ALGEBRAIC GRAPH THEORY, STRONGLY REGULAR GRAPHS, AND CONWAY'S 99 PROBLEM

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Contents

1. Introduction

Strongly regular graphs have long been one of the core topics of interest in algebraic graph theory. A k-regular graph of order n is strongly regular with parameters (n, k, λ, μ) if every pair of adjacent vertices has exactly λ common neighbors and every pair of non-adjacent vertices has exactly μ common neighbors. As explained in [16], the theory of strongly regular graphs was originally introduced by Bose [6] in 1963 in relation to partial geometries and 2-class association schemes. This theory has been expanded over the years with many connections to various incidence structures and designs. The graphs also often give rise to interesting and large automorphism groups and have various spectral properties that we will describe in detail later on.

One problem of recurring interest is to prove the existence or non-existence of strongly regular graphs with given parameters. While there are ways to easily rule out some parameters, it is a highly non-trivial problem to prove whether some sets of parameters are realizable. Brouwer maintains a list of the existence and non-existence of small strongly regular graphs with feasible parameters at [7]. There are still nine feasible parameters for strongly regular graphs on less than 100 vertices for which the existence of the graph is unknown. Of these, maybe the most interesting one is (99,14,1,2) since it is the simplest to explain. Conway [9] has offered \$1,000 for a proof of the existence or non-existence of the graph. He states the problem very elegantly as:

Is there a graph with 99 vertices in which every edge (i.e. pair of joined vertices) belongs to a unique triangle and every nonedge (pair of unjoined vertices) to a unique quadrilateral?

Since solving this problem by brute force is beyond current computational capabilities, a solution will likely require novel methods.

The goal of this essay is to present the techniques that have been used to prove the existence or non-existence of various strongly regular graphs and to show why these methods have not worked for the potential graph with parameters (99,14,1,2). Section 2 presents preliminary definitions and background necessary to understand the problem. Section 3 goes through the classical ways to determine which parameters are feasible, some combinatorial constructions of graphs that do exist, and the techniques used in some recent papers to prove the non-existence of some strongly regular graphs. Section 4 presents some other possible techniques (largely computational) that have not yet yielded any results and section 5 concludes.

Acknowledgements. First, I would like to thank Prof. Pat Devlin for suggesting this beautiful and tantalizing piece of mathematics as a topic for this paper and advising me throughout the completion of this project. While the problem has often been frustrating as a solution seems so attainable and yet so out of reach, Pat has provided useful guidance and intuition along the way. Moreover, I would like to thank all of the math teachers I have had at Yale. Their teaching has given me the tools to be able to understand the literature required for this topic and prepared me to not only complete my degree, but to leave Yale ready to begin conducting my own research.

2. Preliminaries

Here we will present the basic definitions and set up of the problem. This is not meant to be a complete introduction to algebraic graph theory by any means. Rather, we will focus on the definitions and basic theorems needed to understand the techniques that have been used to prove the existence or non-existence of certain strongly regular graphs (SRGs). For a more complete introduction, see [13, 8, 4, 10].

For notation, let $G = (V, E)$ be an undirected graph with vertices V and edges E. Let $n = |V|$. Take A to be the $n \times n$ adjacency matrix of G, where A_{ij} is 1 if there is an edge between vertices i and j and 0 otherwise. We will let $i \sim j$ denote that i and j are adjacent.

Definition 2.1. *The graph G on n vertices is strongly regular with parameters (n, k, λ, μ) (denoted srg(n, k, λ, μ)) if*

1. *G is k-regular, such that every vertex in V has k neighbors*
2. *Each pair of adjacent vertices has exactly λ common neighbors*
3. *Each pair of non-adjacent vertices has exactly μ common neighbors.*

One immediate consequence of the definition is that if $G = \mathrm{srg}(n, k, \lambda, \mu)$ then $\overline{G}$ (the complement of G where edges and non-edges are switched) is also strongly regular and $\overline{G} = \mathrm{srg}(n, n-k-1, n-2-2k-\mu, n-2k+\lambda)$.

Another note to make is that a SRG need not be connected. However, the disconnected (sometimes called imprimitive) SRGs all have the same structure. Let G be a disconnected

SRG. Then $\mu = 0$, since vertices from different components cannot share an edge or any common neighbors. So, each component must be complete since there cannot be any vertices whose minimum distance between each other is two. Thus, G must be the union of connected components on $k+1$ vertices, which is too restrictive to give us much interesting structure. So, from here on, we will generally assume that any SRG we define is connected. Now we can look at some examples of SRGs.

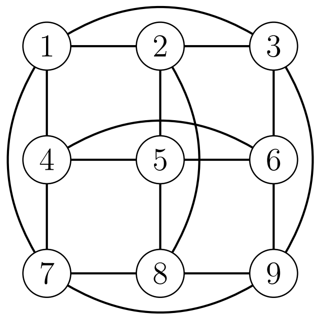

FIGURE 2.2
The strongly regular graph (9,4,1,2).

Example 2.3.

- *The* 5*-cycle* $C_5 = srg(5, 2, 0, 1)$*.*
- *The Petersen graph is* $srg(10, 3, 0, 1)$*.*
- *The graph shown in Figure 2.3 is* $srg(9, 4, 1, 2)$*. This graph is especially interesting to our case since it is the only non-trivial SRG with* $n < 99$ *and* $\lambda = 1, \mu = 2$*.*

Theorem 2.4. *If* $G = srg(n, k, \lambda, \mu)$ *then*

$$k(k - \lambda - 1) = (n - k - 1)\mu.$$

Proof. We fix a vertex u and count the number of edges between the neighbors and non-neighbors of u in two different ways. First, u has a set N of k neighbors where each $v \in N$ is adjacent to u, to λ neighbors of u, and thus to $k - \lambda - 1$ non-neighbors of u. Second, each of the $n - k - 1$ non-neighbors of u is adjacent to μ neighbors of u, giving us the result. □

Example 2.5. *Theorem 2.4 tells us that the graph defined by Conway is 14 regular. Conway's problem definition says that* $n = 99, \lambda = 1, \mu = 2$*, thus, we have* $k(k - 1 - 1) = (99 - k - 1)2$*. Solving yields* $k = 14$*.*

This theorem gives us a first condition for which sets of parameters are feasible. The next section will examine more advanced techniques for determining whether these feasible parameter sets correspond to realizable graphs.

3. Approaches to proving the (non-)existence of strongly regular graphs

3.1. **Eigenvalue multiplicities.** Letting A be the adjacency matrix of G, the i, j entry in the matrix A^2 is the number of paths of length two from vertex i to vertex j. In a SRG the number of paths of length two only depends on whether i and j are equal, adjacent, or non-adjacent. There are k paths if $i = j$, λ paths if $i \sim j$, and μ paths if $i \not\sim j$. So, letting J be the matrix of all 1's, we have the equation:

$$A^2 = kI + \lambda A + \mu(J - I - A)$$

This can be rewritten as the following Theorem:

Theorem 3.1. *If $G = srg(n, k, \lambda, \mu)$, then*

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

Now we can find the eigenvalues of A. First, since we know that G is k-regular, k is an eigenvalue with eigenvector $\mathbf{1}$. Then, A is a real symmetric matrix, so $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of A. Thus, any other eigenvector z with eigenvalue θ of A is orthogonal to $\mathbf{1}$. So,

$$\begin{aligned} A^2z + (\mu - \lambda)Az + (\mu - k)Iz &= \mu Jz = 0 \\ \theta^2 + (\mu - \lambda)\theta + (\mu - k) &= 0. \end{aligned}$$

Let $D = \sqrt{(\mu - \lambda)^2 - 4(\mu - k)}$, then solving the above quadratic equation yields two solutions:

$$\theta = \frac{\lambda - \mu + D}{2}, \qquad \tau = \frac{\lambda - \mu - D}{2}.$$

Next we find the multiplicities of these eigenvalues. Let m_θ and m_τ be the respective multiplicities. We get two linear equations in the multiplicities in terms of the parameters. First, the sum of the multiplicities of all three eigenvalues (including k) must be n since A is real and symmetric. Second, the sum of the eigenvalues with multiplicity must be 0, the trace of A. So,

$$m_\theta + m_\tau + 1 = n, \qquad m_\theta\theta + m_\tau\tau + k = 0.$$

Thus,

$$m_\theta = -\frac{(n-1)\tau + k}{\theta - \tau}, \qquad m_\tau = \frac{(n-1)\theta + k}{\theta - \tau}.$$

Note that $\theta - \tau = D$, so

$$m_\theta = \frac{1}{2}\left(n - 1 - \frac{2k + (n-1)(\lambda - \mu)}{D}\right), \qquad m_\tau = \frac{1}{2}\left(n - 1 + \frac{2k + (n-1)(\lambda - \mu)}{D}\right).$$

Now, since the multiplicities must be integers, we have another strong feasibility condition, namely that m_θ and m_τ must be integers for srg(n, k, λ, τ) to exist.

Theorem 3.2. *If $G = srg(n, k, \lambda, \mu)$ exists, then*

$$\frac{1}{2}\left(n - 1 + \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\mu - \lambda)^2 - 4(\mu - k)}}\right)$$

is an integer.

Example 3.3. *We will show that the (99,14,1,2) satisfy the multiplicity condition in Theorem 3.2. This is because $D = \sqrt{(2-1)^2 - 4(2-14)} = 7$, and thus $\theta = 3, \tau = -4, m_\theta = 54, m_\tau = 44$.*

As an application, we will show how this corollary drastically limits the possible parameters of Moore graphs of diameter 2, as in the classical paper by Hoffman and Singleton [15]. A Moore graph with diameter 2 is defined by having girth (shortest cycle) 5. This is equivalent to being a SRG with $\lambda = 0$ and $\mu = 1$. Also, note that Theorem 2.4 gives us $k^2 + 1 = n$. Now, this breaks into two cases. If $\theta, \tau \notin \mathbb{Z}$ then $m_\theta = m_\tau = \frac{n-1}{2}$ since the matrix A is rational. Using the trace being 0, and the fact that $\theta + \tau = \lambda - \mu = -1$, we get that

$$k + \frac{n-1}{2}(\theta + \tau) = k + \frac{k^2}{2} = 0$$

So that $k = 0, 2$. Here $k = 0$ gives G is a single node, which does not have diameter 2. And $k = 2$ gives us $G = \mathrm{srg}(5, 2, 0, 1)$, the 5-cycle. Now the second case is $\theta, \tau \in \mathbb{Z}$. Now note that Theorem 3.1 gives us $\theta^2 + \theta + 1 = k$, so that $\theta = \frac{-1+\sqrt{4k-3}}{2}, \tau = \frac{-1-\sqrt{4k+3}}{2}$. Since these are integers, there must be some $s \in \mathbb{Z}$ such that $s^2 = 4k - 3$. So, the trace formula gives us

$$k + m_\theta \frac{s-1}{2} + (n - 1 - m_\theta)\frac{-s-1}{2} = 0$$

Substituting $k = \frac{s^2+3}{4}$ and $n - 1 = k^2$, we get

$$s^5 + s^4 + 6s^3 - 2s^2 + (9 - 32m_\theta)s - 15 = 0$$

Since the s that solve this must be integers, we know that s divides 15. This gives us $s = 1, 3, 5, 15$ and thus $k = 1, 3, 7, 57$ and thus $n = 2, 10, 50, 3250$. Again the smallest graph does not have diameter 2. So, the only Moore graphs of diameter 2, ie $\mathrm{srg}(n, k, 0, 1)$ have $n = 5, 10, 50, 3250$. These graphs are the 5-cycle, the Petersen graph, the Hoffman-Singleton graph, and it is still an open problem whether the 3250-graph exists. This example demonstrates the usefulness of the multiplicity condition derived above.

3.2. **Krein bounds.** Another test of the realizability of a given set of parameters is called the Krein bounds. We refer the reader to [13, section 10.7] for a proof of this result. The central ideas of the proof are to find expressions for the eigenvalues of the adjacency matrix of the induced subgraph of the neighbors of a single vertex and using the formulas for the trace and the trace of the square in terms of the eigenvalues set up an application of Cauchy-Schwarz.

Theorem 3.4. *If $G = srg(n, k, \lambda, \mu)$ and A has eigenvalues k, θ, τ, then*

$$\theta^2\tau - 2\theta\tau^2 - \tau^2 - k\tau + k\theta^2 + 2k\theta \geq 0$$

and if the inequality is tight, $k \geq m_\tau$. Since we do not specify which non-k eigenvalue is θ and which is τ here, the inequality also holds interchanging the roles of θ and τ.

To demonstrate the utility of this bound we provide the following examples. One where the bound is useful and one where it is not.

Example 3.5. *Take the parameter set (28,9,0,4). This is feasible and solving for the eigenvalues yields 9 with multiplicity 1, 1 with multiplicity 21 and -5 with multiplicity 6. So, this passes the eigenvalue multiplicity test from the previous section. However, letting $\theta = 1, \tau = -5$, the Krein bound gives us*

$$(1)^2(-5) - 2(1)(-5)^2 - (-5)^2 - 9(-5) + 9(1)^2 + 2(9)(1) = -8$$

which violates the bound. Thus, there is no SRG with parameters (28,9,0,4).

Example 3.6. *An example where the Krein bound is less useful is for the parameters (99,14,1,2). Then, the bound is easily satisfied. Letting* $\theta = 3, \tau = -4$*, we get*

$$(3)^2(-4) - 2(3)(-4)^2 - (-4)^2 - 14(-4) + 14(3)^2 + 2(14)(3) = 118 > 0$$

and interchanging those roles we get

$$(-4)^2(3) - 2(-4)(3)^2 - (3)^2 - 14(3) + 14(-4)^2 + 2(14)(-4) = 181 > 0.$$

Thus, the Krein bounds are another very algebraic technique that eliminate the feasibility of certain sets of parameters.

3.3. **Combinatorial constructions.** The last two sections provided some classical ways of proving the non-existence of SRGs with certain parameters. This section aims to demonstrate some classical constructions of SRGs and classes of SRGs. This treatment is far from complete, but just meant to give a flavor for how some constructive arguments work. More information on constructions of various SRGs can be found in [13, 8, 16].

3.3.1. *Hoffman-Singleton graph.* In Section 3.1, we provided the Hoffman-Singleton graph as an example of a Moore graph with diameter 2 which is a SRG with parameters (50,7,0,1). A simple construction found in [8, section 9.1.7] is as follows. Take 25 vertices (i, j) and 25 vertices $(i, j)'$ with $i, j \in \mathbb{Z}/5\mathbb{Z}$. Now for each $i, j, k \in \mathbb{Z}/5\mathbb{Z}$ we connect: (i, j) to $(i, j + 1)$; $(i, j)'$ to $(i, j + 2)'$; and (i, j) to $(k, ik + j)'$. Thus, each $(i, \cdot)$ and $(i, \cdot)'$ is a 5-cycle and each (i, j) is connected to 5 $(x, y)'$ vertices. Moreover, each union of $(i, \cdot)$ and $(j, \cdot)'$ induces a Petersen graph as a subgraph of the Hoffman-Singleton graph.

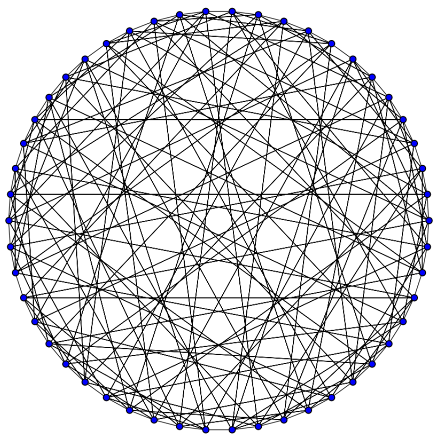

FIGURE 3.7
The Hoffman-Singleton graph.

3.3.2. *Payley graphs.* Let q be a prime power congruent to 1 mod 4. Then, the Payley graph, Payley(q) is the graph on the vertex set $\mathbb{F}_q$ where two vertices are adjacent when their difference is a non-zero square. This exists as an undirected graph since -1 is a square due to the congruence condition.

Proposition 3.8. *Payley(q) with* $q = 4t+1$ *is a SRG with parameters* $(4t+1, 2t, t-1, t)$.

Proof. Recall that of the non-zero elements in a finite field of order equivalent to 1 mod 4, half are quadratic residues and half are non-residues. Thus, it is clear that the Payley graphs are regular with $k = 2t$. Now let $\chi : \mathbb{F}_q \to \{-1, 0, 1\}$ be the quadratic residue character defined as $\chi(0) = 0, \chi(x) = 1$ if x is a non-zero square and -1 otherwise.

Now, because χ is 1 when there is an edge, we can sum over all of the common neighbors of two vertices x, y:

$$\begin{aligned}4 \sum_{z, x\sim z\sim y} 1 &= \sum_{z\neq x,y} (\chi(z-x)+1)(\chi(z-y)+1) \\ &= \sum_{z\neq x,y} \chi(z-x)\chi(z-y) + \chi(z-x) + \chi(z-y) + 1 \\ &= (q-2) - 2\chi(x-y) + \sum_{z\neq x,y} \chi(z-x)\chi(z-y)\end{aligned}$$

where the last line comes from the fact that $\sum_z \chi(z) = 0$. To finish it off, note that $\sum_{z\neq x,y} \chi(z-x)\chi(z-y) = \sum_{z\neq x,y} \chi((z-x)/(z-y)) = \sum_{w\neq 0,1} \chi(w) = -1$. So, we get that

$$\begin{aligned}4\lambda &= 4t+1-2-2(1)-1 = 4t-4 \\ 4\mu &= 4t+1-2-2(-1)-1 = 4t.\end{aligned}$$

□

The Payley graphs give us one of the simplest examples of an infinite family of SRGs. The construction also illustrates how ideas from number theory can be used to prove the existence of combinatorial structures.

3.3.3. *Quasi-symmetric designs.* A t-(v, j, l) design is a set of v points along with a collection $\mathcal{B}$ of j-element subsets called blocks such that every set of t points lies in exactly l blocks. A quasi-symmetric 2-design sets $t = 2$ and is defined by the intersection numbers (ℓ_1, ℓ_2) such that each pair of distinct blocks have exactly ℓ_1 or ℓ_2 points in common. Now we will show that any quasi-symmetric 2-design defines a strongly regular graph.

Proposition 3.9. *Let (D) be a quasisymmetric 2-(v, j, l) design with n blocks and intersection numbers (ℓ_1, ℓ_2). If the graph G with a vertex for each block and an edge when two blocks have ℓ_1 points in common is connected, then it is a SRG.*

Proof. Let $r = nj/v$ be the number of blocks each point sits in. Let N be the $v \times n$ incidence matrix of $\mathcal{D}$ such that N_{ij} is 1 when point i is in block j and 0 otherwise. Then, we have that

$$NN^T = (r-l)I + lJ, \qquad NJ = rJ, \qquad N^TJ = jJ.$$

Note that each of these is a linear combination of I and J. Then, we also have that

$$N^TN = jI + \ell_1 A + \ell_2(J - I - A)$$

Solving for A and squaring both sides, we get that A^2 can be written as a linear combination of A, I, J, NN^T, NJ, N^TJ, which reduces to a linear combination of A, I, J making G a SRG. □

Creating such designs is sometimes easier than constructing graphs directly. For example, in [13, Section 10.11] the Witt design on 23 points is used to define a SRG with paramters (253, 112, 36, 60).

3.4. **Star complements.** Just this year, it was shown in [2] that there does not exist a SRG with parameters (75,32,10,16). This proof relies on the technique of star complements developed by Cvetkovic, Rowlinson, and Simic in the 1990's and explained in [10, Chapter 7] as well as some computational brute force and clever algorithms to find the maximal clique. Here we will describe the way these methods connect to the study of SRGs and briefly explain why these techniques were fruitful for (75,32,10,16) but would be less effective for (99,14,1,2).

Definition 3.10. *A star complement of a graph G and eigenvalue s is an induced subgraph $H \subseteq G$ on $n - m_s$ vertices such that s is not an eigenvalue of A_H where m_s is the multiplicity of s and A_H is the adjacency matrix of H.*

Lemma 3.11. *If G is a graph with s an eigenvalue of its adjacency matrix A then there exists a star complement of G for s.*

Proof. Let G have q eigenvalues σ_j with multiplicities m_j for $j \in \{1, \dots, q\}$ and let $\sigma_1 = s$. Let E_j be the eigenspace associated with the jth eigenvalue and P_j the orthogonal projection onto that space. Let $\{x_i\}_{i=1}^n$ be an orthonormal basis for $\mathbb{R}^n$ of eigenvectors of A and $\{e_i\}_{i=1}^n$ be the standard basis. Let R_1 be an m_1 element subset of $\{1, \dots, n\}$ such that $\{x_i\}_{i \in R_1}$ spans E_1.

Now take T to be the transition matrix from the x_i basis to the e_i basis so that $e_i = \sum_{k=1}^n T_{ki} x_k$. Projecting orthogonally yields $P_1 e_i = \sum_{k \in R_1} T_{ki} x_k$. Now, for any other m_1 element subset of $\{1, \dots, n\}$ we define T_1 to be the $m_1 \times m_1$ submatrix of T whose rows are indexed by R_1 and columns by C_1 and T_1' be the $(n - m_1) \times (n - m_1)$ submatrix indexed by their complements. Then, note that $\det T = \sum_{C_1} \pm \det T_1 \det T_1' \neq 0$ since T is invertible. So, there exists C_1 for which $\det T_1 \neq 0$. This T_1 is invertible so it defines a bijection between $\langle P_1 e_j : j \in C_1 \rangle$ and E_1.

Now define $V_1 = \langle e_j : j \notin C_1 \rangle$. I claim that $\mathbb{R}^n = E_1 \oplus V_1$. Because E_1 is m_1 dimensional and V_1 is $n - m_1$ dimensional, it suffices to show that their intersection is 0. So, take $x \in E_1 \cap V_1$. Then $x = P_1 x$ and $x^T e_i = 0$ for $i \in C_1$. So, $x^T(P_1 e_i) = x^T(P^T e_i) = (P_i x)^T e_i = x^T e_i = 0$. This means that $x \in \langle P_1 e_i : i \in C_1 \rangle^\perp = E_1^\perp$. But, since $x \in E_1$, we have $x = 0$, proving the claim.

To finish the proof, define $H = G - C_1$, the subgraph induced on vertices not in C_1. Let A_H be its adjacency matrix and assume by contradiction that $A_H x = \sigma_1 x$. Now define $x' = \begin{pmatrix} 0 \\ x \end{pmatrix} \in V_1$ and then $Ax' = \begin{pmatrix} \cdot & \cdot \\ \cdot & A_H \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} \cdot \\ \sigma_1 x \end{pmatrix}$, where the dots are placeholders for values that will not matter. Now we want to prove that x must be 0. So, take $y' \in V_1$ then

$$y'^T A x' = \begin{pmatrix} 0 & y \end{pmatrix} \begin{pmatrix} \cdot \\ \sigma_1 x \end{pmatrix} = \sigma_1 y'^T x'$$

So, $(A - \sigma_1 I)x' \in V_1^\perp$. However, if we take $y' \in E_1$, then $y'^T A x' = (Ay')^T x' = \sigma_1 y'^T x'$ so $(A - \sigma_1 I)x' \in E_1^\perp$. Thus by the previous paragraph $(A - \sigma_1 I)x' = 0$ so that $x' \in E_1$, but we started with $x' \in V_1$. So, again by the previous paragraph, we have that $x' = 0$ so that $x = 0$, and thus we have proven that H is indeed a star complement for G and $s = \sigma_1$. □

Finding a star complement can be easier than finding the whole graph since one eigenvalue may have relatively low multiplicity. Moreover, we know that for any subgraph we find with s not as an eigenvalue, it is contained in a star complement, which is stated below.

Lemma 3.12. *Let G be a graph with eigenvalue s and H' be an induced subgraph without s as an eigenvalue, then there exists a star complement H of G for s that contains H' as an induced subgraph.*

The proof can be found in [17], but follows from similar techniques to those used above. Thus, star complements can be constructed in a greedy manner if one is able to combinatorially prove the existence of a relatively large /induced subgraph without s as an eigenvalue.

In practice, constructing these star complements is not enough to complete the proof that there is no SRG with parameters (75,32,10,16). Instead, one must use each of many candidate star complements to reconstruct a full graph. To do this, we need the following theorem from [10, Theorem 7.4.1].

Theorem 3.13. *Let H be the star complement of G for eigenvalue s. Define K to be the subgraph of G induced on the vertices of $G - H$. Then G has adjacency matrix of the form:*

$$A = \begin{pmatrix} A_K & B^T \\ B & A_H \end{pmatrix}$$

and moreover

$$sI - A_K = B^T(sI - A_H)^{-1}B.$$

So, knowing the star complement and the matrix B we can reconstruct the entire graph. However, in our scenario we may not have all of B, so instead we introduce the notion of the compatibility graph.

Definition 3.14. *Let H be a star complement to G for eigenvalue s. Define an inner product:*

$$\langle u, v\rangle = u^T(sI - A_H)^{-1}v.$$

Now the compatibility graph of H and s, denoted $Comp(H, s)$ is the graph with vertices:

$$V(Comp(H, s)) = \{u \in \{0, 1\}^{n-m_s} : \langle u, u\rangle = s, \langle u, \mathbf{1}\rangle = -1\}$$

and adjacency

$$u \sim v \iff \langle u, v\rangle \in \{-1, 0\}.$$

This compatibility graph basically is using the above reconstruction theorem to find candidates for the matrix B. The following proposition from [2, Proposition 3] directly states when the compatibility graph tells us that G is strongly regular.

Proposition 3.15. *If G is strongly regular and H is a star complement for G for eigenvalue s with mutltiplicity m_s, then $Comp(H, s)$ has an m_s-clique.*

Thus, we can now recap the technique of [2]. Since each graph must have a star complement, choose the eigenvalue s with the largest multiplicity m_s and attempt to prove the existence of an subgraph H on $n - m_s$ vertices. Since finding such a graph is difficult note that if there exists some H' on less vertices without s as an eigenvalue, there must exist some H containing H', so we can build the subgraph H is a somewhat greedy manor. This step may yield many candidate subgraphs H which can be reduced by the technique in the next section. Then, for each of the candidate subgraphs, we can attempt to reconstruct the graph by building the compatibility graph and searching for its largest clique. This last step is computationally intensive. Here we have skipped the many combinatorial arguments need to prove the result that the (75,32,10,16) graph does not exist and instead focused on the techniques more generally.

The reason that this technique does not translate well to the (99,14,1,2) graph is that the eigenvalue of highest multiplicity is 3 with multiplicity 54 (see Example 3.3). This means that a star complement must have 45 vertices, which is still likely too large to derive combinatorially or with brute force search.

3.4.1. *Interlacing eigenvalues.* Another interesting technique used in [2] that deserves being mentioned here is that of interlacing eigenvalues. Interlacing eigenvalues is one of the classic ideas in algebraic graph theory and [2] uses it to eliminate many candidate star complement graphs to reduce their search space. Below, we will outline the basic ideas of interlacing eigenvalues and how they help in this case.

Definition 3.16. *Let $n > m$. Take $s_1 \geq \cdots \geq s_n$ and $r_1 \geq \cdots \geq r_m$, two sequences of real numbers. Then $\{r_i\}_{i=1}^m$ interlaces $\{s_i\}_{i=1}^n$ if*

$$s_i \geq r_i \geq s_{n-m+i} \qquad \text{for} \quad i \in \{1, \ldots, m\}.$$

The classical result from algebraic graph theory found in [14] says that the eigenvalues of an induced subgraph interlace the eigenvalues of the graph.

Theorem 3.17. *If H is an induced subgraph of G then the eigenvalues of H interlace the eigenvalues of G.*

Proof. Let $s_1, \ldots, s_n$ be the eigenvalues of A and $r_1, \ldots, r_m$ be the eigenvalues of A_H. Without loss of generality, assume that $A = \begin{pmatrix} A_H & B^T \\ B & C \end{pmatrix}$. Let $\{x_1, \ldots, x_n\}$ be the orthonormal eigenvectors of A and $\{y_1, \cdots, y_m\}$ be the orthonormal eigenvectors of A_H. For any index i, define vector spaces $V = \langle\{x_i, \ldots, x_n\}\rangle$, $W = \langle\{y_1, \ldots, y_i\}\rangle$ and $\widetilde{W} = \left\{\begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathbb{R}^n : w \in W\right\}$. So, $\dim V = n - i + 1$ and $\dim \widetilde{W} = \dim W = i$. Thus, there exists nonzero $\widetilde{w} \in V \cap \widetilde{W}$ and

$$\widetilde{w}^T A \widetilde{w} = \begin{pmatrix} w & 0 \end{pmatrix} \begin{pmatrix} A_H & B^T \\ B & C \end{pmatrix} \begin{pmatrix} w \\ 0 \end{pmatrix} = w^T A_H w.$$

Then by Rayleigh's principle for eigenvalues we have that

$$s_i \geq \frac{\widetilde{w}^T A \widetilde{w}}{\widetilde{w}^T \widetilde{w}} = \frac{w^T A_H w}{w^T w} \geq r_i.$$

The other inequality follows by redefining $V = \langle\{x_1, \ldots, x_{n-m+i}\}\rangle$ and $W = \langle\{y_i, \ldots, y_m\}\rangle$. $\square$

There is also another version of interlacing that uses a matrix defined by a partition of the vertices of A rather than the adjacency matrix.

Definition 3.18. *Let $\mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_k)$ be a partition of the vertices of G. Let $e(\mathcal{V}_i, \mathcal{V}_j)$ be the number of edges between partitions i, j if $i \neq j$ and the number of edges in the* *induced graph on partition i if $i = j$. Then, define the partitioned adjacency matrix $A_{\mathcal{V}}$ to be the $k \times k$ matrix such that*

$$(A_{\mathcal{V}})_{ij} = \begin{cases} \frac{e(\mathcal{V}_i,\mathcal{V}_j)}{|\mathcal{V}_i|} & if \quad i \neq j \\ \frac{2e(\mathcal{V}_i,\mathcal{V}_i)}{|\mathcal{V}_i|} & if \quad i = j. \end{cases}$$

Theorem 3.19. *If $\mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_k)$ is a partition of the vertices of G, then the eigenvalues of $A_{\mathcal{V}}$ interlace the eigenvalues of G.*

This theorem follows from the above and a more detailed proof can be found in [14]. In practice, the partitioned version was found to do a better job of eliminating candidate star complements of G for the parameters (75, 32,10,16) where the partition was defined as one partition for each vertex of the candidate star complement and one partition for the remainder of the graph.

3.5. **Euclidean representation.** Another technique that has been used this year in [5] to disprove the existence of a SRG with parameters (76,30,8,14) is to use a Euclidean representation for the graph and apply some techniques from analysis before again using combinatorial arguments and computer search to conclude a proof. This representation is closely related to the ideas of spherical designs introduced in [11] and is also presented in [4, Chapter 8].

To conduct the Euclidean representation of a SRG, take G to be strongly regular with parameters (n, k, λ, μ). Then, the adjacency matrix A has 3 eigenvalues: k with multiplicity 1, $\theta \geq 0$ with multiplicity m_θ, and $\tau \leq 0$ with multiplicity m_τ. The goal will be to construct a representation of G in $\mathbb{R}^{m_\tau}$.

Let $\{y_i\}_{i=1}^n$ be the column vectors from the matrix $A - \theta I$. Define z_i and x_i as:

$$z_i = y_i - \frac{1}{n}\sum_{j=1}^{n} y_j \qquad x_i = \frac{z_i}{\|z_i\|}.$$

So, $\text{rank}(\langle x_1, \ldots x_n \rangle) = m_\tau$ since that is the rank of $A - \theta I$. Thus, our representation can be viewed as elements of $\mathbb{R}^{m_\tau}$. And, the values of dot products of the x_i are determined by adjacency in G:

$$x_i^T x_j = \begin{cases} 1, & i = j \\ p = \frac{\tau}{k}, & i \sim j \\ q = -\frac{\tau+1}{n-k-1}, & otherwise. \end{cases}$$

The proofs of the values of p, q follow from some algebraic manipulations. But, it is easier to see that such values must exist when we note that $A = kP_k + \tau P_\tau + \theta P_\theta$, where the P are orthogonal projections onto the appropriate eigenspaces. Then our x_i are like projections of the standard basis vectors onto one of the non-trivial eigenspaces.

In addition to this property of being a *two-distance* set (since the dot products assume two values), the x_i also form a *spherical 2-design*. This means that

$$\sum_{i=1}^{n} x_i = 0 \qquad and \qquad \sum_{i=1}^{n} (x_i^T y)^2 = \frac{n}{m_\tau} \quad \forall y : \|y\| = 1.$$

For the (76,30,8,14) graph considered in [5], this makes $(p, q) = (\frac{-4}{15}, \frac{7}{15})$. And for the (99,14,1,2) graph we could construct a projection onto $\mathbb{R}^{44}$ with $(p, q) = (\frac{-4}{15}, \frac{7}{15})$. Also, note that we can also construct a projection onto $\mathbb{R}^{m_\theta}$ by considering the complement of G, which has the eigenvalue multiplicities exchanged.

Proposition 3.20. *Let G be a SRG with eigenvalues k, θ, τ with multiplicities $1 m_\theta, m_\tau$, then*

$$n \leq \binom{m_\theta + 2}{2} - 1, \qquad n \leq \binom{m_\tau + 2}{2} - 1.$$

This proposition is actually the result of a much stronger result from [11] about maximal two-distance sets of points on the unit sphere. This result is sometimes referred to as the absolute bound on the multiplicities of a SRG. The paper about the (76,30,8,14) graph also uses the following proposition.

Proposition 3.21. *Let G be a SRG and $\{x_i\}_{i=1}^n$ the euclidean representation in $\mathbb{R}^{m_\tau}$. For any subset $U \subseteq V$ the Gram matrix $(x_i^T x_j)_{i,j\in U}$ is positive semidefinite with rank equal to rank$(\langle x_i : i \in U\rangle)$. Moreover, if A_U is the adjacency matrix of the subgraph induced on U, then*

$$(x_i^T x_j)_{i,j\in U} = pA + I + q(J - I - A).$$

The proof in [5] uses the above ideas about euclidean representations to prove that there must exist a 4-clique in the (76,30,8,14) graph. This proof relies on ideas from analysis about spherical harmonic polynomials acting on the x_i. Armed with a bound on the umber of 4-cliques the authors can reduce the existence of the graph to 4 cases and check each of them with some combination of combinatorial arguments and brute force computation. There is no clear way to use arguments like these for the (99,14,1,2) graph since the clique number is 3, but the idea of euclidean representations is still an interesting one that can create connections between seemingly disparate branches of mathematics.

3.6. **Automorphism groups.** Many strongly regular graphs have interesting automorphism groups and learning about automorphisms can also shed light on the existence of graphs with certain parameters. Recent work in [3] has proven various results of which primes divide the automorphism groups of SRGs which are unknown to exist. This includes a result that for the (99,14,1,2) graph, should it exist, if p is a prime such that G has an order p automorphism, then p is 2 or 3.

The main idea used to prove this result is to define orbit matrices and make note of some conditions that these matrices must satisfy. Then, the authors of [3] run an exhaustive computer search that proves that automorphisms cannot exist for other values of p because the associated orbit matrices cannot exist. The results do not disprove automorphisms of some small prime orders for largely computational reasons as the method is not necessarily so efficient.

3.6.1. *Orbit matrices.* First, we will define the necessary notation to describe the technique. Assume that G is a SRG with parameters (n, k, λ, μ) that has an automorphism of order p for p prime. Assume that the automorphism defines b orbits $O_1, \dots, O_b$ of sizes $n_1, \dots, n_b$ where each n_i must be either 1 or p. The orbits of order 1 are called fixed points. The adjacency matrix A of G can be subdivided into matrices $[A_{ij}]$ representing the adjacency between orbits O_i, O_j. So, we define $b \times b$ matrices C, R, N such that:

$$C_{ij} = \text{ column sum of } [A_{ij}] \qquad R_{ij} = \text{ row sum of } [A_{ij}] \qquad N = \mathrm{diag}(n_1, \dots, n_b).$$

This gives us the relationships $r_{ij}n_i = c_{ij}n_j$ and $C = R^T$, and we will refer to C as the orbit matrix. Rows and columns of C corresponding to orbits of size 1 are called fixed and the other rows and columns are called non-fixed.

Now we can subdivide A^2 similarly into $[A^2_{ij}]$ to divide the squared adjacency matrix by orbits O_i, O_j. Then, we define the $b \times b$ matrix S such that

$$S_{ij} = \text{ sum of all entries in } [A^2_{ij}].$$

By Theorem 3.1 we get that

$$\begin{aligned} A^2 &= (\lambda - \mu)A + \mu J + (k - \mu)I \\ S_{ij} &= (\lambda - \mu)C_{ij}n_j + \mu n_i n_j + \mathbb{1}_{i=j}(k - \mu)n_j. \end{aligned}$$

We also can show that

$$S = CNR. \tag{1}$$

To see this let α_i be the n dimensional vector that is 1 at the vertices in O_i and 0 elsewhere. Then $S_{ij} = \alpha_i^T A^2 \alpha_j = (\alpha_i^T A)(A\alpha_j)$, another expression for CNR. And then equation (1) gives us equations for the possible entries in row r of the matrix C. For the diagonal element:

$$(k - \mu)n_r + \mu n_r^2 + (\lambda - \mu)c_{rr}n_r = S_{rr} = \sum_{i=1}^{b} C_{ri}R_{ir}n_i = \sum_{i=1}^{n} C_{ri}^2 n_i. \tag{2}$$

Now, we let ϕ be the number of fixed points under our automorphism and ψ be the number of orbits of size p such that

$$\phi = n - p\psi. \tag{3}$$

The approach is now to construct all possible orbit matrices for each feasible ϕ and then attempt to expand the orbit matrices into SRGs. This strategy allows us to restrict the possibilities for entries in row r. First, we disregard the order of the element and define a *prototype* for a row as the distribution of integer values in the row. The rows now break into two cases.

Case 1: r is a fixed row of C, so that $n_r = 1$. Then every entry in the row is either 0 or 1. Let x_0, x_1 be the numbers of 0's and 1's in the fixed-column portion of the row and y_0, y_1 be the numbers of 0's and 1's in the non-fixed-column portion of the row. Then

$$x_0 + x_1 = \phi, \qquad y_0 + y_1 = \psi, \qquad x_1 + py_1 = k.$$

Case 2: r is a non-fixed row of C, so that $n_r = p$. Now, in the fixed-column portion of the row we find either 0's or p's and in the non-fixed-column portion we find integers between 0 and p. So, we define x_0, x_p to be the numbers of 0's and p's in the fixed-column portion of row r and $y_0, \ldots, y_p$ to be the numbers of i's in the non-fixed-column portion of row r for each integer i between 0 and p. Then, applying equation (2) we get the following system:

$$\begin{aligned} x_0 + x_p &= \phi \\ y_0 + y_1 + y_2 + \cdots + y_p &= \psi \\ x_p + y_1 + 2y_2 + 3y_3 + \cdots + py_p &= k \\ px_p + y_1 + 4y_2 + 9y_3 + \cdots + p^2 y_p &= S_{rr}/p. \end{aligned}$$

Now, C can be constructed row by row as follows. First, we find the possible prototypes for row r for both the fixed and non-fixed column portions of the row, if there are none then this candidate cannot be an orbit matrix. Then, as each row is added, there are conditions on its inner product with all the other rows based on equation (1) and the above result of Theorem 3.1. Isomorphic orbit matrices are also eliminated along the way to reduce computation.

An important result to improve computational efficiency is that by equation (3) we know that ϕ is equivalent to n mod p. So, the minimal value for ϕ denoted z is the remainder of n/p. Then $\phi = z, z+p, z+2p, \ldots$. However, if there is a prototype for $\phi \geq 2p$ there exists a prototype for $\phi - p$ and thus we only need to check z and $z+p$ each time. The proof can be found in [3, Theorem 2.2] and is based on manipulating the equations derived above for cases 1 and 2.

3.6.2. *Bounds on fixed points.* Some fairly simple bounds on ϕ also eliminate many possible parameters from the brute force search described above. For the rest of this section we will assume that G is a SRG with parameters (n, k, λ, μ) and eigenvalues $k > \theta > \tau$.

Lemma 3.22. *Let H be the subgraph induced of G by the non-fixed vertices of a non-trivial automorphism ρ. Define $\delta(H)$ to behte minimum degree of vertices in H. Then*

$$\delta(H) \geq k - \max(\lambda, \mu).$$

Proof. Let x be a non-fixed vertex of G and $y = \rho(x)$. A fixed vertex adjacent to x must be adjacent to y. Since the common neighbors of x, y is bounded by $\max(\lambda, \mu)$, that value bounds the number of fixed vertices adjacent to x. Since G is regular of degree k, x has k neighbors so that every non-fixed vertex of G has at least $k - \max(\lambda, \mu)$ non-fixed neighbors. □

Theorem 3.23. *The number of fixed points is bounded by:*

$$\phi \leq \frac{\max(\lambda, \mu)}{k - \theta} n$$

Proof. Let $m = n - \phi$ be the number of non-fixed vertices in G and A' be the $m \times m$ adjacency matrix of the subgraph induced by the non-fixed vertices. Then, let $E_\tau = \frac{1}{\tau - \theta}(A - \theta I + \frac{\theta - k}{n} J)$ be the positive semidefinite idempotent associated with the eigenvalue τ. Then E'_τ, the principal submatrix of E_τ associated with the non-fixed vertices is also positive semidefinite. Let $\alpha = \max(\lambda, \tau)$, then applying the lemma, we get that

$$(\tau - \theta) \sum_{i,j}^{m} E'_{ij} = \sum_{i,j}^{m} A'_{ij} - \theta m + \frac{\theta - k}{n} m^2 \geq (k - \alpha)m - \theta m + \frac{\theta - k}{n} m^2.$$

Then. since $\tau - \theta < 0$ and the sum of elements of a positive semidefinite matrix is non-negative and $m > 0$:

$$\frac{1}{\tau - \theta}\left((k - \alpha)m - \theta m + \frac{\theta - k}{n} m^2\right) \geq \sum_{i,j}^{m} E'_{ij} \geq 0$$

$$-\alpha + \frac{\theta - k}{n}(-\phi) = (k - \alpha) - \theta + \frac{\theta - k}{n} m \leq 0$$

$$\phi \leq \frac{\alpha n}{k - \theta}.$$

□

For the 99-graph, this theorem implies that $\phi \leq \frac{\max(1,2)}{14-3}99 = 18$. Such a result eliminates some computation of orbit matrices in the proof that a prime order automorphism must have order 2 or 3. We conclude this section with one more bound.

Proposition 3.24. *If $p > k$ and $\mu \neq 0$ then $\phi = 0$.*

Proof. If x is a fixed vertex that is adjacent to a non-fixed vertex y, then x must be adjacent to all other vertices in the orbit of y. So, if $p > k$, no fixed vertex is adjacent to a non-fixed vertex which would make the graph imprimitive, since the fixed and non-fixed vertices would form separate parts of the graph. But, this contradicts $\mu \neq 0$ unless there are no fixed vertices, ie $\phi = 0$. □

This result implies that if G is the (99,14,1,2) graph, then any prime order automorphism would have order at most 13 since primes larger than $k = 14$ do not divide 99. This result is strengthened by the computational results of [3] which prove that such a prime must be either 2 or 3. Since we might expect such a G to be fairly symmetric if it exists, this seems to suggest that it may be less likely for such a graph to exist. However, this is in no way a rigorous thought and the question remains open. Some more techniques to approach the question of whether such a graph exists are addressed in the next section.

4. Approaches to the 99 graph

As explained above, the traditional techniques do not seem to have much use for proving the existence or non-existence of the (99,14,1,2) strongly regular graph. However, there is structure that we can see in the graph, should it exist. For example, we can derive subconstituents of such a graph, assuming it exists, and use computational force to rule out certain "nice" structures that one may think the graph contains. This approach is described below.

4.1. **Construction from subconstituents.** In this section, assume that G is a SRG with parameters $(99, 14, 1, 2)$. First, we can show fairly quickly how the graph must look if we start at one vertex v. We know that v must have 14 neighbors, because the graph is 14 regular. Moreover, since $\lambda = 1$, we know that each edge must be a part of one triangle and each vertex is thus on 7 triangles. So, we can label the 14 neighbors of v as $\pm 1, \ldots, \pm 7$ where $i \sim -i$. Then, since $\mu = 2$, each of the 84 vertices remaining in the graph must be connected to two of the vertices $\pm 1, \cdots \pm 7$ and cannot be connected to i and $-i$ since that would violate $\lambda = 1$ for i and $-i$. Thus, for each $|i| \neq |j|$ there exist a vertex $a \sim i, j$. This construction is depicted below in Figure 4.1.

From this construction we can derive the first subconstituent of the adjacency matrix of G. By subconstituent, we mean that A can be divided as

$$A = \begin{pmatrix} 0 & \mathbf{1}^T & 0 \\ \mathbf{1} & A_1 & B^T \\ 0 & B & A_2 \end{pmatrix}.$$

Now we can use the subconstituents to express A^2 as:

$$A^2 = \begin{pmatrix} 14 & \mathbf{1}^T A_1 & \mathbf{1}^T B^T \\ A_1\mathbf{1} & J + A_1^2 + B^T B & A_1 B^T + B^T A_2 \\ B\mathbf{1} & BA_1 + A_2 B & A_2^2 + BB^T \end{pmatrix}$$

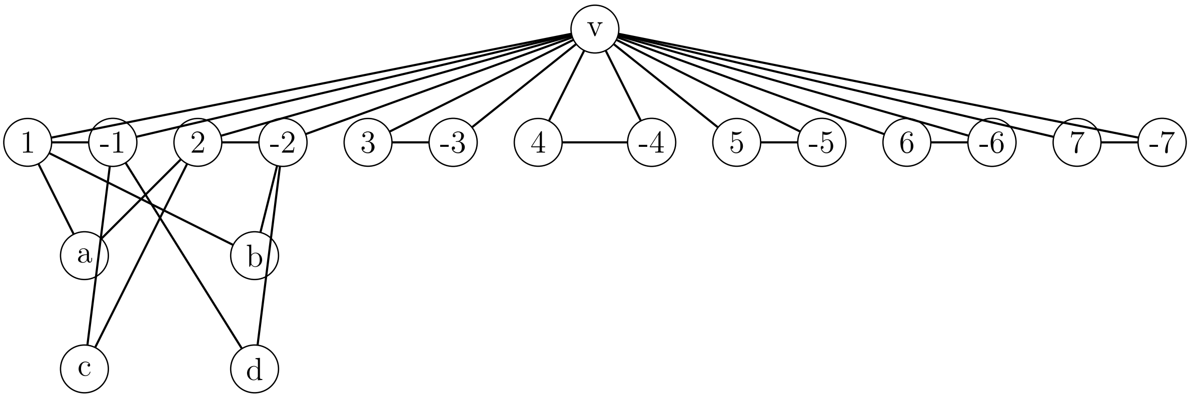

FIGURE 4.1
The subgraph induced by a subconstituent of (99,14,1,2) at a vertex v, with 4 vertices a, b, c, d drawn in to illustrate how the subconstituent connects to the other 84 vertices.

where J is the matrix of all ones of the appropriate dimensions. Thus, by Theorem 3.1 we can derive the following identities:

$$A_1^2 + A_1 - 12I + B^T B = J \tag{4}$$

$$A_2^2 + A_2 - 12I + BB^T = 2J \tag{5}$$

$$BA_1 + A_2 B + B = 2J \tag{6}$$

$$A_1 B^T + B^T A_2 + B^T = 2J. \tag{7}$$

These identities help us to set up a (potentially) simpler problem. Rather than looking for a graph on 99 vertices, we need only find the induced subgraph on 84 vertices. This would be a 12-regular graph with adjacency matrix A_2 that satisfies the above identities.

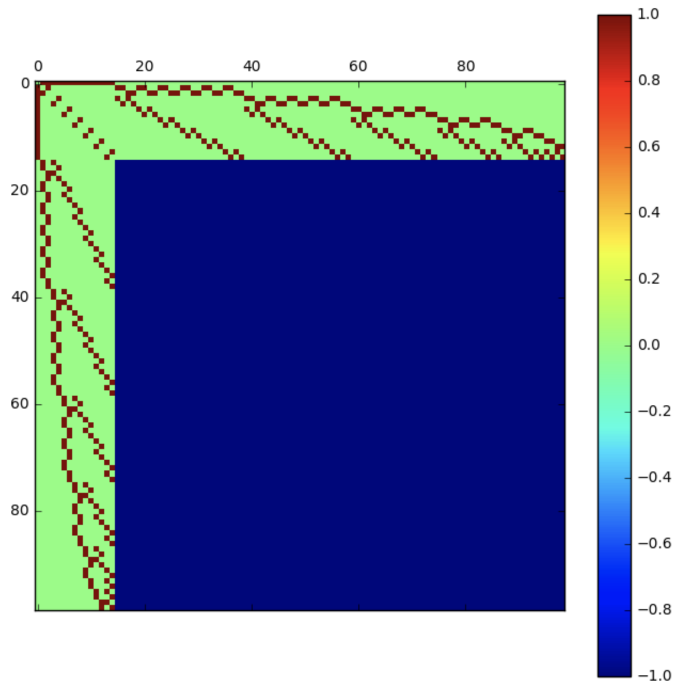

FIGURE 4.2
The adjacency matrix A with A_1, B filled in and A_2 set to $-J$.

4.1.1. *Local eigenvalues.* One way to understand A_2 is to understand its eigenvalues. We can actually deduce these values from the eigenvalues of A and the following lemmas based off of [13].

Definition 4.3. *We define a local eigenvalue of A_1 or A_2 to be an eigenvalue λ such that λ is not an eigenvalue of A and the eigenvector v_λ associated with lambda is orthogonal to $\mathbf{1}$.*

Lemma 4.4. *Let y be an eigenvector of A_2 with eigenvalue σ, such that $\mathbf{1}^T y = 0$. If $B^T y = 0$ then $\sigma \in \{-4, 3\}$.*

Proof. First, let $B^T y = 0$. Then using identity (3) and $\mathbf{1}^T y = 0$, we have

$$\begin{aligned}
A_2^2 + A_2 - 12I + BB^T &= 2J \\
A_2^2 y + A_2 y - 12Iy + BB^T y &= 2Jy \\
\sigma^2 y + \sigma y - 12y &= 0 \\
(\sigma - 3)(\sigma + 4)y &= 0
\end{aligned}$$

Thus, we have $\sigma \in \{-4, 3\}$.

□

Lemma 4.5. *The value σ is a local eigenvalue of A_2 if and only if $-1-\sigma$ is a local eigenvalue of A_1, with equal multiplicities.*

Proof. ($\Rightarrow$) Suppose σ is a local eigenvalue of A_2 with eigenvector y with $\mathbf{1}^T y = 0$. By (7) we have

$$\begin{aligned}
A_1 B^T + B^T A_2 + B^T &= 2J \\
A_1 B^T y + B^T A_2 y + B^T y &= 2Jy \\
A_1 B^T y + B^T \sigma y + B^T y &= 0 \\
A_1(B^T y) &= (-\sigma - 1)(B^T y)
\end{aligned}$$

Since σ is a local eigenvalue, it is not -4 or 3 so Lemma 0.2 tells us that $B^T y \neq 0$. Thus, $-\sigma - 1$ is an eigenvalue of A_1. Moreover, since $\mathbf{1}^T B = (12)\mathbf{1}$ we have that $\mathbf{1}^T By = 0$. And if $-\sigma - 1 = 3, -4$ then $\sigma = -4, 3$, which is not the case. Thus, $-\sigma - 1$ is a local eigenvalue of A_1.

($\Leftarrow$) The reverse direction follows a similar argument, but uses (6) instead of (7) and an analog to Lemma 4.4 that uses (4) instead of (5).

Lastly, to show that the multiplicities of σ and $-1-\sigma$ are the same, note that the mapping B from the $(-1-\sigma)$-eigenspace of A_1 to the σ-eigenspace of A_2 is injective. Similarly, B^T from the σ-eigenspace of A_2 to the $(-1-\sigma)$-eigenspace of A_1 is injective. Thus, the two spaces have the same dimension. □

Proposition 4.6. *The eigenspectrum of A_2 is $\{12^1, 3^{40}, 0^7, -2^6, -4^{30}\}$ where the exponents represent multiplicities.*

Proof. The eigenvalues of A_1 are 1 and -1 each with multiplicity 7 and with $\mathbf{1}$ being one of the eigenvectors associated with 1. Thus, the local eigenvalues of A_1 are 1 with multiplicity 6 and -1 with multiplicity 7. By Lemma 4.5 we have that -2 with multiplicity 6 and 0 with multiplicity 7 are eigenvalues of A_2. Additionally, since A_2 is an adjacency matrix for a 12-regular induced subgraph, 12 is an eigenvalue with multiplicity 1 and eigenvector **1**. So, the remaining 70 eigenvalues must be either 3 or -4. Since the trace of A_2 is zero, so is the sum of the eigenvalues. Let x be the multiplicity of 3 and y the multiplicity of -4, then we have the following equations:

$$12 + 6(-2) + 7(0) + 3x - 4y = 0, \qquad\qquad x + y = 70.$$

Solving these yields the multiplicities 40 and 30. □

Note that these eigenvalues interlace the eigenvalues of A, as would be expected. If you create a partition of the vertices as described in Section 3.4.1 where each vertex in A_1 and v defines a partition per vertex and the remaining 84 vertices define another partition, the eigenvalues also interlace (but with so few eigenvalues the interlacing is less interesting).

Unfortunately, it is not clear how to prove whether such an A_2 exists. Having an induced subgraph A_1 that doesn't share eigenvalues with A could help construct a star complement as described in section 3.4 since the A_1 graph must be a subgraph of a star complement. But, a star complement would need 45 vertices and our subgraph only has 15. One idea to gain some more traction is to attempt to assume some additional structure of A_2, this approach is addressed below.

4.1.2. *Squares assumption.* One assumption that could be logical is to create a square out of the vertices a, b, c, d in Figure 4.1. This assumption is related to transitivity or symmetry of the graph since under the assumption any of the points $\pm 1, \dots, \pm 7$ can be viewed as v and the same structure is induced (but there may be another way to do this without the squares, it is unclear). The assumption also means that the subgraph induced on $v, 1, -1, 2, -2, a, b, c, d$ is a srg$(9, 4, 1, 2)$. Under this assumption it is straightforward to show that there can only be edges between two vertices in two different squares if the squares are defined by disjoint pairs $\pm i, \pm j$ and $\pm p, \pm q$. And in that case, the edges define a bijection between the two squares. Thus, viewing each square as a vertex, the graph A_2 could now be defined by a labeling of the Kneser graph $K_{7,2}$ by elements of S_4. The adjacency matrix defined by this assumption is shown in Figure 4.7. However, as will be explained in the next section, this assumption has proven to be incorrect.

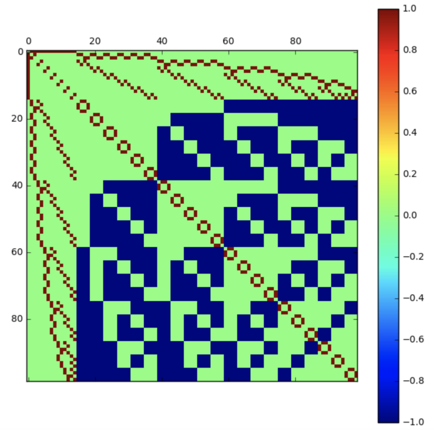

FIGURE 4.7
The adjacency matrix A under the squares assumption with unknown entries set to -1.

4.2. **Gröbner bases.** Equations (5) and (6) above define a system of polynomial equations (all of degree 1 or 2) in the entries of A_2. Also, we know that A_2 is the adjacency matrix of a 12 regular graph. This gives us many more equations than variables when the matrix A_2 is viewed as a matrix of 84^2 variables. If this system has no solutions over the integers, then no such graph exists. The problem is that the system is quite large so solving it is not necessarily computationally feasible.

One way to solve a system of polynomial equation is via Gröbner bases [1]. The basis calculates the ideal generated by the polynomials over a polynomial ring (I considered the rationals adjoined the variables and experimented without improved results with the integers and various finite fields as well). In SageMath [12], I attempted to solve the enormous systems of equations defined by the matrix A_2 via the Gröbner bases. As might be expected, with no assumptions reducing the number of variables, this program did not finish. However, with the squares assumption, there are now only $16 * 105 = 1630$ variables to solve for. In this case, the method revealed that there are no solutions, however printing a certificate of which polynomials combine to 1 did not finish running.

This technique is appealing since it must be able to solve the problem if given enough time. However, much like brute force search of all graphs on 99 vertices, this approach does not seem to be computationally efficient enough to be a promising method. Additionally, if this were to yield a solution it would likely not provide any sort of intuition for why the result (either existence or non-existence) is true.

5. Conclusion

The question of existence of strongly regular graphs is a tantalizing one. The problem is easy to understand and the solutions to certain cases are simple. Yet, some relatively small graphs are still unknown. The (99,14,1,2) graph is especially interesting because of its simple explanation and ability to elude all of the currently known techniques for proving that such a graph does not exist. Here we have presented those techniques beginning with the classical feasibility conditions, eigenvalue multiplicity conditions and Krein bounds. We saw some combinatorial constructions of graphs that do exist. Additionally, we introduced the methods of papers from this year that have resolved some of the previously unknown cases with novel techniques and computational approaches. We then presented some facts about the automorphism group of the graph should it exist. Lastly, we presented some failed computational strategies to find the 99 graph. While I can't conjecture whether or not the 99-graph exist, hopefully this paper provides a useful introduction to the problem of the existence of strongly regular graphs, a problem that may have be solved in the not-so-distant future.

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