

# Fixed-Axis Rotation

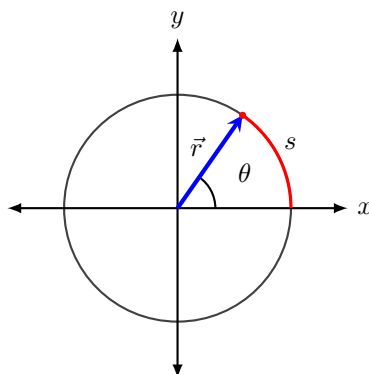
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## 10.1 Rotational Variables

### Angular Velocity

Uniform circular motion is motion in a circle at constant speed, although this is the simplest case of rotational motion, it is used here to introduce rotational variables.

The figure shows a particle moving in a circle. Its position vector from the origin of the circle to the particle sweeps out the angle  $\theta$ , which increases in the counterclockwise direction as the particle moves along its path. The angle  $\theta$  is called the angular position of the particle. As the particle moves, it traces an arc length  $s$ .



The angle is related to the radius of the circle and the arc length by

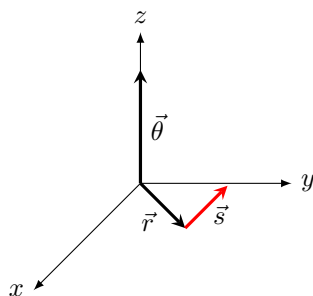
$$\theta = \frac{s}{r} \quad (1)$$

The angle  $\theta$ , the angular position of the particle moving along its path has units of radians (rad). As the particle moves along its circular path, its angular position changes and it undergoes angular displacements  $\Delta\theta$ .

We can assign vectors to the quantities in equation 1, the angle  $\vec{\theta}$  is a vector out of the page. The angular position vector  $\vec{r}$  and the arc length vector  $\vec{s}$  both lie in the plane of the page, they are related by:

$$\vec{s} = \vec{\theta} \times \vec{r} \quad (2)$$

The arc length is the cross product of the angle vector and the position vector



The magnitude of the angular velocity, denoted by  $\omega$ , is the time rate of change of the angle  $\theta$  as the particle moves in a circular path. The instantaneous angular velocity, defined as the limit as  $\Delta t \rightarrow 0$  of the average angular velocity  $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad (3)$$

Where  $\theta$  is the angle of rotation. The units of angular velocity are radians per second ( $\text{rad s}^{-1}$ ). Angular velocity can also be referred to as the rotation rate in radians per second. In many cases, rotation rate is given in revolutions/s or cycles/s, to find angular velocity, multiply revolutions/s by  $2\pi$  (since there are  $2\pi$  radians per revolution). Since a positive angle in a circle is counterclockwise, we take counterclockwise rotations as being positive and clockwise rotations as negative.

We can see how angular velocity is related to the tangential speed of the particle by differentiating equation 1 with respect to time. Equation 1 can be rewritten as:

$$s = \theta r$$

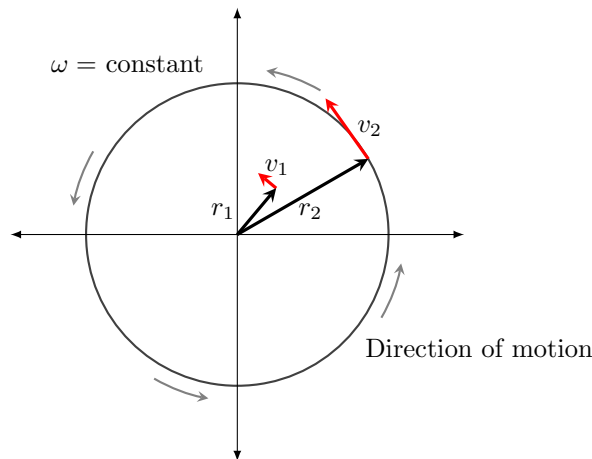
Taking the derivative with respect to time and noting that the radius  $r$  is constant gives:

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\frac{d\theta}{dt}$$

Where  $\theta\frac{dr}{dt} = 0$ . Here,  $\frac{ds}{dt}$  is just the tangential speed  $v_t$  of the particle moving in a circular path. Using equation 3 we arrive at:

$$v_t = r\omega \quad (4)$$

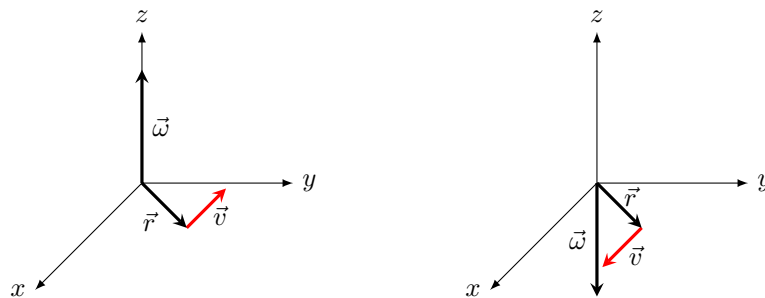
The tangential speed of the particle is its angular velocity times the radius of the circle. The tangential speed of the particle increases with its distance from the axis of rotation for a constant angular velocity. The figure shows two particles placed at different radii on a rotating disk with constant angular velocity. As it rotates, the tangential speed increases linearly with the radius from the axis of rotation. We see that  $v_1 = r_1\omega_1$  and  $v_2 = r_2\omega_2$ . The disk has a constant angular velocity so  $\omega_1 = \omega_2$ . This means that  $\frac{v_1}{r_1} = \frac{v_2}{r_2}$  or  $v_2 = \left(\frac{r_2}{r_1}\right)v_1$ . Thus, since  $r_2 > r_1$ ,  $v_2 > v_1$



Similar to equation 2, one can state a cross product relation to the vector of the tangential velocity as stated in equation 4, therefore:

$$\vec{v} = \vec{\omega} \times \vec{r} \quad (5)$$

The tangential velocity is the cross product of the angular velocity and the position vector as shown below. On the left we see that with the angular velocity in the  $+z$  direction, the rotation in the  $xy$  plane is counterclockwise. On the right, the angular velocity is in the  $-z$  direction, which gives a clockwise rotation in the  $xy$  plane.



### Example 10.1: Rotation of a Flywheel

A flywheel rotates such that it sweeps out an angle at the rate of  $\theta = \omega t = (45.0 \text{ rad s}^{-1})t$  radians. The wheel rotates counterclockwise when viewed in the plane of the page.

- (a) What is the angular velocity  $\omega$  of the flywheel?

$$\omega = \frac{d\theta}{dt} = 45 \text{ rad s}^{-1}, \text{ angular velocity is constant}$$

- (b) What direction is the angular velocity?

The direction of rotation is counterclockwise, so the direction of angular velocity is  $+z$

- (c) How many radians does the flywheel rotate through in 30 s?

$$\theta(t) = \omega t \rightarrow \Delta\theta = \theta(30 \text{ s}) - \theta(0 \text{ s}) = \theta(30 \text{ s}) \rightarrow (45.0 \text{ rad s}^{-1})(30 \text{ s}) = 1350.0 \text{ rad}$$

- (d) What is the tangential speed of a point on the flywheel 10 cm from the axis of rotation

$$v_t = r\omega = (0.1 \text{ m})(45.0 \text{ rad s}^{-1}) = 4.5 \text{ ms}^{-1}$$

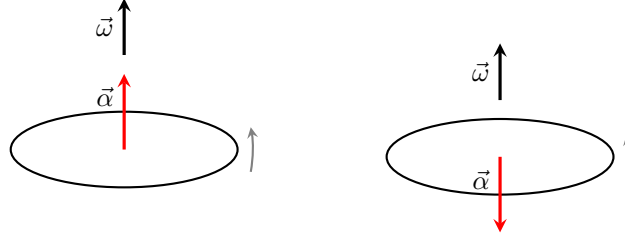
## Angular Acceleration

For describing situations where  $\omega$  changes, we need to define angular acceleration. The faster the change in  $\omega$ , the greater the angular acceleration. Instantaneous angular acceleration  $\alpha$  is defined as the derivative of angular velocity with respect to time:

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad (6)$$

Where we have taken the limit of the average angular acceleration  $\bar{\alpha} = \frac{\Delta \omega}{\Delta t}$  as  $\Delta t \rightarrow 0$ . The units of angular acceleration are radians/s per second, or  $\text{rad s}^{-2}$ .

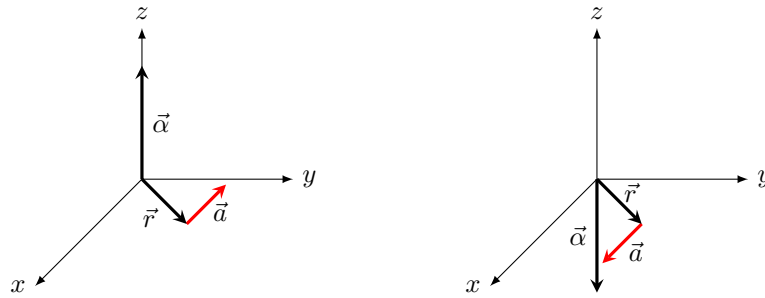
In the same way that the vector associated with angular velocity  $\vec{\omega}$  was defined, we can define  $\vec{\alpha}$ , the vector associated with angular acceleration. If the angular velocity is along the  $+z$  axis and  $\frac{d\omega}{dt}$  is positive, the angular acceleration  $\vec{\alpha}$  is positive and points along the  $+z$  axis, if  $\frac{d\omega}{dt}$  is negative, the angular acceleration is negative and points along the  $-z$  axis.



The tangential acceleration vector can be expressed as a cross product of the angular acceleration and position vectors. This equation can be found by taking the derivative of equation 5

$$\vec{a} = \vec{\alpha} \times \vec{r} \quad (7)$$

The vector relationships for angular acceleration and tangential acceleration are shown below



Tangential acceleration of a point on a rotating body at a distance from the axis of rotation can be related in the same way as tangential velocity and angular velocity. Differentiating equation 4 with respect to time (the radius  $r$  is constant) gives:

$$a_t = r\alpha \quad (8)$$

The tangential acceleration  $a_t$  is the radius times the angular acceleration

### Example 10.2: A Spinning Bike Wheel

A bicycle mechanic mounts a bicycle on the repair stand and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s.

- (a) Calculate the average angular acceleration in  $\text{rad s}^{-2}$

$$\bar{\alpha} = \frac{\Delta \omega}{\Delta t} = \frac{250 \text{ rpm}}{5.00 \text{ s}}$$

Converting from rpm to  $\text{rad s}^{-1}$ :

$$\Delta \omega = 250 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 26.2 \text{ rad s}^{-1}$$

Entering this back into the expression for  $\alpha$  gives:

$$\alpha = \frac{\Delta \omega}{\Delta t} = \frac{26.2 \text{ rad s}^{-1}}{5.00 \text{ s}} = 5.24 \text{ rad s}^{-2}$$

- (b) If the brakes are hit, causing an angular acceleration of  $-87 \text{ rad s}^{-2}$ , how long does it take the wheel to stop?  
Angular velocity decreases from  $26.2 \text{ rad s}^{-1}$  to zero so  $\Delta \omega = -26.2 \text{ rad s}^{-1}$ , and  $\alpha$  is given to be  $-87.3 \text{ rad s}^{-2}$

$$\Delta t = \frac{\Delta \omega}{\alpha} = \frac{-26.2 \text{ rad s}^{-1}}{-87.3 \text{ rad s}^{-2}} = 0.300 \text{ s}$$

**Example 10.3:** Wind Turbine

A wind turbine in a wind farm is being shut down for maintenance. It takes 30 s for the turbine to go from its operating angular velocity to a complete stop in which the angular velocity function is  $\omega(t) = \left[ \frac{(ts^{-1} - 30.0)^2}{100.0} \right] \text{rad s}^{-1}$ , where  $t$  is the time in seconds. If the turbine is rotating counterclockwise looking into the page:

- (a) What are the directions of the angular velocity and acceleration vectors?

Since the turbine is rotating counterclockwise, angular velocity  $\vec{\omega}$  points towards  $+z$ . Since the angular velocity is decreasing, the angular acceleration  $\vec{\alpha}$  points towards  $-z$ .

- (b) What is the average angular acceleration?

At  $t = 0$ , the initial angular velocity of the turbine is  $\omega = 9.0 \text{ rad s}^{-1}$ , the final angular velocity is zero, so the average angular velocity  $\bar{\alpha}$  is:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t - t_0} = \frac{0 - 9.0 \text{ rad s}^{-1}}{30.0 - 0 \text{ s}} = -0.3 \text{ rad s}^{-2}$$

- (c) What is the instantaneous angular acceleration at  $t = 0.0, 15.0, 30.0$  s?

Taking the derivative of angular velocity with respect to time gives

$$\alpha = \frac{d\omega}{dt} = \left[ \frac{(t - 30.0)}{50.0} \right] \text{rad s}^{-2}$$

Thus:  $\alpha(0.0 \text{ s}) = -0.6 \text{ rad s}^{-2}$ ,  $\alpha(15.0 \text{ s}) = -0.3 \text{ rad s}^{-2}$ , and  $\alpha(30.0 \text{ s}) = 0 \text{ rad s}^{-2}$

## 10.2 Rotation With Constant Angular Acceleration

In this section, the definitions from the previous section are used to derive relationships among these variables, and use these relationships to analyze rotational motion for a rigid body about a fixed axis under a constant angular acceleration, forming the basis for rotational kinematics. If angular acceleration is constant, the equations of rotational kinematics simplify, similar to the equations of linear kinematics.

### Kinematics of Rotational Motion

In the previous section we saw that if a flywheel has an angular acceleration in the same direction as its angular velocity, its angular velocity increases with time and its angular displacement also increases. If the angular acceleration is opposite to the angular velocity vector, its angular velocity decreases with time. Under a constant angular acceleration, we can describe these physical situations with a consistent set of rotational kinematic equations.

If the system is rotating under a constant acceleration, then the average angular velocity follows a simple relation because the angular velocity is increasing linearly with time. The average angular velocity is just half of the sum of the initial and final values:

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2} \quad (9)$$

Using the definition of average angular velocity, an equation that relates the angular position, average angular velocity, and time can be found:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Solving for  $\theta$  gives:

$$\theta_f = \theta_0 + \bar{\omega}t \quad (10)$$

Where  $t_0 = 0$ . This equation can be useful when the average angular velocity of the system is known. Then the angular displacement over a given period of time could be found. To determine an equation relating  $\omega, \alpha$ , and  $t$ , we start with the definition of angular acceleration:

$$\alpha = \frac{d\omega}{dt}$$

This is rearranged to  $\alpha dt = d\omega$ , then we integrate both sides of the equation from initial to final values, from  $t_0$  to  $t$  and from  $\omega_0$  to  $\omega_f$ . (angular acceleration is constant and can be pulled outside)

$$\alpha \int_{t_0}^t dt = \int_{\omega_0}^{\omega_f} d\omega$$

Setting  $t_0 = 0$  gives:

$$\alpha t = \omega_f - \omega_0$$

This is rearranged to obtain

$$\omega_f = \omega_0 + \alpha t \quad (11)$$

Where  $\omega_0$  is the initial angular velocity. This equation is the rotational counterpart to the linear kinematic equation  $v_f = v_0 + at$ . With equation 11, the angular velocity of an object at any specified time  $t$  can be found given the initial angular velocity and angular acceleration.

Doing a similar thing to the equation  $\omega = \frac{d\theta}{dt}$ , rearranging it to  $\omega dt = d\theta$  and integrating both sides from initial to final values, noting that angular acceleration is constant and does not have a time dependence. This time angular velocity is not constant, so equation 11 is substituted in:

$$\begin{aligned} \int_{t_0}^{t_f} (\omega_0 + \alpha t) dt &= \int_{\theta_0}^{\theta_f} d\theta \\ \int_{t_0}^t \omega_0 dt + \int_{t_0}^t \alpha t dt &= \int_{\theta_0}^{\theta_f} d\theta \\ \left[ \omega_0 t + \frac{1}{2} \alpha t^2 \right]_{t_0}^t &= \omega_0 t + \frac{1}{2} \alpha t^2 = \theta_f - \theta_0 \end{aligned}$$

Where  $t_0 = 0$ . Now we rearrange to obtain:

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \quad (12)$$

This is the rotational counterpart to the linear kinematic equation  $s_f = s_0 + v_0 t + \frac{1}{2} at^2$ . This equation gives the angular position of a rotating rigid body at any time  $t$  given the initial conditions ( $\theta_0$  and  $\omega_0$ ) and the angular acceleration

We can find an equation that is independent of time by solving for  $t$  in equation 11 and substituting into equation 12:

$$\begin{aligned}
 \theta_f &= \theta_0 + \omega_0 \left( \frac{\omega_f - \omega_0}{\alpha} \right) + \frac{1}{2} \alpha \left( \frac{\omega_f - \omega_0}{\alpha} \right)^2 \\
 &= \theta_0 + \frac{\omega_0 \omega_f}{\alpha} - \frac{\omega_0^2}{\alpha} + \frac{1}{2} \frac{\omega_f^2}{\alpha} - \frac{\omega_0 \omega_f}{\alpha} + \frac{1}{2} \frac{\omega_0^2}{\alpha} \\
 &= \theta_0 + \frac{1}{2} \frac{\omega_f^2}{\alpha} - \frac{1}{2} \frac{\omega_0^2}{\alpha} \\
 \theta_f - \theta_0 &= \frac{\omega_f^2 - \omega_0^2}{2\alpha}
 \end{aligned}$$

This rearranges to:

$$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta) \quad (13)$$

Equations 10 - 13 describe fixed-axis rotation for constant acceleration and are summarized below

Rotational	Linear
$\theta_f = \theta_0 + \bar{\omega}t$	$s_f = s_0 + \bar{v}t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$s_f = s_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta s)$

### **Example: 10.4/5** Calculating the Acceleration of a Fishing Pole

A deep-sea fisherman hooks a big fish that swims away from the boat, pulling the fishing line from his fishing reel. The whole system is initially at rest, and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of  $110 \text{ rad s}^{-2}$  for 2.00 s

- (a) What is the final velocity of the reel after 2 s?

Since  $\alpha$  and  $t$  are given, the most straightforward equation to use to find  $\omega$  is  $\omega_f = \omega_0 + \alpha t$ . Because the system starts from rest,  $\omega_0 = 0$ , so:

$$\omega_f = 0 + (110 \text{ rad s}^{-2})(2.00 \text{ s}) = 220 \text{ rad s}^{-1}$$

- (b) How many revolutions does the reel make?

To find the number of revolutions, find  $\theta$  in radians (1 rev =  $2\pi$  rad).  $\alpha$  and  $t$  are given, and  $\omega_0 = 0$ , so  $\theta$  can be obtained by using:

$$\begin{aligned}
 \theta_f &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
 &= 0 + 0 + \frac{1}{2} (110 \text{ rad s}^{-2})(2.00 \text{ s})^2 \\
 &= 200 \text{ rad}
 \end{aligned}$$

Converting from radians to revolutions:

$$(200 \text{ rad}) \left( \frac{1 \text{ rev}}{2\pi \text{ rad}} \right) = 35.0 \text{ rev}$$

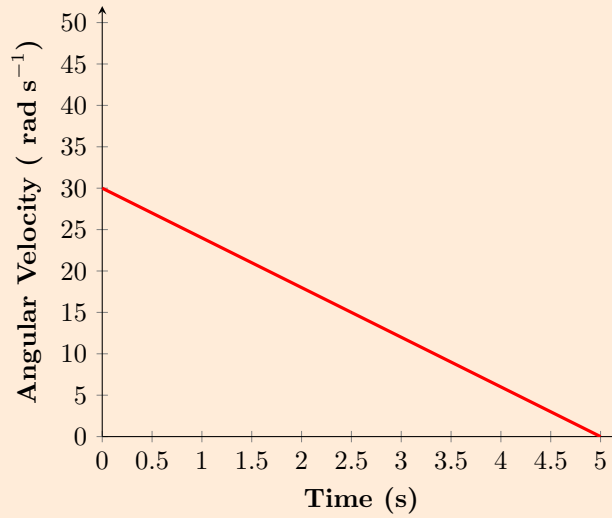
- (c) Now the fisherman applies a brake to the spinning wheel, achieving an angular acceleration of  $-300 \text{ rad s}^{-2}$ . How long does it take for the reel to come to a stop?

Solving the equation  $\omega_f = \omega_0 + \alpha t$  for  $t$ , and substituting in known values gives:

$$t = \frac{\omega_f - \omega_0}{\alpha} = \frac{0 - 220.0 \text{ rad s}^{-1}}{-300 \text{ rad s}^{-2}} = 0.733 \text{ s}$$

**Example 10.6:** Angular Acceleration of a Propeller

The figure shows a graph of the angular velocity of a propeller on an aircraft as a function of time. Its angular velocity starts at  $30 \text{ rad s}^{-1}$  and drops linearly to  $0 \text{ rad s}^{-1}$  over the course of 5 s.



- (a) Find the angular acceleration of the object and verify the result using kinematic equations

Because angular velocity varies linearly with time, angular acceleration is constant and not dependent on time. The angular acceleration is the derivative of angular velocity,  $\alpha = \frac{d\omega}{dt}$ . At  $t = 0 \text{ s}$ ,  $\omega_0 = 30 \text{ rad s}^{-1}$  and at  $t = 5 \text{ s}$ ,  $\omega_f = 0 \text{ rad s}^{-1}$

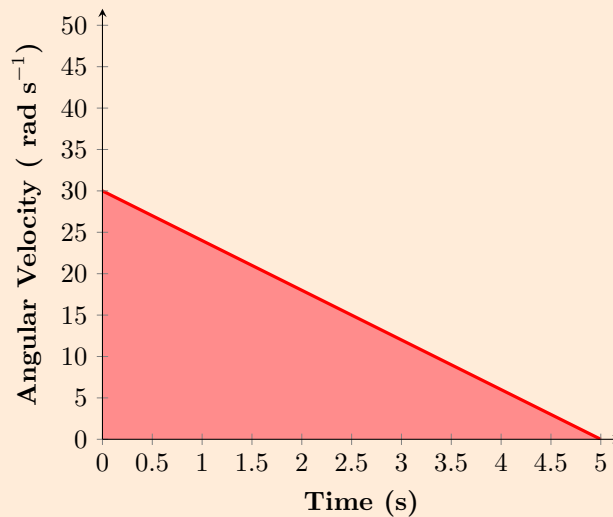
$$\alpha = \frac{\omega - \omega_0}{t - t_0} = \frac{(0 - 30.0) \text{ rad s}^{-1}}{(5.0 - 0) \text{ s}} = -6.0 \text{ rad s}^{-2}$$

- (b) Find the angle through which the propeller rotates during these 5 seconds and verify your result using kinematic equations

Since  $\omega = \frac{d\theta}{dt}$ , angular displacement  $\Delta\theta$  can be calculated by integrating the angular velocity

$$\int_{\theta_0}^{\theta_f} d\theta = \theta_f - \theta_0 = \int_{t_0}^{t_f} \omega(t) dt$$

The area under the curve can be found by calculating the area of the right triangle shown:



$$\Delta\theta = A(\text{triangle}) = \frac{1}{2}l \times h$$

$$\Delta\theta = \frac{1}{2}(30 \text{ rad s}^{-1})(5 \text{ s})$$

This is verified using equation 12 ( $\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ ), and setting  $\theta_0 = 0$  gives:

$$\theta_f = (30.0 \text{ rad s}^{-1})(5.0 \text{ s}) + \frac{1}{2}(-6.0 \text{ rad s}^{-2})(5.0 \text{ rad s}^{-1})^2 = 150.0 - 75.0 = 75.0 \text{ rad}$$

## 10.3 Relating Angular and Translational Quantities

### Angular vs. Linear Variables

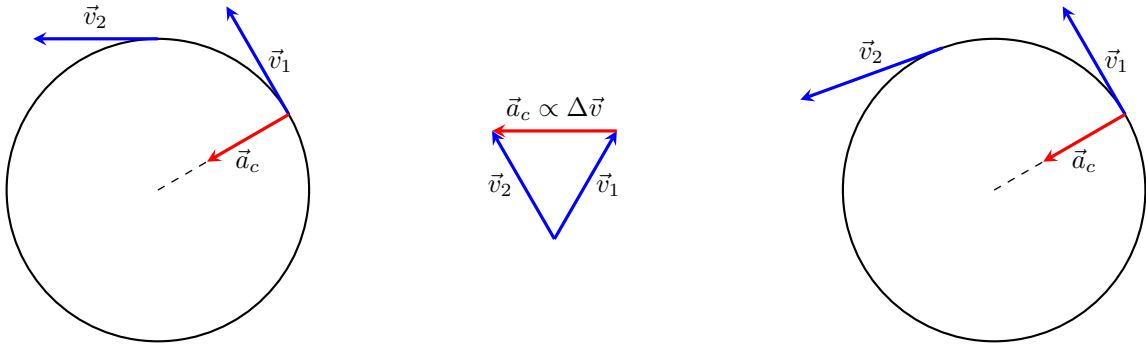
Comparing the definitions of the rotational variables with the definitions of linear kinematic variables shows that there is a mapping of the linear variables to the rotational ones. Linear position, velocity, and acceleration have their rotational counterparts, as seen below.

	Linear	Rotational
Position	$x$	$\theta$
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

In uniform and nonuniform circular motion, there exists a centripetal acceleration  $a_c$ . The centripetal acceleration vector points inward from the particle toward the axis of rotation. The magnitude of centripetal acceleration is:

$$a_c = \frac{v_t^2}{r} \quad (14)$$

In uniform circular motion, when  $\omega = \text{constant}$  and  $\alpha = 0$ , there is a linear acceleration - centripetal acceleration - since  $v_t = \text{constant}$ . If nonuniform circular motion is present, the rotating system has an angular acceleration, and there is both a linear centripetal acceleration that is changing ( $\Delta v_t \neq 0$ ) as well as a linear tangential acceleration.



The centripetal acceleration is due to the change in the direction of tangential velocity, whereas the tangential acceleration is due to any change in the magnitude of the tangential velocity. The tangential and centripetal acceleration vectors  $\vec{a}_t$  and  $\vec{a}_c$  are always perpendicular to each other (the direction of  $\vec{v}_t$  = the direction of  $\vec{a}_t$ ).

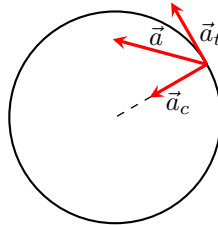
We can assign a total linear acceleration vector to a point on a rotating rigid body or a particle executing circular motion at a radius  $r$  from a fixed axis. The total linear acceleration vector  $\vec{a}$  is the vector sum of the centripetal and tangential accelerations:

$$\vec{a} = \vec{a}_c + \vec{a}_t \quad (15)$$

The total linear acceleration vector in the case of nonuniform circular motion points at an angle between the centripetal and tangential acceleration vectors. Since  $\vec{a}_c \perp \vec{a}_t$ , the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2}$$

If the angular acceleration is zero, the total linear acceleration is equal to the centripetal acceleration



### Relationship Between Rotational & Translational Motion

The relationship between the rotational and translational kinematic equations was mentioned above. The second relationship between rotational and translational motion relates linear and rotational variables in the special case of circular motion

Rotational	Translational	Relationship
$\theta$	$s$	$\theta = \frac{s}{r}$
$\omega$	$v_t$	$\omega = \frac{v_t}{r}$
$\alpha$	$a_t$	$\alpha = \frac{a_t}{r}$
	$a_c$	$a_c = \frac{v_t^2}{r}$



**Example 10.7** Linear Acceleration of a Centrifuge

A centrifuge has a radius of 20 cm and accelerates from a maximum rotation rate of 10,000 rpm to rest in 30 seconds under a constant angular acceleration, rotating counterclockwise. What is the magnitude of the total acceleration for a point at the tip of the centrifuge at  $t = 29.0$  s? What is the direction of the total acceleration vector?

Angular acceleration is:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - (1 \cdot 10^4)2\pi/60.0 \text{ s}(\text{ rad s}^{-1})}{30.0 \text{ s}} = -34.9 \text{ rad s}^{-2}$$

Tangential acceleration is  $a_t = r\alpha$ , therefore:

$$a_t = 0.2 \text{ m}(-34.9 \text{ rad s}^{-2}) = -7.0 \text{ ms}^{-2}$$

The angular velocity at  $t = 29.0$  s can be found using the equation  $\omega = \omega_0 + \alpha t$ :

$$\begin{aligned}\omega &= \left[ (1.0 \cdot 10^4 \text{ rpm}) \left( \frac{2\pi}{60 \text{ s}} \right) \right] \text{ rad s}^{-1} + (-34.9 \text{ rad s}^{-2})(29.0 \text{ s}) \\ &= (1047.2 - 1012.71) \text{ rad s}^{-1} = 35.1 \text{ rad s}^{-1}\end{aligned}$$

The tangential velocity  $v_t$  is given by  $v_t = r\omega$ , therefore  $v_t$  at  $t = 29.0$  s is:

$$v_t = 0.2 \text{ m}(35.1 \text{ rad s}^{-1}) = 7.0 \text{ ms}^{-1}$$

The centripetal acceleration  $a_c$  at  $t = 29.0$  s can now be calculated using  $a_c = \frac{v^2}{r}$ , therefore:

$$a_c = \frac{(7.0 \text{ ms}^{-1})^2}{0.2 \text{ m}} = 245.0 \text{ ms}^{-2}$$

Since the two acceleration vectors are perpendicular to each other, the magnitude of the total linear acceleration is:

$$\begin{aligned}|\vec{a}| &= \sqrt{a_c^2 + a_t^2} = \sqrt{(245.0 \text{ ms}^{-2})^2 + (-7.0 \text{ ms}^{-2})^2} \\ &= 245.1 \text{ ms}^{-2}\end{aligned}$$

Since the centrifuge has a negative angular acceleration, it is slowing down. The angle of the total acceleration vector with respect to the centripetal acceleration vector is:

$$\theta = \tan^{-1} \frac{-7.0 \text{ ms}^{-2}}{245.0 \text{ ms}^{-2}} = -1.6^\circ$$

## 10.4 Moment of Inertia and Rotational Kinetic Energy

This section introduces two new quantities that are helpful for analyzing properties of rotating objects: moment of inertia and rotational kinetic energy

### Rotational Kinetic Energy

The energy associated with rotational motion is the same as kinetic energy in translational motion however, because kinetic energy is given by  $K = \frac{1}{2}mv^2$ , and velocity is different for every point on a rotating body, it makes sense to write kinetic energy in terms of  $\omega$ . For a single particle rotating around a fixed axis, we relate the angular velocity to the magnitude of the translational velocity using the relation  $v_t = \omega r$ , where  $r$  is the distance of the particle from the axis of rotation and  $v_t$  is its tangential velocity. Substituting into the equation for kinetic energy gives:

$$K = \frac{1}{2}mv_t^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}(mr^2)\omega^2$$

In the case of a rigid rotating body, the total body can be divided up into a large number of smaller masses, each with mass  $m_j$  and distance from the axis of rotation  $r_j$ , such that the total mass of the body  $M$  is equal to the sum of the original masses  $M = \sum_j m_j$ . Each smaller mass has a tangential velocity  $v_j$ . The total kinetic energy of the rigid rotating body is:

$$K = \sum_j \frac{1}{2}m_j v_j^2 = \sum_j \frac{1}{2}m_j (r_j \omega_j)^2$$

Since  $\omega_j = \omega$  for all masses:

$$K = \frac{1}{2} \left( \sum_j m_j r_j^2 \right) \omega^2 \quad (16)$$

Like translational kinetic energy, the units of this equation are joules (J)

### Moment of Inertia

The quantity  $\sum_j m_j r_j^2$  is the counterpart for mass in the equation for rotational kinetic energy. This quantity is called the moment of inertia  $I$  and has units of  $\text{kg m}^{-2}$

$$I = \sum_j m_j r_j^2 \quad (17)$$

For now the expression is left in summation form, representing the moment of inertia of a system of point particles rotating about a fixed axis. The moment of inertia of a single point particle about a fixed axis is simply  $mr^2$ .

The moment of inertia is the quantitative measure of rotational inertia, just as in translational motion, mass is the quantitative measure of linear inertia - that is, the more massive an object is, the more inertia it has, and the greater is its resistance to change in linear velocity. Similarly, the greater the moment of inertia of a rigid body or system of particles, the greater its resistance to change in angular velocity about a fixed axis of rotation. Rigid bodies and systems of particles with more mass concentrated at a greater distance from the axis of rotation have greater moments of inertia than bodies/systems of the same mass bt concentrated near the axis of rotation. Substituting equation 17 into equation 16, the expression for the kinetic energy of a rotating rigid body becomes:

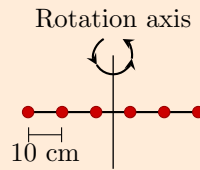
$$K = \frac{1}{2}I\omega^2 \quad (18)$$

The kinetic energy of a rotating rigid body is directly proportional to the moment of inertia and the square of the angular velocity. The rotational and translational quantities for kinetic energy are summarized below:

Rotational	Translational
$I = \sum_j m_j r_j^2$	$m$
$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$

**Example 10.8:** Moment of Inertia of a System of Particles

Six small washers are spaced 10 cm apart on a rod of negligible mass and 0.5 m in length. The mass of each washer is 20 g. The rod rotates about an axis located at 25 cm as shown.



- (a) What is the moment of inertia of the system?

Use the definition of moment of inertia for a system of particles:  $I = \sum_j m_j r_j^2$

$$\begin{aligned} I &= (0.02 \text{ kg}) [2(0.25 \text{ m})^2 + 2(0.15 \text{ m})^2 + 2(0.05 \text{ m})^2] \\ &= 3.5 \cdot 10^{-3} \text{ kg m}^{-2} \end{aligned}$$

- (b) If the two washers closest to the axis are removed, what is the moment of inertia of the remaining four washers?  
Repeat the steps from part (a):

$$\begin{aligned} I &= (0.02 \text{ kg}) [2(0.25 \text{ m})^2 + 2(0.15 \text{ m})^2] \\ &= 3.4 \cdot 10^{-3} \text{ kg m}^{-2} \end{aligned}$$

- (c) If the system with six washers rotates at 5 rev/s, what is its rotational kinetic energy?  
Insert the result from part (a) into the expression for rotational kinetic energy:

$$\begin{aligned} K &= \frac{1}{2} I \omega^2 \\ &= \frac{1}{2} (3.5 \times 10^{-3} \text{ kg m}^{-2}) [5.0(2\pi \text{ rad s}^{-1})]^2 \\ &= 1.73 \text{ J} \end{aligned}$$

## Applying Rotational Kinetic Energy

### Problem Solving Strategy: Rotational Energy

1. Determine that energy or work is involved in the rotation
2. Determine the system of interest
3. Analyze the situation to determine the types of work and energy involved
4. If there are no losses of energy due to friction and other nonconservative forces, mechanical energy is conserved:  
 $K_0 + U_0 = K_f + U_f$
5. If nonconservative forces are present, mechanical energy is not conserved, determine what they are and calculate them as necessary
6. Eliminate terms wherever possible to simplify the algebra
7. Evaluate the numerical solution to see if it makes sense in the physical situation presented in the wording of the problem

### Example 10.9: Calculating Helicopter Energies

A typical small rescue helicopter has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg.

- (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm

The rotational kinetic energy is given by  $\frac{1}{2}I\omega^2$ . First, the angular velocity must be converted to  $\text{rad s}^{-1}$ , the angular velocity  $\omega$  is

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \text{ rad s}^{-1}$$

The moment of inertia of one blade is that of a thin rod rotated about its end, the total  $I$  is 4 times this because there are 4 blades.

$$I = 4 \frac{ML^2}{3} = 4 \left( \frac{(50.0 \text{ kg})(4.00 \text{ m})^2}{3} \right) = 1067.0 \text{ kg m}^{-2}$$

Entering  $\omega$  and  $I$  into the expression for rotational kinetic energy gives:

$$\begin{aligned} K &= \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}(1067 \text{ kg m}^{-2})(31.4 \text{ rad s}^{-1})^2 \\ &= 5.26 \cdot 10^5 \text{ J} \end{aligned}$$

- (b) Calculate the translational kinetic energy of the helicopter when it flies at  $20.0 \text{ ms}^{-1}$ , and compare it with the rotational energy in the blades

Entering the given values into the equation for translational kinetic energy gives:

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}(10^3 \text{ kg})(20 \text{ ms}^{-1})^2 \\ &= 2.00 \cdot 10^5 \text{ J} \end{aligned}$$

To compare kinetic energies take the ratio of translational kinetic energy to rotational kinetic energy:

$$\frac{2.00 \cdot 10^5 \text{ J}}{5.26 \cdot 10^5 \text{ J}} = 0.380$$

This tells us that most of the energy of the helicopter is in its spinning blades

**Example 10.10:** Energy in a Boomerang

A person hurls a boomerang into the air with a velocity of  $30.0\text{ ms}^{-1}$  at an angle of  $40.0^\circ$  with respect to the horizontal. It has a mass of  $1.0\text{ kg}$  and it rotating at  $10.0\text{ rev/s}$ . The moment of inertia of the boomerang is given as  $I = \frac{1}{12}mL^2$  where  $L = 0.7\text{ m}$ .

- (a) What is the total energy of the boomerang when it leaves the hand?

The total kinetic energy of the boomerang is:  $K_{tot} = K_R + K_T$

The moment of inertia is:  $I = \frac{1}{12}mL^2 = \frac{1}{12}(1.0\text{ kg})(0.7\text{ m})^2 = 0.041\text{ kg m}^2$

The angular velocity is:  $\omega = 2\pi(10.0\text{ rev/s}) = 62.83\text{ rad s}^{-1}$

Therefore the rotational and translational kinetic energies are:

$$K_R = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}(0.041\text{ kg m}^2)(62.83\text{ rad s}^{-1})^2$$

$$= 80.93\text{ J}$$

$$K_T = \frac{1}{2}mv^2$$

$$= \frac{1}{2}(1.0\text{ kg})(30.0\text{ ms}^{-1})^2$$

$$= 450.0\text{ J}$$

Therefore, the total kinetic energy of the boomerang is  $K_{tot} = (80.93 + 450.0)\text{ J} = 530.93\text{ J}$

- (b) How high does the boomerang go from the elevation of the hand, neglecting air resistance?

The maximum height can be found using the conservation of mechanical energy ( $E_0 = E_f$ ). Since the boomerang is launched at an angle, the total energy of the system must be written in terms of its linear kinetic energies using velocity in the  $x$  and  $y$  directions.

The energy when the boomerang leaves the hand is:  $E_0 = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}I\omega^2$

The total energy at the maximum height  $h$  is:  $E_f = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2 + mgh$

Equating and canceling like terms gives:

$$\begin{aligned} \cancel{\frac{1}{2}mv_x^2} + \frac{1}{2}mv_y^2 + \cancel{\frac{1}{2}I\omega^2} &= \cancel{\frac{1}{2}mv_x^2} + \cancel{\frac{1}{2}I\omega^2} + mgh \\ \frac{1}{2}mv_y^2 &= mgh \end{aligned}$$

Rearranging and entering known values gives:

$$h = \frac{\cancel{mv_y^2}}{2\cancel{mg}} \rightarrow h = \frac{v_y^2}{2g}$$

$$h = \frac{(19.28\text{ ms}^{-1})^2}{2(9.8\text{ ms}^{-2})} = 18.97\text{ m}$$