Solution Manual to Elementary Analysis, 2nd Ed., by Kenneth A. Ross

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December 18, 2018

Contents

1 Basic Properties of the Derivative

Note: In this section, we make routine use of the fact that $\lim_{x\to a}$ is evaluated on sets $J = I \setminus \{a\}$ so that, for example, $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x$ is allowed, despite the cancellation not being valid at $x = 0$.

28.1
a) $\{0\}$
b) $\{0\}$
c) $\{\pi n | n \in \mathbb{Z}\}$
d) $\{0, 1\}$
e) $\{-1, 1\}$
f) $\{2\}$

28.2
a)$(x^3 - 8) = (x - 2)(x^2 + 4x + 4)$,
so, $\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2}(x^2 + 2x + 4) =$ (by 20.4) $4 + 4 + 4 = 12$
b)$\lim_{x\to a} \frac{(x+2)-(a+2)}{x-a} = \lim_{x\to a} 1 = 1$
c)$\lim_{x\to 0} \frac{x^2\cos(x)-0}{x-0} = \lim_{x\to 0} x\cos(x) = 0 * 1 = 0$
d)$\lim_{x\to 1} \frac{\frac{3x+4}{2x-1}-7}{x-1} = \lim_{x\to 1} \frac{(3x+4)-(14x-7)}{(2x-1)(x-1)} = \lim_{x\to 1} \frac{(-11)(x-1)}{(2x-1)(x-1)} = -11$

28.3
a)$(x - a) = (\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a})$,
so, $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$ for $a > 0$.
Note: For $\lim_{x\to a} f(x)$ to exist, we require that the limit converge for all (x_n) in a set $J = I \setminus \{a\}$ for some open interval I containing a. However, $\sqrt{x}$ is not defined on an open interval around 0, so the limit does not exist there.
b) Similarly to part (a), $\lim_{x\to a} \frac{x^{1/3}-a^{1/3}}{x-a} = \lim_{x\to a} \frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{1}{3a^{2/3}}$ for $a \neq 0$.
c) $f(x) = x^{1/3}$ is not differentiable at $x = 0$. $\lim_{x\to 0} \frac{x^{1/3}-0^{1/3}}{x-0} = \lim_{x\to 0} \frac{1}{x^{2/3}}$ and this limit does not exist, because the limits approaching from the left and right are distinct ($-\infty$ and ∞).

28.4
a) First, notice $\sin(x)$ is differentiable on $\mathbb{R}$ and therefore on $\mathbb{R} \setminus \{0\}$.
Since 1 and x are differentiable on $\mathbb{R}$ and $x \neq 0$ on $\mathbb{R} \setminus \{0\}$, $\frac{1}{x}$, by theorem 28.3, is differentiable on $\mathbb{R} \setminus \{0\}$.
Clearly x^2 is differentiable on $\mathbb{R} \setminus \{0\}$.
Therefore, by 28.3, $f(x) = x^2 \sin(\frac{1}{x})$ is differentiable on $\mathbb{R} \setminus \{0\}$ and $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$.
b) By def 28.1 $f'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x-0} = \lim_{x\to 0} x \sin(\frac{1}{x})$. By the squeeze theorem, this limit is 0. c) Since our choice of sequence $(x_n) \to 0$ can influence $\lim n \to \infty f'(x_n)$, we conclude $\lim n \to 0 f'(x)$ does not exist, so $f'(x)$ is discontinuous there.
Note: Recall theorem 28.2 showed that, if f is differentiable at a, then it is continuous there. This exercise has shown that while any f differentiable at a must be continuous at a, the corresponding derivative function, f', need not be.

28.5
a) f is differentiable on $\mathbb{R}$ by exercise 28.4. g is alos differentiable on $\mathbb{R}$, obviously.
b) 0.
c) We can find x arbitrarily close to 0 for which the argument of the limit is not defined. Hence, there is no open interval I around 0 on which the argument of the limit is defined, so the limit cannot be evaluated.

28.6
a) f is continuous, since $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem, and $f(0) = 0$.
b) f is not differentiable at $x = 0$.
$\lim_{x\to 0} \frac{x \sin(1/x) - f(0)}{x-0} = \lim_{x\to 0} \frac{x \sin(1/x)}{x} = \lim_{x\to 0} \sin(1/x)$ which does not exist.

28.7
a) [Graph Not Shown]
b) $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ is not obvious, but $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = 0$ and $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x = 0$, so, by theorem, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. Therefore, f is differentiable at $x = 0$.
c) By section 28, example 3, we have $f'(x) = 2x$ for $x > 0$. In part (b), we showed $f'(0) = 0$, and clearly $f'(x) = 0$ for $x < 0$.
d) Continuous: Yes. Differentiable: No.

28.8
a) $f(0) = 0$. Let $\epsilon > 0$. Let $\delta = \sqrt{\epsilon}$. $|f(x) - 0| = ||x|^2 - 0|$ or $|0 - 0|$. Clearly, $||x|^2 - 0| = |x^2 - 0|$. If $|x - 0| < \delta$ then $|x^2 - 0| < |\delta^2| = \epsilon$. $|0 - 0| < \epsilon$ trivially. Hence, $|f(x) - 0| < \epsilon$.
b) Since the rationals and irrationals are dense, there are rational numbers arbitrarily close to each irrational number, and there are irrational numbers arbitrarily close to each rational number. Let $x_0 \neq 0$, we know $x^2 > 0$, so there is an open interval I around x^2 such that $x^2 > \epsilon$ for some $\epsilon > 0$. Since x^2 is continuous, there is a corresponding open interval G around x_0 that is the inverse image of I. Select rational x_r from G. For arbitrarily close irrational numbers $x_i \in G$, $f(x_r) - f(x_i) > \epsilon - 0$ so f is discontinuous for all $x \neq 0$.
c) $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} f(x)x$
$\lim_{x\to 0^{\mathbb{Q}}} \frac{f(x)}{x} = \lim_{x\to 0^{\mathbb{Q}}} \frac{x^2}{x} = \lim_{x\to 0^{\mathbb{Q}}} x = 0$
$\lim_{x\to 0^{\mathbb{R}\setminus\mathbb{Q}}} \frac{f(x)}{x} = 0$
So, by theorem, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0 = f'(0)$.

28.9
a)$h'(x) = 7(x^4 + 13x)^6(4x^3 + 13)$
b)$h(x) = g \circ f(x)$ where $g(x) = x^7$ and $f(x) = x^4 + 13x$

28.10
a)$h'(x) = 12(\cos(x) + e^x)^{11}(-\sin(x) + e^x)$
b)$h(x) = g \circ f(x)$ where $g(x) = x^{12}$ and $f(x) = \cos(x) + e^x$

28.11
$(h \circ g \circ f)'(a)$: Extended Chain Rule
By theorem 28.4, since f is differentiable at x, and g is differentiable at $f(a)$, $g \circ f$ is differentiable at a. This, along with the fact that h is differentiable at $g \circ f(a)$, shows that, by theorem 28.4, $h(g \circ f) = h\ circg \circ f$ is differentiable at a, and $(h \circ g \circ f)'(a) = h'(g \circ f(a))(g \circ f)'(a)$. As stated before, by 28.4 we have $g \circ f$ differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$. Hence, $(h \circ g \circ f)'(a) = h'(g \circ f(a))g'(f(a))f'(a)$.

28.12
a) $-\sin(e^{x^5-3x})e^{x^5-3x}(5x^4 - 3)$
b) $\cos(e^{x^5-3x}) = h \circ g \circ f(x)$ where $h(x) = \cos(x)$, $g(x) = e^x$, and $f(x) = x^5 - 3x$.

28.13
Let I be an open interval containing $f(a)$ on which g is defined. Naturally, we can find some $\epsilon > 0$ such that $(f(a) - \epsilon, f(a) + \epsilon) \subseteq I$. Since f is continuous at a, there exists δ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Since f is defined on some open interval G containing a, it is defined on $G \cap (a - \delta, a + \delta)$. Notice that $x \in G \cap (a - \delta, a + \delta)$ implies $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$, so g is defined at $f(x)$. Therefore, we conclude $g \circ f$ is defined on $x \in G \cap (a - \delta, a + \delta)$.

28.14

a) For every sequence (h_n) in $\mathbb{R}$ that converges to 0, there exists (x_n) in $\mathbb{R}$ such that $x_n = h_n + a$. Clearly, $(x_n) \to 0 + a = a$. Since f is differentiable at a, we know $\lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n-a}$ exists and is finite. So, for all $(h_n) \to 0$,

$$\begin{aligned} \lim_{n\to\infty} \frac{f(a+h_n)-f(a)}{h_n} &= \lim_{n\to\infty} \frac{f(a+h_n)+f(a)}{a+h_n-a} \\ &= \lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n-a} \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= f'(a). \end{aligned}$$

Therefore, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$.

b) From part (a), $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$. So, by renaming h as $-h$, $\lim_{-h\to 0} \frac{f(a-h)-f(a)}{-h} = f'(a)$. Further, $-h \to 0$ as $h \to -0$, so $\lim_{h\to -0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a)-f(a-h)}{h} = f'(a)$. Thus,

$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{2h} + \frac{f(a)-f(a-h)}{2h} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{2h} + \lim_{h\to 0} \frac{f(a)-f(a-h)}{2h} \\ &= f'(a)/2 + f'(a)/2 \\ &= f'(a). \end{aligned}$$

28.15

Assume f and g have n derivatives at a. By the product rule (theorem 28.3 iii) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$. Suppose now that $(fg)^{(n-1)}(a) = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)}(a)g^{(n-1-k)}(a)$. Then,

$$\begin{aligned} (fg)^{(n)}(a) &= [(fg)^{(n-1)}]'(a) \\ &= \sum_{k=0}^{n-1} \left(\binom{n-1}{k} f^{(k)}(a)g^{(n-k)}(a) + \binom{n-1}{k} f^{(k+1)}(a)g^{(n-k-1)}(a) \right) \\ &= f(a)g^{(n)}(a) + \left(\sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] f^{(k)}(a)g^{(n-k)}(a) \right) + f^{(n)}(a)g(a) \\ &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a) \end{aligned}$$

So the theorem holds true by mathematical induction.

28.16
Let f be a function defined on an open interval I containing a.

First, assume $f'(a)$ exists (in other words, f is differentiable at a). Define $\epsilon(x) = f'(a) - \frac{f(x)-f(a)}{x-a}$ for $x \neq a$ and $\epsilon(a) = f'(a)$. Clearly, since f is defined on I, ϵ is defined on I. When $x = a$, $f(x) - f(a) = (x-a)[f'(a) - \epsilon(a)]$ trivially. When $x \neq a$,

$$\begin{aligned} f(x) - f(a) &= (x-a)f'(a) - \big[(x-a)f'(a) - [f(x) - f(a)]\big] \\ &= f(x) - f(a). \end{aligned}$$

Finally,

$$\begin{aligned} \lim_{x \to a} \epsilon(x) &= \lim_{x \to a} f'(a) - \frac{f(x)-f(a)}{x-a} \\ &= f'(a) - \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \\ &= f'(a) - f'(a) \\ &= 0. \end{aligned}$$

So we have identified $\epsilon(x)$ defined on I that exhibits the desired properties.

Now assume $\epsilon(x)$ exists on I with the above properties, and we will show $f'(a)$ exists. So $\frac{f(x)-f(a)}{x-a} = f'(a) - \epsilon(x)$, and

$$\lim_{x \to a^{I \setminus \{a\}}} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^{I \setminus \{a\}}} f'(a) - \epsilon(x) = f'(a)$$

So f is differentiable at a.

2 The Mean Value Theorem

29.1
a) $x = 1/2$
b) $x = \pi/2$
c) Not differentiable at $x = 0$
d) Not continuous or differentiable at $x = 0$
e) $x = \sqrt{3}$ f) Not continuous or differentiable at $x = 0$

29.2
Suppose there exist $x, y \in \mathbb{R}$, $x > y$ such that $|\cos(x) - \cos(y)| > |x - y|$. This implies $|\frac{\cos(x)-\cos(y)}{x-y}| > 1$. By the Mean Value Theorem, we must have at least one $x_0 \in (x, y)$ such that the derivative of $\cos(x)$, $-\sin(x)$, is greater than 1 or less than -1. However, $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, so we have a contradiction.

29.3
a) $\frac{f(2)-f(0)}{2-0} = 1/2$, so, by the mean value theorem, $f'(x) = 1/2$ for some $x \in (0, 2)$.
b) $\frac{f(2)-f(1)}{2-1} = 0$, so, by the mean value theorem, $f'(x_1) = 0$ for some $x_1 \in (0, 2)$. From part (a) we saw there is an $x_2 \in (0, 2)$ such that $f'(x_2) = 1/2$. Hence, by the intermediate value theorem for derivatives, $f'(x) = 1/7$ for some $x \in (x_1, x_2) \subseteq (0, 2)$.

29.4
Define $h(x) = f(x)e^{g(x)}$. Hence, $h(x)$ is differentiable on the same open interval I. Hence, h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$. Therefore, by Rolle's Theorem, there exists $x_0 \in (a, b)$ such that $h'(x_0) = 0$. Note that $h'(x) = (f'(x) + f(x)g'(x))e^{g(x)}$ and $e^{g(x_0)} > 0$ for any $g(x_0) \in \mathbb{R}$, so $h'(x_0) = 0$ implies $f'(x_0) + f(x_0)g'(x_0) = 0$.

29.5
$|f(x) - f(y)| \leq (x - y)^2$ implies $|\frac{f(x)-f(y)}{x-y}| \leq |x - y|$. Since this holds for all $x, y \in \mathbb{R}$, we have $\lim_{x \to y} |\frac{f(x)-f(y)}{x-y}| = f'(y) = 0$, because $|x - y| < \epsilon$ guarantees $|\frac{f(x)-f(y)}{x-y}| < \epsilon$. Thus, f is differentiable for all $y \in \mathbb{R}$ and $f'(y) = 0$, so by corollary 29.4, f is a constant function.

29.6
$L(x) = \frac{f(b)-f(a)}{b-a}(x - a) + f(a)$

29.7
a) $f''(x) = 0$ on I, so $f'(x) = a$ for constant a by corollary 29.4. The function $g(x) = ax$ has derivative $g'(x) = a$. Since g and f have the same derivative on I, then, by corollary 29.5, there exists constant b such that $f(x) = g(x) + b = ax + b$ on I. b) Similarly to part (a), we can show $f(x) = ax^2 + bx + c$.

29.8
ii) Suppose $a < x_1 < x_2 < b$, yet $f(x_2) \geq f(x_1)$. Thus, by the Mean Value Theorem, there exists x_0 such that $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \geq 0$, but $f'(x) < 0$ for all $x \in (a, b)$, so we have a contradiction.
iii) Suppose $a < x_1 < x_2 < b$, yet $f(x_2) < f(x_1)$. Thus, by the Mean Value Theorem, there exists x_0 such that $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1} < 0$, but $f'(x) \geq 0$ for all $x \in (a, b)$, so we have a contradiction.
iv) Suppose $a < x_1 < x_2 < b$, yet $f(x_2) > f(x_1)$. Thus, by the Mean Value Theorem, there exists x_0 such that $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$, but $f'(x) \leq 0$ for all $x \in (a, b)$, so we have a contradiction.

29.9
Let $f(x) = e^x - ex$. Notice that $f(1) = 0$. $f'(x) = e^x - e$, and $f''(x) = e^x$. $f''(x) = e^x$ is positive for all $x \in \mathbb{R}$, so, by corollary 29.7, f' is strictly increasing. Since f' is strictly increasing and $f'(1) = 0$, $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. Hence, by corollary 29.7, f is strictly decreasing for $x < 1$ and strictly increasing for $x > 1$. Thus, for all $x < 1$, $f(x) > f(1) = 0$ and for all $x > 1$, $f(x) > f(1) = 0$, so $f(x) \geq 0$ for all $x \in \mathbb{R}$. Therefore, $ex \leq e^x$ for all $x \in \mathbb{R}$.