Bayesian Statistics - an Introduction

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- The interpretation of interval estimates is more natural.
- Nuisance parameter and constraints on parameters can be easily dealt with.

Bayes theorem

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- The theorem was published posthumously in 1763 in 'An essay towards solving a problem in the doctrine of chances'.

Bayes theorem

- Let Ω be a sample space and $B_1, B_2, \ldots, B_k$ be mutually exclusive and exhaustive events in Ω (i.e. $B_i \cap B_j = \emptyset, i \neq j, \cup_{i=1}^{k} B_i = \Omega$; the B_i form a partition of Ω.)

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- Let A be any event with $\Pr[A] > 0$.
- $$\Pr[B_i|A] = \frac{\Pr[B_i]\Pr[A|B_i]}{\Pr[A]} = \frac{\Pr[B_i]\Pr[A|B_i]}{\sum_{j=1}^{k}\Pr[B_j]\Pr[A|B_j]}$$

Example

- A diagnostic test for a disease gives a correct result 99% of the time. Suppose 2% of the population have the disease. If a person selected at random from the population is given the test and produces a positive result what is the probability that the person has the disease? Suppose they are given a second independent test and it is also positive, what is the probability that they have the disease now?

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- Let '+' and '−' denote the events that a test is positive or negative, respectively. Let D denote the event that the person has the disease.
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- By Bayes theorem

$$\begin{aligned} p(D|+) &= \frac{p(+|D)p(D)}{p(+|D)p(D) + p(+|\bar{D})p(\bar{D})} \\ &= \frac{0.99 \times 0.02}{0.99 \times 0.02 + 0.01 \times 0.98} \\ &= 0.6689 \end{aligned}$$

Solution for a second test

- Suppose that we have a second positive test result

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- Thus although the test is very accurate we need two positive results before we can say with confidence that the person has the disease.

Density form of Bayes Theorem

- Let X, θ be two continuous r.v.'s (possibly multivariate).

$$f(\theta|x) = \frac{f(\theta)f(x|\theta)}{f(x)} = \frac{f(\theta)f(x|\theta)}{\int f(\theta')f(x|\theta')d\theta'}$$

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$$p(\theta|x) \propto p(x|\theta)p(\theta)$$

The Likelihood Principle

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The Likelihood Principle

- To illustrate the difference between the classical and Bayesian approaches we start with an example.
- Suppose we toss a drawing pin and get 9 'ups' and 3 'downs'. We denote 'up' by U and 'down' by D. Is the pin unbiased?
- Classically we might test $H_0 : p = \frac{1}{2}$ versus $H_1 : p > \frac{1}{2}$ where $p = p(U)$. The probability of the observed result or something more extreme (tail area) if H_0 is true is

$$\left\{ \binom{12}{3} + \binom{12}{2} + \binom{12}{1} + \binom{12}{0} \right\} \left(\frac{1}{2}\right)^{12}$$

which $= \frac{299}{4096} \approx 7.3\%$. Thus we would accept H_0 at the 5% level.

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- What if we decided to toss the pin until we achieved 3 D's?
- Now the probability of the observed result or something more extreme if H_0 is true is

$$\binom{11}{2}\left(\frac{1}{2}\right)^{12} + \binom{12}{2}\left(\frac{1}{2}\right)^{13} + \binom{13}{2}\left(\frac{1}{2}\right)^{14} + \cdots$$

We may calculate the probability of the complement of this event by

$$\binom{10}{2}\left(\frac{1}{2}\right)^{11} + \binom{9}{2}\left(\frac{1}{2}\right)^{10} + \cdots + \binom{2}{2}\left(\frac{1}{2}\right)^{3}$$

It follows that the P-value is $\frac{134}{4096} \approx 3.25\%$. So we would reject H_0 at the 5% level.

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- Possibilities for the drawing pin are:
  (i) $\{(u, d) : u + d = 12\}$,
  (ii) $\{(u, d) : d = 3\}$,
  or if I carry on tossing the pin until my coffee is ready, so that there is a random stopping point
  (iii) $\{\text{all } (u, d)\}$.

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$$\begin{aligned} p(\theta|\text{data}) &\propto p(\text{data}|\theta)p(\theta) \\ &\quad \theta^9(1-\theta)^3 p(\theta) \end{aligned}$$

The sampling rule is irrelevant.

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- This roughly says that if two experiments give the same likelihoods then the inferences we make should be the same in each case.
- Classical hypothesis tests or confidence intervals violate the likelihood principle.

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- Suppose we take

$$p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1} \quad a, b > 0$$

ie a beta distribution with mean $a/(a+b)$ and variance

$$\frac{a}{a+b}\frac{b}{a+b}\frac{1}{a+b+1}.$$

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- It then follows that

$$p(\theta|\text{data}) \propto \theta^{9+a-1}(1-\theta)^{3+b-1}.$$

That is $Be(9+a, 3+b)$. Thus if we take a beta prior for θ we shall obtain a beta posterior.

- The choice $a = b = 1$ gives a uniform distribution as the prior, ie we think any value of θ is equally likely. More realistically for a pin we might take $a = b = 2$, reflecting a belief that we think θ is more likely to be near 0.5 than 0 or 1 but not being very sure.

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- Others might choose an asymmetric prior, perhaps arguing that a pin with a very long point would very likely land down so a pin with any point would land down more often than up. I am not convinced by this argument but it shows that different people do have different beliefs and one of the advantages of the Bayesian approach is that the analysis can reflect these.

- If we were throwing a coin rather than a pin then we would almost certainly choose a different prior. We have much more experience throwing coins than pins and are much more sure that θ, the chance the coin will land heads, is close to 0.5. Thus we might take $a = b = 50$ say.

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- The posteriors we get for the pin and the coin will be very different. For the pin the posterior mean will be $11/16 = 0.6875$ and the posterior variance $11/16 \times 5/16 \times 1/17 = 0.0126$. For the coin the posterior mean will be $59/112 = 0.527$ which is close to 0.5.

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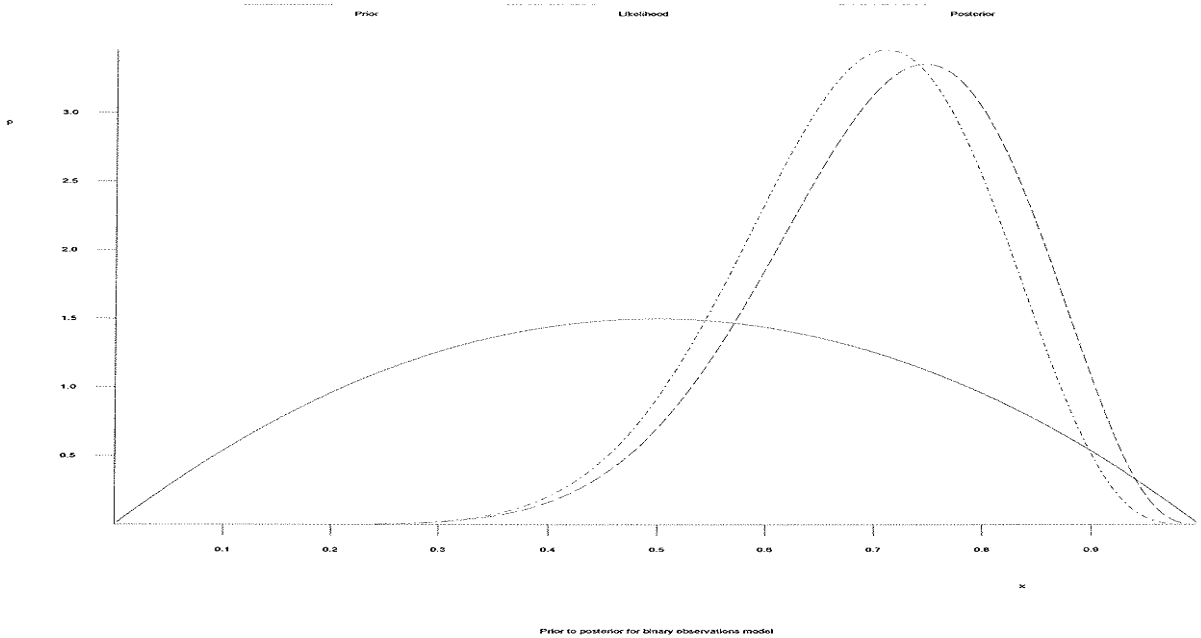
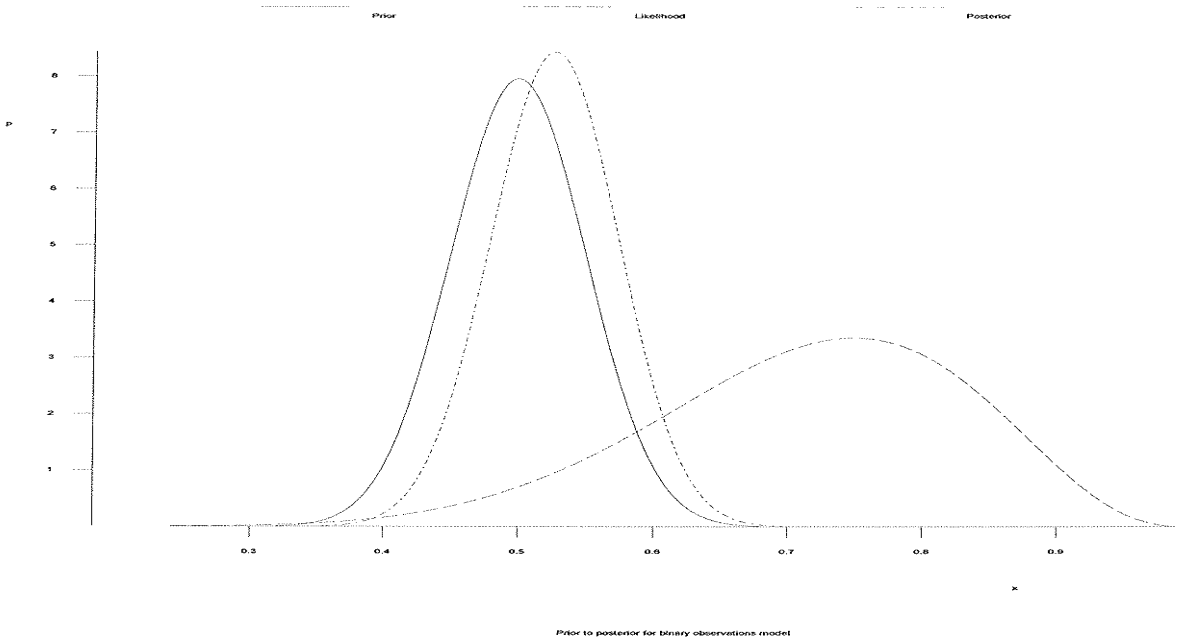
- The classical unbiased estimate is $9/12 = 0.75$ if the number of throws is fixed, or $9/11 = 0.818$ if we continue until we have three 'failures'. The classical answers are the same whether for pins or coins, ignoring the extra information that we have.

Plot for pins



Plot for coins



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- As an example consider the result when a coin is spun on its edge. Experience has shown that when spinning a coin the proportion of heads is more likely to be 1/3 or 2/3 than 1/2. Therefore a bimodal prior seems appropriate. Since spinning coins will be Bernoulli trials the beta distribution will be conjugate.

- By taking mixtures of conjugate priors we can represent more realistic beliefs.
- As an example consider the result when a coin is spun on its edge. Experience has shown that when spinning a coin the proportion of heads is more likely to be $1/3$ or $2/3$ than $1/2$. Therefore a bimodal prior seems appropriate. Since spinning coins will be Bernoulli trials the beta distribution will be conjugate.
- Therefore we take the following prior

$$p(\theta) = \frac{1}{2} Be(10, 20) + \frac{1}{2} Be(20, 10)$$

i.e

$$p(\theta) = \frac{1}{2} \frac{\Gamma(30)}{\Gamma(10)\Gamma(20)} \theta^{10-1}(1-\theta)^{20-1} + \frac{1}{2} \frac{\Gamma(30)}{\Gamma(20)\Gamma(10)} \theta^{20-1}(1-\theta)^{10-1}$$

where θ is the chance that the coin comes down heads

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- The weight on the first posterior, α', is

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Now

$$p_1(\underline{x}) = \frac{\Gamma(30)}{\Gamma(10)\Gamma(20)} \frac{\Gamma(13)\Gamma(27)}{\Gamma(40)}$$

and similarly

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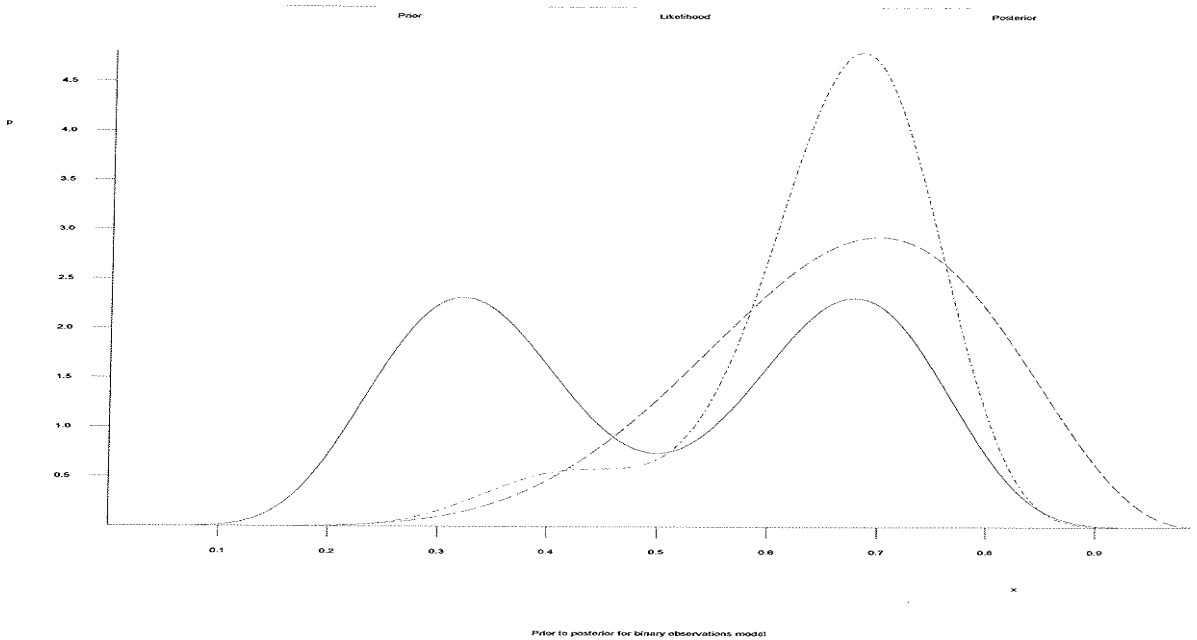
A little calculation shows that $\alpha' = 0.89$ and $\beta' = 0.11$. Hence the posterior is

$$p(\theta|x) = 0.89Be(13, 27) + 0.11Be(23, 17)$$

Plot for mixture prior



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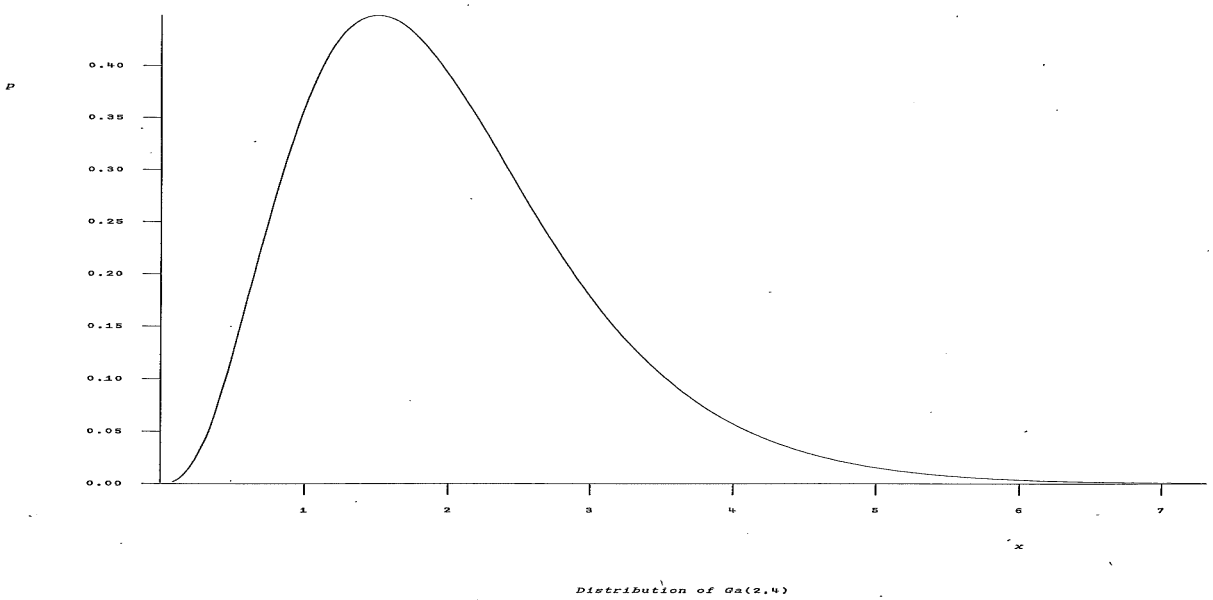
- Prior?
- May expect that on average there would be 2 claims per month, with a variance of 1.
- Conjugate prior for θ is Gamma(4,2):

$$p(\theta) \propto \theta^{4-1} \exp(-2\theta)$$

Prior density - Gamma(4,2)



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- Posterior

$$p(\theta|\underline{x}) \propto \theta^{22-1} \exp(-8\theta)$$

which we can recognise is a $Ga(22, 8)$.

Calculation of the posterior

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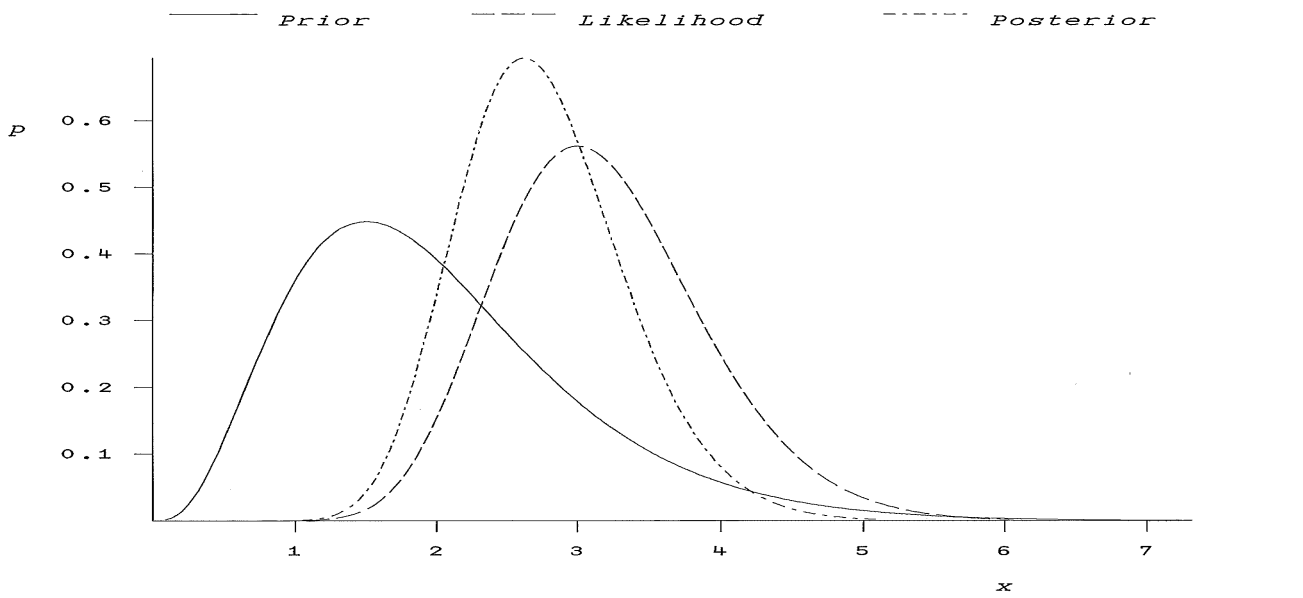
- Weights are related to the prior and data variances.
- This holds for other common distributions.

Prior/Posterior density



Prior to posterior for Poisson observations (time parameter 1) model

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- Prior $p(\theta)$. Posterior $p(\theta|\underline{x})$.
- y - independent observation from the same distribution. Want $p(y|\underline{x})$ - predictive distribution of y.

Predictive density

Now by extension of the argument

$$\begin{aligned} p(y|\underline{x}) &= \int p(y|\theta, \underline{x}) p(\theta|\underline{x}) d\theta \\ &= \int p(y|\theta) p(\theta|\underline{x}) d\theta \end{aligned}$$

since y is independent of $\underline{x}$ given θ.

Poisson predictive example

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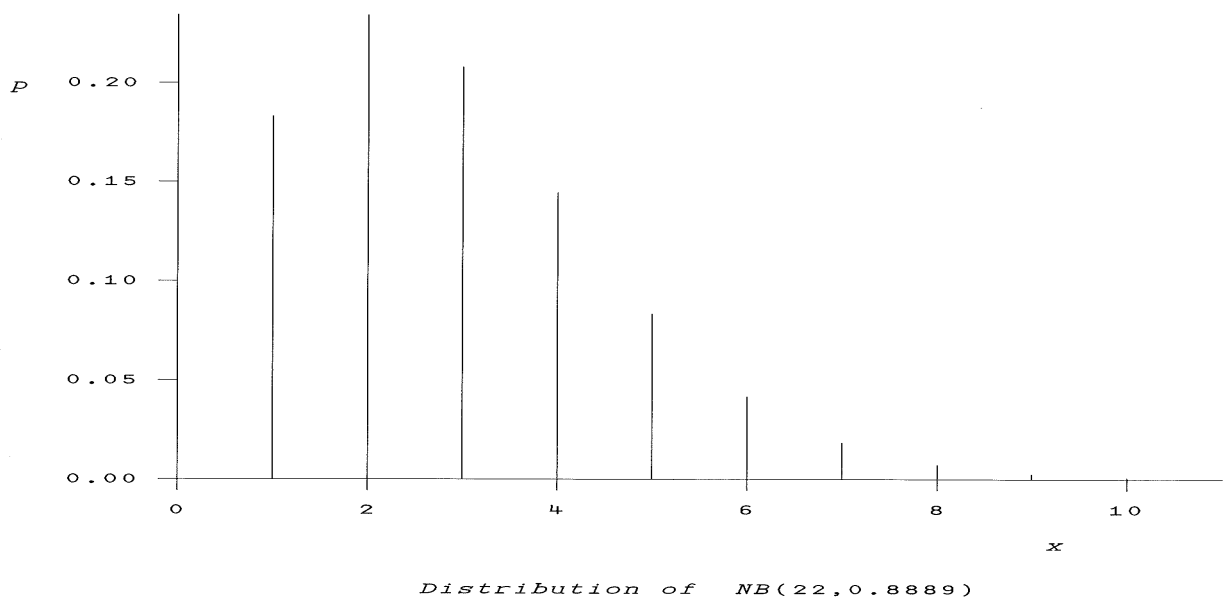
- $x_1, \ldots, x_n \sim$ Poisson(θ), prior for θ is Gamma(4,2)
  Posterior $\Rightarrow$ Gamma(22,8)
- Suppose $y \sim$ Poisson(θ)

$$
\begin{aligned}
p(y|\underline{x}) &= \int p(y|\theta)p(\theta|\underline{x})d\theta \\
&= \int \frac{\theta^y \exp(-\theta)}{y!} \frac{8^{22}\theta^{21}\exp\{-8\theta\}}{\Gamma(22)} d\theta \\
&= \frac{8^{22}}{\Gamma(22)y!} \int \theta^{21+y} \exp\{-\theta(9)\} d\theta \\
&= \frac{8^{22}}{\Gamma(22)y!} \frac{\Gamma(y+22)}{9^{22+y}} \qquad y = 0, 1, 2, 3, \ldots
\end{aligned}
$$

Predictive density of y



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- As an extra ingredient we specify a loss function and choose an estimate to minimise expected loss.

- For example, quadratic loss

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- Absolute error loss

$$l(d, \theta) = |d - \theta|$$

leads to the posterior median as the estimator.

Stochastic Simulation - Gibbs sampling

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- use the simulated sample to make inferences.

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- Likelihood - $p(\underline{x}|\theta, \delta, \phi)$.
- Prior - $p(\theta, \delta, \phi)$.
- Suppose we can find the conditional distributions:

$$p(\theta|\delta, \phi, \underline{x}), \quad p(\delta|\phi, \theta, \underline{x}), \quad p(\phi|\theta, \delta, \underline{x})$$

that we can simulate from.

Gibbs Sampling

- The Gibbs sampler is an iterative scheme:
  - Step 1 Choose initial estimates $\theta^0, \delta^0, \phi^0$
  - Step 2 Given current estimates $\theta^i, \delta^i, \phi^i$ simulate new values
    θ^{i+1} from $p(\theta|\delta^i, \phi^i, \underline{x})$
    δ^{i+1} from $p(\delta|\phi^i, \theta^{i+1}, \underline{x})$
    ϕ^{i+1} from $p(\phi|\theta^{i+1}, \delta^{i+1}, \underline{x})$
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    ϕ^{i+1} from $p(\phi|\theta^{i+1}, \delta^{i+1}, \underline{x})$
  - Step 3 Return to step 2
- The sequence $(\theta, \delta, \phi)^i \quad i = 1, 2, \ldots$ is a realisation of a Markov chain which, under mild regularity conditions, has equilibrium distribution $p(\theta, \delta, \phi|\underline{x})$, the joint posterior distribution of θ, δ, ϕ.

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- The predictive density of a new observation y can be estimated as

$$p(y|\underline{x}) = \frac{1}{t}\sum_{i=L+1}^{L+t} P(y|\theta^i, \delta^i, \phi^i)$$

Poisson hierarchical model

- 1st stage - observations $s_j \sim$ Poisson with mean $t_j \lambda_j$ for $j = 1, 2, \ldots, p$, t_j known

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- $p + 1$ unknown parameters - λ's and β.
- Joint posterior of all the parameters:

$$\prod_{j=1}^{p} \frac{(\lambda_j t_j)^{s_j} \exp(-\lambda_j t_j)}{s_j!} \times \prod_{j=1}^{p} \Gamma(\alpha)^{-1} \beta^{\alpha} \lambda_j^{\alpha-1} \exp(-\beta\lambda_j)$$

$$\times \Gamma(\gamma)^{-1} \delta^{\gamma} \beta^{\gamma-1} \exp(-\delta\beta)$$

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$$p(\beta|\underline{\lambda}, \text{data}) \propto \beta^{p\alpha} \exp(-\sum \lambda_j \beta)\beta^{\gamma-1} \exp(-\delta\beta)$$

$$\Rightarrow Ga(p\alpha + \gamma, \sum \lambda_j + \delta)$$

Gibbs runs

- For suitable starting values for the unknown parameters the Gibbs sampler proceeds by drawing

$$\begin{aligned} \lambda_j^{i+1} &\sim Ga(\alpha + s_j, \beta^i + t_j) \quad j = 1, \ldots, p \\ \beta^{i+1} &\sim Ga(p\alpha + \gamma, \sum \lambda_j^{i+1} + \delta) \end{aligned}$$

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- Use large sample of λ's and β's to estimate posterior and predictive quantities of interest.