# Zero-Bids in a Two-Player Product-Mix Auction

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# Outline

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# Setup

## Assumptions

- Every bidder is assigned a valuation for each good being auctioned. Each value is drawn from a common distribution of values.
- Note: Because of diminishing marginal returns, each additional unit is valued less than or equal to the previous one for each bidder.

# Second Price Auction

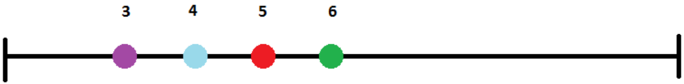
## Setup

- One unit of one good is up for auction
- Each bidder places a bid for the good
- The good is sold to the highest bidder
- He pays the SECOND highest price

# Example

## Valuations

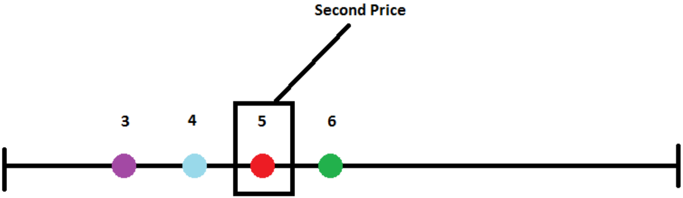
Purple $=3$, Blue $=4$, Red $=5$, Green $=6$



# Example

## Result

Green wins and pays 5 for the good



# Advantages of Second Price Auction

**Theorem - Vickrey (1961)**

In a second price auction, the weakly dominant strategy for each bidder is to bid their valuation. So, it is an equilibrium for each bidder to bid their value.

# What if we have multiple units?

## Auction them all at once

- Avoid Signaling (Dishonest incentives)
- Time Constraints

## Uniform Price

- Truthful bidding
- Fairness
- Ease of Participation

# N+1 Price Auction

## Setup

- There are N identical units of a good up for auction
- The N highest bids win one unit each
- They all pay the uniform price of the N+1 highest bid
- Note: It is possible to win multiple goods

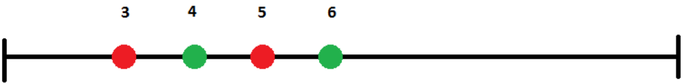
## Problem

It is possible to set the price that you pay for the good

# Example

## Valuations

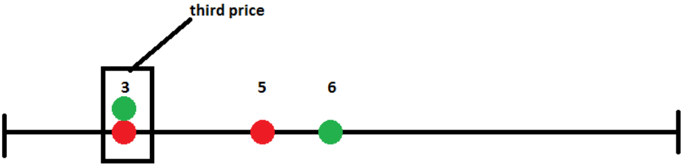
Green = (6,4), Red = (5,3), both players win one unit



# Example

## Bid Below Value

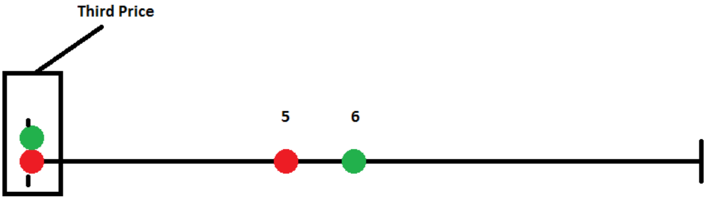
- Green bids 3 instead of 4
- Now Green still wins one unit, Red still wins one unit. However, they both pay 3 instead of 4.



# Example

## Both Bid Below Value

- Both Green and Red bid 0 for their second bid
- Now Green still wins one unit, Red still wins one unit. However, they both pay 0.



# Theorem

## Due to Englebrecht-Wiggans and Kahn (1998)

If two bidders with values drawn from the same uniform distribution are bidding for two units of a single good in a third price auction, then the Nash equilibrium is for each bidder to bid (v, 0). Where v is their value of one unit and they bid zero for the second unit.

# Outline

# Motivation

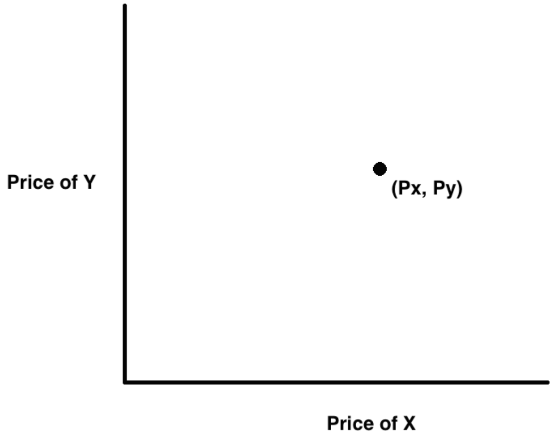
- The auctioneer has n indivisible units to apportion between two substitute goods that have not been produced yet
- Example: A car company determining the product mix between two different models of cars
- Klemperer and Baldwin created an auction mechanism to solve this problem (in their case for the Bank of England providing loans against strong or weak collateral)

# Product-Mix Bids

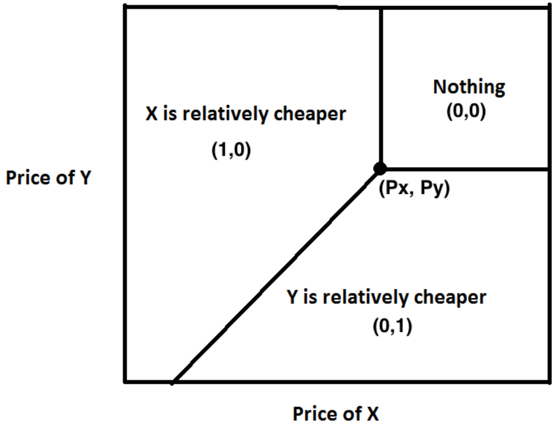
## Description

Each bid placed is a set of prices $(p_x, p_y)$ such that if the prices were set at $(p_x, p_y)$ the bidder would be indifferent between receiving quantities (1, 0), (0,1) and (0,0). Where each of these ordered pairs is a quantity of Good X and a quantity of Good Y

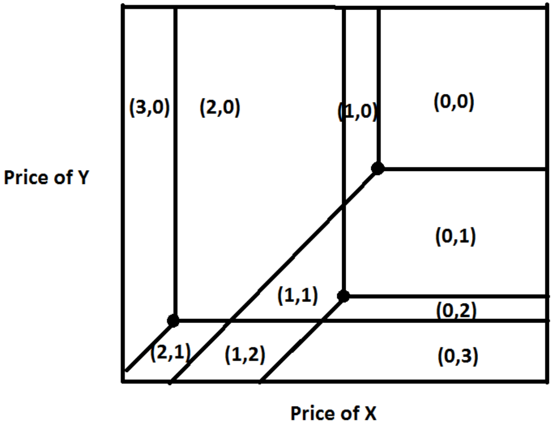
# Visualizing Bids



# Visualizing Bids



# Aggregating Bids

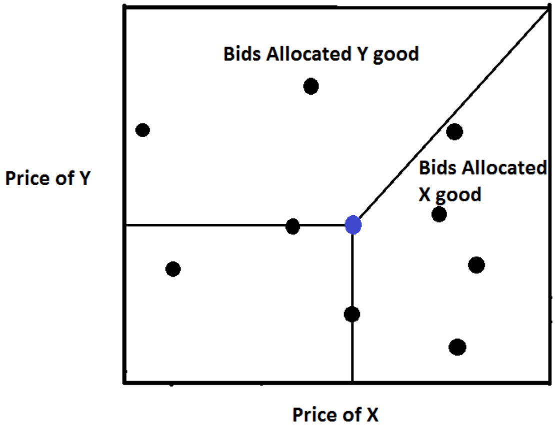


# Product-Mix Prices

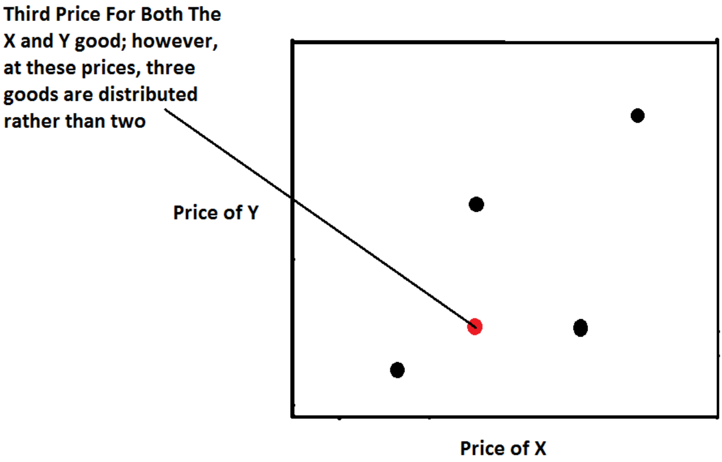
## Description

The auctioneer chooses a uniform price for each of the two types of goods. Then he allocates one good to every bid above the price in either coordinate. Which type of good is allocated to each winning bid is determined by which good that bid values more relative to the market price.

# Visualizing Choosing Prices



# This is NOT a 3rd Price Auction



# Outline

# Setup

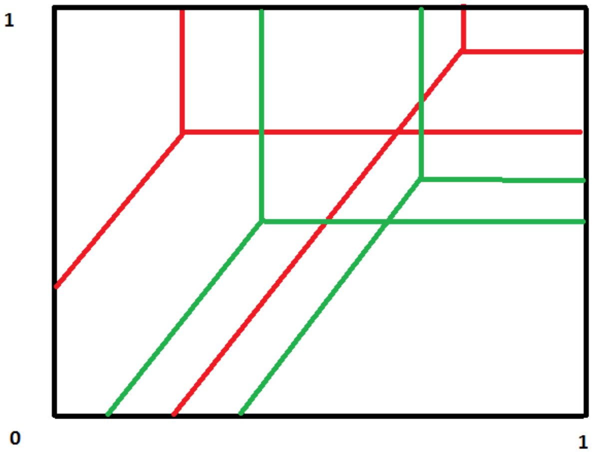
## Assumptions

- 2 bidder, auctioneer selling 2 units, so each bidder places 2 bids
- Both bidders have their values drawn from the uniform distribution on $[0,1] \times [0,1]$

## Goal

Find a strategy and or equilibrium for the two player product mix game.

# Determining Prices by Bids



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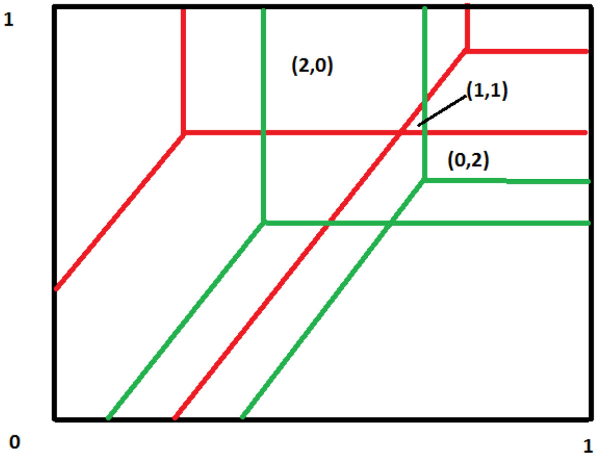

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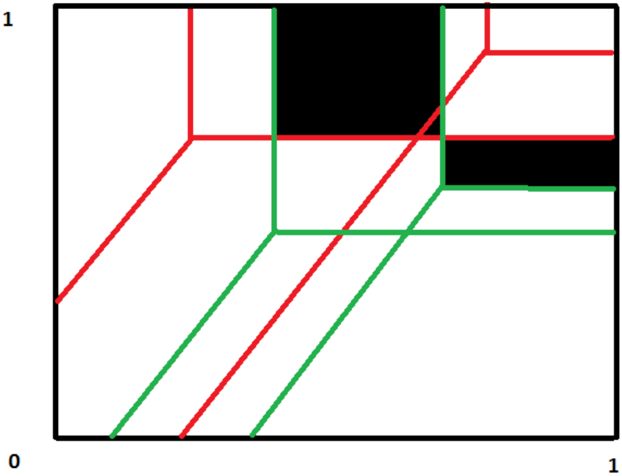

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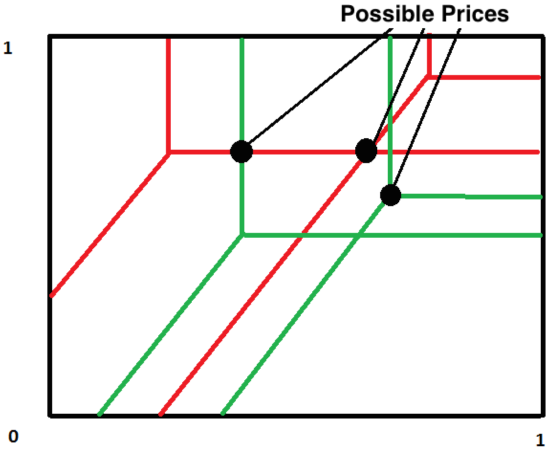


# Definition of Increasing Values

# Definition of Decreasing Values

# Result

**Theorem**

The strategy that maps increasing values to the bids (0,0) and $(v_{1x}, v_{1y})$ and decreasing values to the bids $(v_{1x}, 0)$ and $(0, v_{1y})$, is a Nash equilibrium for this game.

# The Decreasing Values Case

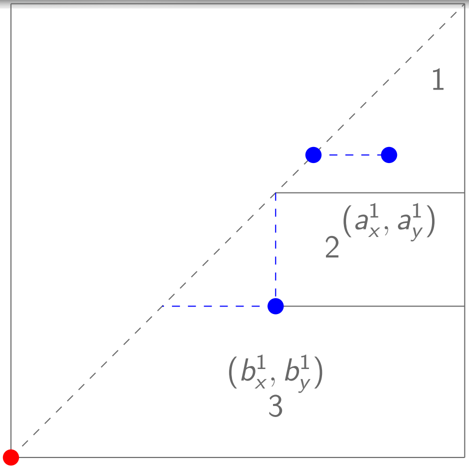
# The Decreasing Values Case

# The Decreasing Values Case

# The Decreasing Values Case

# Sketch of Increasing Case

Price of Y good




Price of X good

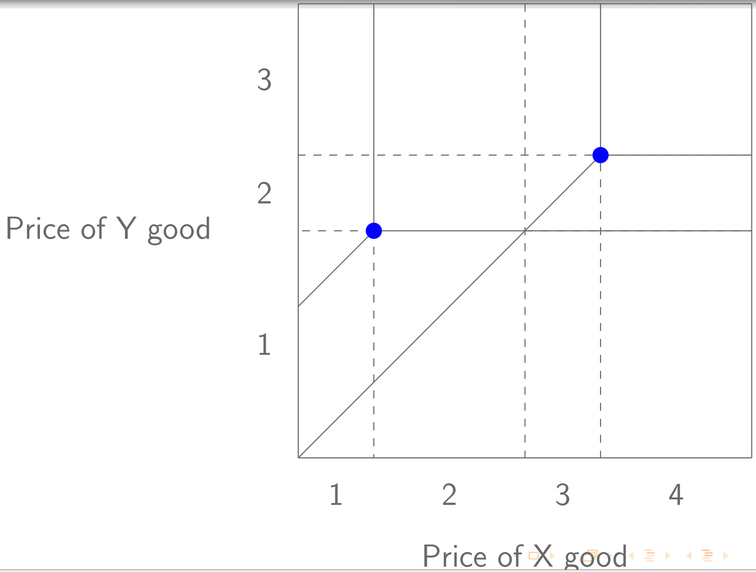
# Sketch of Increasing Case

$$Utility = \int_{c_x^1}^{1}\int_{y}^{1}(a_y^1 - c_x^1)f(x)f(y)dxdy \tag{1}$$

$$+ \int_{c_y^1}^{c_x^1}\int_{y}^{1}(a_x^1 + b_x^1 - 2c_y^1)f(x)f(y)dxdy \tag{2}$$

$$+ \int_{0}^{c_y^1}\int_{y}^{1}(a_x^1 + c_x^1 - y)dxdy \tag{3}$$

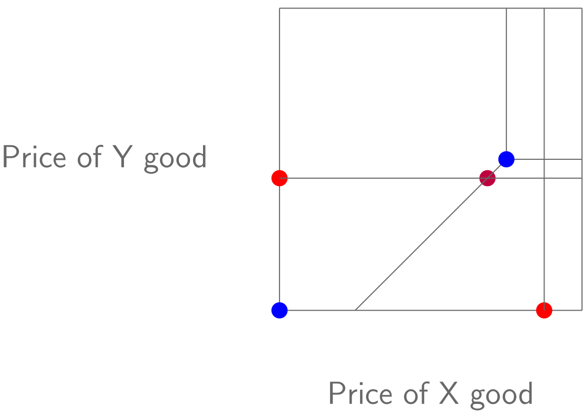
# Sketch of Increasing Case



# Sketch of Increasing Case

$$
\begin{aligned}
Utility = & \int_0^{c_y^1} \int_0^{c_x^1 - c_y^1 + y} (b_y^1 - y + a_x^1 - c_x^1 + c_y^1 - y) f(x) f(y) dxdy \\
& + \int_0^{c_y^1} \int_{c_x^1 - c_y^1 + y}^{c_x^1} (b_y^1 - y + a_x^1 - c_x^1) f(x) f(y) dxdy \\
& + \left[ (a_x^1 - c_x^1)(1 - F(c_y^1) F(c_x^1) \right] \\
& + \left[ (a_y^1 - c_y^1)(1 - F(c_x^1)) F(c_y^1) \right] \\
& + \left[ (a_y^1 - c_y^1)(F(a_x^1 - a_y^1 + c_y^1) - F(c_x^1))(1 - F(c_y^1)) \right] \\
& + \left[ \int_{a_x^1 - a_y^1 + c_y^1}^{a_x^1} (a_x^1 - x) f(x) dx (1 - F(a_x^1)) \right] \\
& + \left[ \int_{c_y^1}^{a_y^1} (a_y^1 - y) f(y) dy (1 - F(a_x^1)) \right] \\
& + \left[ \int_{c_y^1}^{a_y^1} \int_{a_x^1 - a_y^1 + c_y^1}^{a_x^1 - a_y^1 + y} (a_x^1 - x) f(x) f(y) dxdy + \int_{c_y^1}^{a_y^1} \int_{a_x^1 - a_y^1 + y}^{a_y^1} (a_y^1 - y) f(x) f(y) dxdy \right]
\end{aligned}
$$

# Observation



# Future Directions

- Finding a Nash equilibrium for n bidders and k units.
- Proving the equilibrium shown in the 2 units and 2 players case is unique.
- Classifying all distributions under which the results hold

# Acknowledgments


- Matthew Calvin
- Elizabeth Baldwin
- Sam Payne
- Yale University

# Questions ?