Deriving the Work Energy Theorem

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This is a small synopsis on the derivation of the work energy theorem!

The key step is to convert the calculus definition for acceleration into an expression that is a derivative of x.

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$$

We start with Newton's second law of motion, which states that the sum of all forces acting on a object are equal to a resultant force that is the object's mass times the objects acceleration.

$$\sum \vec{F} = ma$$

$$\sum \vec{F} = mv\frac{dv}{ds}$$

The integral of the force irrespective of position, is the definition of work

$$W = \int_{s_i}^{s_f} \sum \vec{F}\, ds$$

$$W = \int_{s_i}^{s_f} ma\, ds$$

Now we can substitute using our identity for acceleration.

$$a = v\frac{dv}{ds}$$

$$W = \int_{s_i}^{s_f} ma\, ds$$

$$W = \int_{s_i}^{s_f} mv\frac{dv}{ds}\,ds$$

$$W = \int_{v_i}^{v_f} mv\,dv$$

$$W = \tfrac{1}{2}s^2\Big|_{v_i}^{v_f}$$

And so we end up with a function for work that's completely irrespective of position!

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$