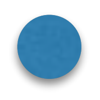
Points and Vectors

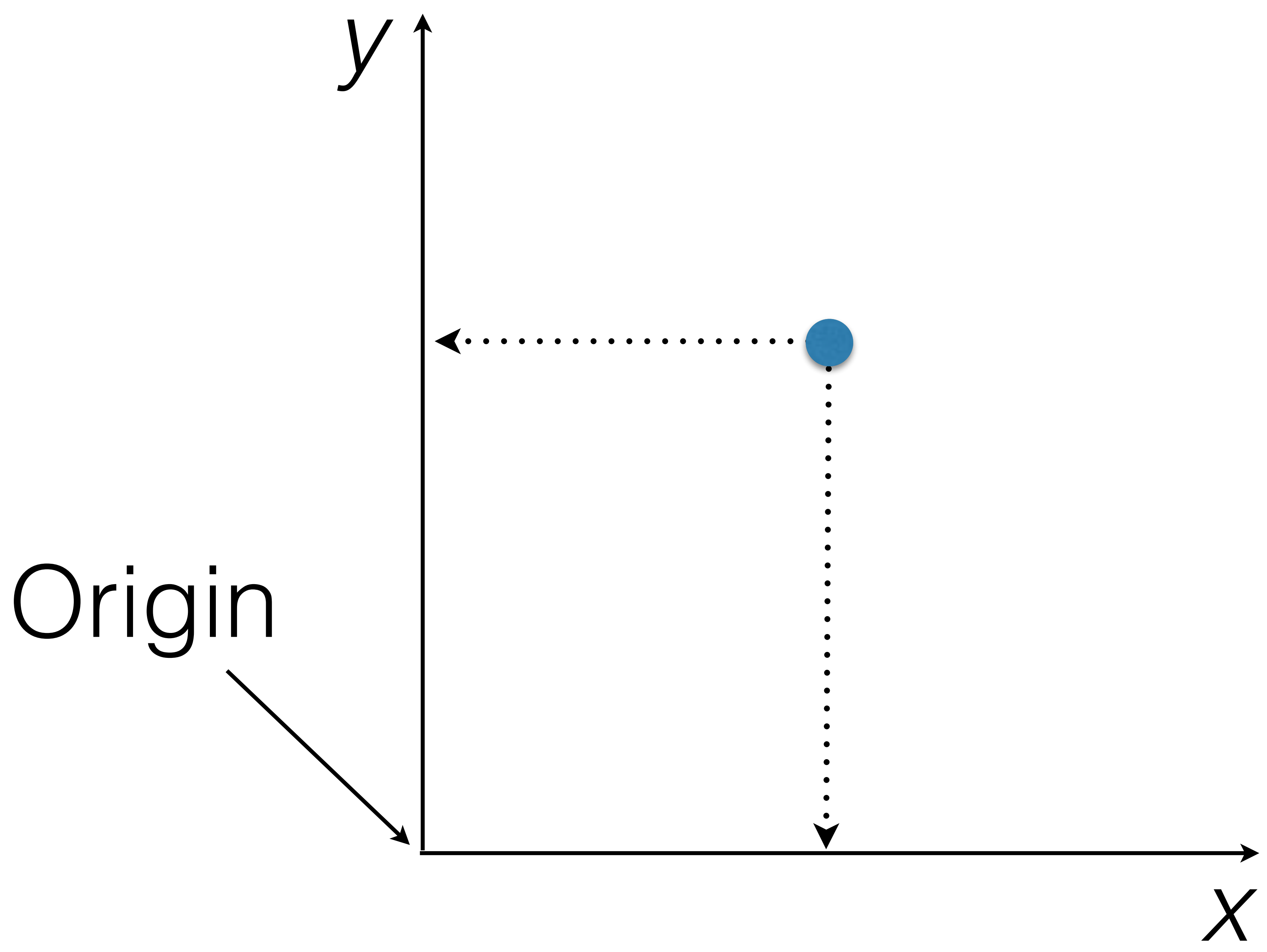
CS 355: Introduction to Graphics and Image Processing

Describing Points



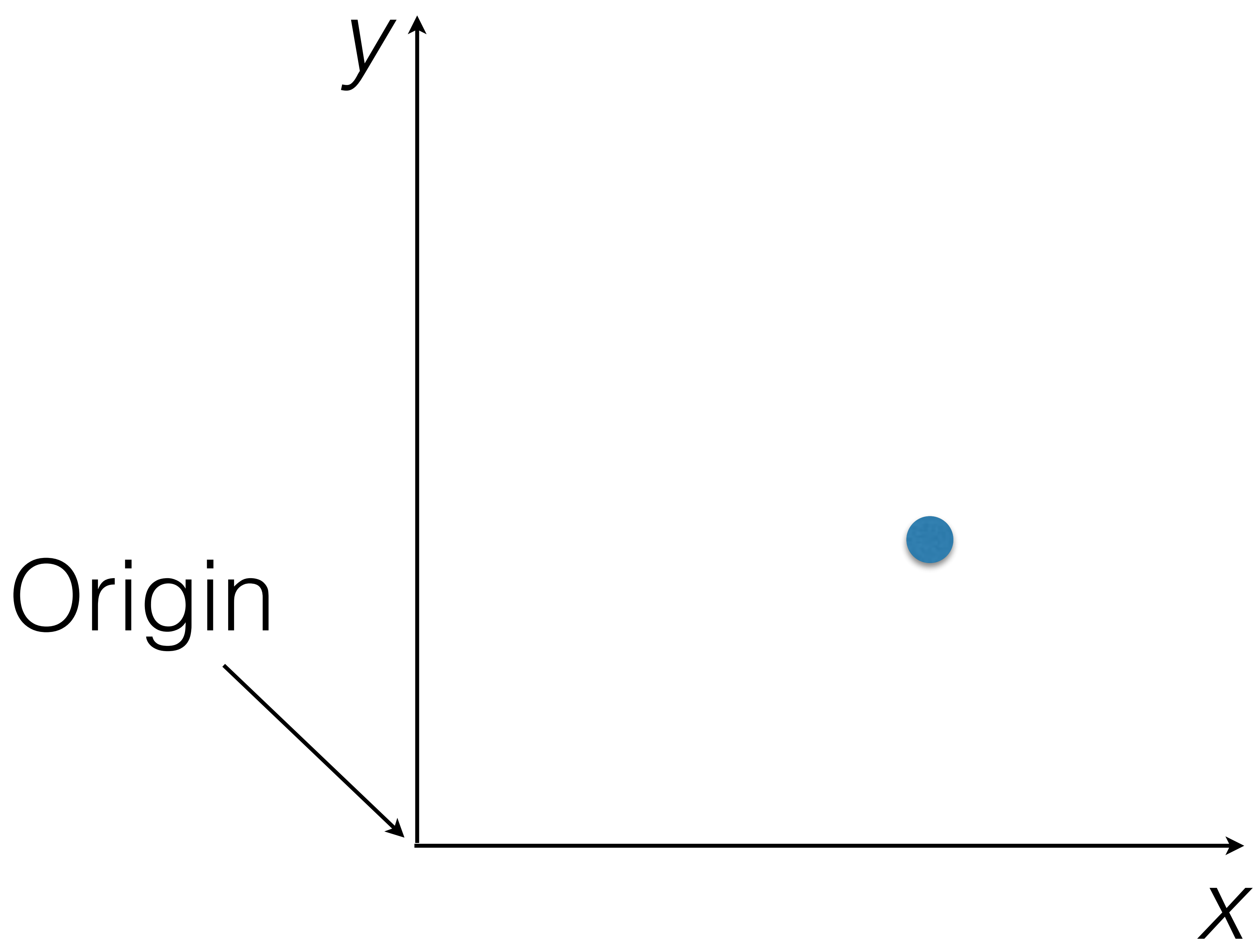
How do you describe this point <u>numerically</u>?

Coordinate Systems



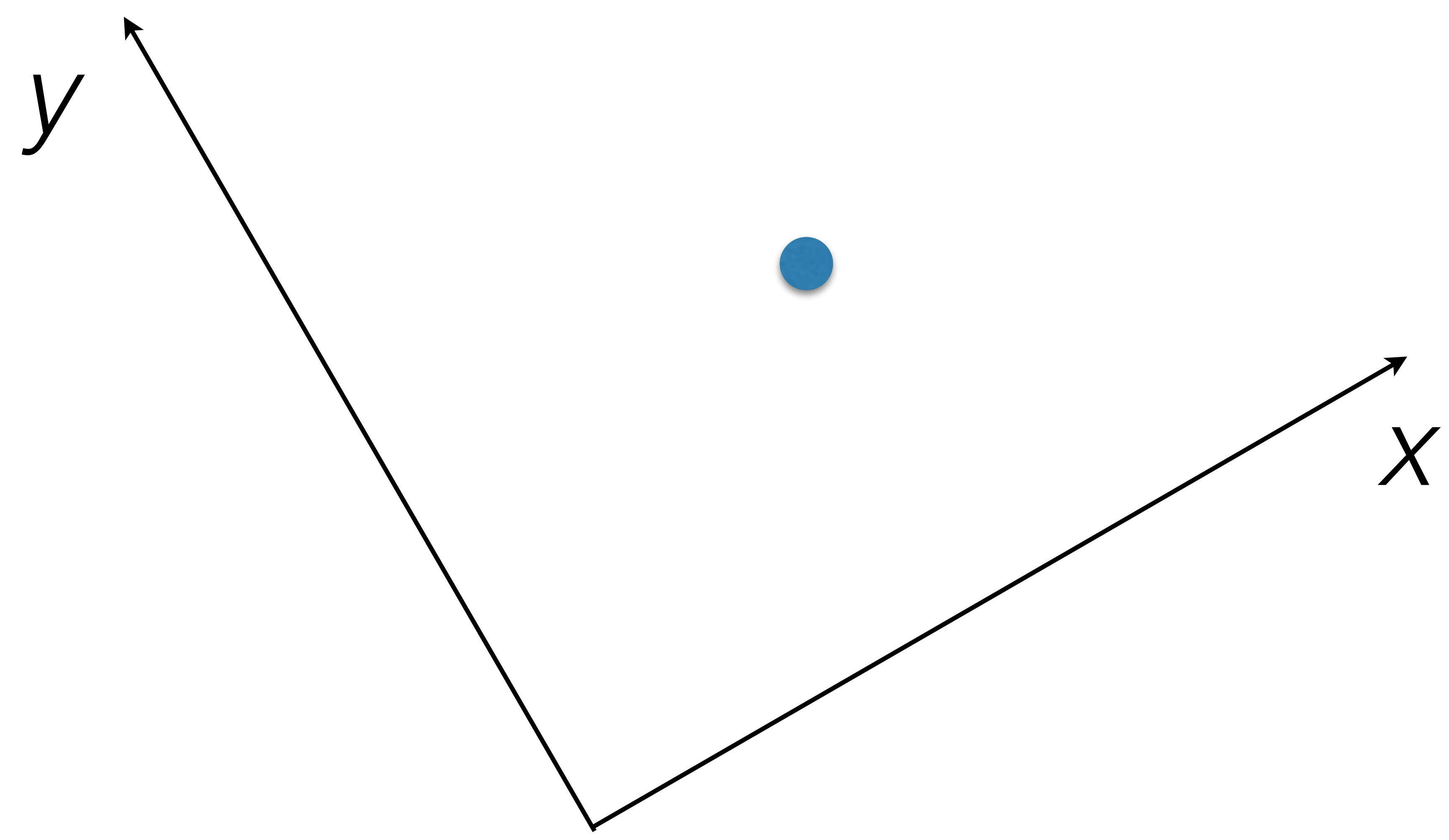
How do you describe this point <u>numerically</u>?

Coordinate Systems



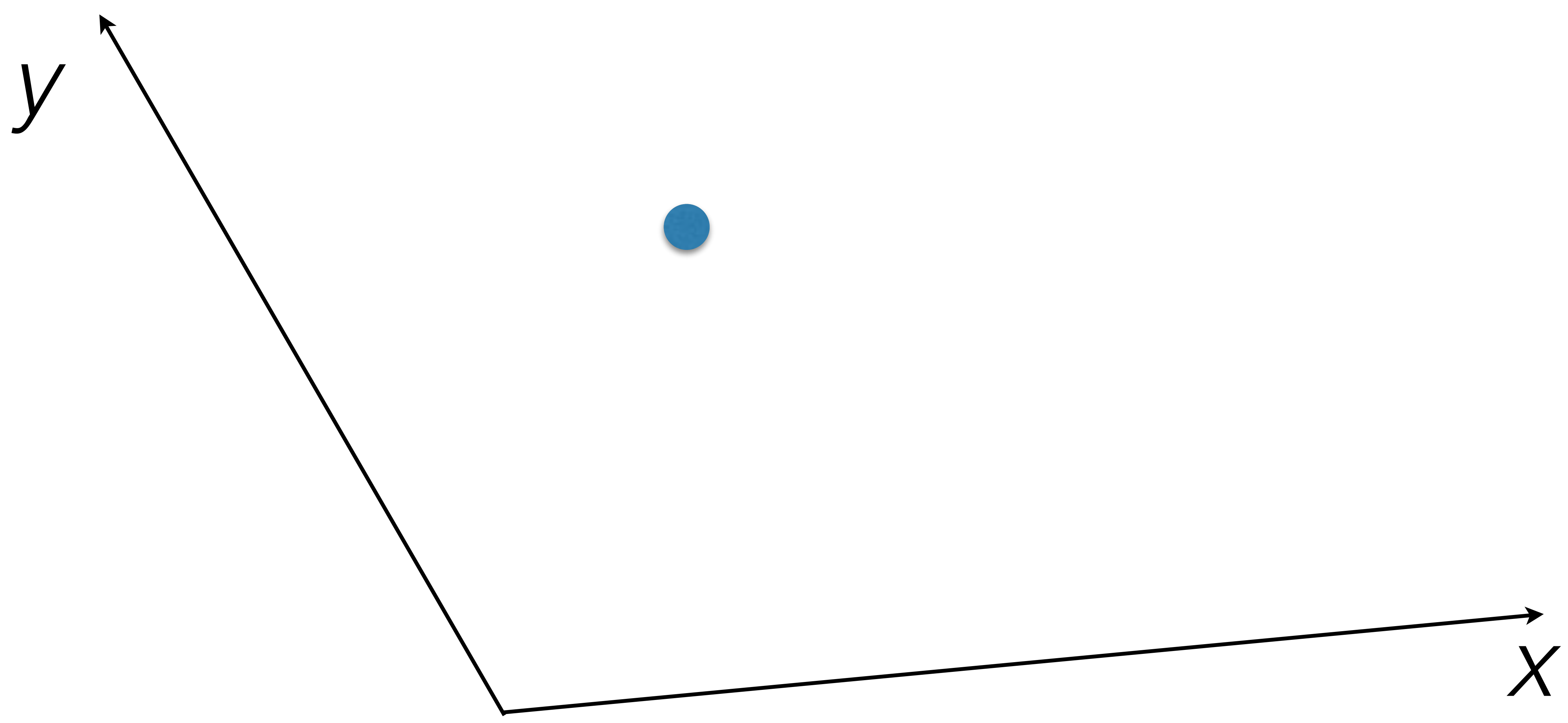
How about this coordinate system?

Coordinate Systems



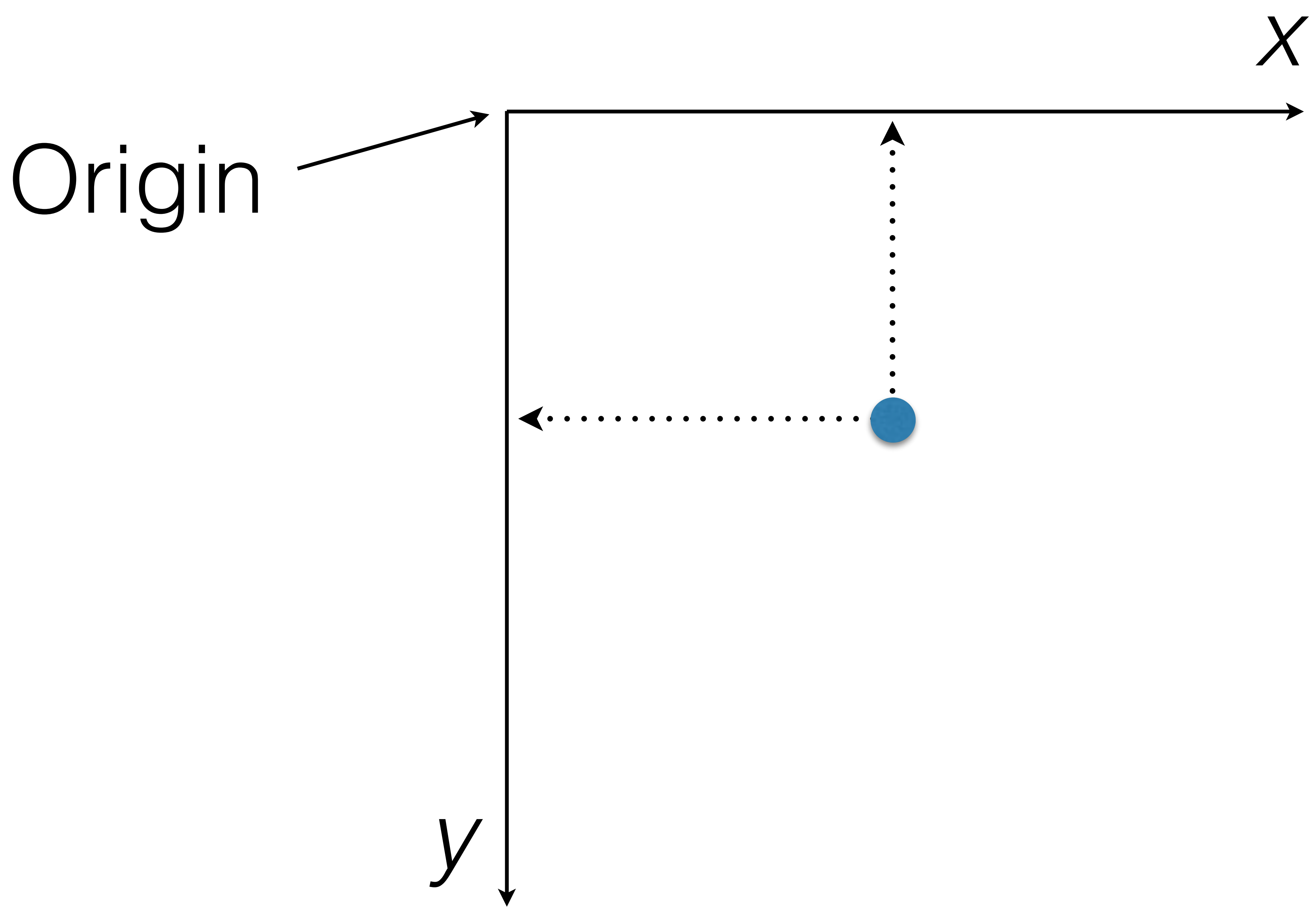
Or this one?

Coordinate Systems



What about this one?

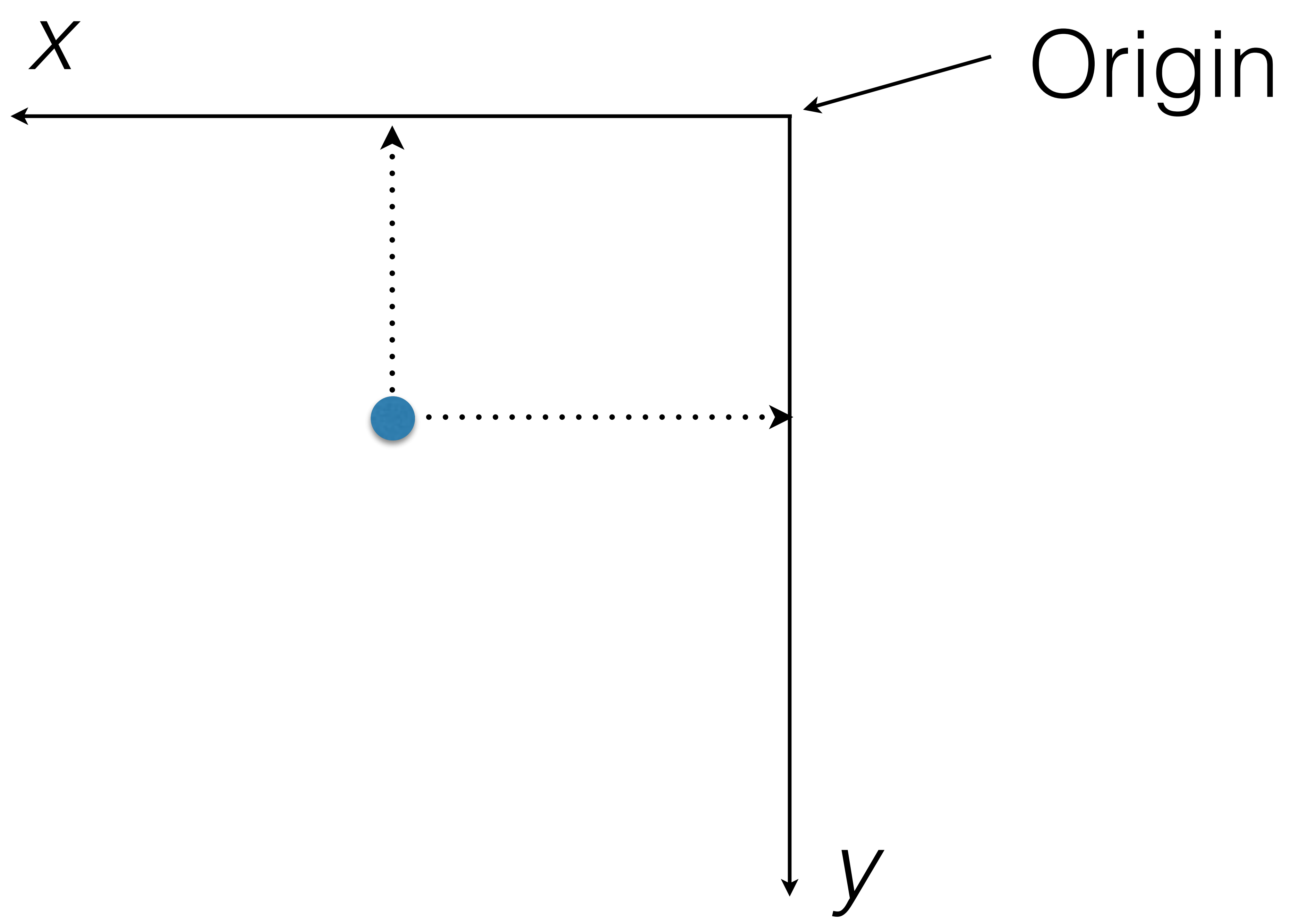
Coordinate Systems



(This is what most screens use)

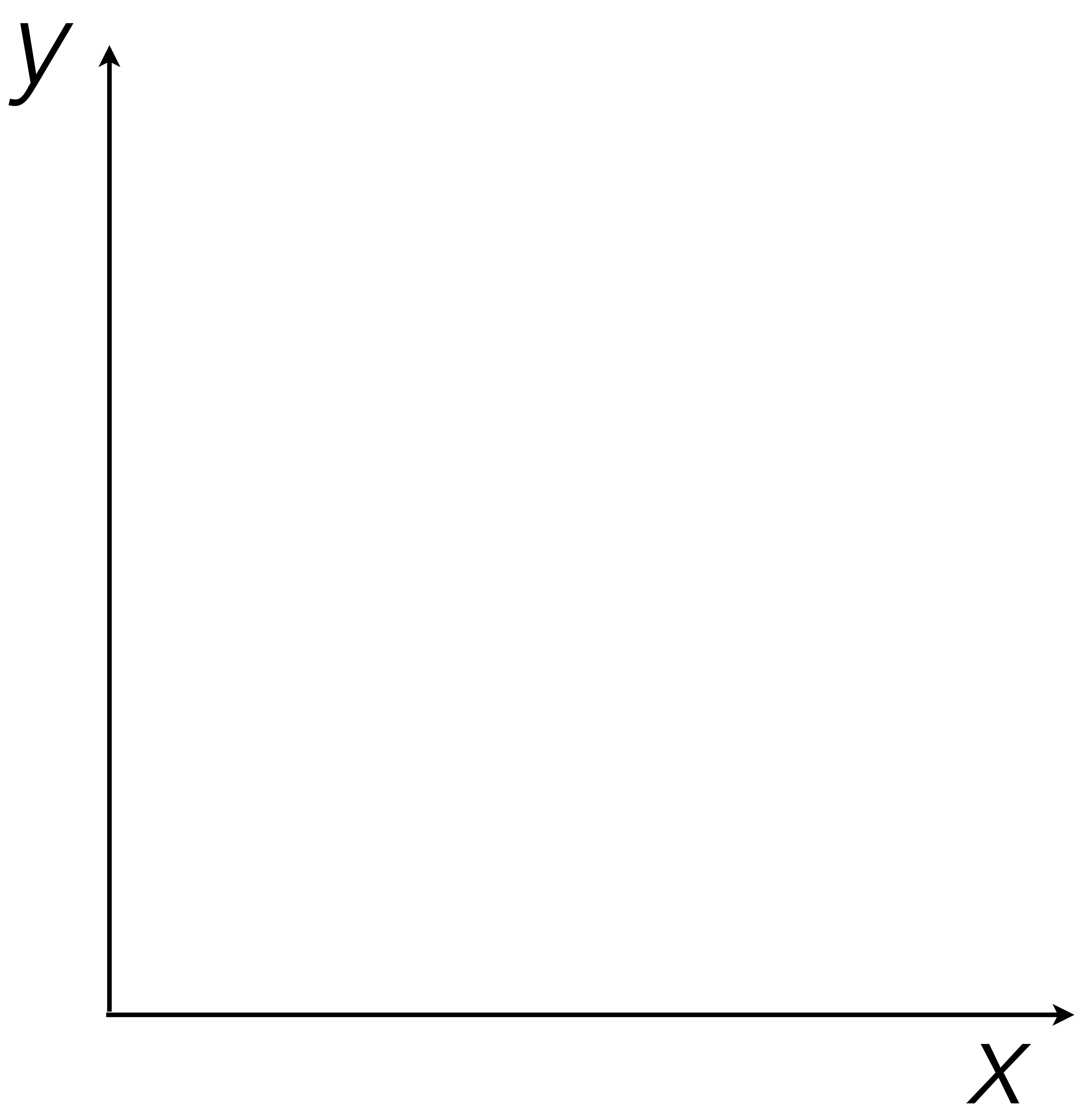
Why not this?

Coordinate Systems

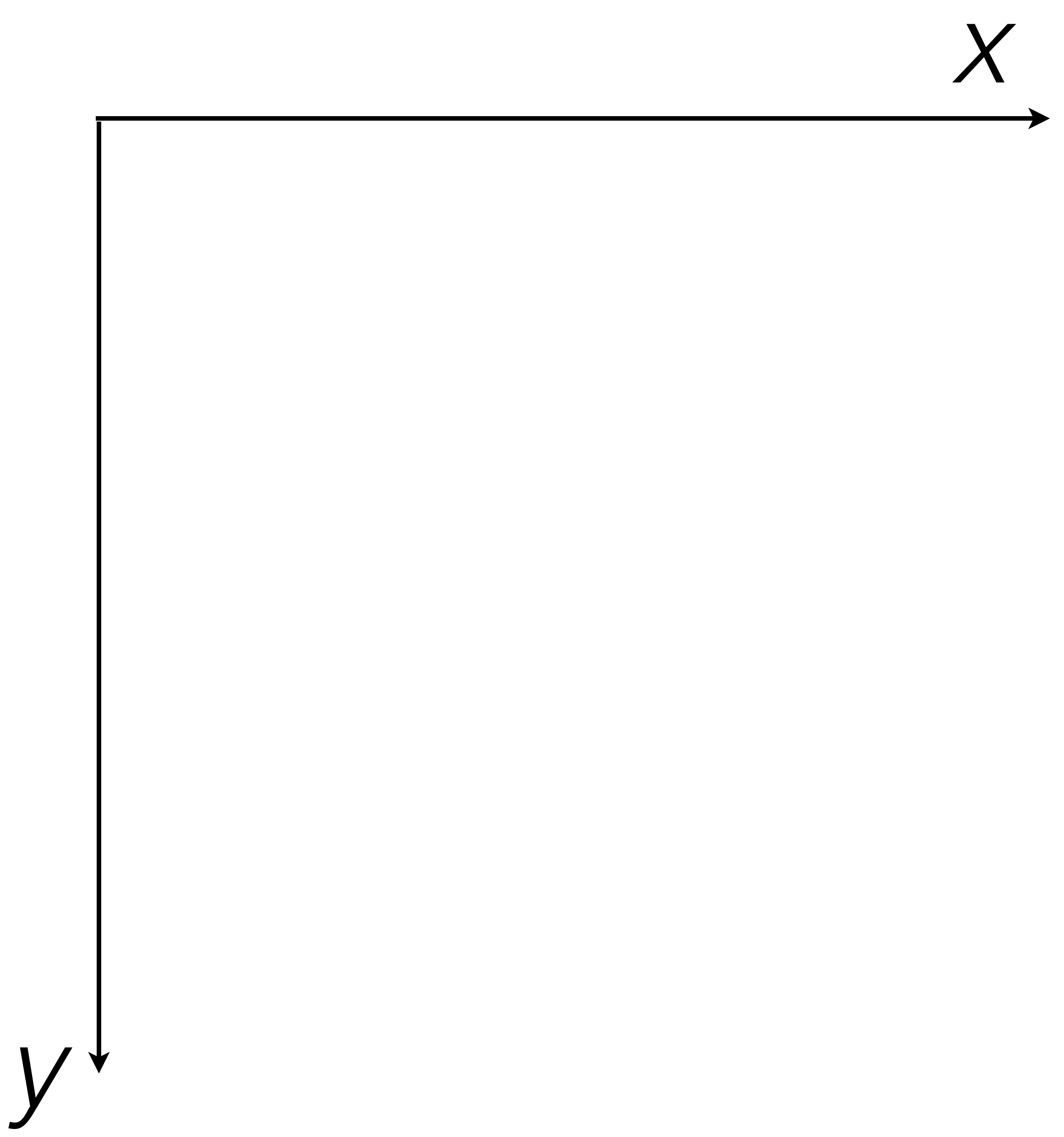


Or this?

Coordinate Systems

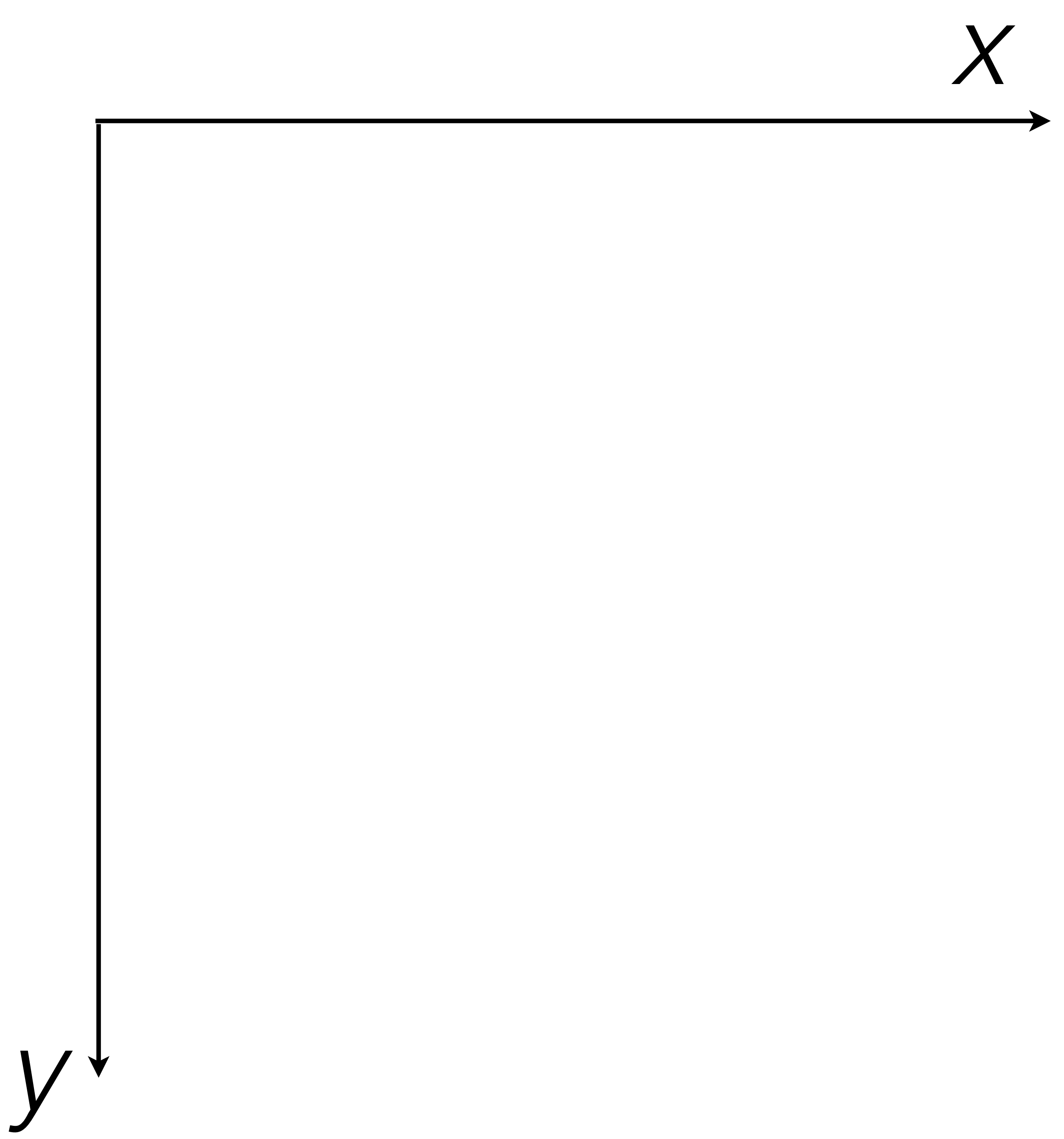


Math teachers
("right handed")

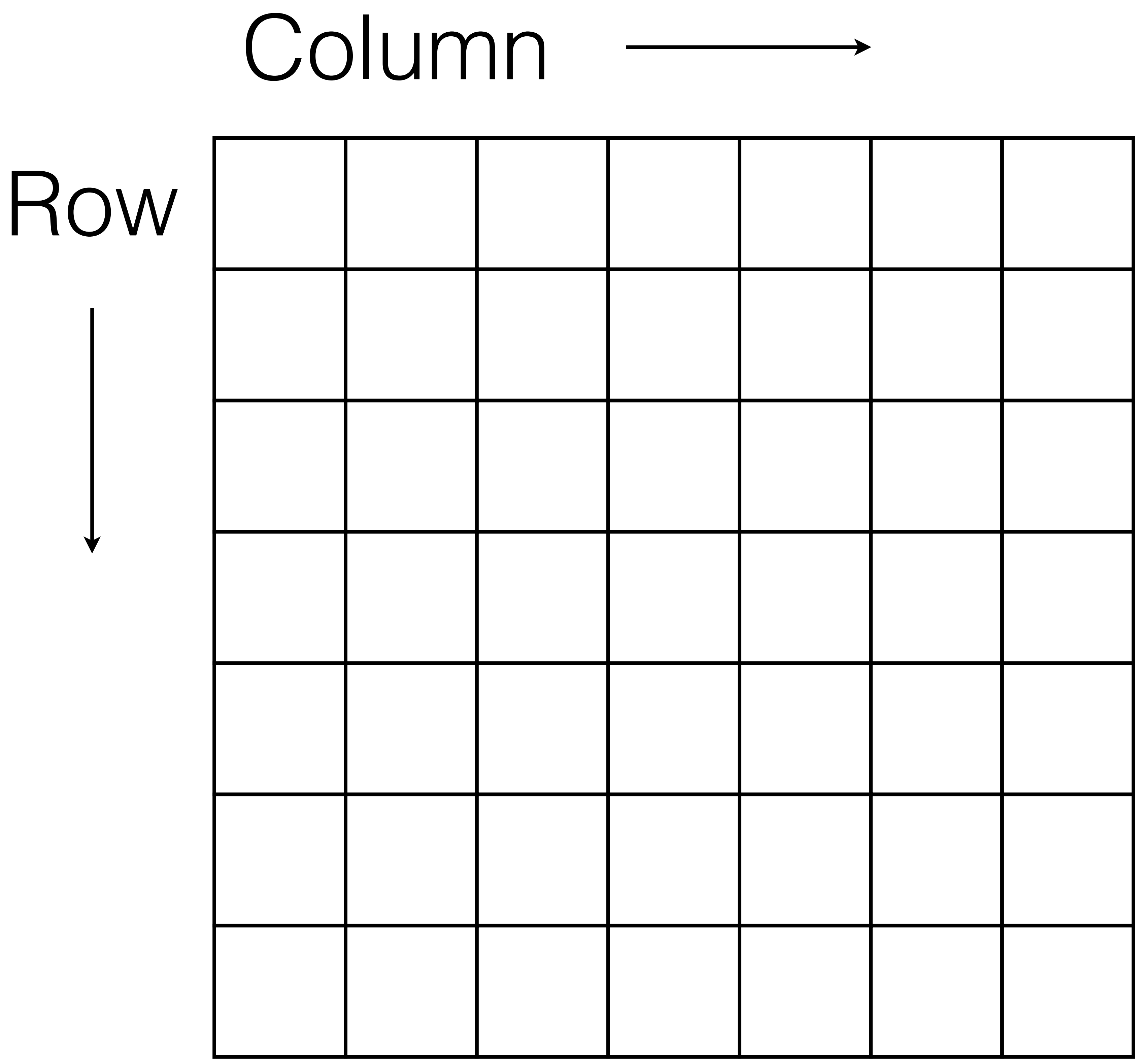


Computer screens
("left handed")

Math vs. Code

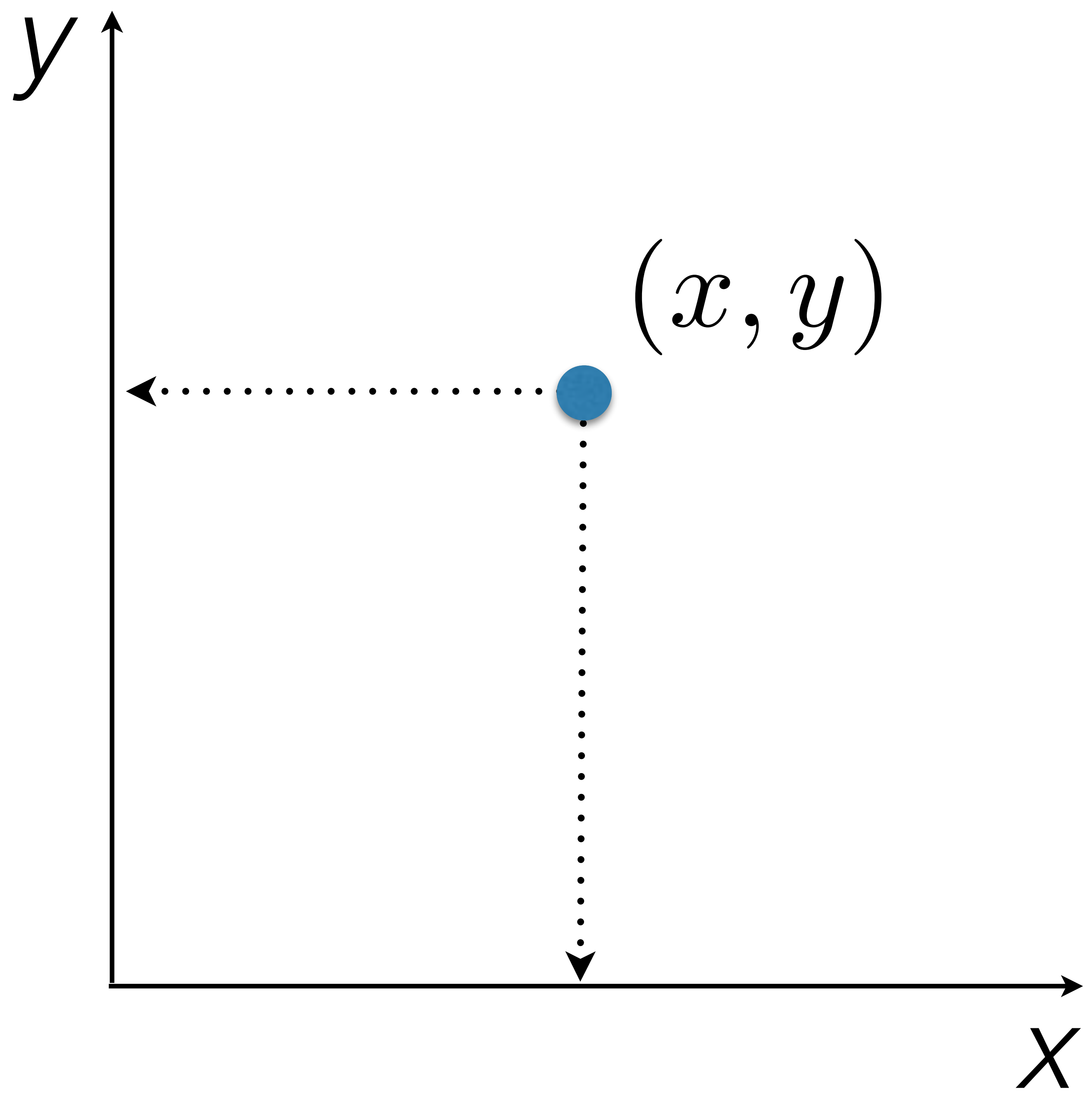


How we draw
(x,y)



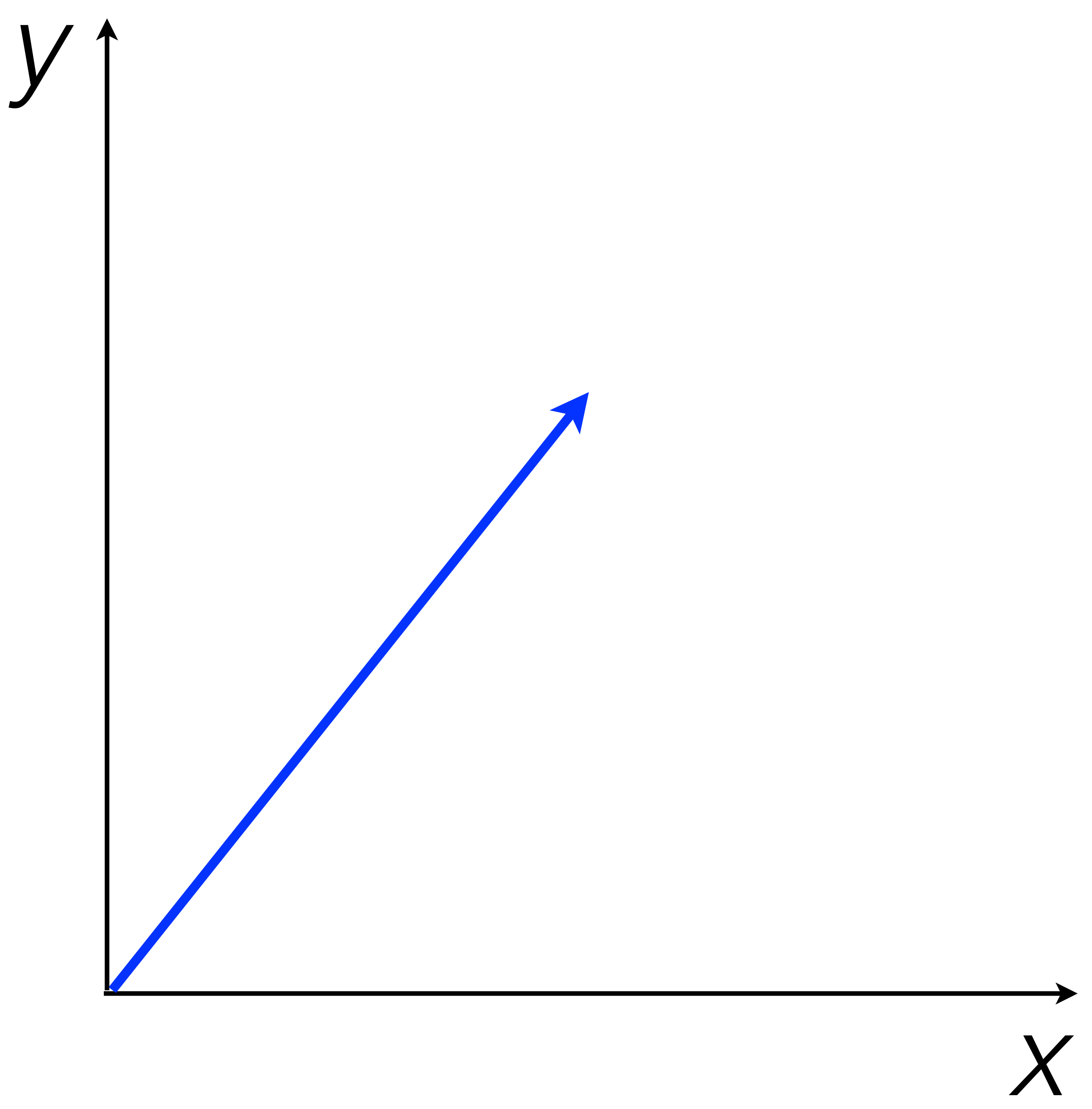
How we store
(row,col)

Points



Points can be described by their Cartesian coordinates.
Coding: use short arrays or n-tuples

Vectors



Vectors can also be thought of
in terms of Cartesian coordinates

Vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

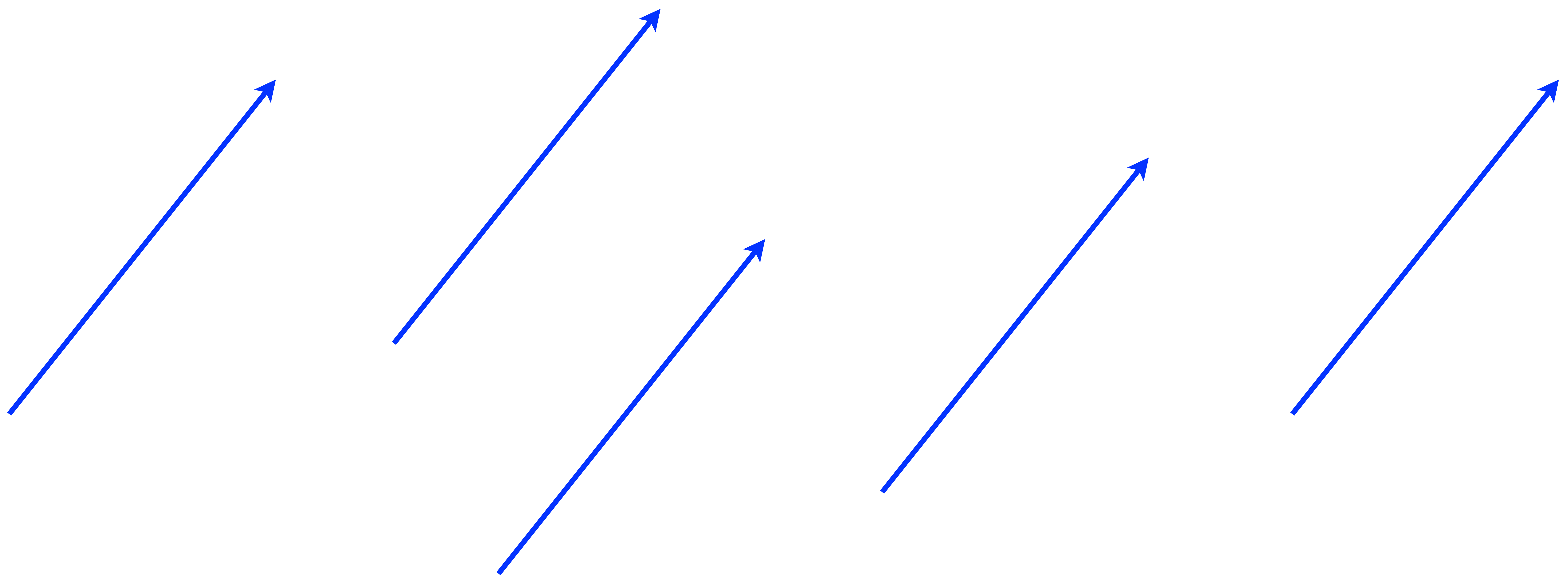
Coding: just store in an array

Vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

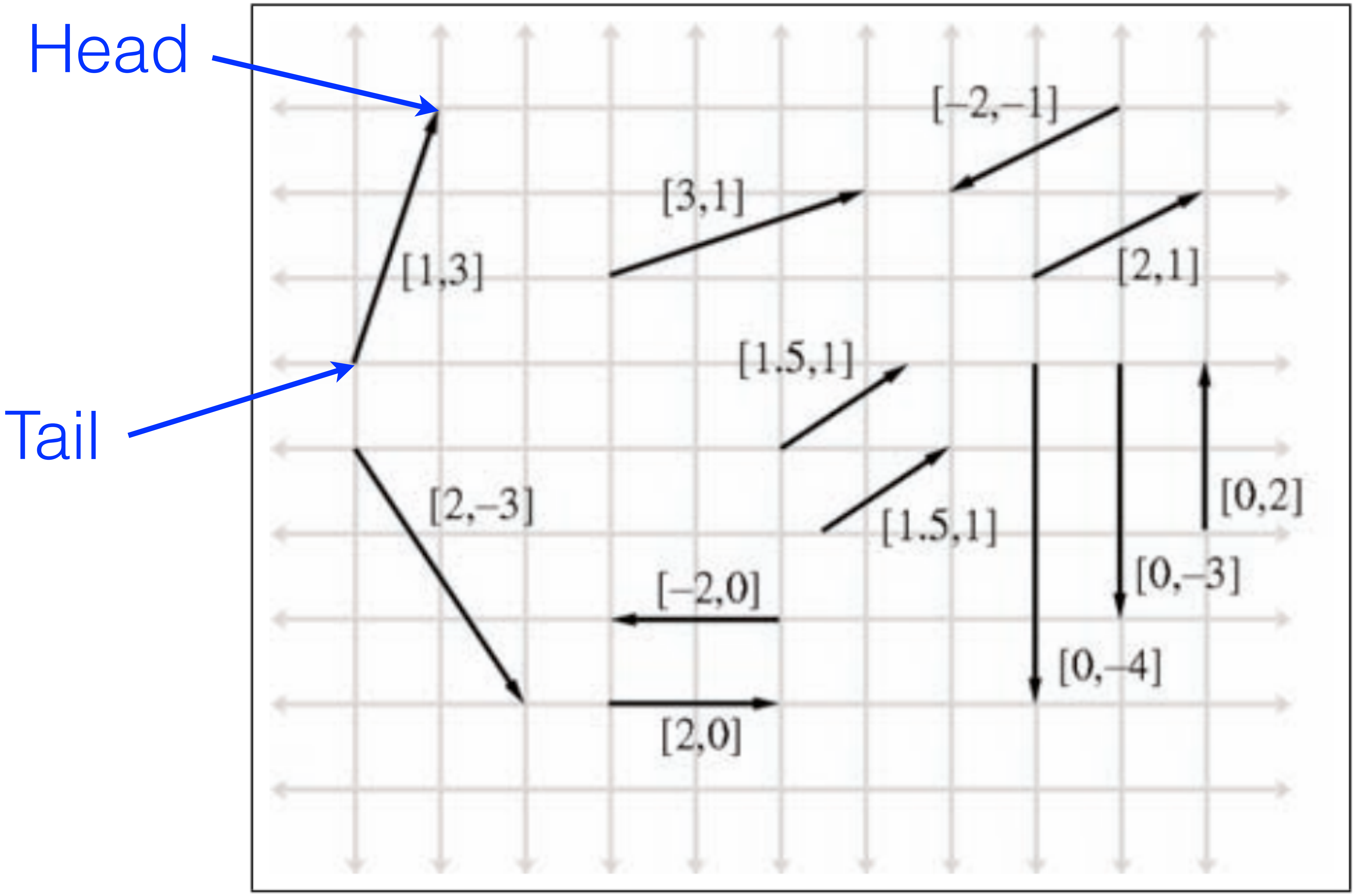
Often use subscripts to denote elements of a vector.
Coding: use `a[i]`

Vectors



Vectors are directional quantities
without a specific location

Vectors as Displacements

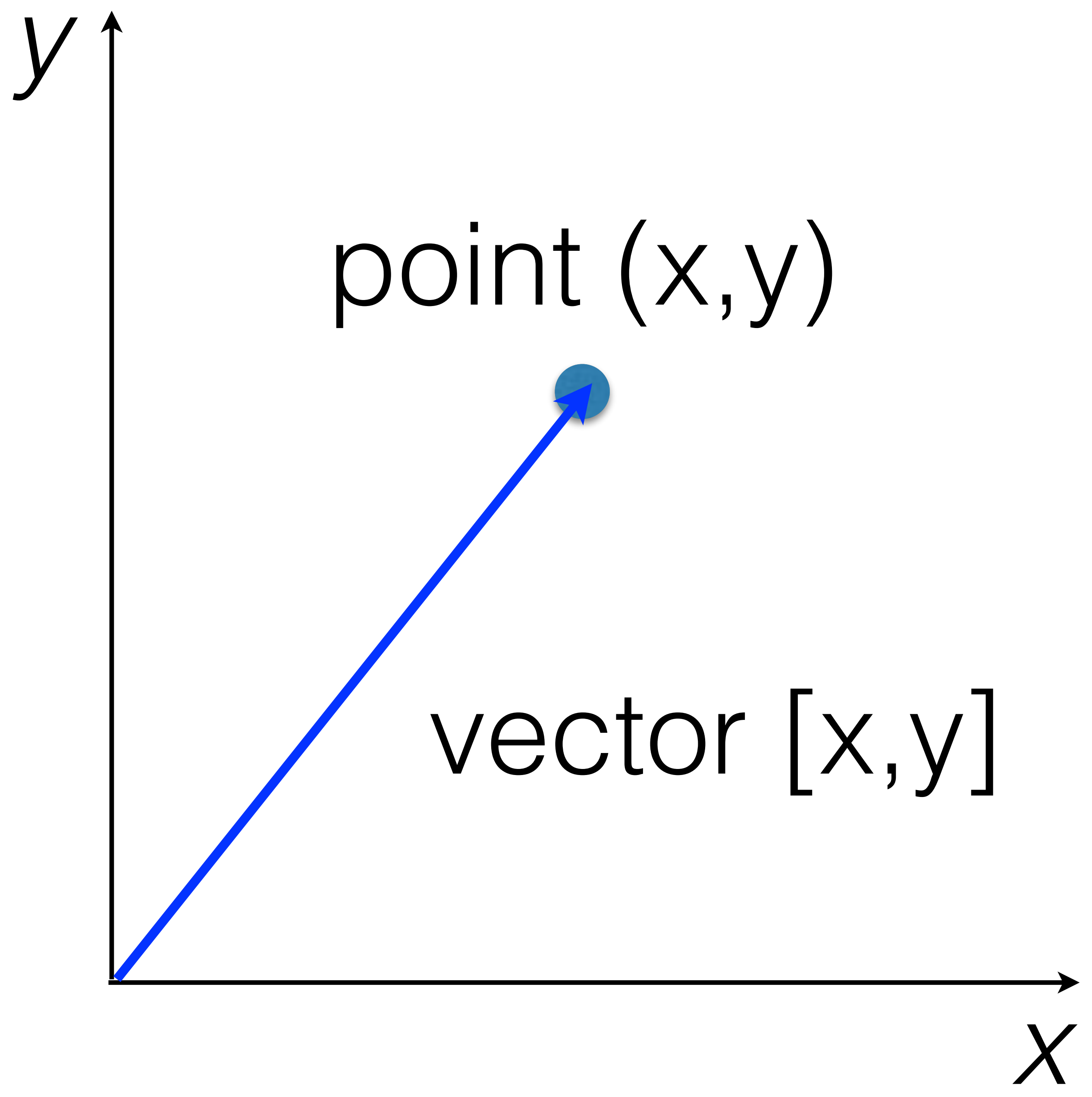


Zero Vector

- The *zero vector* is all zeroes
- No displacement
- Magnitude is 0
- Direction is undefined

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Points and Vectors



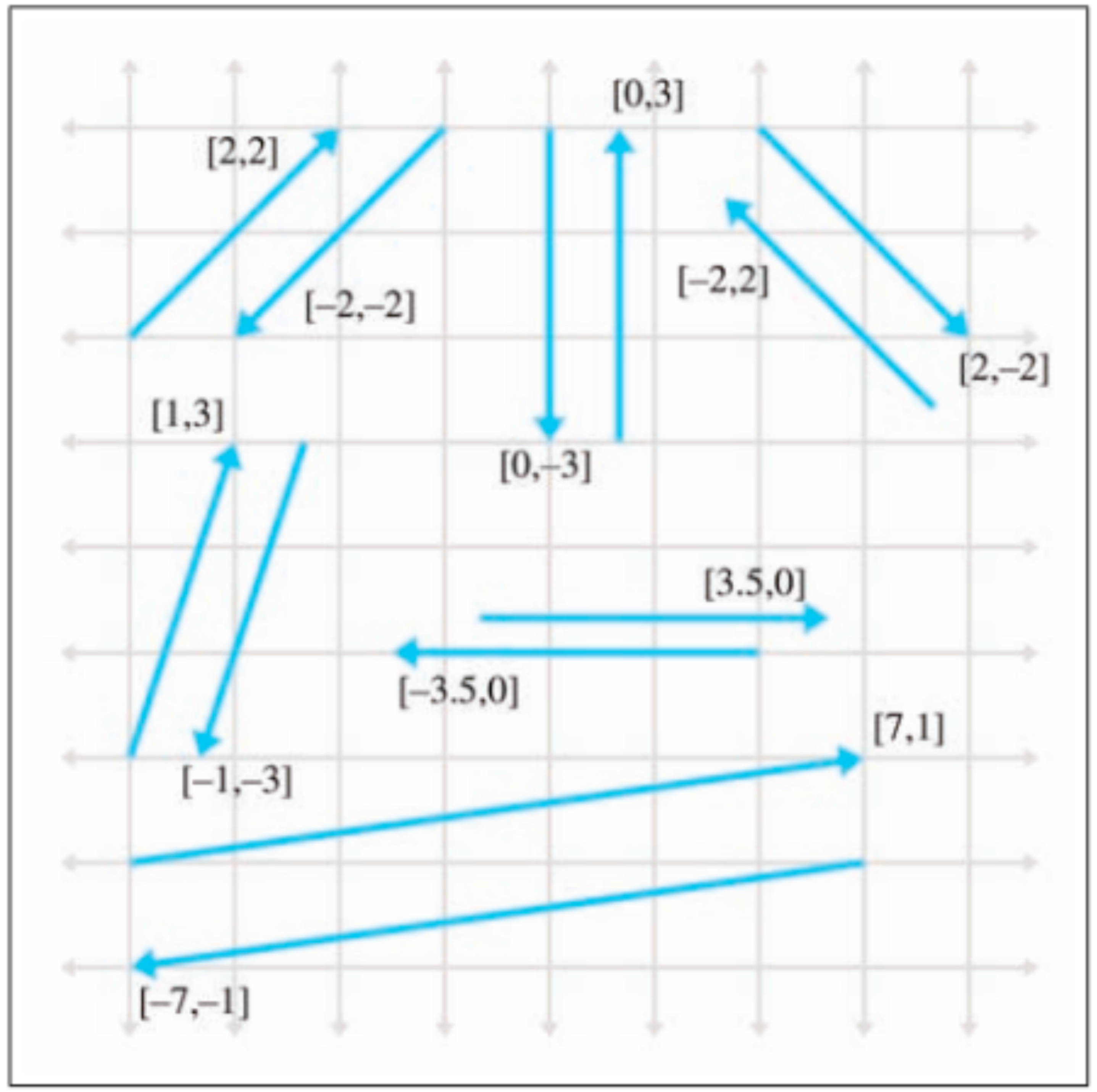
Points and vectors are different but related

Interchangeable (but be careful)

Negating Vectors

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$-\mathbf{v} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$$

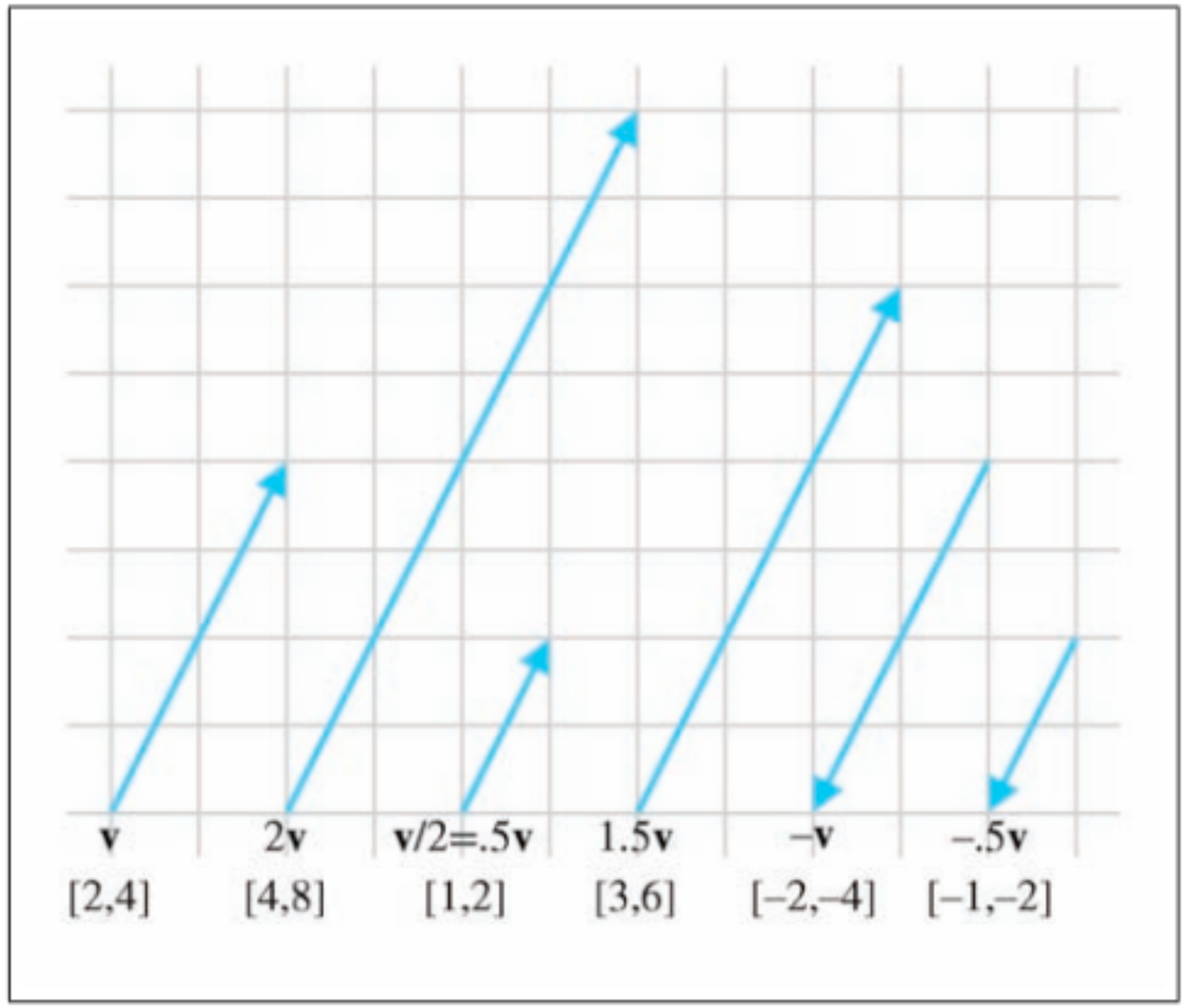


The negative of a vector has the same magnitude
in the opposite direction

Scaling Vectors

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$k\ \mathbf{v} = \begin{bmatrix} k\ x \\ k\ y \\ k\ z \end{bmatrix}$$



Multiplying by a constant multiplies each element
-- multiplies magnitude, same (or opposite) direction

Adding Vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

Point-wise add the elements

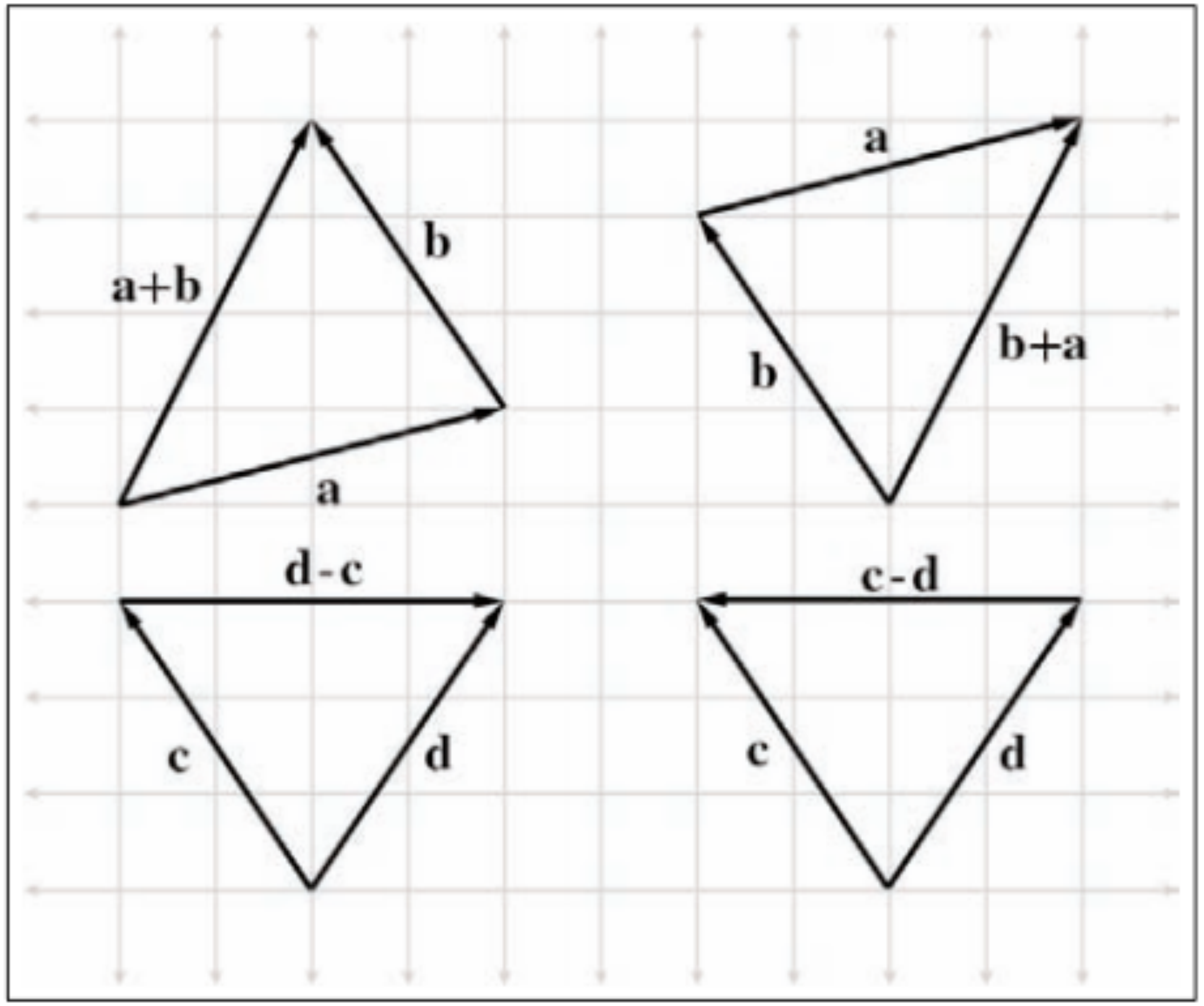
Subtracting Vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathbf{a} - \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}$$

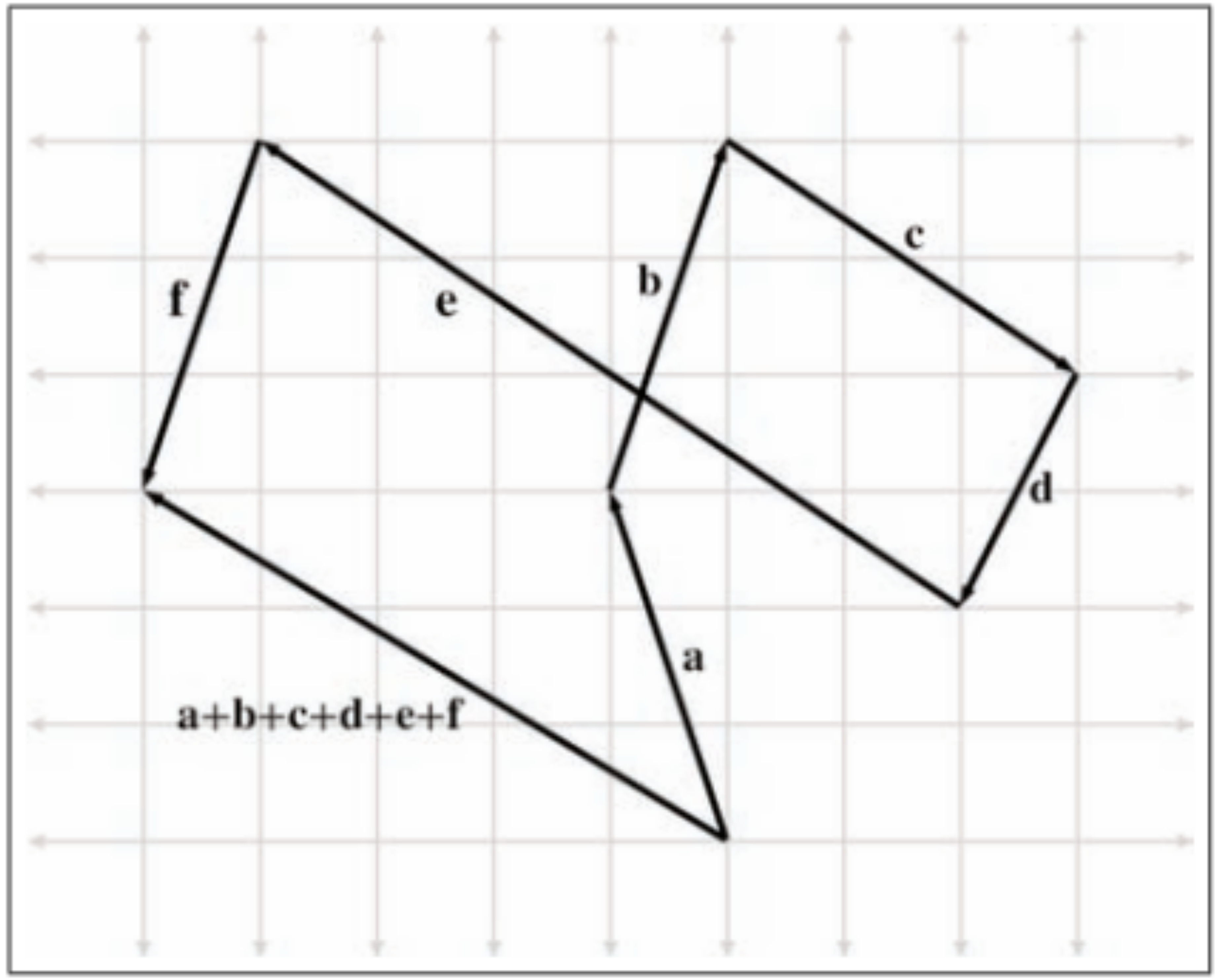
Point-wise subtract the elements

Geometric Interpretation

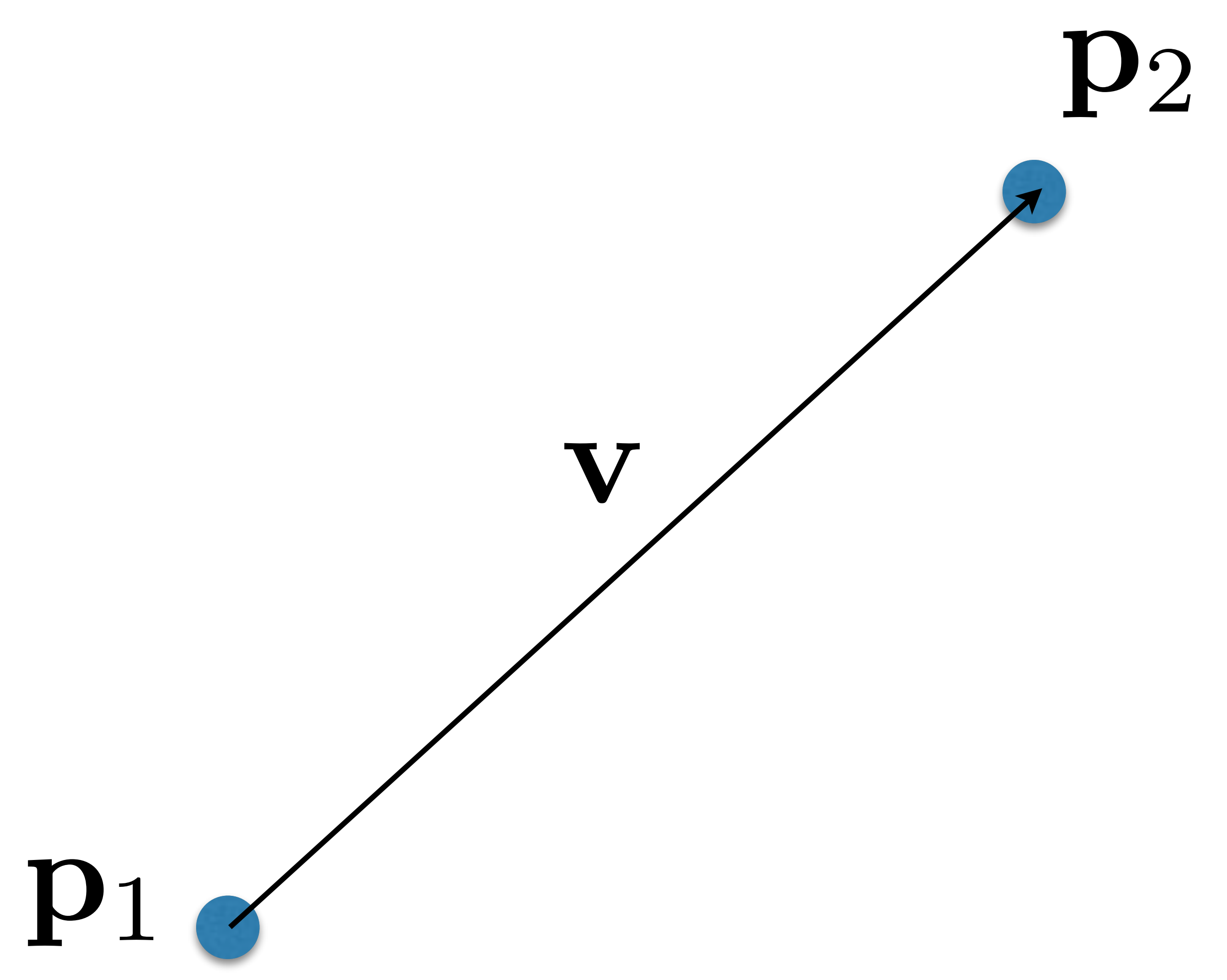


"Go this way, then go that way"

Geometric Interpretation



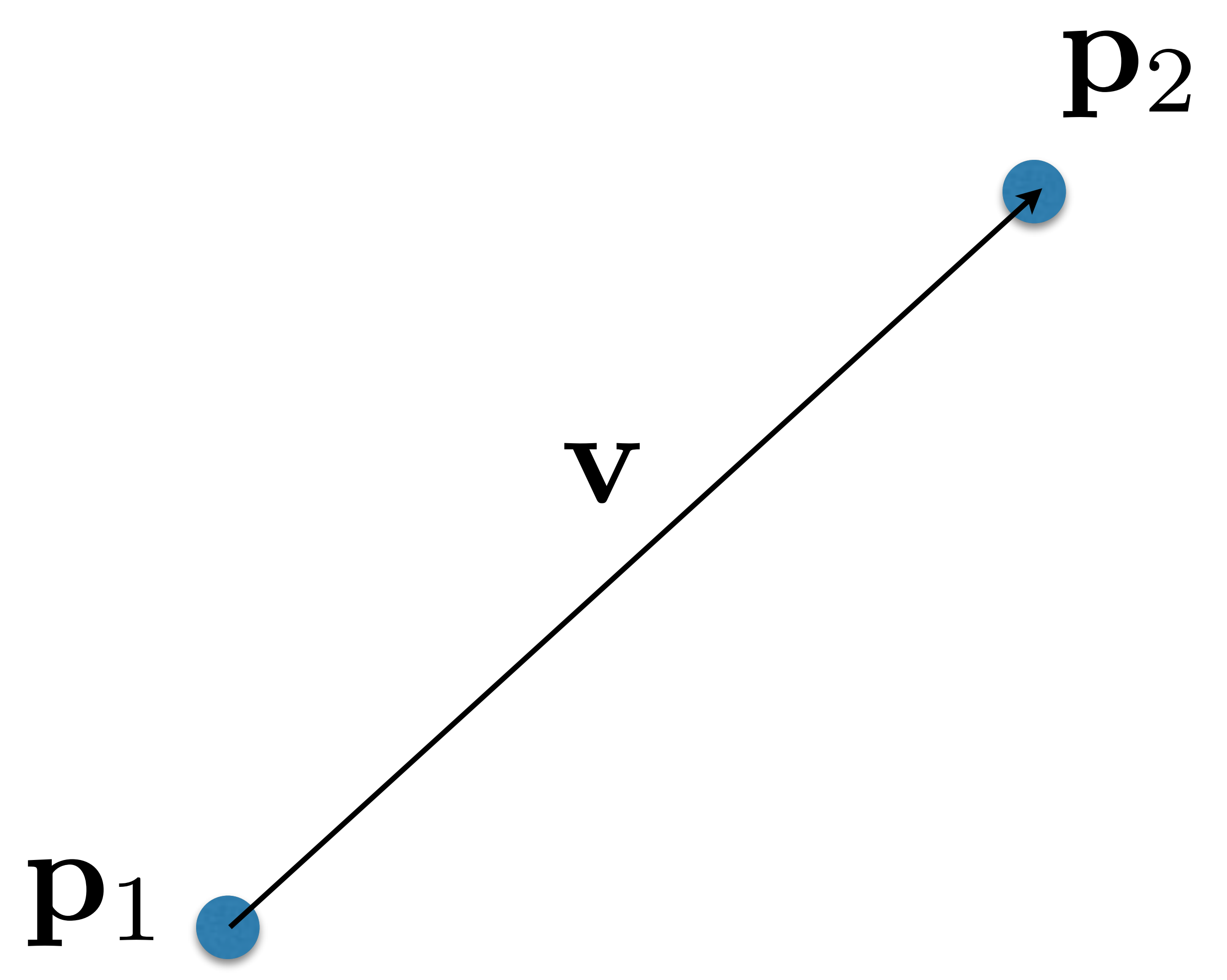
Vectors as Displacements



$$\mathbf{p}_1 + \mathbf{v} = \mathbf{p}_2$$

Type system: point + vector = point

Vectors as Displacements

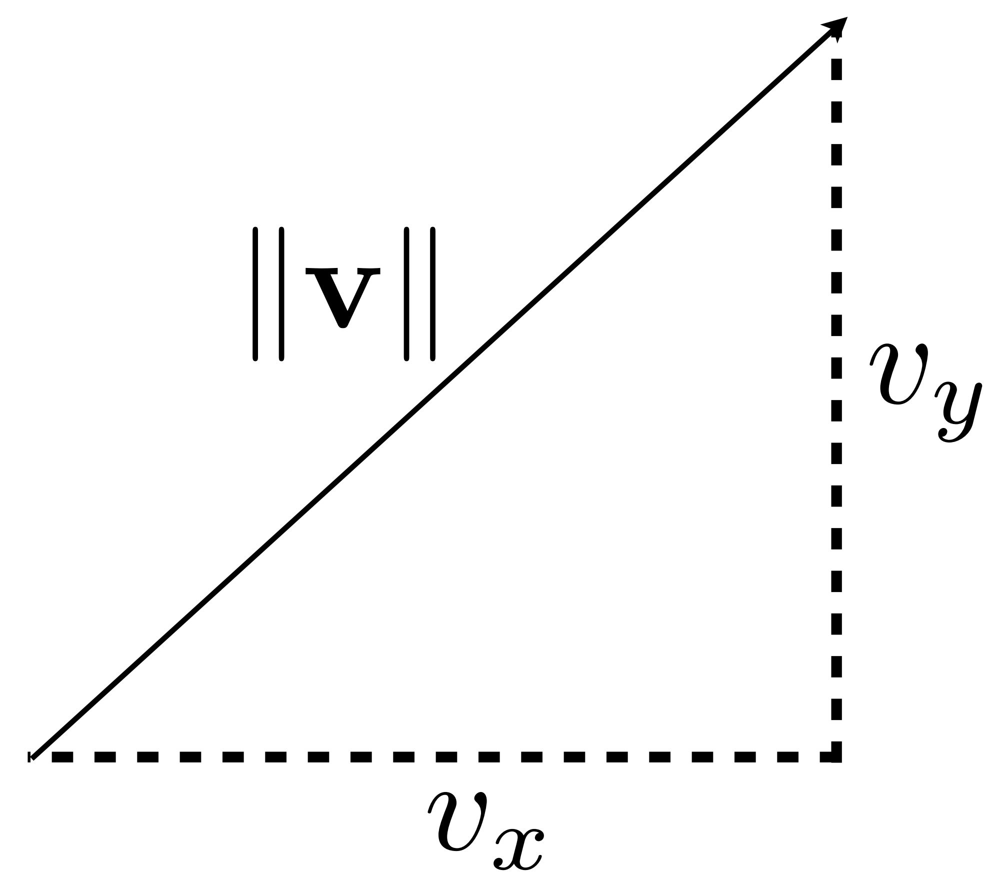


$$\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$$

Type system: point - point = vector

Magnitude

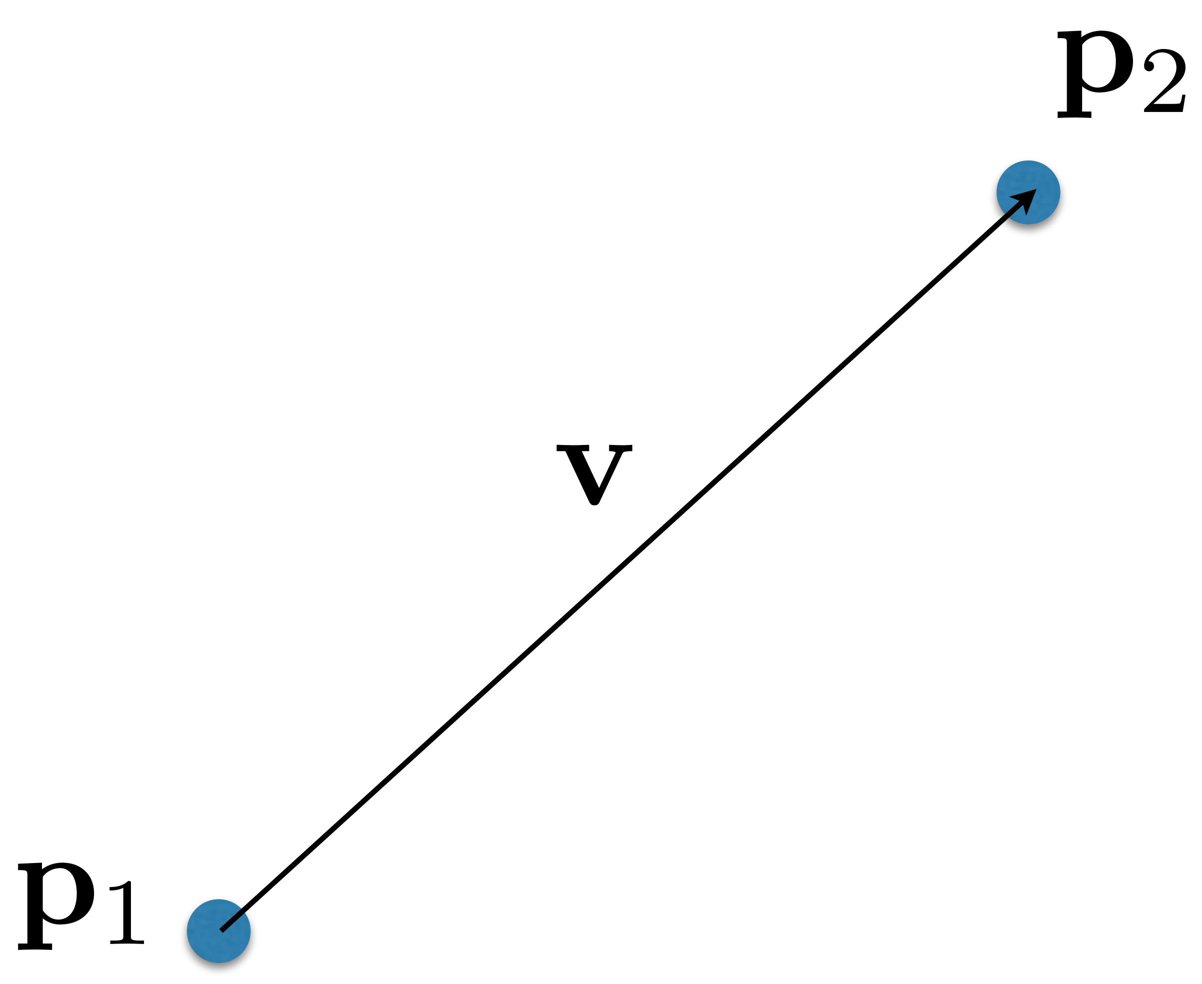
- The magnitude (length) of a vector can be calculated using Pythagorean theorem
- Sometimes called the *norm* of the vector
- Note: there are other vector norms, but assume this unless stated otherwise



$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

“magnitude”, “length”, or “norm” of $\mathbf{v}$

Distance



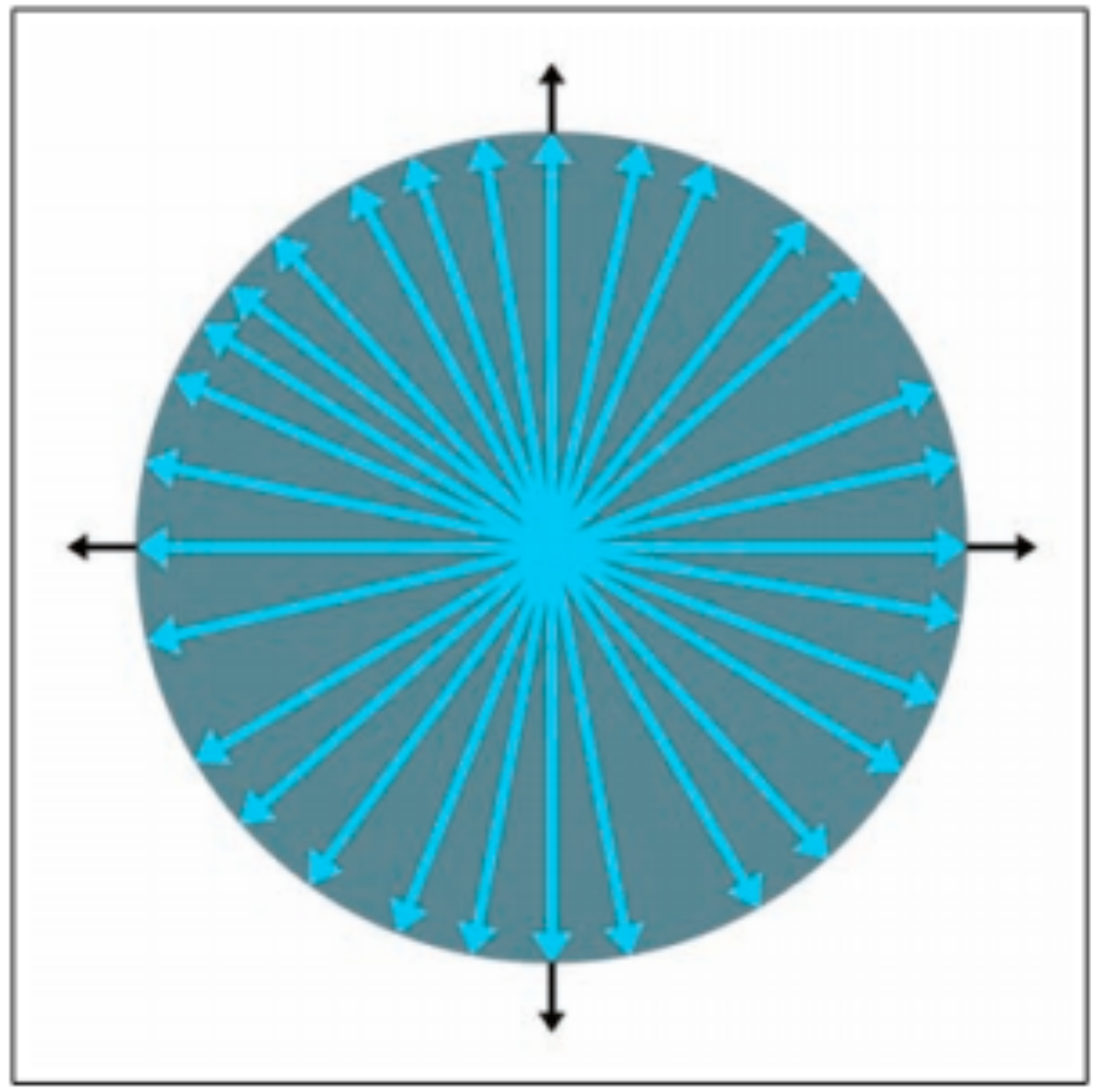
$$\|\mathbf{v}\| = \|\mathbf{p}_2 - \mathbf{p}_1\| = \sqrt{(p_2[x] - p_1[x])^2 + (p_2[y] - p_1[y])^2}$$

Convenient way to write / calculate distance between points

Unit Vectors

$$\|v\| = 1$$

- A “unit vector” has a **length of one**
- Useful to describe direction when we don’t care about magnitude



Normalizing

- Sometimes we want to **normalize** a vector to have the same direction but unit length
- Key: just divide it by its own length
- Can't do this for the zero vector

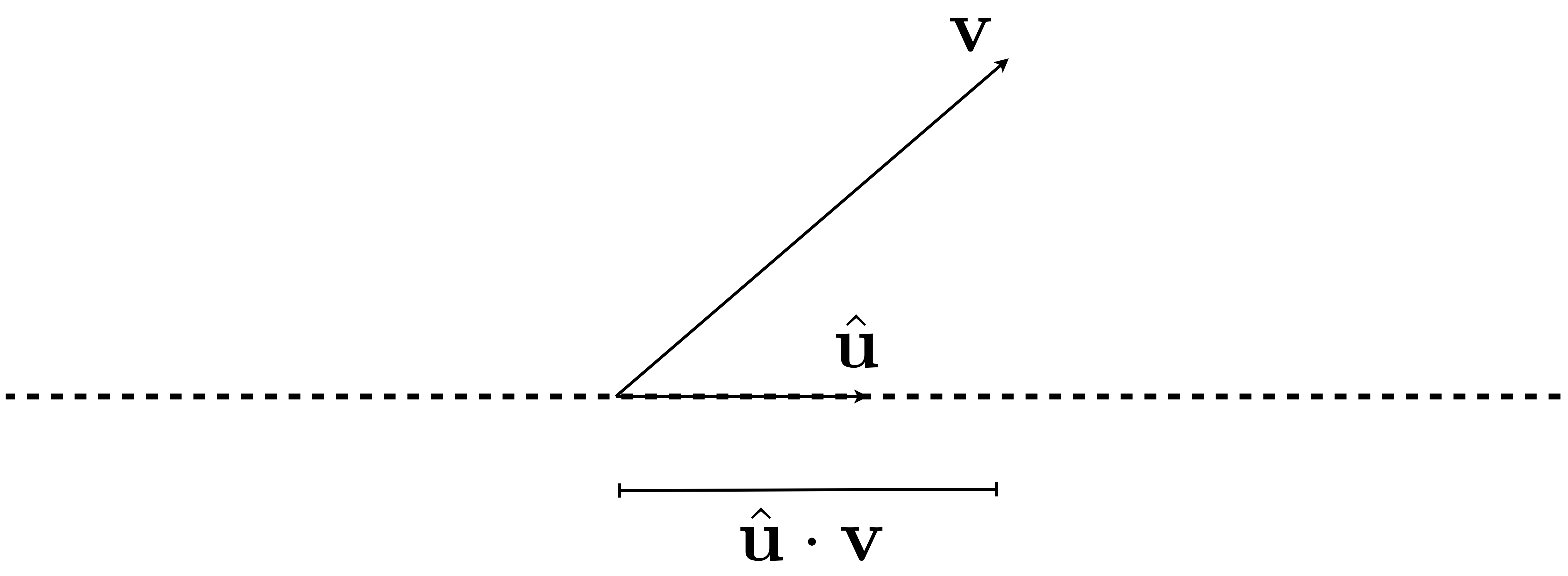
$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}}{\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$$

Vector Dot Products

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i$$

One of the most commonly used
vector operations in graphics

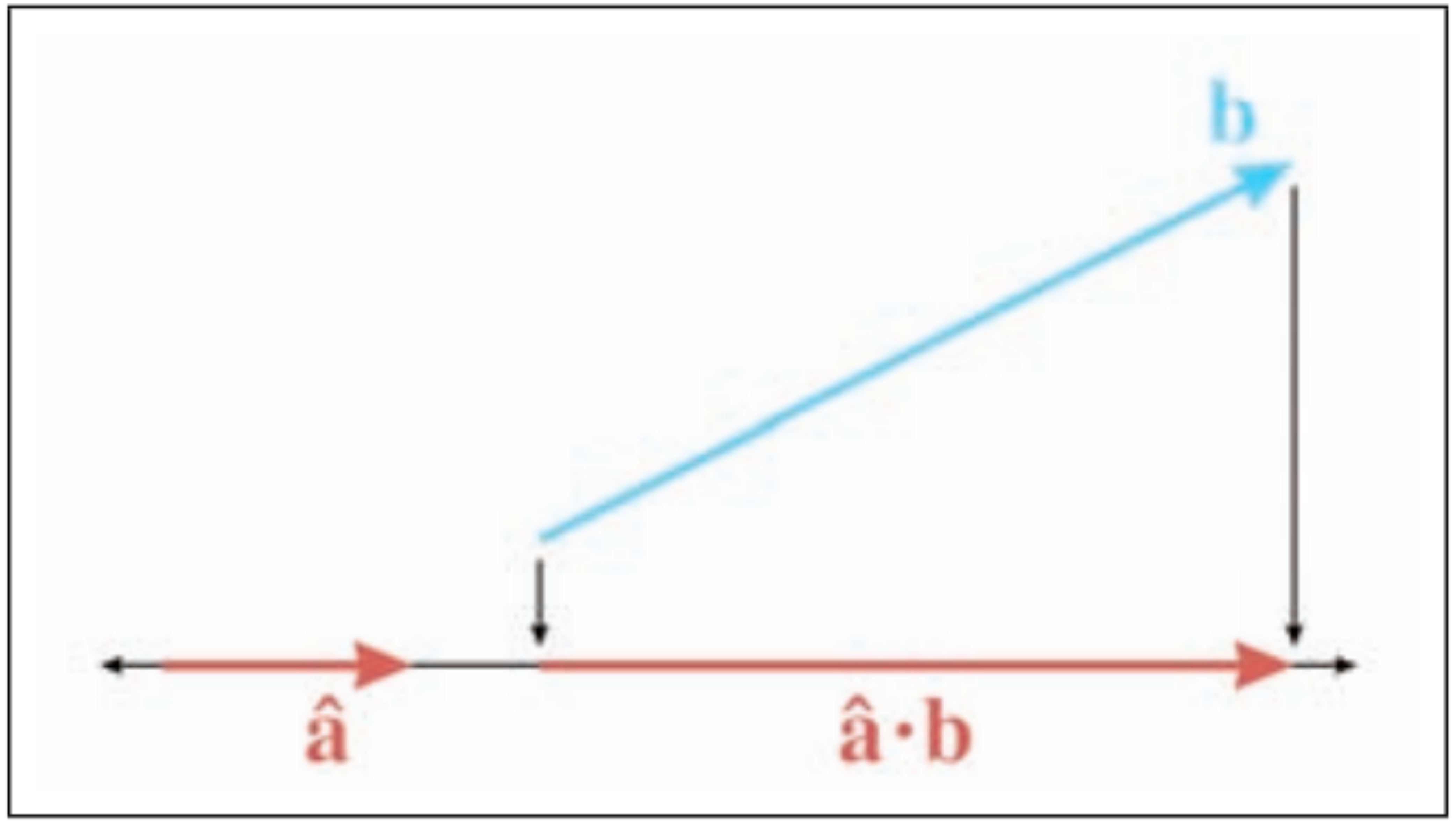
Geometric Interpretation



The dot product of a vector and a <u>unit</u> vector
is the length of the projection onto that unit vector

"How much of this vector lies in that direction?"

Geometric Interpretation



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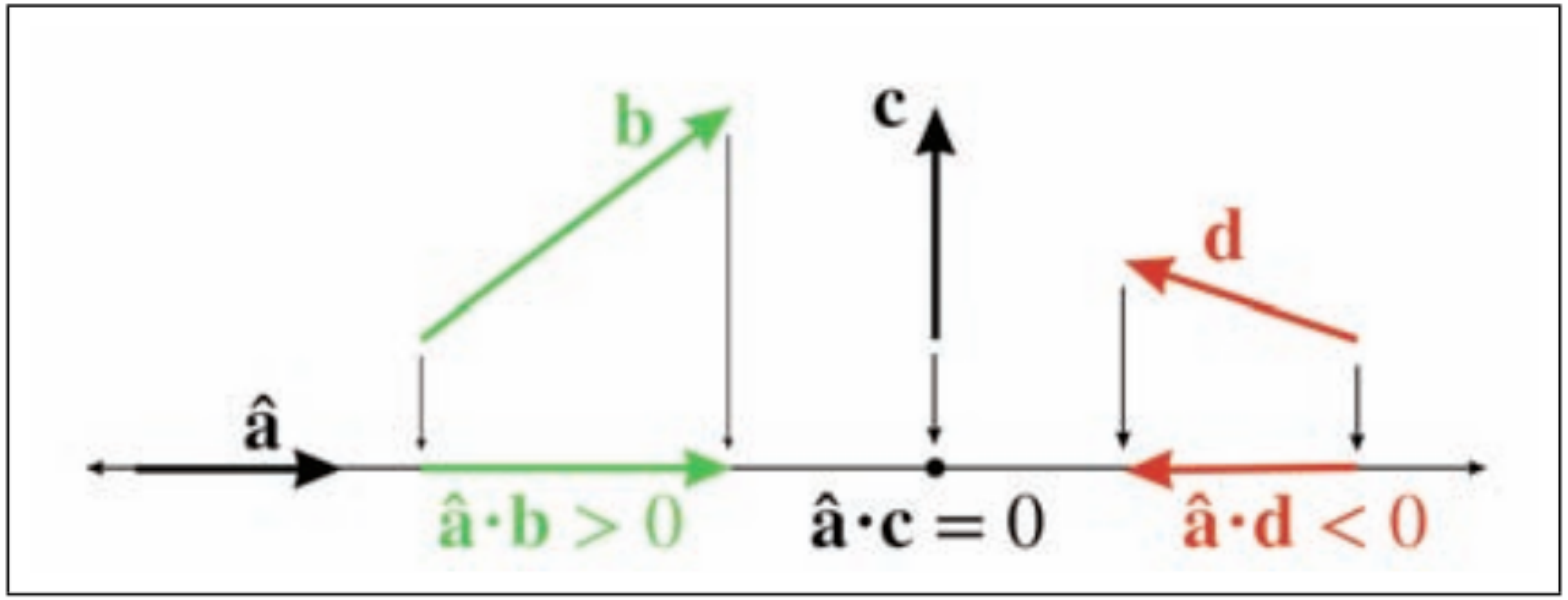
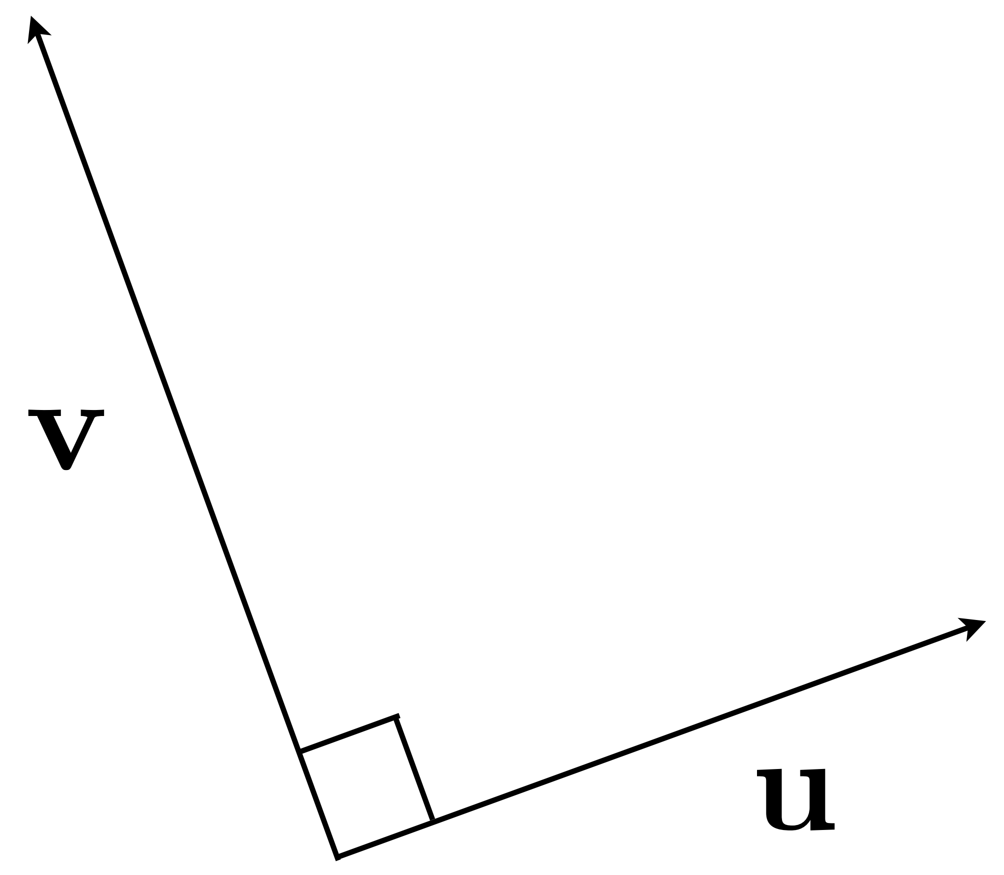
Sign of the Dot Product



The sign of the dot product between two vectors tells whether the projection is in the same direction

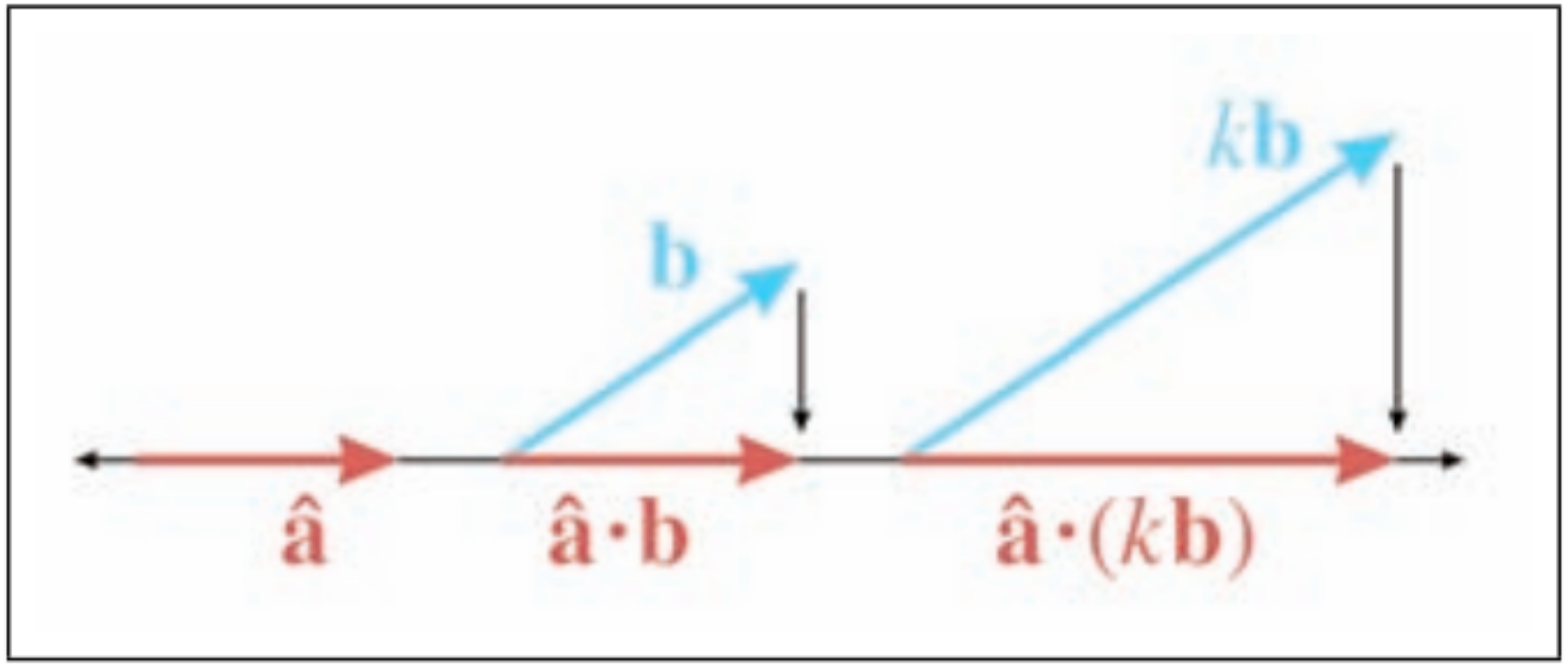
Orthogonality

- Vectors whose dot product is zero are said to be “orthogonal”
- “Right angle” to each other (regardless of length)
- The zero vector is trivially orthogonal to everything else



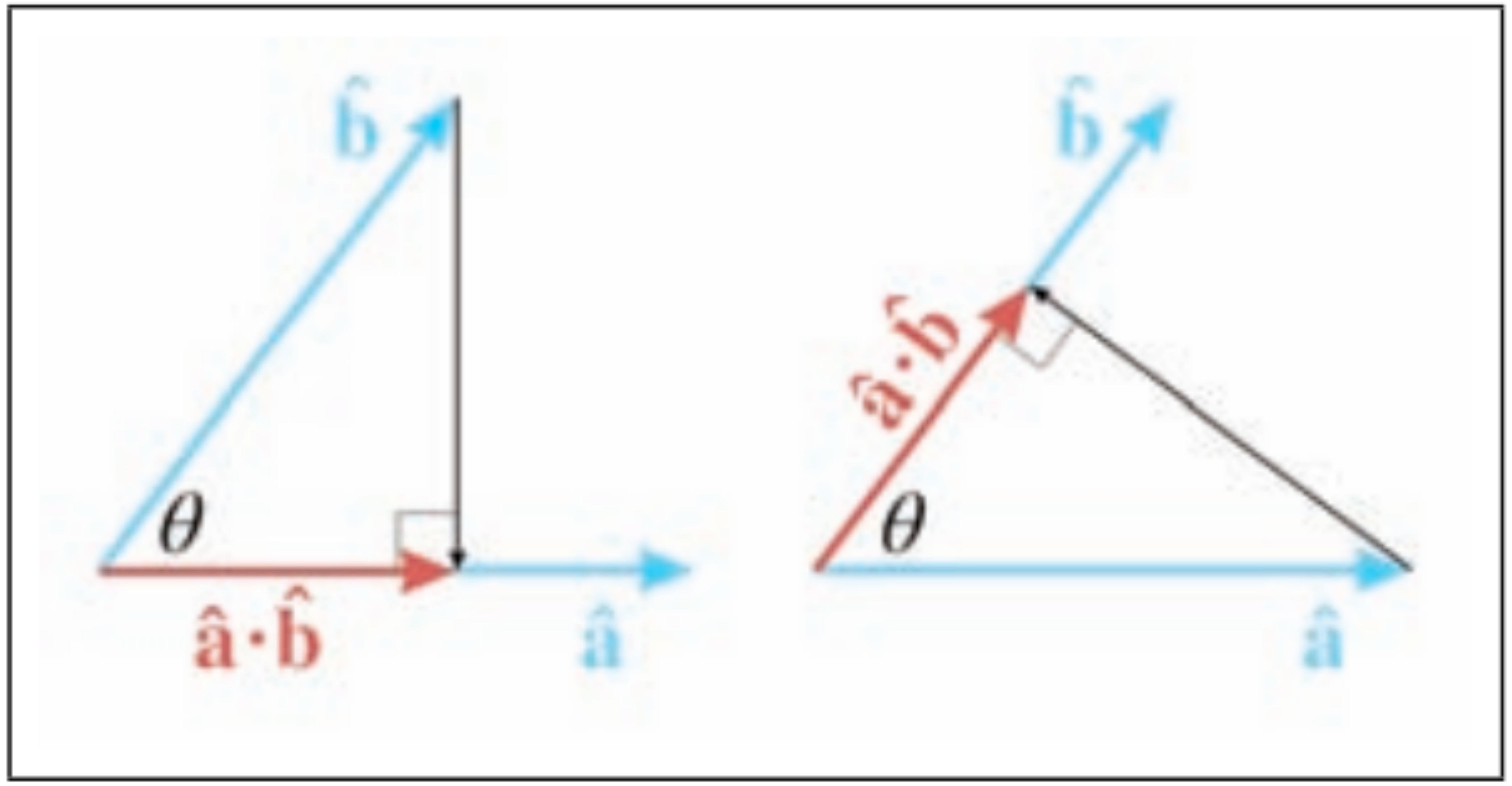
$$\mathbf{u} \cdot \mathbf{v} = 0$$

Scalar Multiplication



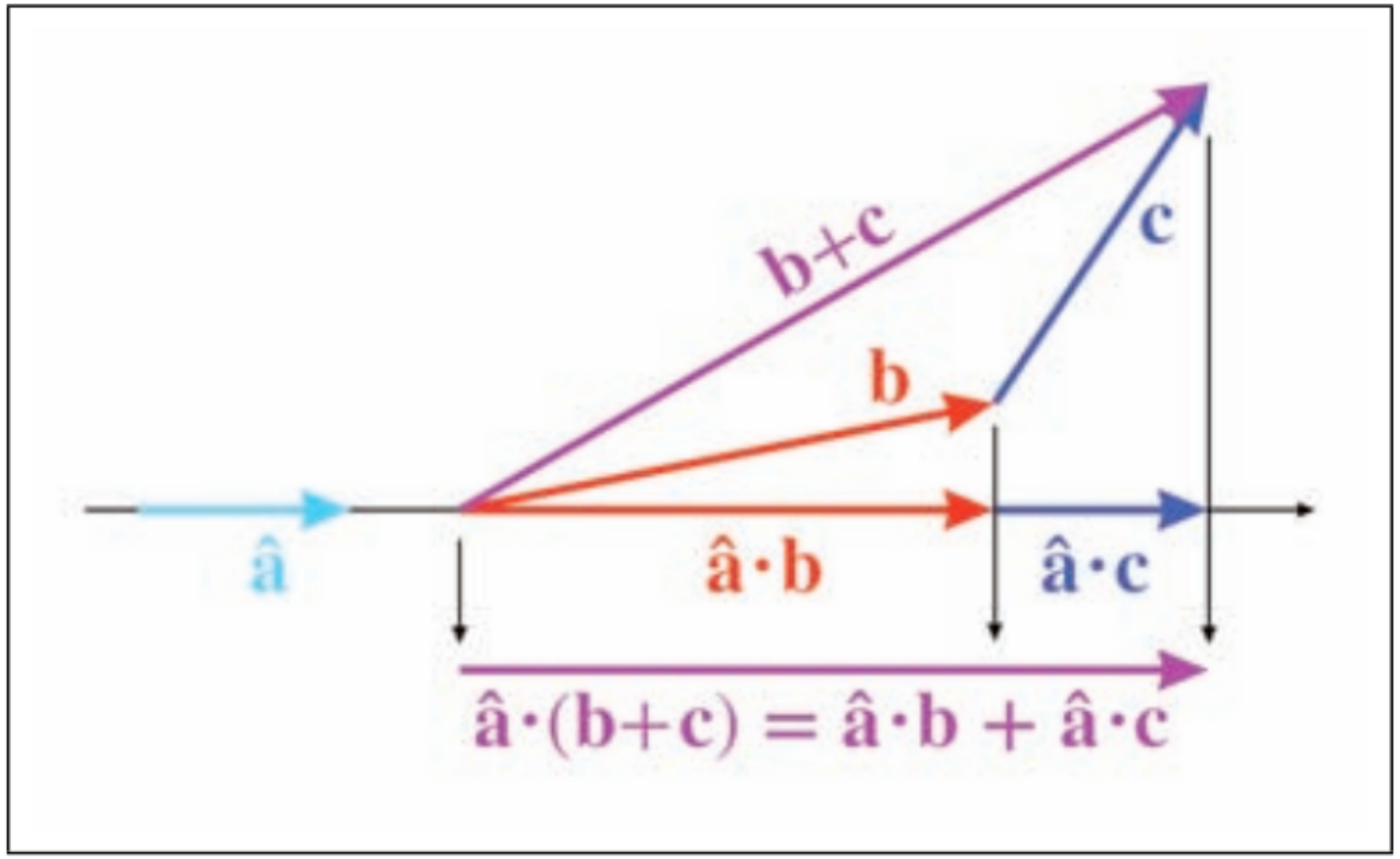
$$\mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b}$$

Commutative



$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Distributes Over Addition



$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$$

Angles

$$\mathbf{u} = \|\mathbf{u}\| \;\; \hat{\mathbf{u}}$$

$$\mathbf{v} = \|\mathbf{v}\| \;\; \hat{\mathbf{v}}$$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} \quad &= \quad \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}) \\ &= \quad \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta_{uv} \end{aligned}$$

The dot product of two vectors is the product of their lengths times the cosine of the angle between them

Lengths

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

The dot product of something with itself
is its own length squared

Tip: lots of “distance” tests only need squared distance

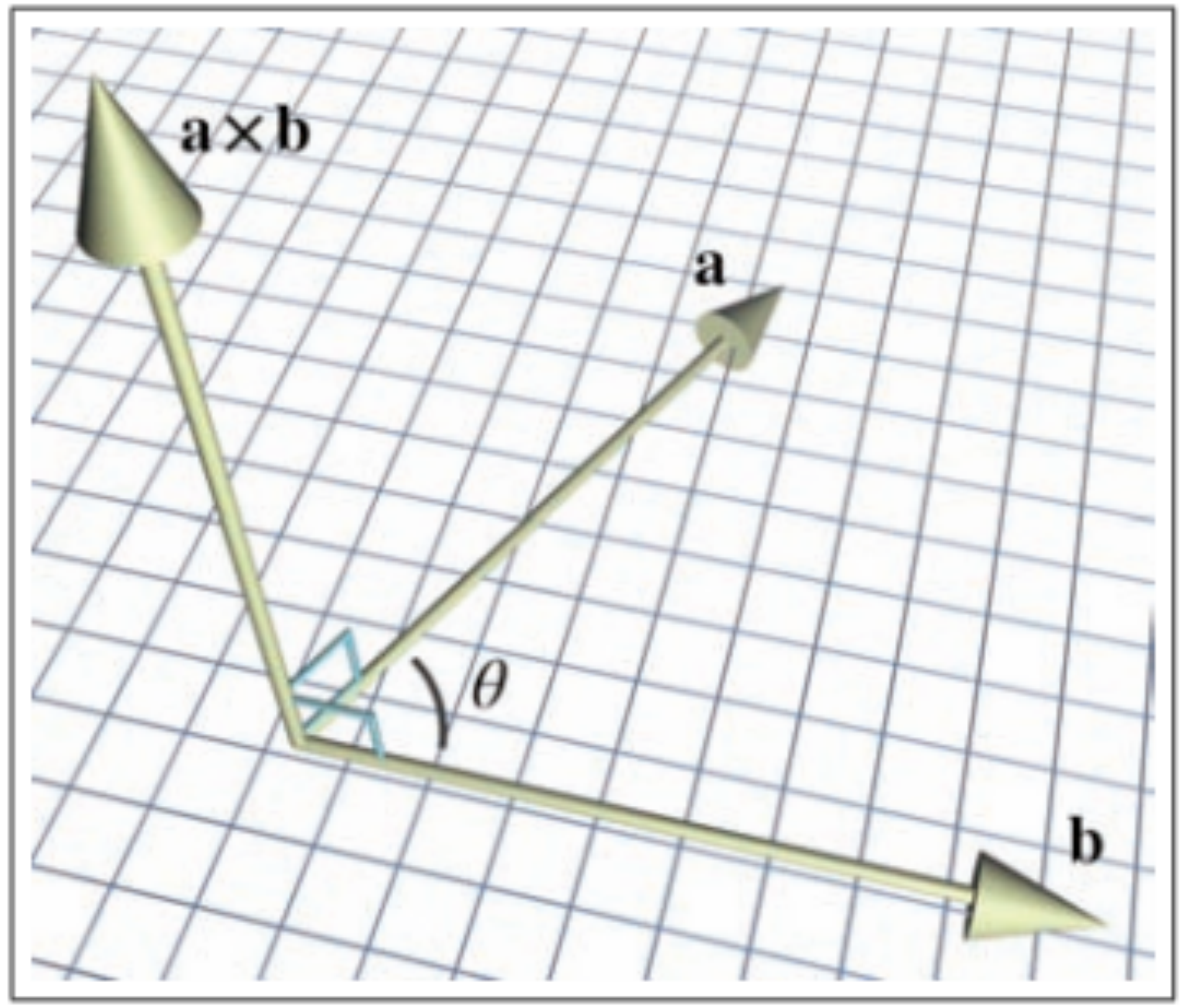
Cross Product

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

Result is a vector

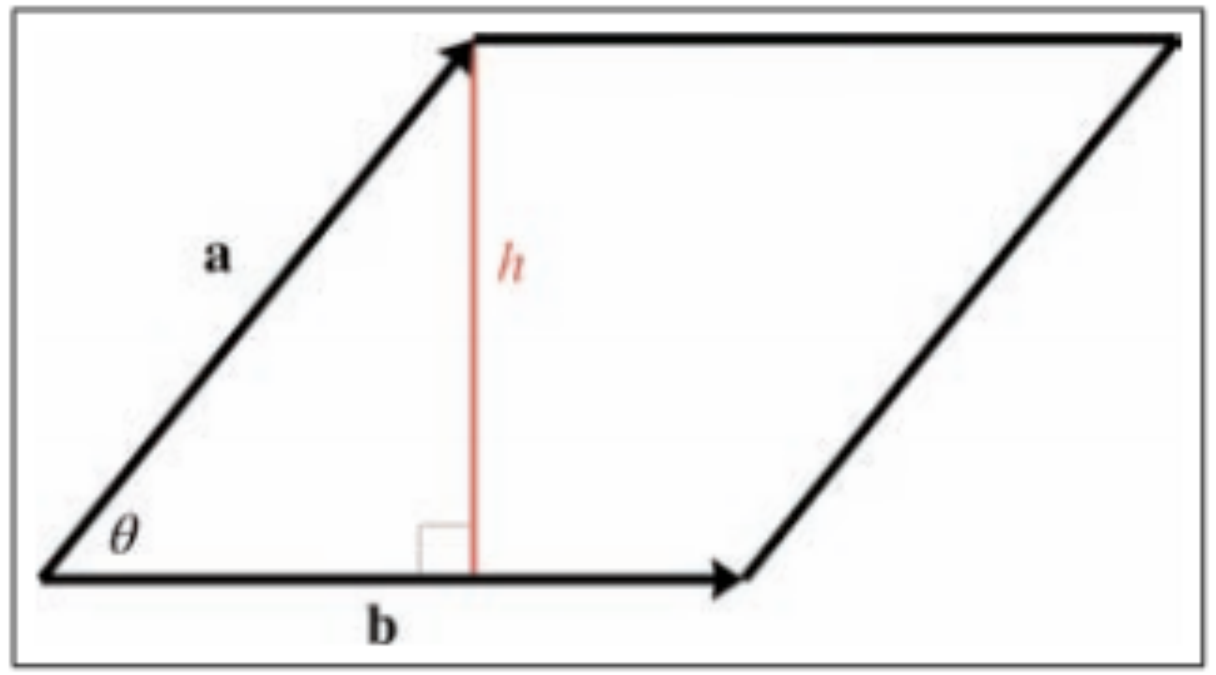
Only done in 3D

Geometric Interpretation



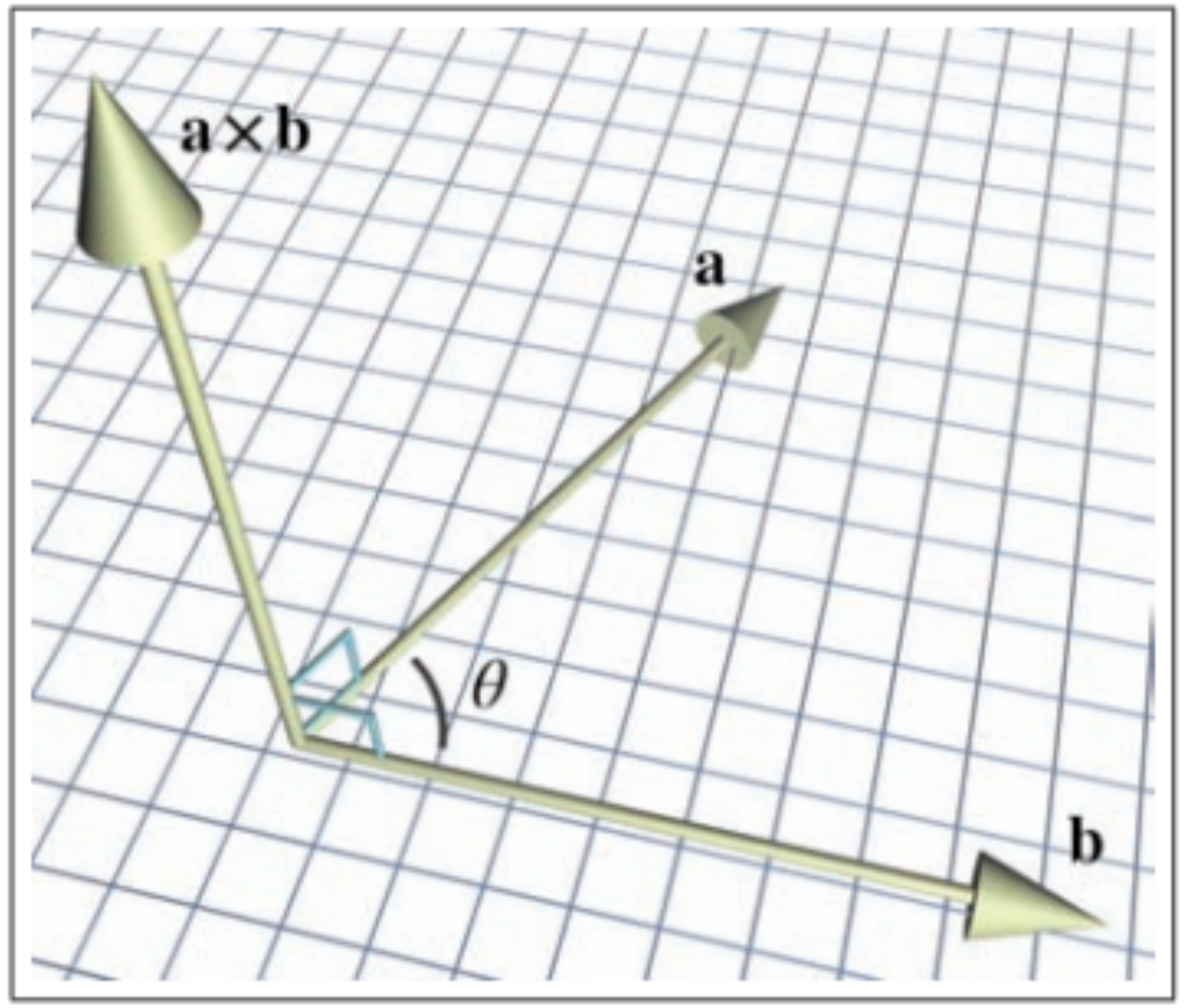
The cross product of two vectors is
another vector orthogonal to the two
(really useful property in 3D geometry!)

Geometric Interpretation



The length of the cross product of two vectors
is the area of the parallelogram spanned by the two

Geometric Interpretation



$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta_{ab}$$

Linear Algebra Identities

Identity	Comments
$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	Commutative property of vector addition
$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$	Definition of vector subtraction
$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$	Associative property of vector addition
$s(t\mathbf{a})=(st)\mathbf{a}$	Associative property of scalar multiplication
$k(\mathbf{a}+\mathbf{b})=k\mathbf{a}+k\mathbf{b}$	Scalar multiplication distributes over vector addition
$\\|k\mathbf{a}\\|=\|k\|\\|\mathbf{a}\\|$	Multiplying a vector by a scalar scales the magnitude by a factor equal to the absolute value of the scalar
$\\|\mathbf{a}\\|\geq 0$	The magnitude of a vector is nonnegative
$\\|\mathbf{a}\\|^2+\\|\mathbf{b}\\|^2=\\|\mathbf{a}+\mathbf{b}\\|^2$	The Pythagorean theorem applied to vector addition.
$\\|\mathbf{a}\\|+\\|\mathbf{b}\\|\geq\\|\mathbf{a}+\mathbf{b}\\|$	Triangle rule of vector addition. (No side can be longer than the sum of the other two sides.)
$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$	Commutative property of dot product
$\\|\mathbf{a}\\|=\sqrt{\mathbf{a}\cdot\mathbf{a}}$	Vector magnitude defined using dot product
$k(\mathbf{a}\cdot\mathbf{b})=(k\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(k\mathbf{b})$	Associative property of scalar multiplication with dot product
$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$	Dot product distributes over vector addition and subtraction
$\mathbf{a}\times\mathbf{a}=\mathbf{0}$	The cross product of any vector with itself is the zero vector. (Because any vector is parallel with itself.)
$\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$	Cross product is anticommutative.
$\mathbf{a}\times\mathbf{b}=(-\mathbf{a})\times(-\mathbf{b})$	Negating both operands to the cross product results in the same vector.
$k(\mathbf{a}\times\mathbf{b})=(k\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(k\mathbf{b})$	Associative property of scalar multiplication with cross product.
$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$	Cross product distributes over vector addition and subtraction.

Coming up…

- Transformations