Khintchine Inequality

David Brandfonbrener
Math 731

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Aleksandr Khinchin

1894 - 1959

Moscow State University

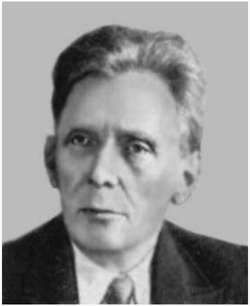
Studied under Luzin

Worked with Kolmogorov

Contributed to law of iterated logarithm, central limit theorem, and information theory

Won the Stalin Prize (1941)

Aleksandr Khinchin (fun facts)



Interested in literature and theater, acted in plays and wrote poetry

His student Gelfond proved that a^b transcendental if a, b algebraic and b irrational

Not a member of the communist party

Friends with Mayakovsky

Theorem (Khinchin, 1923)

Generalized by Littlewood (1930) and Paley and Zygmund (1930).
Sometimes called Khinchin-Kahane after a paper of Kahane (1964)

Let $\{\varepsilon_n\}_{n=1}^N$ be independent random variables taking values ± 1 with probability $1/2$ and $\{a_n\}_{n=1}^N$ be real numbers. Then, for $0 < p < \infty$ there exist constants A_p, B_p depending only on p such that

$$A_p \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2} \leq \left(\mathbb{E} \left[\left| \sum_{n=1}^N \varepsilon_n a_n \right|^p \right] \right)^{1/p} \leq B_p \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2}.$$

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*Note: Actually holds for a_n complex and has analogs in many other spaces.

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Take $x \in \mathbb{R}$. Then

$$\frac{e^x + e^{-x}}{2} \le e^{\frac{x^2}{2}}.$$

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Proof

Taylor series.

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \le 1 + \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} + \dots$$

Since $(2n)! \ge 2^n \cdot n!$ for $n \ge 0$. $\square$

Lemma 2

Lemma

Take f a real valued function on a probability space (Ω, μ) and $0 < p < \infty$. Then

$$\mathbb{E}(|f|^p) = \int_\Omega |f(x)|^p d\mu = p \int_0^\infty \lambda^{p-1} \mathbb{P}(|f| \geq \lambda)\, d\lambda.$$

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Proof

First note that $|f(x)|^p = p \int_0^{|f(x)|} \lambda^{p-1} d\lambda$.

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Proof

First note that $|f(x)|^p = p \int_0^{|f(x)|} \lambda^{p-1} d\lambda$. Now by Fubini,

$$\begin{aligned}
\int_\Omega |f(x)|^p d\mu &= \int_\Omega \left(p \int_0^{|f(x)|} \lambda^{p-1} d\lambda \right) d\mu \\
&= \int_0^\infty p\lambda^{p-1} \left(\int_\Omega \mathbb{1}_{\lambda \leq |f(x)|} d\mu \right) d\lambda \\
&= \int_0^\infty \lambda^{p-1} \mathbb{P}(|f| \geq \lambda)\, d\lambda \quad \square
\end{aligned}$$

Proof (Upper Bound)

By independence of the ε_n we have that for $t > 0$,

$$\mathbb{E}\left(e^{t\sum_n a_n\varepsilon_n}\right) = \prod_n \mathbb{E}\left(e^{ta_n\varepsilon_n}\right) = \prod_n \left(\frac{e^{-ta_n} + e^{ta_n}}{2}\right).$$

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Which by Lemma 1 gives us

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So, like in the proof of Chernoff, and using Markov,

$$\mathbb{P}\left(\sum_n a_n \varepsilon_n \geq \lambda\right) = \mathbb{P}\left(e^{t\sum_n a_n \varepsilon_n} \geq e^{t\lambda}\right) \leq \frac{e^{\frac{t^2 \sum_n a_n^2}{2}}}{e^{\lambda t}}$$

Proof (Upper Bound)

Now we set $t = \frac{\lambda}{\sum_n a_n^2}$ and we get

$$\mathbb{P}\left(\sum_n a_n \varepsilon_n \geq \lambda\right) \leq e^{\frac{-\lambda^2}{2\sum_n a_n^2}} \quad \Rightarrow \quad \mathbb{P}\left(\left|\sum_n a_n \varepsilon_n\right| \geq \lambda\right) \leq 2e^{\frac{-\lambda^2}{2\sum_n a_n^2}}.$$

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Now we consider to the case $\sum_n a_n^2 = 1$. Then we have

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$$\mathbb{P}\left(\left|\sum_n a_n \varepsilon_n\right| \geq \lambda\right) \leq 2e^{\frac{-\lambda^2}{2}}$$

By Lemma 2, this gives us

$$\left(\mathbb{E}(|\sum_n a_n \varepsilon_n|^p)\right)^{1/p} \leq \left(p\int_0^\infty \lambda^{p-1} 2e^{\frac{-\lambda^2}{2}} d\lambda\right)^{1/p} = B_p$$

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Proof (Upper Bound)

Now, note that

$$\left(\mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^p\right)\right)^{1/p} \leq B_p = B_p\left(\sum_n a_n^2\right)^{1/2}$$

The upper bound follows by scaling the a_n.
If $\sum_n a_n^2 = S$, then define $b_n = \frac{a_n}{\sqrt{S}}$ and $\sum_n b_n^2 = 1$.
Apply the inequality to the b_n, then multiply both sides by $\sqrt{S}$ to get the desired result.

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Proof

Assume $p_0 < p_1$. Set $F = |f|^{p_0}$ and $G = 1$. Set $p = p_1/p_0 > 1$ and $1/p + 1/q = 1$ then by Hölder,

$$\int_\Omega |f|^{p_0} d\mu = \int_\Omega |FG| d\mu \leq \left(\int_\Omega |F|^p d\mu \right)^{1/p} \cdot 1 = \left(\int_\Omega |f|^{p_1} d\mu \right)^{p_0/p_1}$$

Then take the p_0 root. $\square$

Proof (Lower Bound)

First consider $2 \leq p < \infty$. Then, since we are in a probability space, by independence of the ε_n and Lemma 3 we have

$$\left(\sum_n (a_n)^2\right)^{1/2} = \left(\mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^2\right)\right)^{1/2}$$
$$\leq \left(\mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^p\right)\right)^{1/p}$$

Proof (Lower Bound)

Now, take $0 < p < 2$ and assume that not all of the a_k are 0. Then, by Cauchy-Schwarz we have

$$\begin{aligned}
\sum_n (a_n)^2 &= \mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^2\right) \\
&= \mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^{p/2}\left|\sum_n a_n \varepsilon_n\right|^{2-p/2}\right) \\
&\leq \left(\mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^p\right)\right)^{1/2}\left(\mathbb{E}\left(\left|\sum_n a_n \varepsilon_n\right|^{4-p}\right)\right)^{1/2}
\end{aligned}$$

Proof (Lower Bound)

Now we apply the upper bound for $4-p$ to get that

$$\sum_n (a_n)^2 \leq \left(\mathbb{E}\left(\left|\sum_n a_n\varepsilon_n\right|^p\right)\right)^{1/2}\left(\left(B_p\sum_n (a_n)^2\right)^{\frac{4-p}{2}}\right)^{1/2}$$

$$= B_p^{2-p/2}\left(\mathbb{E}\left(\left|\sum_n a_n\varepsilon_n\right|^p\right)\right)^{1/2}\left(\sum_n (a_n)^2\right)^{1-p/4}$$

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This yields:

$$\left(\sum_n (a_n)^2\right)^{p/4} \le B_p^{2-p/2}\left(\mathbb{E}\left(\left|\sum_n a_n\varepsilon_n\right|^p\right)\right)^{1/2}$$

Taking both sides to the $2/p$ power gives the result. □

Sharp Constant (Haagerup, 1982)

Let $p_0 \in (1,2)$ solve $\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}$, so $p_0 \approx 1.847$. Then

$$A_p = \begin{cases} 2^{1/2-1/p} & 0 < p \le p_0 \\ \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{1/p} & p_0 < p < 2 \\ 1 & 2 \le p < \infty \end{cases}$$

$$B_p = \begin{cases} 1 & 0 < p \le 2 \\ \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{1/p} & 2 < p < \infty \end{cases}$$

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The proof is 50 pages and relies on a lot of casework and inequalities involving the function:

$$F(s) = \frac{2}{\pi}\int_0^\infty (1 - |\cos(\frac{t}{\sqrt{s}})|^s)t^{-2}dt$$

Sharp Constant ($p = 1$, Szarek, 1976)

Let $\{\varepsilon_n\}_{n=1}^N$ be independent random variables taking values ± 1 with probability $1/2$ and $\{a_n\}_{n=1}^N$ be real numbers.

$$\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{N}|a_n|^2\right)^{1/2} \le \mathbb{E}\left[\left|\sum_{n=1}^{N}\varepsilon_n a_n\right|\right] \le \left(\sum_{n=1}^{N}|a_n|^2\right)^{1/2}.$$

Proof ($p = 1$) (Latala and Oleszkiewicz, 1994)

This proof works for a_n in any normed vector space F.

Notation

$\sigma = (\sigma_1, \ldots, \sigma_N) \in \{0,1\}^N$, $\eta = (\eta_1, \ldots, \eta_N) \in \{-1,1\}^N$,
$x = (x_1, \ldots, x_N) \in \mathbb{R}^N$
$|\sigma| = \sum_{n=1}^N \sigma_n$
$x^\sigma = \prod_{n=1}^N x_n^{\sigma_n}$
$L_\varepsilon = \|\sum_{n=1}^N \varepsilon_n a_n\|$
$d(\varepsilon, \eta) = \mathrm{card}\{n : \varepsilon_n \neq \eta_n\}$

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Statement

Let $\{a_n\}_{n=1}^N$ be vectors in a normed vector space F and $\{\varepsilon_n\}_{n=1}^N$ be random variables in ± 1 each with probability 1/2. Define $S = \sum_{n=1}^N a_n \varepsilon_n$, then

$$\frac{1}{\sqrt{2}} (\mathbb{E}(\|S\|^2))^{1/2} \leq \mathbb{E}\|S\| \leq (\mathbb{E}(\|S\|^2))^{1/2}$$

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Upper bound holds by Lemma 3, tight if $a_1 = 1, N = 1$.

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Lower bound. Let $t \in \mathbb{R}$, then we have the equality

$$t^2 \prod_{n=1}^{N} (1 + t^{-1} x_i) = \sum_{\sigma \in \{0,1\}^N} t^{2-|\sigma|} x^\sigma$$

Differentiating in t and then setting $t = 1$ we get

$$2 \prod_{n=1}^{N} (1 + x_n) - \sum_{n=1}^{N} x_n \prod_{i=1, i \neq n}^{N} (1 + x_i) = \sum_{\sigma \in \{0,1\}^N} (2 - |\sigma|) x^\sigma$$

Now we substitute $x_n = \varepsilon_n \eta_n$ and sum both sides over all possible η and ε.

Proof ($p = 1$)

$$\sum_{\varepsilon,\eta\in\{-1,1\}^N}\left(2\prod_{n=1}^{N}(1+\varepsilon_n\eta_n)-\sum_{n=1}^{N}\varepsilon_n\eta_n\prod_{i=1,i\neq n}^{N}(1+\varepsilon_i\eta_i)\right)L_\varepsilon L_\eta$$

$$=\sum_{\varepsilon,\eta\in\{-1,1\}^N}\sum_{\sigma\in\{0,1\}^N}(2-|\sigma|)\varepsilon^\sigma\eta^\sigma L_\varepsilon L_\eta$$

$$=\sum_{\sigma\in\{0,1\}^N}(2-|\sigma|)\left(\sum_{\varepsilon\in\{-1,1\}^N}\varepsilon^\sigma L_\varepsilon\right)^2$$

$$\leq 2\left(\sum_{\varepsilon\in\{-1,1\}^N}\varepsilon^\sigma L_\varepsilon\right)^2$$

The inequality holds because $L_\varepsilon = L_{-\varepsilon}$ so that for each σ with $|\sigma| = 1$ we have $\sum_{\varepsilon\in\{-1,1\}^N}\varepsilon^\sigma L_\varepsilon = 0$.

Proof ($p = 1$)

Note that $\prod_n (1+\varepsilon_n\eta_n) \neq 0$ iff $\varepsilon = \eta$ and $\prod_{i, i\neq n}(1+\varepsilon_i\eta_i) \neq 0$ iff $\varepsilon_i = \eta_i$ for all $i \neq n$. So, we can rewrite the previous inequality as:

$$2^{N+1} \sum_{\varepsilon\in\{-1,1\}^N} L_\varepsilon^2 - N2^{N-1} \sum_{\varepsilon\in\{-1,1\}^N} L_\varepsilon^2 + 2^{N-1} \sum_{\varepsilon,\eta\in\{-1,1\}^N, d(\varepsilon,\eta)=1} L_\varepsilon L_\eta$$

$$= 2^N \sum_{\varepsilon\in\{-1,1\}^N} L_\varepsilon^2 + 2^{N-1} \sum_{\varepsilon\in\{-1,1\}^N} L_\varepsilon \left(\sum_{\substack{\varepsilon,\eta\in\{-1,1\}^N,\\ d(\varepsilon,\eta)=1}} L_\eta - (N-2)L_\varepsilon \right)$$

$$\leq 2 \left(\sum_{\varepsilon\in\{-1,1\}^N} \varepsilon^\sigma L_\varepsilon \right)^2$$

Proof ($p = 1$)

By the triangle inequality, for fixed ε we have

$$(N-2)L_\varepsilon \leq \sum_{\varepsilon,\eta\in\{-1,1\}^N, d(\varepsilon,\eta)=1} L_\eta$$

So, we get that the long term in the previous sum is positive, yielding:

$$2^N \sum_{\varepsilon\in\{-1,1\}^N} L_\varepsilon^2 \leq 2\left(\sum_{\varepsilon\in\{-1,1\}^N} \varepsilon^\sigma L_\varepsilon\right)^2$$

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Dividing by $(2^N)^2$ yields

$$\mathbb{E}\|S\|^2 = \frac{1}{2^N}\sum_{\varepsilon\in\{-1,1\}^N} L_\varepsilon^2 \leq 2\left(\frac{1}{2^N}\sum_{\varepsilon\in\{-1,1\}^N} \varepsilon^\sigma L_\varepsilon\right)^2 = 2(\mathbb{E}\|S\|)^2$$

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Note this is tight since we have equality if $a_1 = a_2, N = 2.$ $\square$

Example Application

Take $1 \leq p < \infty$ and $T : L^p(X,\mu) \to L^p(Y,\nu)$ a bounded linear operator. Then, there exists a constant $C_{p,\|T\|} > 0$ such that

$$\left\| \left(\sum_{n=1}^{N} Tf_n \right)^{1/2} \right\|_{L^p(Y,\nu)} \leq C_{p,\|T\|} \left\| \left(\sum_{n=1}^{N} f_n \right)^{1/2} \right\|_{L^p(X,\mu)}$$

Rademacher Functions

These results are often formulated in Rademacher functions instead of ε_n.
The Rademacher functions $r_k(t)$ for $k \in \mathbb{N}_+$ and $t \in [0, 1]$ are defined as

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These functions can be treated as independent random variables taking on values ± 1 with probability 1/2. Since

$$\int_0^1 r_k(t) r_j(t) dt = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

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$$\int_0^1 r_k(t) r_j(t) dt = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

Then the inequality can be restated as

$$A_p \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2} \leq \left(\int_0^1 \left[\left| \sum_{n=1}^N r_k(t) a_n \right|^p \right] dt \right)^{1/p} \leq B_p \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2}$$

The End