Angular Momentum

OpenStax University Physics Vol. 1

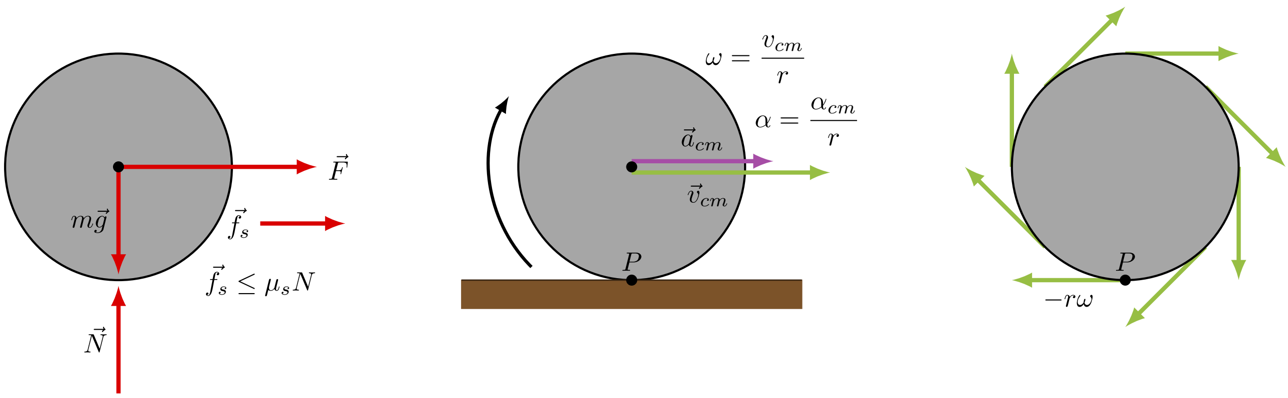
11.1 Rolling Motion

Rolling motion is the common combinatipno of rotational and translational motion seen every day, such as wheels moving on a car along a highway or wheels on a plane landing on a runway.

Rolling Motion Without Slipping

Rolling motion without slipping has been observed since the invention of the wheel. For example, consider the interaction of a car's tires and the surface of the road. If the driver floors the accelerator such that the tires spin without moving the car, there must be kinetic friction between the wheels and the surface of the road. If the driver depresses the accelerator slowly causing the car to move forward, then the tires roll without slipping. In fact the bottom of the wheel is at rest with respect to the ground, indicating there must be static friction between the tires and the road surface.

To analyze rolling without slipping, we first derive the linear variables of velocity and acceleration of the center of mass of the wheel in terms of the angular variables that describe the wheel's motion, shown below.



(a) Forces on the wheel (b) The wheel rolls without slipping (c) Point P has negative velocity

In (a), we see the force vectors involved in preventing the wheel from slipping. In (b), the point P that touches the surface is at rest relative to the surface. Relative to the center of mass, point P has velocity $-r\omega\hat{i}$, where r is the radius of the wheel and ω is the wheel's angular velocity about its axis. Since the wheel is rolling, the velocity of P with respect to the surface is its velocity with respect to the center of mass plus the velocity of the center of mass with respect to the surface:

$$\vec{v}_P = -r\omega\hat{i} + v_{cm}\hat{i}$$

Since the velocity of P relative to the surface is zero, $v_P = 0$, this says that:

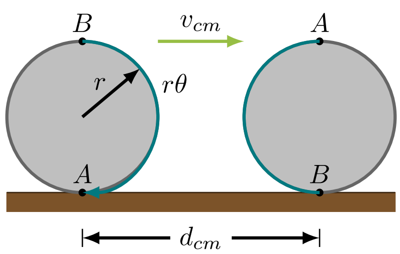
$$v_{cm} = r\omega \tag{1}$$

The velocity of the wheel's center of mass is its radius times the angular velocity about its axis. Differentiating the left side of the equation gives an expression for the linear acceleration of the center of mass, on the right side, r is constant and since $\alpha = \frac{d\omega}{dt}$:

$$a_{cm} = r\alpha \tag{2}$$

Further, the distance the wheel travels can be found in terms of angular variables. As the wheel rolls from point A to point B, its outer surface maps onto the ground by exactly the distance traveled d_{cm}. The length of the outer surface that maps onto the ground is the arc length $r\theta$. Equating the two distances gives:

$$d_{cm} = r\theta \tag{3}$$



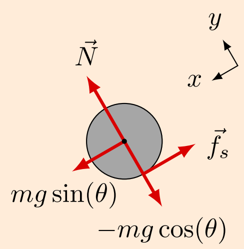
Example 11.1: Rolling Down an Inclined Plane

A solid cylinder rolls down an inclined plane without slipping, starting from rest. It has mass m and radius r.

(a) What is its acceleration?

There is barely enough friction to keep the cylinder rolling without slipping. Since there is no slipping, the magnitude of the friction force is $f_s \leq \mu_s N$. Writing down Newton's laws in the x and y direction we have:

$$\sum F_x = ma_x \quad \text{and} \quad \sum F_y = ma_y$$



Substituting in from the free-body diagram:

$$mg\sin(\theta) - f_s = m(a_{cm})$$
$$N - mg\cos(\theta) = 0$$

The linear acceleration can then be solved for the linear acceleration of the center of mass using the equation

$$a_{cm} = g\sin(\theta) - \frac{f_s}{m}.$$

However, it is useful to express the linear acceleration in terms of the moment of inertia, for this, use Newton's second law for rotation

$$\sum \tau_{cm} = I_{cm}\alpha$$

The torques are calculated about the axis through the center of mass of the cylinder. The only nonzero torque is provided by the friction force:

$$f_s r = I_{cm}\alpha$$

Finally, the linear acceleration is related to the angular acceleration by $a_{cm,x} = r\alpha$. These equations can be used to solve for a_{cm}, α, and f_s in terms of the moment of inertia. a_{cm} is rewritten in terms of the vertical component of gravity and the friction force, and the following substitutions are made:

$$f_s = \frac{I_{cm}\alpha}{r} = \frac{I_{cm}a_c m}{r^2}$$

Substituting this in for f_s in the equation above, gives:

$$a_{cm} = g\sin(\theta) - \frac{I_{cm}a_{cm}}{mr^2}$$
$$= \frac{mg\sin(\theta)}{m + ({}^{I_{cm}}/_{r^2})}$$

Therefore:

$$\alpha = \frac{a_{cm}}{r} = \frac{2}{3r}g\sin(\theta)$$

(b) What condition must the coefficient of static friction μ_s satisfy so the cylinder does not slip?

Because slipping does not occur, $f_s \leq \mu_s N$. Solving for the friction force f_s,

$$f_s = I_{cm}\frac{\alpha}{r} = I_{cm}\frac{a_{cm}}{r^2} = \frac{I_{cm}}{r^2}\left(\frac{mg\sin(\theta)}{m + ({}^{I_{cm}}/_{r^2})}\right) = \frac{mgI_{cm}\sin(\theta)}{mr^2 + I_{cm}}$$

Substituting in the condition for no slipping and noting that $N = mg\cos(\theta)$, gives

$$\frac{mgI_{cm}\sin(\theta)}{mr^2 + I_{cm}} \leq \mu_s mg\cos(\theta) \quad \text{or} \quad \mu_s \geq \frac{\tan(\theta)}{1 + ({}^{mr^2}/_{I_{cm}})}$$

For the solid cylinder this becomes:

$$\mu_s \geq \frac{\tan(\theta)}{1 + ({}^{2mr^2}/_{mr^2})} = \frac{1}{3}\tan(\theta)$$

It is worthwile to repeat the equation derived in the example for the acceleration of an object rolling without slipping:

$$a_{cm} = \frac{mg\sin(\theta)}{1 + \left({}^{I_{cm}}/_{r^2}\right)} \tag{4}$$

This equation is very useful for solving problems involving rolling without slipping. Note that the acceleration is less thana that of an object sliding down a frictionless plane with no rotation. The acceleration will also be different for two rotating objects with different rotational inertias.