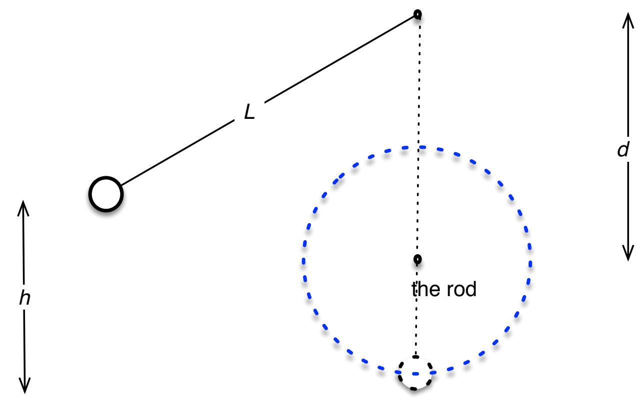
Physics 11 Problem Set 7
Hammer Due March 14 at 11:00

Problem set 7

1) Last week we spent time in lecture on a question about whether a pendulum bob would reach the height, if the string hit a rod on the way. Talking about it, someone suggested if the rod were lower, the pendulum might wrap around it. Good problem!



So: A pendulum of length L swings and hits a rod at a distance d below the pivot.

Let's suppose $d > L/2$: The rod is lower than the halfway point of the string. We'll release the bob at a height h above the bottom of the swing. If h is high enough, the pendulum will hit the rod and make a complete loop, as it did in lecture.

The question: How high does h have to be for that to happen? Assume the string's mass is negligible, no loss to friction or air resistance, and the rod is very thin. *I'll put hints on Piazza if you want help for how to work on this. And "bob" is what we call the mass at the end of a pendulum.*

First, just to say, it almost always helps to try it — get a string, a keychain or something else with a little mass to use as a bob, tie it somewhere and use a pen as the rod. It could even help just to swing something in a vertical circle. You can see that it has to go fast enough to make it all the way around, and when it's just barely fast enough, the string is just barely pulling. If it's faster than that, the string exerts a force down. If it's too slow, the acceleration due to gravity by the earth has it fall below the path of the circle.

So if we want to know the *slowest* it can be moving and still go in a circle, we should set the centripetal force at the top of the circle to be mg — just the pull of the earth:

$$\frac{mv^2}{r} = mg$$

at the top of the circle, where $r = L - d$ is the radius of the circle the bob makes. So

$v^2 = rg$, or, more useful, the bob's kinetic energy at the top of the swing is

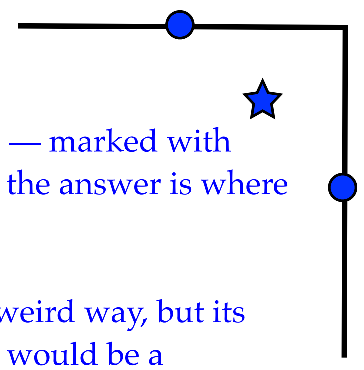
$$\tfrac{1}{2} mv^2 = \tfrac{1}{2} mrg.$$

The potential energy at that instant is $mg(2r)$, calling the bottom of the swing $h = 0$. That make the total mechanical energy $= \tfrac{1}{2} mrg + 2mgr = 2.5mgr$.

When we first release the pendulum, it isn't moving. So the initial total mechanical energy is entirely gravitational potential energy, mgh. That means:

$$mgh = 2.5mgr$$

So $h = 2.5r$, which is one way to say the answer: 2.5 times the distance between the peg and the bottom of the swing. That's $r = L - d$, so it's $h = 2.5\ (L - d)$.

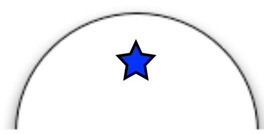


2) The figure shows a thin L square — two straight edges held together at right angles. Each has length a, and the total mass of the pair is M. Find the center of mass.

The center of mass of each of the sides is just at its respective center — marked with dots. And the center of mass between those dots is just midway, so the answer is where I put the star.

If you were to toss an L square in the air, it would flop around in a weird way, but its center of mass would move in a nice smooth parabola. The motion would be a combination of the motion of the center of mass and the L-square's rotation around the c.o.m.

If that's too easy, find the center of mass of a thin bar bent into a semi-circle of radius a and total mass M.



We want to find $\dfrac{\int (x,y)\,dm}{\int dm}$. Orient the semi-circle symmetrically across the y-axis, and it's clear $\int x\,dm = 0$, because for every piece of mass on the left there's one on the right, the same distance from the center. So really we only need to find $\int y\,dm$.

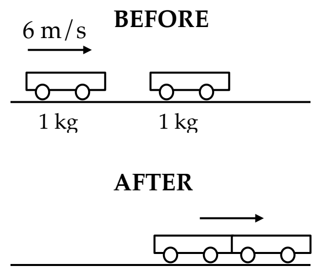
One piece of mass here, dm, can be a small length of the bar, along a small bit of angle $d\theta$. A semicircle has π radians in all, so $d\theta/\pi$ is a fraction of the bar that has a mass $dm = M\,d\theta/\pi$.

The y coordinate of any piece of mass is is $a\sin\theta$. So we have:

$\int y\,dm = \int_0^\pi (a\sin\theta)M\dfrac{d\theta}{\pi} = -\dfrac{Ma}{\pi}\left[\cos(\pi) - \cos(0)\right] = 2\dfrac{Ma}{\pi}$. Divided by M, and $y = \dfrac{2}{\pi}a$.

$2/\pi$ is a little less than ⅔, so the center of mass is about where I put the star.

3) A 1 kg cart, rolling at 6 m/s, collides with and sticks to an identical cart that's initially at rest. So, after colliding, the carts roll together as a single, 2 kg unit. (See the picture on the next page.) How fast does the pair of carts roll, stuck together? (Suppose friction is very small, compared to other forces in the problem.)



a) What would someone say, who's never had a physics course? (Feel free to ask someone, if you're not comfortable to guess!)

I have asked people, who haven't had physics, and they typically say "3 m/s." (Usually I use miles per hour as the unit of speed instead of m/s, so as not to sound so physics-y.) I think they're just reasoning that if you suddenly make the cart twice as big, it must go half as fast.

b) Now try for an answer consistent with Newton's Laws. One way to do that is by conservation of momentum: Since there are no external forces acting on the two-carts system, their total momentum is conserved.
Check what happens with your answer from part a: Is momentum conserved? If not, something's wrong — go figure out how to reconcile the contradiction.

Before the collision, the momentum was (1 kg) (6 m/s) = 6 kg-m/s. If I go with the guess in part a, then after it's (2 kg) (3 m/s) = 6 kg-m/s. Momentum was conserved, so what people say works.

Another way to think about it, just in terms of forces and accelerations: When Cart 1 hits Cart 2, it exerts force on C2, making it accelerate. By the Third Law, C2 exerts and equal and opposite force on C1. Since the masses of the two carts are the same, that means C1 has an equal and opposite acceleration, so the change in its velocity must be equal and opposite the change in C2's velocity. And that's what happens: C2 gains 3 m/s; C1 loses 3 m/s.

Well, really that's saying that whatever momentum C1 loses, C2 gains… so momentum is conserved, so this second way ends up actually the same as the first.

c) Find the total kinetic energy before and after the collision.

Before the collision, the energy was $\frac{1}{2}(1kg)(6m/s)^2 = 18\,\text{Joules}$. After the collision, it was $\frac{1}{2}(2kg)(3m/s)^2 = 9\,\text{Joules}$. So 9 *J* of kinetic energy was "lost," meaning it converted to some form other than the kinetic energy of the carts. (It goes into heat — the carts are a bit warmer — and sound, which ends up heating the air a tiny bit, and maybe damage to the carts.)

d) Again, except make the stationary cart mass 2 kg; the moving cart still 1 kg. Find the speed and total kinetic energy of the carts after the collision.

Before the collision, the momentum was (1 kg) (6 m/s) = 6 kg-m/s, and after it's (3 kg) v which must also be 6 kg-m/s, because momentum must be conserved. So: v = 2 m/s. That's also the speed of the center of mass, before and after the collision, and you can notice that the magnitude of the change in velocity of the 1 kg cart (6 --> 2 is a change of -4) is twice the magnitude of the change in velocity for the 2 kg cart (0 --> 2).

As for energy, before the collision, it's is still 18 Joules. After the collision, it's now $\frac{1}{2}(3kg)(2m/s)^2 = 6\text{Joules}$. More energy is "lost" to heat, sound, damage to the carts. And if you make the stationary cart more massive still, that will be even more kinetic energy going into heat and damage.

It makes sense that more energy is lost, I think: If you had to collide with something, wouldn't you rather it be something light? The more kinetic energy you keep, the less of it can go into doing damage.

4) More of those carts. Another way to think about them is to consider the motion of the center of mass.

a) For the 1 kg carts, find the speed and direction of the center of mass.

Since the carts have equal mass, the center-of-mass is half-way between them. With one of the carts is moving at 6 m/s and the other sitting still, that point half-way between them must be moving 3 m/s. And of course that's how fast it's moving afterward, if indeed they move at 3 m/s, as common sense suggests.

Which is yet another way to solve that first question, #3 part a, and again giving that same answer people usually guess.

b) Imagine you were *there* somehow, at the location of the center of mass, watching the collision. Describe the motion of the carts, before and after the collision, from your perspective there at the center of mass. That is, how fast are they coming toward you, before they collide?

If you were moving at 3 m/s, at the location of the center of mass of the two carts (both 1 kg for this part), you'd see the two carts coming toward you from opposite directions, each at 3 m/s, zero total momentum, and then when they hit they'd just be sitting still, right where you are.

c) Now suppose that instead of sticking together, the carts bounce off each other, without any loss of kinetic energy (a "perfectly elastic collision"). What would that look like to you, at the location of the center of mass?

So now, if they bounce back out, you'd see just that! Each cart moving away from you now, at 3 m/s. Notice that in this reference frame, the center of mass reference frame, the center of mass doesn't move — it's always halfway between the two carts. That's the same as saying the total momentum stays zero.

Notice also that the kinetic energy isn't the same in one reference frame as another. There's less total kinetic energy in the center-of-mass reference frame.

(And if the carts were to stick together, then in the center of mass reference frame, they would end up not moving — no kinetic energy left at all: *all of the kinetic energy in the center-of-mass reference frame would be lost.* In the room reference frame, you found in problem 3 part d, they still have kinetic energy.)

d) Explain what the collision in part b would look like if you were watching from the room (the original perspective).

In the center of mass frame, that first cart is now moving backward at 3 m/s. So in the frame of reference of the room, it's not moving at all.

The second cart, though, is now moving at 6 m/s in the original direction. Notice that the center of mass of the two carts is still moving 3 m/s, in the room frame.

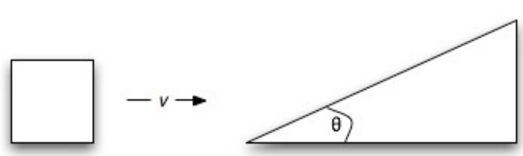
e) And again, repeat all that for the 1 kg cart hitting the 2 kg cart.

A bit harder, but not so bad: The center of mass is moving 2 m/s. So in the center of mass reference frame you see the 1 kg cart coming at you at 4 m/s, and you see the 2 kg cart coming at you at 2/m/s from the other side. If they stick together everything's done; the center of mass never moves, and all the kinetic energy is lost.

If they bounce out, then each cart just turns around: The 1 kg cart is now moving away from you 4 m/s, and the 2 kg cart is moving away 2 m/s. (There's no other way to keep the center of mass stationary and conserve kinetic energy.)

And so, back in the room: The 1 kg cart is moving 2 m/s to the left, and the 2 kg cart is moving 4 m/s to the right. And check it out: Their total kinetic energy is
$\frac{1}{2}(1kg)(2m/s)^2 + \frac{1}{2}(2kg)(4m/s)^2 = 18$ Joules.

5) A block of mass m and a ramp of the same mass m are both free to slide along a surface, without friction. The ramp is at an incline θ. So: the block slides along the surface at speed



v hits the ramp, which is initially at rest, and slides up, also without any friction. The question is to find the maximum height h that the block rises on the ramp, in terms of the other variables. Remember, both the block and the ramp are free to slide along the surface, so the ramp moves.

That's enough information to solve the problem. For more help, see Piazza!

At that instant, the block and the ramp aren't moving relative to each other, so they might as well be stuck together, just like the carts in problem 2. So they are moving at the speed of the center of mass, which has been constant throughout (there's no external force in the x-direction). And for the same reasoning as in problem 2, we know that speed is $v_{cm} = v/2$.

The kinetic energy of the block and ramp at that instant, when the block is at its highest point and they're both moving at a speed $v_{cm} = v/2$,

$$½\,(2m)(v_{cm})2 = ½(2m)(v/2)^2 = ¼\,mv^2.$$

The initial kinetic energy of the block ,of course, is $½mv^2$, so the total kinetic energy is half what it was at the start… all that energy went into lifting the block on the ramp.

The potential energy of the block on the ramp is mgh and we just found out that

$mgh = ¼\,mv^2$. That gives us $h = \frac{1}{4g}v^2$.

As I explained on WebEx, this is it's a nice way to think about an elastic collision, sort of happening in slow-motion: That block sitting at the top of the ramp has potential energy, half-way through a collision between it and the ramp. Let it slide back down and we get the rest of the collision.

And, yes, the answer doesn't depend on the angle of the ramp. That seems off, doesn't it? If the angle is near 90°, the situation seems pretty different from when the angle is small! How could the block even go up the ramp, if the ramp is like a wall?

The set up of the problem assumes that the block will slide up the ramp with no friction, so no loss of energy. That assumption wouldn't work very well for large values of the angle: For one, the size of the normal force between the block and the ramp would get to be very large, and unless the coefficient of kinetic friction is very close to 0, it would be hard to assume there's no friction! And for another, that normal force could start to dent the surface. The assumption wouldn't work for very small angles, either, because that would mean a very long ramp, which would again need the kinetic friction to be very close to 0.