Fixed-Axis Rotation

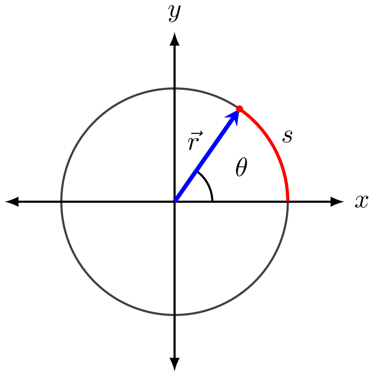
OpenStax University Physics Vol. 1

10.1 Rotational Variables

Angular Velocity

Uniform circular motion is motion in a circle at constant speed, although this is the simplest case of rotational motion, it is used here to introduce rotational variables.

The figure shows a particle moving in a circle. Its position vector from the origin of the circle to the particle sweeps out the angle θ, which increases in the counterclockwise direction as the particle moves along its path. The angle θ is called the angular position of the particle. As the particle moves, it traces an arc length s.



The angle is related to the radius of the circle and the arc length by

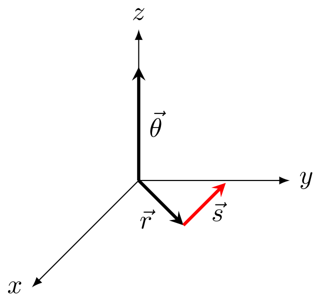
$$\theta = \frac{s}{r} \tag{1}$$

The angle θ, the angular position of the particle moving along its path has units of radians (rad). As the particle moves along its circular path, its angular position changes and it undergoes angular displacements $\Delta\theta$.

We can assign vectors to the quantities in equation 1, the angle $\vec{\theta}$ is a vector out of the page. The angular position vector $\vec{r}$ and the arc length vector $\vec{s}$ both lie in the plane of the page, they are related by:

$$\vec{s} = \vec{\theta} \times \vec{r} \tag{2}$$

The arc length is the cross product of the angle vector and the position vector



The magnitude of the angular velocity, denoted by ω, is the time rate of change of the angle θ as the particle moves in a circular path. The instantaneous angular velocity, defined as the limit as $\Delta t \to 0$ of the average angular velocity $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{3}$$

Where θ is the angle of rotation. The units of angular velocity are radians per second (rad s^{-1}). Angular velocity can also be referred to as the rotation rate in radians per second. In many cases, rotation rate is given in revolutions/s or cycles/s, to find angular velocity, multiply revolutions/s by 2π (since there are 2π radians per revolution). Since a positive angle in a circle is counterclockwise, we take counterclockwise rotations as being positive and clockwise rotations as negative.

We can see how angular velocity is related to the tangential speed of the particle by differentiating equation 1 with respect to time. Equation 1 can be rewritten as:

$$s = \theta r$$

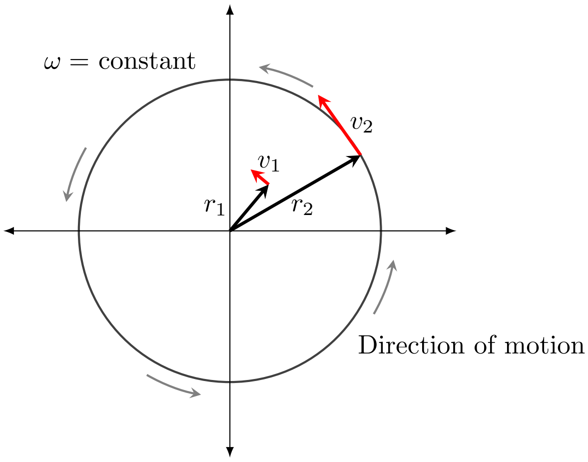
Taking the derivative with respect to time and noting that the radius r is constant gives:

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = \theta\frac{dr}{dt} + r\frac{d\theta}{dt} = r\frac{d\theta}{dt}$$

Where $\theta\frac{dr}{dt} = 0$. Here, $\frac{ds}{dt}$ is just the tangential speed v_t of the particle moving in a circular path. Using equation 3 we arrive at:

$$v_t = r\omega \tag{4}$$

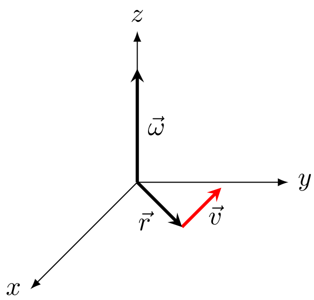
The tangential speed of the particle is its angular velocity times the radius of the circle. The tangential speed of the particle increases with its distance from the axis of rotation for a constant angular velocity. The figure shows two particles placed at different radii on a rotating disk with constant angular velocity. As it rotates, the tangential speed increases linearly with the radius from the axis of rotation. We see that $v_1 = r_1\omega_1$ and $v_2 = r_2\omega_2$. The disk has a constant angular velocity so $\omega_1 = \omega_2$. This means that $\frac{v_1}{r_1} = \frac{v_2}{r_2}$ or $v_2 = \left(\frac{r_2}{r_1}\right)$. Thus, since $r_2 > r_1$, $v_2 > v_1$

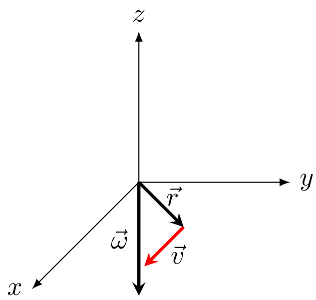


Similar to equation 2, one can state a cross product relation to the vector of the tangential velocity as stated in equation 4, therefore:

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{5}$$

The tangential velocity is the cross product of the angular velocity and the position vector as shown below. On the left we see that with the angular velocity in the $+z$ direction, the rotation in the xy plane is counterclockwise. On the right, the angular velocity is in the $-z$ direction, which gives a clockwise rotation in the xy plane.





Example 10.1: Rotation of a Flywheel

A flywheel rotates such that it sweeps out an angle at the rate of $\theta = \omega t = (45.0 \text{ rad s}^{-1})t$ radians. The wheel rotates counterclockwise when viewed in the plane of the page.

(a) What is the angular velocity ω of the flywheel?

$\omega = \dfrac{d\theta}{dt} = 45 \text{ rad s}^{-1}$, angular velocity is constant

(b) What direction is the angular velocity?

The direction of rotation is counterclockwise, so the direction of angular velocity is $+z$

(c) How many radians does the flywheel rotate through in 30 s?

$\theta(t) = \omega t \rightarrow \Delta\theta = \theta(30 \text{ s}) - \theta(0 \text{ s}) = \theta(30 \text{ s}) \rightarrow (45.0 \text{ rad s}^{-1})(30 \text{ s}) = 1350.0 \text{ rad}$

(d) What is the tangential speed of a point on the flywheel 10 cm from the axis of rotation

$v_t = r\omega = (0.1 \text{ m})(45.0 \text{ rad s}^{-1}) = 4.5 \text{ ms}^{-1}$

Angular Acceleration

For describing situations where ω changes, we need to define angular acceleration. The faster the change in ω, the greater the angular acceleration. Instantaneous angular acceleration α is defined as the derivative of angular velocity with respect to time:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{6}$$

Where we have taken the limit of the average angular acceleration $\bar{\alpha} = \frac{\Delta \omega}{\Delta t}$ as $\Delta t \to 0$. The units of angular acceleration are radians/s per second, or rad s^{-2}.