Techniques of Integration

Calculus: Early Transcendentals 9e

7.1 Integration by Parts

Recall the product rule:

$$\frac{d}{dx}\big[f(x)g(x)\big] = f(x)g'(x) + g(x)f'(x)$$

Written as an integral:

$$\int \big[f(x)g'(x) + g(x)f'(x)\big]dx = f(x)g(x)$$

or

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$$

This can be rearranged to:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \tag{1}$$

This is the equation for integration by parts. To make it easier to read, let $u = f(x)$ and $v = g(x)$. The differentials are $du = f'(x)dx$ and $dv = g'(x)dx$. This makes the formula for integration by parts:

$$\int u\,dv = uv - \int v\,du \tag{2}$$

Example 1

Find $\displaystyle\int x\sin(x)dx$

$$u = x \qquad dv = \sin(x)dx$$

Take the derivative of u and antiderivative of dv, therefore:

$$du = dx \qquad v = -\cos(x)$$

Using formula 1:

$$\begin{aligned}\int x\sin(x)dx &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

In general, the function chosen to be $u = f(x)$ is one that becomes simpler or at least not more complicated when differentiated. u can be determined using the priority list:

Logarithmic	$\ln(x)$
Inverse Trig	$\tan^{-1}(x)$
Algebraic	$5x^2 + 3$
Trigonometric	$\cos(x)$
Exponential	10^x

Example 2

Find $\int \ln(x)dx$

$$\begin{aligned} u &= \ln(x) & dv &= dx \\ du &= \frac{1}{x} & v &= x \end{aligned}$$

Therefore:

$$\begin{aligned} \int \ln(x)dx &= x\ln(x) - \int x\left(\frac{1}{x}\right)dx \\ &= x\ln(x) - x + C \end{aligned}$$

Example 3

Find $\int t^2e^tdt$

$$\begin{aligned} u &= t^2 & dv &= e^tdt \\ du &= 2tdt & v &= e^t \end{aligned}$$

Therefore:

$$\int t^2e^tdt = t^2e^t - 2\int te^tdt$$

The integral $\int te^tdt$ also requires integration by parts

$$\begin{aligned} u_1 &= t & dv &= e^tdt \\ du_1 &= dt & v &= e^t \end{aligned}$$

Therefore:

$$\begin{aligned} \int t^2e^tdt &= t^2e^t - 2\Big(te^t - \int e^tdt\Big) \\ &= t^2e^t - 2te^t + 2e^t \end{aligned}$$