Rödl Nibble

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Definition 1. For $2 \leq l < k < n$, define the covering number $M(n,k,l)$ to be the minimal size of a family $\mathcal{K}$ of k-element subsets of $\{1,..,n\}$ such that every l-element subset of $\{1,..,n\}$ is contained in some $A \in \mathcal{K}$

Theorem 1. For fixed $2 \leq l < k$, where $o(1) \to 0$ as $n \to \infty$:

$$M(n,k,l) \leq (1+o(1))\frac{\binom{n}{l}}{\binom{k}{l}}$$

Definition 2. Let $H = (V,E)$ be an r-uniform hypergraph, with $x \in V$. Then define the degree of x in H, $d(x)$ to be the number of edges in E containing x. And for $x, y \in V$, define $d(x,y)$ to be the number of edges in E containing both x and y. And a covering of H is a set of edges in $C \subseteq E$ such that every vertex in V is in some edge in C.

Lemma 1. For every integer $r \geq 2$ and reals $k \geq 1$, $a > 0$, there are $\gamma = \gamma(r,k,a) > 0$ and $d_0 = d_0(r,k,a)$ such that for every $n \geq D \geq d_0$ the following holds.

For every $H = (V,E)$ an r-uniform hypergraph with n vertices, all with positive degree and satisfying:

1. For all except at most γn vertices $x \in V$, $d(x) = (1 \pm \gamma)D$
2. For all $x \in V$, $d(x) < kD$
3. For any distinct $x, y \in V$, $d(x,y) < \gamma D$.

Then there exists a cover of H with at most $(1+a)\frac{n}{r}$ edges.

Lemma 2. For every integer $r \geq 2$ and reals $K > 0$, $\epsilon > 0$, and every $\delta' > 0$, there are $\delta = \delta(r,K,\epsilon,\delta') > 0$ and $D_0 = D_0(r,K,\epsilon,\delta') > 0$ such that for every $n \geq D \geq D_0$ the following holds.

For every $H = (V,E)$ an r-uniform hypergraph with n vertices, satisfying:

1. For all except at most δn vertices $x \in V$, $d(x) = (1 \pm \delta)D$
2. For all $x \in V$, $d(x) < KD$
3. For any distinct $x, y \in V$, $d(x,y) < \delta D$.

Then there exist a set of edges $E' \subseteq E$ that has the following properties:

4. $|E'| = \frac{\epsilon n}{r}(1 \pm \delta')$
5. Define $V' = V \setminus \cup_{e \in E'} e$. Then we have $|V'| = ne^{-\epsilon}(1 \pm \delta')$
6. For all except at most $\delta'|V'|$ vertices $x \in V'$ the degree $d'(x)$ in the induced hypergraph of H on V' satisfies $d'(x) = De^{-\epsilon(r-1)}(1 \pm \delta')$