Mathematics 204
Lab 3: Dynamical systems
Due: Monday, November 21 at 10pm

Please work on this lab activity in a group of 2 or 3 students. When you are ready, scan your work into a single PDF and submit through Blackboard with only one submission per group. Just make sure everyone's name is included below.

Names:

This lab has a total of 40 points with point totals given in bold next to the questions.

1. We'll begin by considering two species R and S whose populations k years after we begin recording them are R_k and S_k. Suppose that the population change from year to year according to the rule

$$\begin{aligned} R_{k+1} &= 0.9R_k + 0.8S_k \\ S_{k+1} &= 0.2R_k + 0.9S_k. \end{aligned}$$

[2] We will use the *state vector* $\mathbf{x}_k = \begin{bmatrix} R_k \\ S_k \end{bmatrix}$ to record the populations. Find the matrix A such that

$$\mathbf{x}_{k+1} = A\mathbf{x}_k.$$

[2] Verify that $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are eigenvectors of your matrix A and find their respective eigenvalues.

[2] Suppose the initial populations are given by the state vector $\mathbf{x}_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Write $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2] Now use the linearity property of matrix multiplication (that is, $A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2$), along with the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A, to write $\mathbf{x}_1 = A\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. So that you see a general pattern, do not simplify $\mathbf{x}_1$ to be a single vector; leave it expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with weights.

[2] In the same way, express $\mathbf{x}_2 = A\mathbf{x}_1$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2] Find a general expression for $\mathbf{x}_k$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2] What happens to $\mathbf{x}_k$ after a long time?

[3] What is the ratio of R_k to S_k after a long time and how is it related to the eigenvectors of A? Explain your thinking.

[3] What is the eventual growth rate of R_k and S_k after a long time and how is it related to the eigenvalues of A? Explain your thinking.

2. Now imagine that a city has a bike-sharing program where bicycles are rented and returned from two locations P and Q. Your research indicates that

   - 70% of the bicycles rented from location P are returned to P that evening with the remaining 30% returned to Q.
   - 40% of the bicycles rented from location Q are returned to Q that evening with the remaining 60% returned to P.

   (Fun fact: a former Grand Valley math major used to work for the City of Grand Rapids and was responsible for studying and maintaining the data from their bike-sharing program.)

   [2] If there are initially 100 bicycles at P and none at Q, how are they distributed the next day?

   [2] If there are initially 100 bicycles at Q and none at P, how are they distributed the next day?

   [2] We'll record the number of bicycles k days after we start recording data with the state vector $\mathbf{x}_k = \begin{bmatrix} P_k \\ Q_k \end{bmatrix}$ where P_k and Q_k are the number of bicycles at the two locations. Find the matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

[4] Find a linearly independent set of eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of A and their associated eigenvalues.

[2] Suppose that there are initially 800 bicycles at P and 700 at Q. Write the initial state vector $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2] Express $\mathbf{x}_1$, the state vector for the next day, as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2] Express $\mathbf{x}_k$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2] What happens to $\mathbf{x}_k$ after a very long time?

[2] After a very long time, how many bicycles are at P and how many at Q?