Generators of Jacobians of Graphs

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Overview

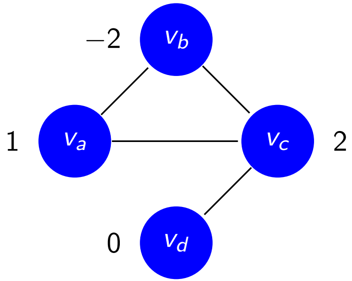
Outline

1. Defining the Jacobian
2. Results
3. Methods
4. Random Graphs

Divisor

Definition (Divisor)

An assignment of integer values to the vertices of a graph G.
$\mathrm{Div}(G) \cong \mathbb{Z}^{|V(G)|} = \mathbb{Z}^n$ denotes the group of all divisors.



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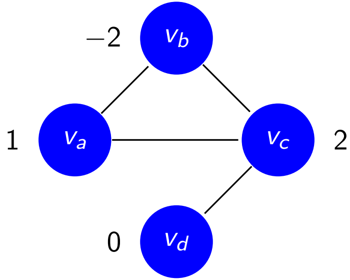
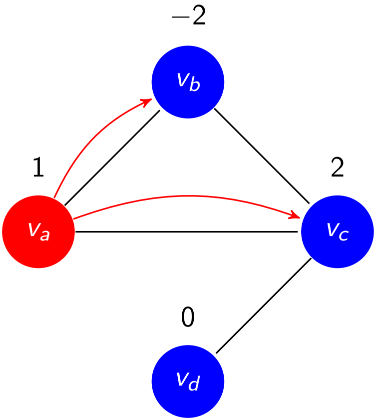


Definition (Degree)

The sum of the integer values in a divisor.
Let $d \in \mathrm{Div}(G)$, then $deg(d) = \sum_{v \in V(G)} d(v)$

Chip Firing Equivalence



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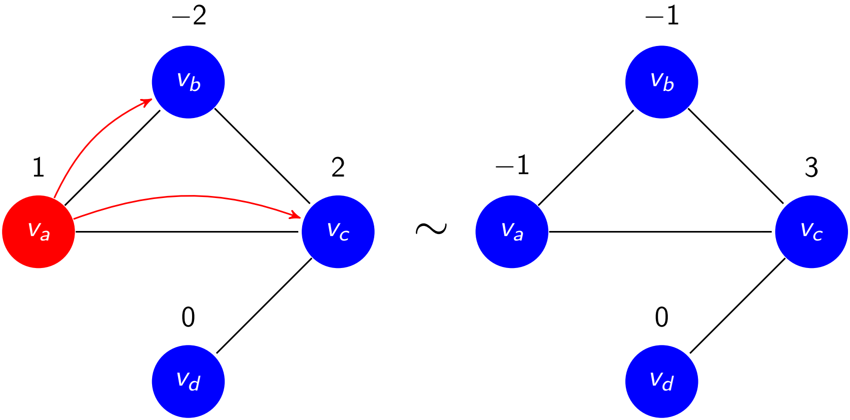

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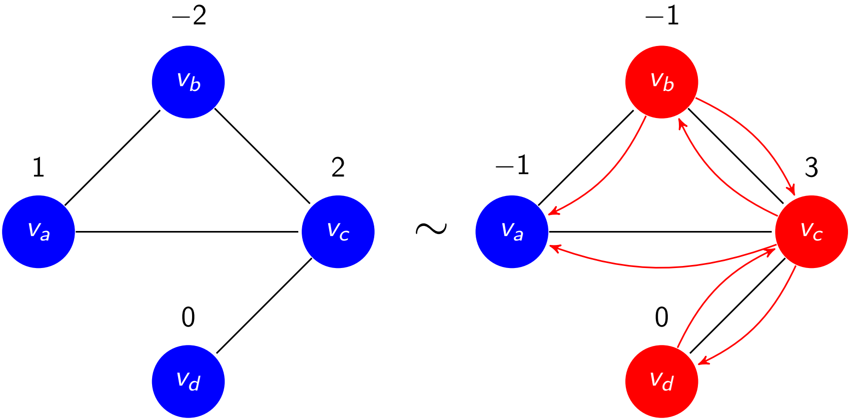

Properties of Chip Firing

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The equivalence respects addition of divisors

If $d_1 \sim d_2$ and $e_1 \sim e_2$, then $d_1 + e_1 \sim d_2 + e_2$.

Jacobian group

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Definition (Jacobian Group)

$\mathrm{Jac}(G) := \mathrm{Div}^0(G)/\sim$

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Current Research

Theorem (Lorenzini, 1989)

Let G be a simple, connected, undirected graph. Let G_1 be the graph obtained by the removal of an edge (x, y). Then if $\gcd(|\text{Jac}(G)|, |\text{Jac}(G_1)|) = 1$, then $\text{Jac}(G)$ is cyclic.

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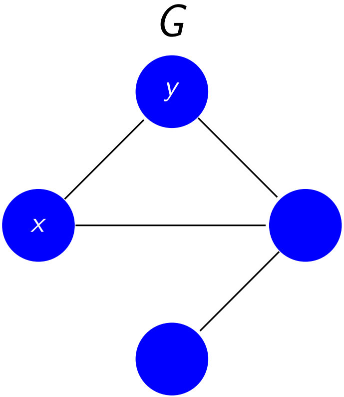

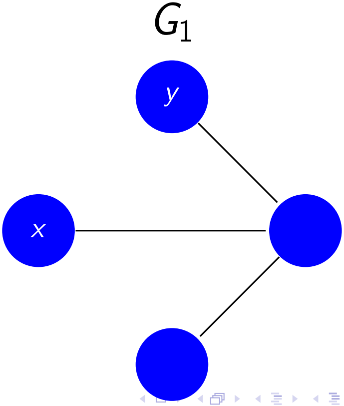

Current Research

Open Question (Becker & Glass, 2014)

Let δ_{xy} be the divisor that takes the value -1 on the vertex x, 1 on the vertex y, and 0 elsewhere. Then if $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = 1$, does δ_{xy} generate $\mathrm{Jac}(G)$?

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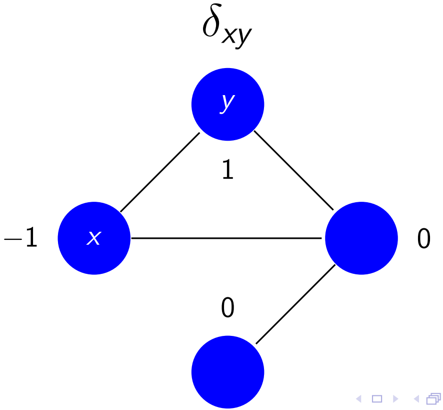

Results – Edge Deletion and Insertion

Theorem (BKRS, 2017)

$$[\mathrm{Jac}(G) : \langle \delta_{xy} \rangle] \mid \gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|)$$

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Corollary (Answer to Open Question)

Suppose that $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = 1$. Then δ_{xy} generates $\mathrm{Jac}(G)$.

Question (Converse of Above)

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If δ_{xy} generates $\mathrm{Jac}(G)$, then $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = 1$.

Results – Edge Contraction

Corollary (BKRS, 2017)

Let G be a graph and let $e = (x, y)$ be an edge. Let G/e denote the graph obtained by identifying the vertices x and y. Then $[\text{Jac}(G) : \langle \delta_{xy} \rangle] \mid \gcd(|\text{Jac}(G)|, |\text{Jac}(G/e)|)$.

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Proposition (BKRS, 2017)

Let G be a graph, and e, G/e defined as above. Let z be the vertex obtained by identifying x and y, and let $\mathrm{val}_G(x)$ denote the valency of x in G. If $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G/e)|$ are relatively prime and

$$D_z(v) = \begin{cases} 0 & (v, x) \notin E(G) \\ (\mathrm{val}_G(x) - 1) & v = z \\ -1 & (v, x) \in E(G), v \neq z, \end{cases}$$

then $\mathrm{Jac}(G/e)$ is cyclic and D_z generates it.

Results – Bounding the Order of δ_{xy} Below

Theorem (GJRWW, SUMRY 2014)

Let G be an undirected, biconnected multigraph, and let (x, y) be an edge of G. Then $|\delta_{xy}| \geq \mathrm{val}(x)$.

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Proposition (BKRS, 2017)

Let G be a graph with $n = |V(G)|$ vertices and $\epsilon = |E(G)|$ edges.

1. There exists some edge (x, y) such that $|\delta_{xy}| \geq \epsilon/(n-1)$.
2. There exists some edge (u, v) such that $|\delta_{uv}| \geq \epsilon/(\epsilon - n + 1)$.

Note that in the first inequality the bound is tight for a spanning tree on n vertices, and in the second the bound is tight for an n-cycle.

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Define G as before. Then if G is biconnected, we have:

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Conjecture – Improved Bound

For any vertex $x \in G$, there exists some vertex $y \in G$ such that $|\delta_{xy}| \geq n$.

Outline

Constructing Maps into $\mathbb{Z}/m\mathbb{Z}$

Goal

Characterize homomorphisms $\phi : \mathrm{Div}^0(G) \to \mathbb{Z}/m\mathbb{Z}$, where $m = |\mathrm{Jac}(G)|$. Determine which ϕ descend to maps from $\mathrm{Jac}(G) \to \mathbb{Z}/m\mathbb{Z}$

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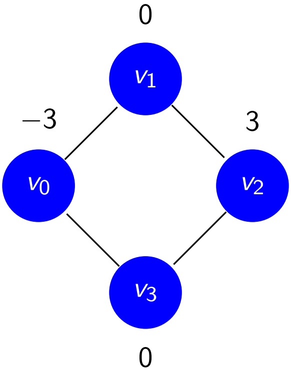
Method

Associate each map ϕ with an assignment of integer weights to the vertices of G.

Example of Vertex Weights

d :



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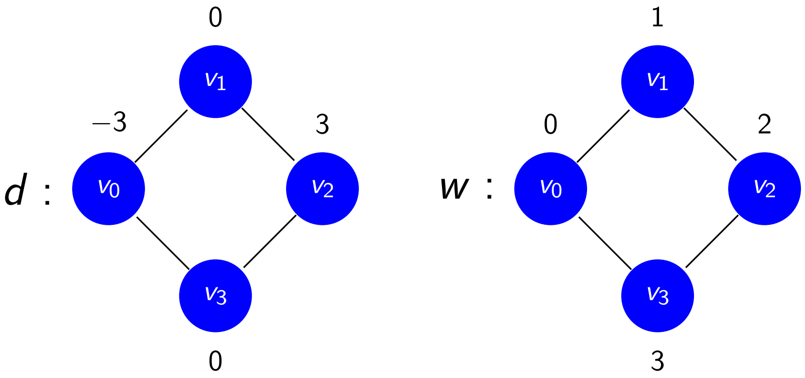

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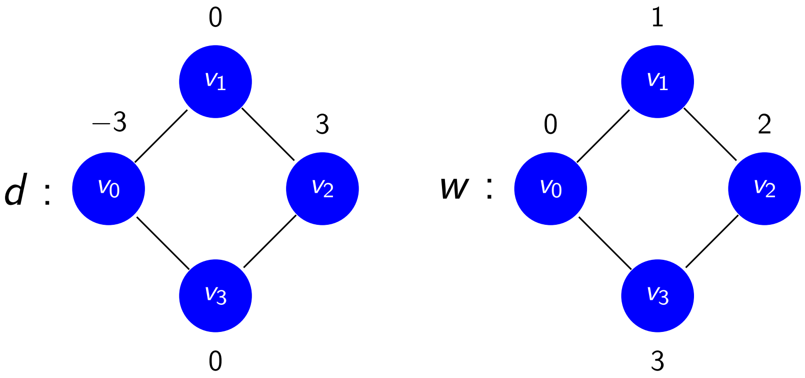
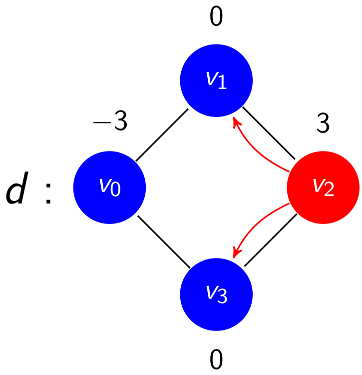


Weight map

$\phi : \mathrm{Div}^0(G) \to \mathbb{Z}/m\mathbb{Z}$

$\phi(d) := w \cdot d \pmod 4 \equiv 0(-3) + 1(0) + 2(3) + 3(0) \equiv 6 \equiv 2$

Example of Vertex Weights - Invariance Under Chip-Firing



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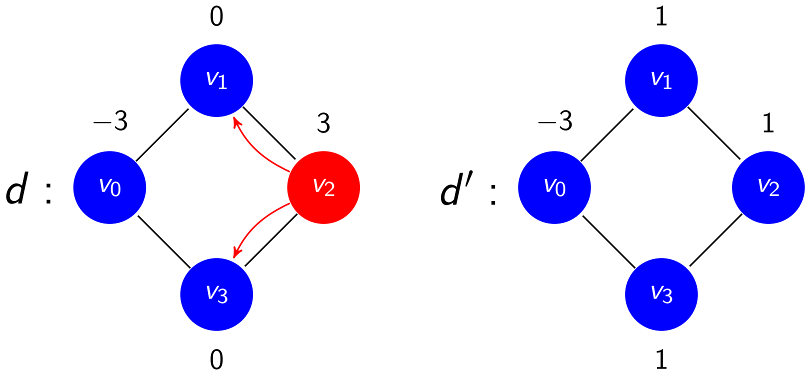

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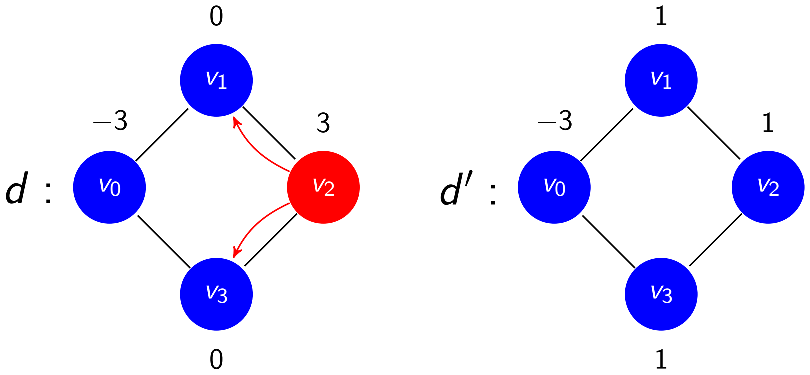

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$\phi(d) = w \cdot d \pmod 4 \equiv 0(-3) + 1(0) + 2(3) + 3(0) \equiv 2$

$\phi(d') = w \cdot d' \pmod 4 \equiv 0(-3) + 1(1) + 2(1) + 3(1) \equiv 2$

Monodromy Weights

Monodromy weight definition

If $w \cdot d = w \cdot d'$ whenever $d \sim d'$, we say that w is a monodromy weight. This corresponds to the associated map $\phi : \mathrm{Div}^0(G) \to \mathbb{Z}/m\mathbb{Z}$ descending to a well-defined map from $\mathrm{Jac}(G) \to \mathbb{Z}/m\mathbb{Z}$.

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Correspondence with $\mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$

If we consider monodromy weights modulo m and require that the weight on the nth vertex of G is zero, we obtain a bijection between monodromy weights and homomorphisms $\phi \in \mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$.

Self-Duality of G

Self-duality of $\mathrm{Jac}(G)$

Since $\mathrm{Jac}(G)$ is a finite abelian group, it is self-dual. i.e. $\mathrm{Jac}(G) \cong \mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$. But, there is no "canonical" isomorphism.

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Canonical isomorphism between Jac(G) and $\mathbb{Z}/m\mathbb{Z}$

We can construct an isomorphism F from Jac(G) to the group of monodromy weights. This can be extended to a canonical isomorphism $\Phi \circ F : \text{Jac}(G) \to \text{Hom}(\text{Jac}(G), \mathbb{Z}/m\mathbb{Z})$.

Using Monodromy Weights to Prove Main Theorems

Lemma (using the $\Phi \circ F$ isomorphism)

If $d \in \mathrm{Jac}(G)$ and $\phi = \Phi \circ F(d)$, then we have $|d| = |\mathrm{Im}(\phi)|$.

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- By linear algebra and the above lemma, we show that if $d = \delta_{xy}$, then $m/|\text{Im}(\phi)| \mid \gcd(|\text{Jac}(G)|, |\text{Jac}(G_1)|)$.
- The converse follows with a bit of extra work (using similar methods).

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Research Direction: Random Graphs

Definition ($G_{n,p}$)

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Conjecture (Clancy, Kaplan, Leake, Payne, Wood, 2014)

$\lim_{n\to\infty} \text{Prob}(\text{Jac}(G_{n,p}) \text{ cyclic}) = \prod_{i=1}^{\infty} \zeta(2i+1)^{-1} \approx 0.7935$

Experimental Results and Conjectures

Probability of fixed δ_{xy} generator

$\mathbb{P}\left(\delta_{xy} \text{ generates } \mathrm{Jac}(G_{n,p}) \mid \mathrm{Jac}(G_{n,p}) \textit{ cyclic}\right) \approx .607 \approx \zeta(2)^{-1}$

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Existence of some δ_{xy} generator

$\lim_{n\to\infty} \mathbb{P}\left(\exists\ x, y \text{ } s.t. \text{ } \delta_{xy} \text{ generates } \mathrm{Jac}(G_{n,p}) \mid \mathrm{Jac}(G_{n,p}) \text{ } cyclic\right) \approx 1$

Thank You

We would like to thank our mentors for their help with our project, SUMRY for funding the research, YMC for hosting us, and you for attending our talk.