmetry operations of the equilateral triangle.

Table 1. Multiplication table of the group of symmetry operations of the equilateral triangle.

	$\boldsymbol{E}$	$\boldsymbol{C_3}$	$\boldsymbol{C_3^2}$	$\boldsymbol{\sigma_A}$	$\boldsymbol{\sigma_B}$	$\boldsymbol{\sigma_C}$
$\boldsymbol{E}$	E	C_3	C_3^2	σ_A	σ_B	σ_C
$\boldsymbol{C_3^2}$	C_3^2	E	C_3	σ_C	σ_A	σ_B
$\boldsymbol{C_3}$	C_3	C_3^2	E	σ_B	σ_C	σ_A
$\boldsymbol{\sigma_A}$	σ_A	σ_B	σ_C	E	C_3^2	C_3
$\boldsymbol{\sigma_B}$	σ_B	σ_C	σ_A	C_3	E	C_3^2
$\boldsymbol{\sigma_C}$	σ_C	σ_A	σ_B	C_3^2	C_3	E

The following matrixes give the regular representation:

$$D^{(R)}(E) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad D^{(R)}(C_3) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D^{(R)}(C_3^2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad D^{(R)}(\sigma_A) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D^{(R)}(\sigma_B) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad D^{(R)}(\sigma_C) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.3. Characters of the regular representation

The character $\chi_{g_i}^{(R)}$ of the element g_i in the regular representation is the trace of the matrix $D^{(R)}(g_i)$. In Table 2 are shown the characters of the regular representation.

Table 2. Characters of the regular representation of symmetry operations of the equilateral triangle

	$\boldsymbol{E}$	$\boldsymbol{C_3}$	$\boldsymbol{C_3^2}$	$\boldsymbol{\sigma_A}$	$\boldsymbol{\sigma_B}$	$\boldsymbol{\sigma_C}$
$\boldsymbol{\chi}$	6	0	0	0	0	0

4.4. Irreducible representations in the regular representation

The characters table for the C_{3v} group is shown in Table 3. The wonderful orthogonality theorem for characters apply to the de regular representation implies that it is possible to obtain the number of irreducible representations in a reducible one. The C_{3v} group have three classes, $\mathcal{C}_1 = \{E\}$, $\mathcal{C}_2 =$

$\{C_3, C_3^2\}$, and $\mathcal{C}_3 = \{\sigma_A, \sigma_B, \sigma_C\}$; note the number of elements in each class are, $N_1 = 1$, $N_2 = 2$, and $N_3 = 3$. Then, the number n_i of times that the irreducible representation Γ_i are in the regular representation is given by,

$$n_i = \frac{1}{h}\sum_k N_k \left[\chi^{(\Gamma_j)}(\mathcal{C}_k)\right]^* \chi^{(R)}(\mathcal{C}_k)$$

Table 3. Characters of the C_{3v} group and the regular representation

	$\mathcal{C}_1$	$2\mathcal{C}_2$	$3\mathcal{C}_3$
$\mathbf{\Gamma_1}$	1	1	1
$\mathbf{\Gamma_2}$	1	1	-1
$\mathbf{\Gamma_3}$	2	-1	0
$\Gamma^{(R)}$	6	0	0

Then, the regular representation can be expressed in irreducible representations as,

$$\Gamma^{(R)} = \Gamma_1 \oplus \Gamma_2 \oplus 2\Gamma_3$$

5. Self-explanation prompts

5.1. To describe why the characters of regular representation are cero except for the identity.

5.2. If two rows are interchanged in the multiplication table, is obtained a regular representation? Explain your answer.

5.3. How is obtained n_i by applying the wonderful orthogonality theorem?

5.4. Apply the wonderful orthogonality theorem for characters to Table 3. What is orthogonality in this context?

5.5. What does the symbol $\oplus$ means in the last expression?

6. References and further reading

(a) https://mattyford.com/blog/2014/1/23/the-feynman-technique-model

(b) Dresselhaus, M and Dresselhaus, G. Group Theory: Application to the Physics of Condensed Matter. Berlin: Springer, 2009, pp 37-40.

Prof. David A. Miranda, Ph.D., proposed this Worked Example on September 12, 2020