

**Solution Manual to Elementary Analysis,
2nd Ed., by Kenneth A. Ross**

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1 Basic Properties of the Derivative

Note: In this section, we make routine use of the fact that $\lim_{x \rightarrow a}$ is evaluated on sets $J = I \setminus \{a\}$ so that, for example, $\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x$ is allowed, despite the cancellation not being valid at $x = 0$.

28.1

- a) $\{0\}$
- b) $\{0\}$
- c) $\{\pi n | n \in \mathbb{Z}\}$
- d) $\{0, 1\}$
- e) $\{-1, 1\}$
- f) $\{2\}$

28.2

- a) $(x^3 - 8) = (x - 2)(x^2 + 4x + 4)$,
so, $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = \lim_{x \rightarrow 2} (x^2 + 4x + 4) = (\text{by 20.4}) 4 + 4 + 4 = 12$
- b) $\lim_{x \rightarrow a} \frac{(x+2) - (a+2)}{x - a} = \lim_{x \rightarrow a} 1 = 1$
- c) $\lim_{x \rightarrow 0} \frac{x^2 \cos(x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \cos(x) = 0 * 1 = 0$
- d) $\lim_{x \rightarrow 1} \frac{\frac{3x+4}{2x-1} - 7}{x - 1} = \lim_{x \rightarrow 1} \frac{(3x+4) - (14x-7)}{(2x-1)(x-1)} = \lim_{x \rightarrow 1} \frac{(-11)(x-1)}{(2x-1)(x-1)} = -11$

28.3

- a) $(x - a) = (\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a})$,
so, $\lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$ for $a > 0$.

Note: For $\lim_{x \rightarrow a} f(x)$ to exist, we require that the limit converge for all (x_n) in a set $J = I \setminus \{a\}$ for some open interval I containing a . However, \sqrt{x} is not defined on an open interval around 0, so the limit does not exist there.

- b) Similarly to part (a), $\lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \rightarrow a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}}$ for $a \neq 0$.
- c) $f(x) = x^{1/3}$ is not differentiable at $x = 0$. $\lim_{x \rightarrow 0} \frac{x^{1/3} - 0^{1/3}}{x - 0} = \lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$ and this limit does not exist, because the limits approaching from the left and right are distinct $(-\infty \text{ and } \infty)$.

1 Basic Properties of the Derivative

28.4

a) First, notice $\sin(x)$ is differentiable on \mathbb{R} and therefore on $\mathbb{R} \setminus \{0\}$.

Since 1 and x are differentiable on \mathbb{R} and $x \neq 0$ on $\mathbb{R} \setminus \{0\}$, $\frac{1}{x}$, by theorem 28.3, is differentiable on $\mathbb{R} \setminus \{0\}$.

Clearly x^2 is differentiable on $\mathbb{R} \setminus \{0\}$.

Therefore, by 28.3, $f(x) = x^2 \sin(\frac{1}{x})$ is differentiable on $\mathbb{R} \setminus \{0\}$ and $f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$.

b) By def 28.1 $f'(0) = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin(\frac{1}{x})$. By the squeeze theorem, this limit is 0. c) Since our choice of sequence $(x_n) \rightarrow 0$ can influence $\lim_{n \rightarrow \infty} f'(x_n)$, we conclude $\lim_{n \rightarrow 0} f'(x)$ does not exist, so $f'(x)$ is discontinuous there.

Note: Recall theorem 28.2 showed that, if f is differentiable at a , then it is continuous there. This exercise has shown that while any f differentiable at a must be continuous at a , the corresponding derivative function, f' , need not be.

28.5

a) f is differentiable on \mathbb{R} by exercise 28.4. g is also differentiable on \mathbb{R} , obviously.

b) 0.

c) We can find x arbitrarily close to 0 for which the argument of the limit is not defined. Hence, there is no open interval I around 0 on which the argument of the limit is defined, so the limit cannot be evaluated.

28.6

a) f is continuous, since $\lim_{x \rightarrow 0} f(x) = 0$ by the squeeze theorem, and $f(0) = 0$.

b) f is not differentiable at $x = 0$.

$\lim_{x \rightarrow 0} \frac{x \sin(1/x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x)}{x} = \lim_{x \rightarrow 0} \sin(1/x)$ which does not exist.

28.7

a) [Graph Not Shown]

b) $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$ is not obvious, but $\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = 0$ and $\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} x = 0$, so, by theorem, $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0$. Therefore, f is differentiable at $x = 0$.

c) By section 28, example 3, we have $f'(x) = 2x$ for $x > 0$. In part (b), we showed $f'(0) = 0$, and clearly $f'(x) = 0$ for $x < 0$.

d) Continuous: Yes. Differentiable: No.

28.8

a) $f(0) = 0$. Let $\epsilon > 0$. Let $\delta = \sqrt{\epsilon}$. $|f(x) - 0| = ||x|^2 - 0|$ or $|0 - 0|$. Clearly, $||x|^2 - 0| = |x^2 - 0|$. If $|x - 0| < \delta$ then $|x^2 - 0| < |\delta^2| = \epsilon$. $|0 - 0| < \epsilon$ trivially. Hence, $|f(x) - 0| < \epsilon$.

b) Since the rationals and irrationals are dense, there are rational numbers arbitrarily close to each irrational number, and there are irrational numbers arbitrarily close to each rational number. Let $x_0 \neq 0$, we know $x^2 > 0$, so there is an open interval I around x^2 such that $x^2 > \epsilon$ for some $\epsilon > 0$. Since x^2 is continuous, there is a corresponding open interval G around x_0 that is the inverse image of I . Select rational x_r from G . For arbitrarily close irrational numbers $x_i \in G$, $f(x_r) - f(x_i) > \epsilon - 0$ so f is discontinuous for all $x \neq 0$.

c) $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} f(x)x$

$\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0$

$\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$

So, by theorem, $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0 = f'(0)$.

28.9

a) $h'(x) = 7(x^4 + 13x)^6(4x^3 + 13)$

b) $h(x) = g \circ f(x)$ where $g(x) = x^7$ and $f(x) = x^4 + 13x$

28.10

a) $h'(x) = 12(\cos(x) + e^x)^{11}(-\sin(x) + e^x)$

b) $h(x) = g \circ f(x)$ where $g(x) = x^{12}$ and $f(x) = \cos(x) + e^x$

28.11

$(h \circ g \circ f)'(a)$: Extended Chain Rule

By theorem 28.4, since f is differentiable at x , and g is differentiable at $f(a)$, $g \circ f$ is differentiable at a . This, along with the fact that h is differentiable at $g \circ f(a)$, shows that, by theorem 28.4, $h(g \circ f) = h \circ (g \circ f)$ is

2 The Mean Value Theorem

29.1

- a) $x = 1/2$
- b) $x = \pi/2$
- c) Not differentiable at $x = 0$
- d) Not continuous or differentiable at $x = 0$
- e) $x = \sqrt{3}$ f) Not continuous or differentiable at $x = 0$

29.2

Suppose there exist $x, y \in \mathbb{R}$, $x > y$ such that $|\cos(x) - \cos(y)| > |x - y|$. This implies $|\frac{\cos(x) - \cos(y)}{x - y}| > 1$. By the Mean Value Theorem, we must have at least one $x_0 \in (x, y)$ such that the derivative of $\cos(x)$, $-\sin(x)$, is greater than 1 or less than -1 . However, $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, so we have a contradiction.

29.3

- a) $\frac{f(2) - f(0)}{2 - 0} = 1/2$, so, by the mean value theorem, $f'(x) = 1/2$ for some $x \in (0, 2)$.
- b) $\frac{f(2) - f(1)}{2 - 1} = 0$, so, by the mean value theorem, $f'(x_1) = 0$ for some $x_1 \in (0, 2)$. From part (a) we saw there is an $x_2 \in (0, 2)$ such that $f'(x_2) = 1/2$. Hence, by the intermediate value theorem for derivatives, $f'(x) = 1/7$ for some $x \in (x_1, x_2) \subseteq (0, 2)$.

29.4

Define $h(x) = f(x)e^{g(x)}$. Hence, $h(x)$ is differentiable on the same open interval I . Hence, h is continuous on $[a, b]$, differentiable on (a, b) , and $h(a) = h(b) = 0$. Therefore, by Rolle's Theorem, there exists $x_0 \in (a, b)$ such that $h'(x_0) = 0$. Note that $h'(x) = (f'(x) + f(x)g'(x))e^{g(x)}$ and $e^{g(x_0)} > 0$ for any $g(x_0) \in \mathbb{R}$, so $h'(x_0) = 0$ implies $f'(x_0) + f(x_0)g'(x_0) = 0$.

29.5

$|f(x) - f(y)| \leq (x - y)^2$ implies $|\frac{f(x) - f(y)}{x - y}| \leq |x - y|$. Since this holds for all $x, y \in \mathbb{R}$, we have $\lim_{x \rightarrow y} |\frac{f(x) - f(y)}{x - y}| = f'(y) = 0$, because $|x - y| < \epsilon$ guarantees $|\frac{f(x) - f(y)}{x - y}| < \epsilon$. Thus, f is differentiable for all $y \in \mathbb{R}$ and $f'(y) = 0$, so by corollary 29.4, f is a constant function.

29.6

$$L(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

2 The Mean Value Theorem

29.7

a) $f''(x) = 0$ on I , so $f'(x) = a$ for constant a by corollary 29.4. The function $g(x) = ax$ has derivative $g'(x) = a$. Since g and f have the same derivative on I , then, by corollary 29.5, there exists constant b such that $f(x) = g(x) + b = ax + b$ on I . b) Similarly to part (a), we can show $f(x) = ax^2 + bx + c$.

29.8