Fixed-Axis Rotation

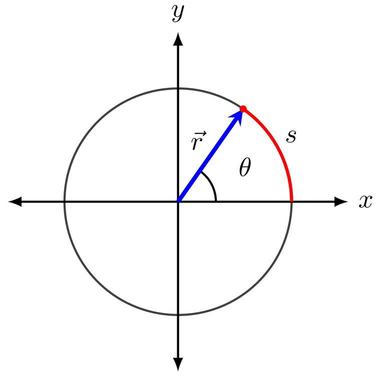
OpenStax University Physics Vol. 1

10.1 Rotational Variables

Angular Velocity

Uniform circular motion is motion in a circle at constant speed, although this is the simplest case of rotational motion, it is used here to introduce rotational variables.

The figure shows a particle moving in a circle. Its position vector from the origin of the circle to the particle sweeps out the angle θ, which increases in the counterclockwise direction as the particle moves along its path. The angle θ is called the angular position of the particle. As the particle moves, it traces an arc length s.



The angle is related to the radius of the circle and the arc length by

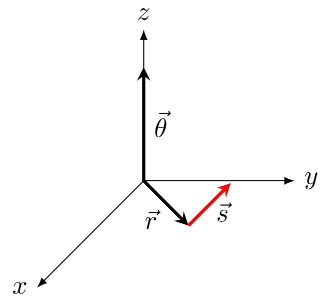
$$\theta = \frac{s}{r} \tag{1}$$

The angle θ, the angular position of the particle moving along its path has units of radians (rad). As the particle moves along its circular path, its angular position changes and it undergoes angular displacements $\Delta\theta$.

We can assign vectors to the quantities in equation 1, the angle $\vec{\theta}$ is a vector out of the page. The angular position vector $\vec{r}$ and the arc length vector $\vec{s}$ both lie in the plane of the page, they are related by:

$$\vec{s} = \vec{\theta} \times \vec{r} \tag{2}$$

The arc length is the cross product of the angle vector and the position vector



The magnitude of the angular velocity, denoted by ω, is the time rate of change of the angle θ as the particle moves in a circular path. The instantaneous angular velocity, defined as the limit as $\Delta t \to 0$ of the average angular velocity $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{3}$$

Where θ is the angle of rotation. The units of angular velocity are radians per second (rad s^{-1}). Angular velocity can also be referred to as the rotation rate in radians per second. In many cases, rotation rate is given in revolutions/s or cycles/s, to find angular velocity, multiply revolutions/s by 2π (since there are 2π radians per revolution). Since a positive angle in a circle is counterclockwise, we take counterclockwise rotations as being positive and clockwise rotations as negative.

We can see how angular velocity is related to the tangential speed of the particle by differentiating equation 1 with respect to time. Equation 1 can be rewritten as:

$$s = \theta r$$

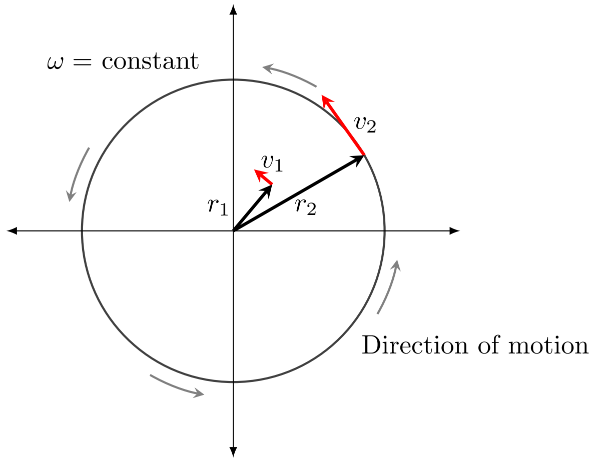
Taking the derivative with respect to time and noting that the radius r is constant gives:

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = \theta\frac{dr}{dt} + r\frac{d\theta}{dt} = r\frac{d\theta}{dt}$$

Where $\theta\frac{dr}{dt} = 0$. Here, $\frac{ds}{dt}$ is just the tangential speed v_t of the particle moving in a circular path. Using equation 3 we arrive at:

$$v_t = r\omega \tag{4}$$

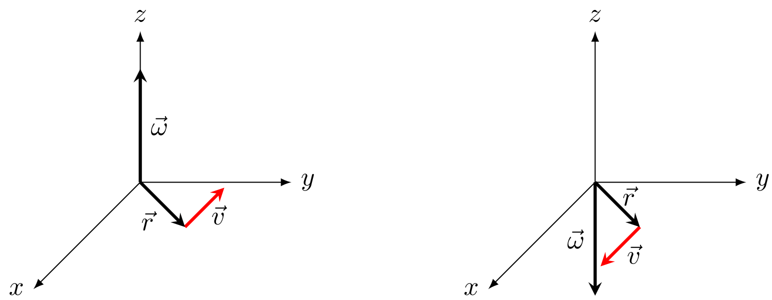
The tangential speed of the particle is its angular velocity times the radius of the circle. The tangential speed of the particle increases with its distance from the axis of rotation for a constant angular velocity. The figure shows two particles placed at different radii on a rotating disk with constant angular velocity. As it rotates, the tangential speed increases linearly with the radius from the axis of rotation. We see that $v_1 = r_1\omega_1$ and $v_2 = r_2\omega_2$. The disk has a constant angular velocity so $\omega_1 = \omega_2$. This means that $\frac{v_1}{r_1} = \frac{v_2}{r_2}$ or $v_2 = \left(\frac{r_2}{r_1}\right)$. Thus, since $r_2 > r_1$, $v_2 > v_1$



Similar to equation 2, one can state a cross product relation to the vector of the tangential velocity as stated in equation 4, therefore:

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{5}$$

The tangential velocity is the cross product of the angular velocity and the position vector as shown below. On the left we see that with the angular velocity in the $+z$ direction, the rotation in the xy plane is counterclockwise. On the right, the angular velocity is in the $-z$ direction, which gives a clockwise rotation in the xy plane.



Example 10.1: Rotation of a Flywheel

A flywheel rotates such that it sweeps out an angle at the rate of $\theta = \omega t = (45.0 \text{ rad s}^{-1})t$ radians. The wheel rotates counterclockwise when viewed in the plane of the page.

(a) What is the angular velocity ω of the flywheel?

$\omega = \dfrac{d\theta}{dt} = 45 \text{ rad s}^{-1}$, angular velocity is constant

(b) What direction is the angular velocity?

The direction of rotation is counterclockwise, so the direction of angular velocity is $+z$

(c) How many radians does the flywheel rotate through in 30 s?

$\theta(t) = \omega t \rightarrow \Delta\theta = \theta(30 \text{ s}) - \theta(0 \text{ s}) = \theta(30 \text{ s}) \rightarrow (45.0 \text{ rad s}^{-1})(30 \text{ s}) = 1350.0 \text{ rad}$

(d) What is the tangential speed of a point on the flywheel 10 cm from the axis of rotation

$v_t = r\omega = (0.1 \text{ m})(45.0 \text{ rad s}^{-1}) = 4.5 \text{ ms}^{-1}$

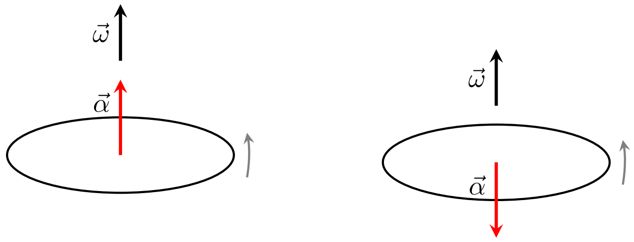
Angular Acceleration

For describing situations where ω changes, we need to define angular acceleration. The faster the change in ω, the greater the angular acceleration. Instantaneous angular acceleration α is defined as the derivative of angular velocity with respect to time:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{6}$$

Where we have taken the limit of the average angular acceleration $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$ as $\Delta t \to 0$. The units of angular acceleration are radians/s per second, or rad s^{-2}.

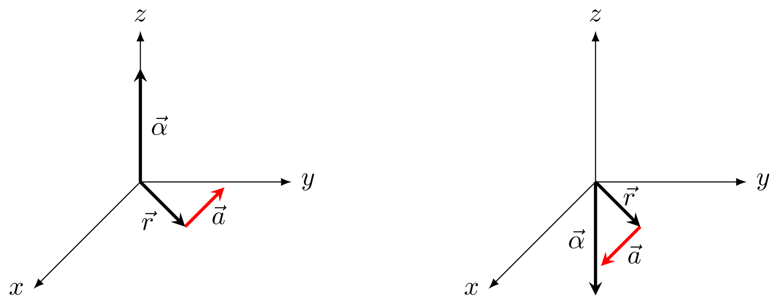
In the same way that the vector associated with angular velocity $\vec{\omega}$ was defined, we can define $\vec{\alpha}$, the vector associated with angular acceleration. If the angular velocity is along the $+z$ axis and $\frac{d\omega}{dt}$ is positive, the angular acceleration $\vec{\alpha}$ is positive and points along the $+z$ axis, if $\frac{d\omega}{dt}$ is negative, the angular acceleration is negative and points along the $-z$ axis.



The tangential acceleration vector can be expressed as a cross product of the angular acceleration and position vectors. This equation can be found by taking the derivative of equation 5

$$\vec{a} = \vec{\alpha} \times \vec{r} \tag{7}$$

The vector relationships for angular acceleration and tangential acceleration are shown below



Tangential acceleration of a point on a rotating body at a distance from the axis of rotation can be related in the same way as tangential velocity and angular velocity. Differentiating equation 4 with respect to time (the radius r is constant) gives:

$$a_t = r\alpha \tag{8}$$

The tangential acceleration a_t is the radius times the angular acceleration

<u>**Example 10.2:** A Spinning Bike Wheel</u>

A bicycle mechanic mounts a bicycle on the repair stand and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s.

(a) Calculate the average angular acceleration in rad s^{-2}

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{250 \text{ rpm}}{5.00 \text{ s}}$$

Converting from rpm to rad s^{-1}:

$$\Delta\omega = 250\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 26.2\text{rad s}^{-1}$$

Entering this back into the expression for α gives:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{26.2 \text{ rad s}^{-1}}{5.00 \text{ s}} = 5.24 \text{ rad s}^{-2}$$

(b) If the brakes are hit, causing an angular acceleration of -87 rad s^{-2}, how long does it take the wheel to stop?

Angular velocity decreases from 26.2 rad s^{-1} to zero so $\Delta\omega = -26.2$ rad s^{-1}, and α is given to be -87.3 rad s^{-2}

$$\Delta t = \frac{\Delta\omega}{\alpha} = \frac{-26.2 \text{ rad s}^{-1}}{-87.3 \text{ rad s}^{-2}} = 0.300 \text{ s}$$

Example 10.3: Wind Turbine

A wind turbine in a wind farm is being shut down for maintenance. It takes 30 s for the turbine to go from its operating angular velocity to a complete stop in which the angular velocity function is $\omega(t) = \left[\frac{(t\mathrm{s}^{-1} - 30.0)^2}{100.0}\right]$rad s^{-1}, where t is the time in seconds. If the turbine is rotating counterclockwise looking into the page:

(a) What are the directions of the angular velocity and acceleration vectors?

Since the turbine is rotating counterclockwise, angular velocity $\vec{\omega}$ points towards $+z$. Since the angular velocity is decreasing, the angular acceleration $\vec{\alpha}$ points towards $-z$

(b) What is the average angular acceleration?

At $t = 0$, the initial angular velocity of the turbine is $\omega = 9.0$ rad s^{-1}, the final angular velocity is zero, so the average angular velocity $\bar{\alpha}$ is:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t - t_0} = \frac{0 - 9.0 \text{ rad s}^{-1}}{30.0 - 0 \text{ s}} = -0.3 \text{ rad s}^{-2}$$

(c) What is the instantaneous angular acceleration at $t = 0.0, 15.0, 30.0$ s?

Taking the derivative of angular velocity with respect to time gives

$$\alpha = \frac{d\omega}{dt} = \left[\frac{(t - 30.0)}{50.0}\right] \text{ rad s}^{-2}$$

Thus: $\alpha(0.0 \text{ s}) = -0.6$ rad s^{-2}, $\alpha(15.0 \text{ s}) = -0.3$ rad s^{-2}, and $\alpha(30.0\text{s}) = 0$ rad s^{-2}

10.2 Rotation With Constant Angular Acceleration

In this section, the definitions from the previous section are used to derive relationships among these variables, and use these relationships to analyze rotational motion for a rigid body about a fixed axis under a constant angular acceleration, forming the basis for rotational kinematics. If angular acceleration is constant, the equations of rotational kinematics simplify, similar to the equations of linear kinematics.

Kinematics of Rotational Motion

In the previous section we saw that if a flywheel has an angular acceleration in the same direction as its angular velocity, its angular velocity increases with time and its angular displacement also increases. If the angular acceleration is opposite to the angular velocity vector, its angular velocity decreases with time. Under a constant angular acceleration, we can describe these physical situations with a consistent set of rotational kinematic equations.

If the system is rotating under a constant acceleration, then the average angular velocity follows a simple relation because the angular velocity is increasing linearly with time. The average angular velocity is just half of the sum of the initial and final values:

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2} \tag{9}$$

Using the definition of average angular velocity, an equation that relates the angular position, average angular velocity, and time can be found:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Solving for θ gives:

$$\theta_f = \theta_0 + \bar{\omega}t \tag{10}$$

Where $t_0 = 0$. This equation can be useful when the average angular velocity of the system is known. Then the angular displacement over a given period of time could be found. To determine an equation relating ω, α, and t, we start with the definition of angular acceleration:

$$\alpha = \frac{d\omega}{dt}$$

This is rearranged to $\alpha dt = d\omega$, then we integrate both sides of the equation from initial to final values, from t_0 to t and from ω_0 to ω_f. (angular acceleration is constant and can be pulled outside)

$$\alpha \int_{t_0}^{t} dt = \int_{\omega_0}^{\omega_f} d\omega$$

Setting $t_0 = 0$ gives:

$$\alpha t = \omega_f - \omega_0$$

This is rearranged to obtain

$$\omega_f = \omega_0 + \alpha t \tag{11}$$

Where ω_0 is the initial angular velocity. This equation is the rotational counterpart to the linear kinematic equation $v_f = v_0 + at$. With equation 11, the angular velocity of an object at any specified time t can be found given the initial angular velocity and angular acceleration.

Doing a similar thing to the equation $\omega = \frac{d\theta}{dt}$, rearranging it to $\omega dt = d\theta$ and integrating both sides from initial to final values, noting that angular acceleration is constant and does not have a time dependence. This time angular velocity is not constant, so equation 11 is substituted in:

$$\int_{t_0}^{t_f} (\omega_0 + \alpha t) dt = \int_{\theta+0}^{\theta_f} d\theta$$

$$\int_{t_0}^{t} \omega_0 dt + \int_{t_0}^{t} \alpha t dt = \int_{\theta_0}^{\theta_f} d\theta$$

$$\left[\omega_0 t + \frac{1}{2}\alpha t^2\right]_{t_0}^{t} = \omega_0 t + \frac{1}{2}\alpha t^2 = \theta_f - \theta_0$$

Where $t_0 = 0$. Now we rearrange to obtain:

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{12}$$

This is the rotational counterpart to the linear kinematic equation $s_f = s_0 + v_0 t + \frac{1}{2}at^2$. This equation gives the angular position of a rotating rigid body at any time t given the initial conditions (θ_0 and ω_0) and the angular acceleration

We can find an equation that is independent of time by solving for t in equation 11 and substituting into equation 12:

$$\begin{aligned}
\theta_f &= \theta_0 + \omega_0\left(\frac{\omega_f - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega_f - \omega_0}{\alpha}\right)^2 \\
&= \theta_0 + \frac{\omega_0\omega_f}{\alpha} - \frac{\omega_0^2}{\alpha} + \frac{1}{2}\frac{\omega_f^2}{\alpha} - \frac{\omega_0\omega_f}{\alpha} + \frac{1}{2}\frac{\omega_0^2}{\alpha} \\
&= \theta_0 + \frac{1}{2}\frac{\omega_f^2}{\alpha} - \frac{1}{2}\frac{\omega_0^2}{\alpha} \\
\theta_f - \theta_0 &= \frac{\omega_f^2 - \omega_0^2}{2\alpha}
\end{aligned}$$

This rearranges to:

$$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta) \tag{13}$$

Equations 10 - 13 describe fixed-axis rotation for constant acceleration and are summarized below

Rotational	Linear
$\theta_f = \theta_0 + \bar{\omega}t$	$s_f = s_0 + \bar{v}t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$s_f = s_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta s)$