Vectors in Space

OpenStax Calculus Vol. 3

2.1 Vectors in the Plane

A vector in a plane is represented by a directed line section, its endpoints are called the initial and terminal points respectively. An arrow from the initial point to the terminal point indicates the direction of the vector and the length of the line segment represents its magnitude. The notation $||\vec{v}||$ or $||\mathbf{v}||$ is used to denote the magnitude of the vector $\vec{v}$ (or $\mathbf{v}$). The zero vector is a vector with the same initial and terminal point denoted $\vec{0}$ or $\mathbf{0}$.

Vectors with the same magnitude and direction are considered equivalent vectors and are treated as equal even if they have different initial points. Two vectors are parallel if they have the same or opposite directions. Vectors are defined by magnitude and direction regardless of the location of the initial point.

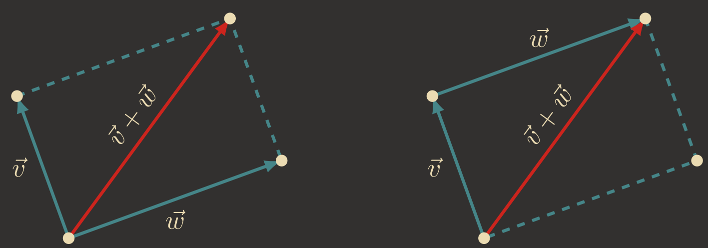
Combining Vectors

Scalars are quantities that have only a magnitude and no direction. Multiplying a vector by a scalar changes the vector's magnitude; this is called scalar multiplication. Changing the magnitude of a vector does not indicate a change in its direction.

Definition of Scalar Multiplication

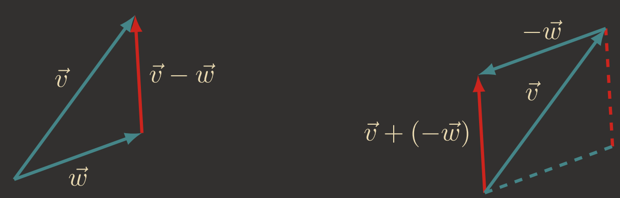
The product $k\vec{v}$ of a vector $\vec{v}$ and scalar k is a vector with a magnitude that is $|k| \cdot ||\vec{v}||$ and a direction that is equal to the direction of $\vec{v}$ if $k > 0$ and opposite the direction of $\vec{v}$ if $k < 0$. If $k = 0$ or $\vec{v} = \vec{0} \implies k\vec{v} = \vec{0}$

Because each vector may have its own direction, the process for adding vectors is different from adding scalars. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first. The sum of the vectors $\vec{v}$ and $\vec{w}$ is the vector with an initial point that coincides with the initial point of $\vec{v}$ and a terminal point that coincides with the terminal point of $\vec{w}$.



For $\vec{u} = \vec{v} + \vec{w}$, the initial point of $\vec{u}$ is the initial point of $\vec{v}$ and the terminal point is the terminal point of $\vec{w}$. These three vectors form a triangle, it follows that the length of any one side is less than the sum of the lengths of the remaining sides, therefore: $||\vec{u}|| \leq ||\vec{v}|| + ||\vec{w}||$

For vector subtraction, $\vec{v} - \vec{w}$ is defined as $\vec{v} + (-\vec{w}) = \vec{v} + (-1)\vec{w}$. The vector $\vec{v} - \vec{w}$ is called the vector difference and is depicted graphically by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$.



Vector Components

A vector with initial point at the origin is called a standard-position vector. Because the initial point of any vector in standard position is (0,0), it can be described by the coordinates of its terminal point.

Definition

The vector with initial point (0,0) and terminal point (x, y) can be written in component form as

$$\vec{v} = \langle x, y \rangle$$

The scalars x and y are called the components of $\vec{v}$

For a vector not already in standard form its component form can be determined algebraically by subtracting the x and y values of the initial point from those of the terminal point.

Let $\vec{v}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t), $\vec{v}$ can be expressed in component form as

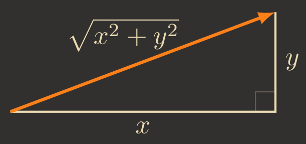
$$\vec{v} = \langle x_t - x_i, y_t - y_i \rangle$$

To find the magnitude of a vector, calculate the distance between its initial and terminal points. The magnitude of vector $\vec{v} = \langle x, y \rangle$ is denoted $||\vec{v}||$ and can be calculated using the formula

$$||\vec{v}|| = \sqrt{x^2 + y^2}$$

Based on this, it is clear that for any vector $\vec{v}$, $||\vec{v}|| \geq 0$, and $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$.

The magnitude of a vector can also be derived using Pythagorean theorem.



Expressing vectors in component form allows scalar multiplication and vector addition to be performed algebraically.

Let $\vec{v} = \langle x_1, y_1 \rangle$ and $\vec{w} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.
Scalar Multiplication: $k\vec{v} = \langle kx_1, ky_1 \rangle$
Vector Addition: $\vec{v} + \vec{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$

Properties of Vector Operations

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in a plane. Let r and s be scalars.

i.	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$	Commutative Property
ii.	$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$	Associative Property
ii.	$\vec{u} + \vec{0} = \vec{u}$	Additive Identity Property
iv.	$\vec{u} + (-\vec{u}) = \vec{0}$	Additive Inverse Property
v.	$r(s\vec{u}) = (rs)\vec{u}$	Associativity of Scalar Multiplication
vi.	$(r + s)\vec{u} = r\vec{u} + s\vec{u}$	Distributive Property
vii.	$r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$	Distributive Property
viii.	$1\vec{u} = \vec{u}$	Identity Property
ix.	$0u = \vec{0}$	Zero Property

Proof of Commutative & Distributive Properties

Let $\vec{u} = \langle x_1, y_1 \rangle$ and $\vec{v} = \langle x_2, y_2 \rangle$

Commutative Property

Apply the commutative property for $\mathbb{R}$

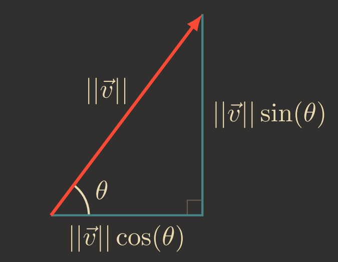
$$\vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2 \rangle = \langle x_2 + x_1, y_2 + y_1 \rangle = \vec{v} + \vec{u}$$

Distributive Property

Apply the distributive property for $\mathbb{R}$

$$\begin{aligned} r(\vec{u} + \vec{v}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\ &= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\ &= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\ &= r\vec{u} + r\vec{v} \end{aligned}$$

In some cases, only the magnitude and direction of a vector are known, not the initial or terminal points. For these vectors, the horizontal and vertical components can be identified using trigonometry.



Consider the angle θ formed by the vector $\vec{v}$ and the positive x-axis. From the triangle above, the components of the vector $\vec{v}$ are $\langle ||\vec{v}||\cos(\theta), ||\vec{v}||\sin(\theta)\rangle$. Therefore, given an angle and the magnitude of a vector, the components can be found using the sine and cosine of the angle.

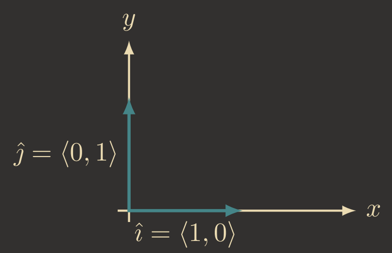
Unit Vectors

A unit vector is a vector with magnitude 1. For any nonzero vector $\vec{v}$, scalar multiplication can be used to find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$ by multiplying the vector by the reciprocal of its magnitude.

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$$

Recall that for scalar multiplication, $||k\vec{v}|| = |k| \cdot ||\vec{v}||$. For $\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$ it follows that $||\vec{u}|| = \frac{1}{||\vec{v}||}(||\vec{v}||) = 1$. We say that $\vec{u}$ is the unit vector in the direction of $\vec{v}$. The process of using scalar multiplication to find a unit vector with a given direction is called normalization.

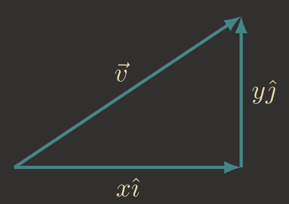
Sometimes it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To do this, the standard unit vectors $\hat{\imath} = \langle 1, 0\rangle$ and $\hat{\jmath} = \langle 0, 1\rangle$ are used.



By applying the properties of vectors, it is possible to express any vector in terms of $\hat{\imath}$ and $\hat{\jmath}$ in what is called a linear combination.

$$\vec{v} = \langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle = x\hat{\imath} + y\hat{\jmath}$$

Thus $\vec{v}$ is the sum of a horizontal vector with magnitude x and a vertical vector with magnitude y as shown below.



2.2 Vectors in Three Dimensions

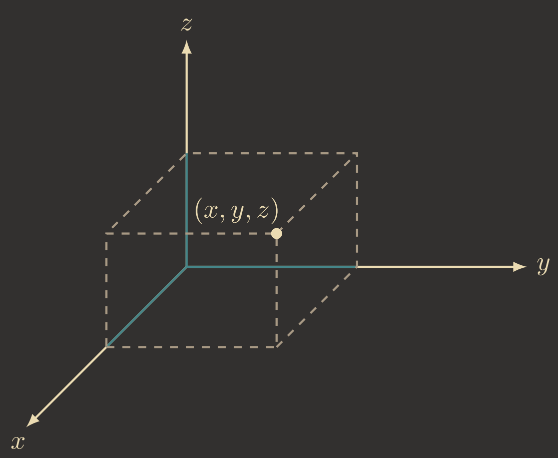
The two-dimensional rectangular coordinate system can be expanded by adding a third dimension, the z-axis which is perpendicular to both the x-axis and the y-axis.

Three-Dimensional Coordinate Systems

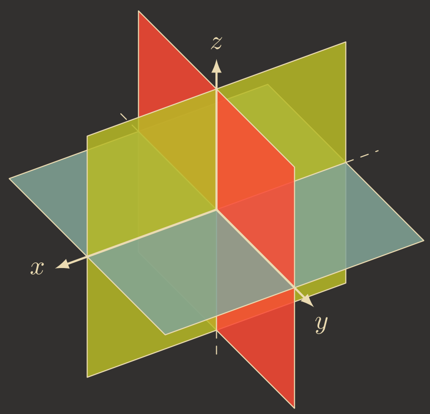
Definition

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-, y-, and z-axes, and an origin at the point of intersection (0) of the axes. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In two dimensions, a point in the plane is described with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate z is appended to indicate alignment with the z-axis: (x, y, z).



In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes, these axes allow any location in the plane to be named. In three dimensions, coordinate planes are defined by the coordinate axes as in two dimensions. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane. The xy-plane is formally defined as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. The xz-plane and yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$ respectively.



If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. The distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The formula for the distance between two points in space is a natural extension of this formula.

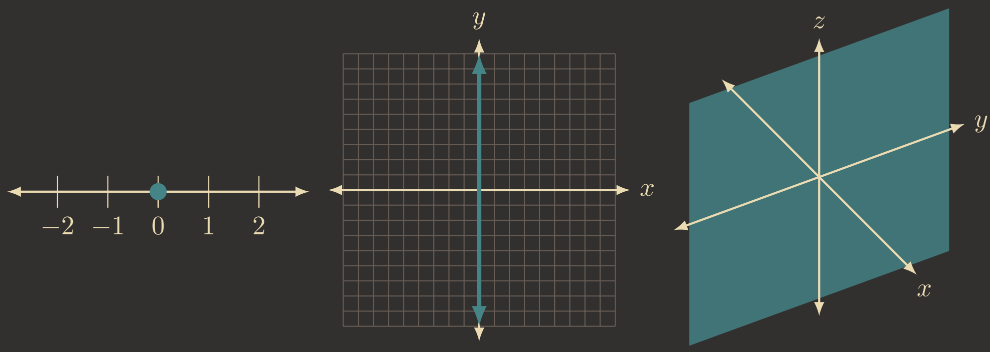
The Distance Between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1}$$

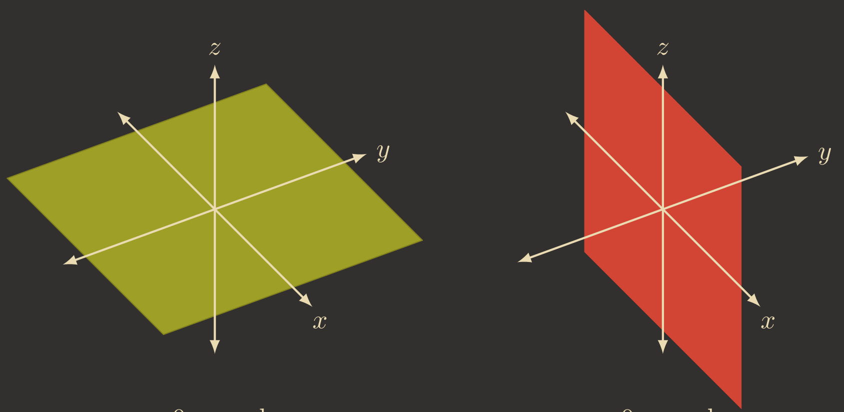
Writing Equations in $\mathbb{R}^3$

Compare the graphs of $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. In the graphs, the same equation describes a point, a line, and a plane.



$\mathbb{R}$ $\mathbb{R}^2$ $\mathbb{R}^3$

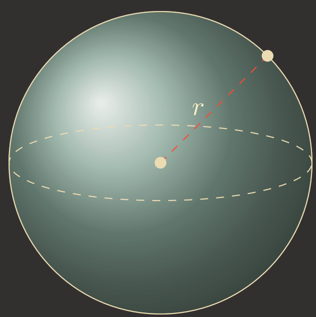
In space, the equation $x = 0$ describes all points $(0, y, z)$ and defines the yz-plane. Similarly, the the xy-plane contains all points of the form $(x, y, 0)$ and is defined by the equation $z = 0$. The equation $y = 0$ defines the xz-plane.



$z = 0$, xy-plane $y = 0$, xz-plane

When a plane is parallel to the xy-plane, for example, the z-coordinate of each point has the same constant value.

A sphere is the set of all points equidistant from a fixed point, the center of the sphere, just as the set of all points in a plane equidistant from the center represents a circle.



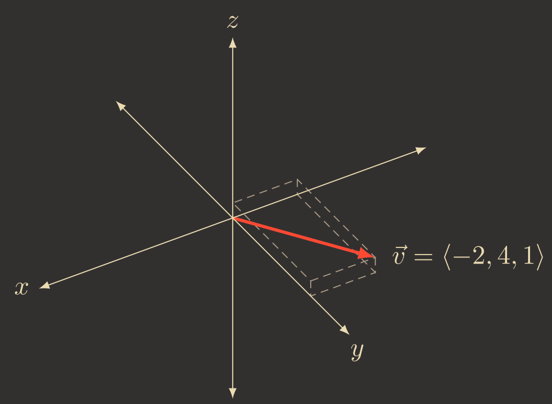
Equation of a Sphere

A sphere with center (a, b, c) and radius r can be represented by the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \tag{2}$$

Working with Vectors in $\mathbb{R}^3$

Three-dimensional vectors can be represented in component form. The notation $\vec{v} = \langle x, y, z\rangle$ extends the two-dimensional case and represents a vector with initial point at the origin and terminal point at (x, y, z). The three-dimensional vector $\vec{v} = \langle -2, 4, 1\rangle$ is represented by a directed line segment from the point $(0, 0, 0)$ to the point $(-2, 4, 1)$.



The standard unit vectors are easily extended to three dimensions, $\hat{\imath} = \langle 1, 0, 0\rangle$, $\hat{\jmath} = \langle 0, 1, 0\rangle$, and $\hat{k} = \langle 0, 0, 1\rangle$, and are used in the same way as in two dimensions. A vector in $\mathbb{R}^3$ can be represented in these ways:

$$\vec{v} = \langle x, y, z\rangle = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

2.3 The Dot Product

The first type of vector multiplication is called the dot product because of the notation used and is defined as follows:

Definition

Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ be vectors, $\vec{u}, \vec{v} \in \mathbb{R}$. The dot product $\vec{u} \cdot \vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \tag{3}$$

When two vectors are added or subtracted, the result is a vector; the dot product of two vectors is a scalar. Like addition and subtraction, the dot product has several algebraic properties.

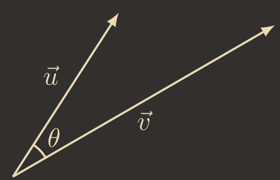
Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let c be a scalar.

i.	$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$	Commutative Property
ii.	$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$	Distributive Property
iii.	$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$	Associative Property
iv.	$\vec{v} \cdot \vec{v} = \|\|\vec{v}\|\|^2$	Property of Magnitude

Using the Dot Product to Find the Angle Between Two Vectors

The dot product provides a way to measure the agle between two nonzero vectors in standard position. This is due to the fact that the dot product can be expressed in terms of the cosine of the angle formed by the two vectors.

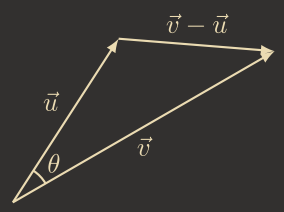


The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta) \tag{4}$$

Proof

Place vectors $\vec{u}$ and $\vec{v}$ in standard position and consider the vector $\vec{v}-\vec{u}$. These three vectors form a triangle with side lengths $|\vec{u}|$, $|\vec{v}|$, and $|\vec{v}-\vec{u}|$.



Recall that the law of cosines describes the relationship between the sides of the triangle and the angle θ. Applying the law of cosines here gives

$$|\vec{v}-\vec{u}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$$

Rewriting the left side of the equation using the dot product:

$$\begin{aligned}
|\vec{v}-\vec{u}|^2 &= (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u}) \\
&= (\vec{v}-\vec{u})\cdot\vec{v} - (\vec{v}-\vec{u})\cdot\vec{u} \\
&= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{u} \\
&= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{u} \\
&= |\vec{v}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{u}|^2
\end{aligned}$$

Substituting this into the law of cosines gives

$$\begin{aligned}
|\vec{v}-\vec{u}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta) \\
|\vec{v}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{u}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta) \\
-2\vec{u}\cdot\vec{v} &= -2|\vec{u}||\vec{v}|\cos(\theta) \\
\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos(\theta)
\end{aligned}$$

This form of the dot product can be used to find the angle between two nonzero vectors. The below equation rearranges equation 3 to solve for the cosine of the angle:

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \tag{5}$$

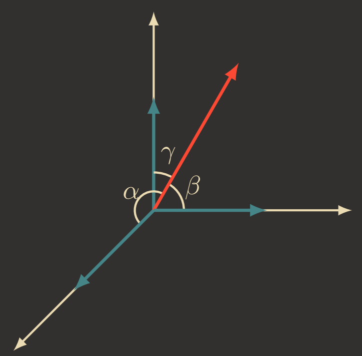
Because we are considering the smallest angle between the vectors, assume $0 \le \theta \le \pi$. Over this range, $\arccos(\theta)$ is unique, and the angle θ can be determined.

Theorem: Orthogonal Vectors

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}\cdot\vec{v} = 0$

Since $\cos(\theta) = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$, $\vec{u}\cdot\vec{v} = 0$ if and only if $\theta = \dfrac{\pi}{2}$

In practical computations, the angle a vector makes with each of the coordinate axes - its direction angles - are very important. Direction angles are often calculated using the dot product and the cosines of the angles, called direction cosines.



For example, the direction cosines of the vector $\vec{v} = \langle 2,3,3\rangle$ are $\cos(\alpha) = \frac{2}{\sqrt{22}}$, $\cos(\beta) = \frac{3}{\sqrt{22}}$, and $\cos(\gamma) = \frac{3}{\sqrt{22}}$. For any vector $\vec{u} = \langle a,b,c\rangle$ its direction cosines are given by $\cos(\alpha) = \frac{a}{|\vec{u}|}$, $\cos(\beta) = \frac{b}{|\vec{u}|}$, and $\cos(\gamma) = \frac{c}{|\vec{u}|}$

Projections

Definition

The vector projection of $\vec{v}$ onto $\vec{u}$ is denoted $\text{proj}_{\vec{u}}\vec{v}$. It has the same initial point as $\vec{u}$ and $\vec{v}$, and the same direction as $\vec{u}$, and represents the component of $\vec{v}$ acting in the direction of $\vec{u}$. If θ is the angle between $\vec{u}$ and $\vec{v}$, then the length of $\text{proj}_{\vec{u}}\vec{v}$ is $|\text{proj}_{\vec{u}}\vec{v}| = |\vec{v}|\cos(\theta)$.

Expressing $\cos(\theta)$ in terms of the dot product, this becomes

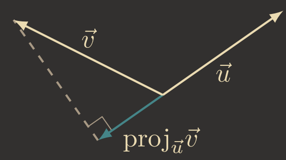
$$\begin{aligned} |\text{proj}_{\vec{u}}\vec{v}| &= |\vec{v}|\cos(\theta) \\ &= |\vec{v}|\left(\frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}||\vec{v}|}\right) \\ &= \frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|} \end{aligned}$$

This is multiplied by a unit vector in the direction of $\vec{u}$ to give $\text{proj}_{\vec{u}}\vec{v}$

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}\left(\frac{1}{|\vec{u}|}\vec{u}\right) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} \tag{6}$$

The magnitude of this vector is known as the scalar projection of $\vec{v}$ onto $\vec{u}$ and is denoted by

$$|\text{proj}_{\vec{u}}\vec{v}| = \text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} \tag{7}$$



It is sometimes useful to decompose vectors, splitting them into sums. This process is called resolution of a vector into components. Projections allow for the identification of two orthogonal vectors with a desired sum. For example, let $\vec{v} = \langle 6, -4\rangle$ and $\vec{u} = \langle 3, 1\rangle$. To decompose the vector $\vec{v}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{u}$, first find the component that has the same direction as $\vec{u}$ by projecting $\vec{v}$ onto $\vec{u}$. Let $\text{proj}_{\vec{u}}\vec{v} = \vec{p}$

$$\begin{aligned} \vec{p} &= \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} \\ &= \frac{7}{5}\vec{u} = \frac{7}{5}\langle 3, 1\rangle = \left\langle \frac{21}{5}, \frac{7}{5}\right\rangle \end{aligned}$$

Consider the vector $\vec{q} = \vec{v} - \vec{p}$

$$\begin{aligned} \vec{q} &= \vec{v} - \vec{p} \\ &= \langle 6, -4\rangle - \left\langle \frac{21}{5}, \frac{7}{5}\right\rangle \\ &= \left\langle \frac{9}{5}, -\frac{27}{5}\right\rangle \end{aligned}$$

By the definition of $\vec{q}$, $\vec{v} = \vec{q} + \vec{p}$, and

$$\begin{aligned} \vec{q}\cdot\vec{p} &= \left\langle \frac{9}{5}, -\frac{27}{5}\right\rangle \cdot \left\langle \frac{21}{5}, \frac{7}{5}\right\rangle \\ &= \frac{9(21)}{25} + \frac{-27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0 \end{aligned}$$

Therefore $\vec{q}$ and $\vec{p}$ are orthogonal.

2.4 The Cross Product

The cross product of two nonzero vectors generates a third vector orthogonal to the first two. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. To find the vector $\vec{w} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{u}$ and $\vec{v}$ - that is $\vec{w}$ such that $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$ - $\vec{w}$ must satisfy the following

$$\begin{aligned} u_1 w_1 + u_2 w_2 + u_3 w_3 &= 0 \\ v_1 w_1 + v_2 w_2 + v_3 w_3 &= 0 \end{aligned}$$

The variable w_3 can be eliminated by multiplying the top equation by v_3 and the bottom equation by v_3 which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0$$

Selecting values for w_1 and w_2 and substituting back into the original equations gives

$$\begin{aligned} w_1 &= u_2 v_3 - u_3 v_2 \\ w_2 &= -(u_1 v_3 - u_3 v_1) \\ w_3 &= u_1 v_2 - u_2 v_1 \end{aligned}$$

Which gives the vector $\vec{w}$ which is orthogonal to both $\vec{u}$ and $\vec{v}$

$$\vec{w} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

Definition

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then the cross product $\vec{u} \times \vec{v}$ is the vector

$$\begin{aligned} \vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2)\hat{\imath} - (u_1 v_3 - u_3 v_1)\hat{\jmath} + (u_1 v_2 - u_2 v_1)\hat{k} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle \end{aligned} \tag{9}$$