More on Matrices as Transformations

CS 355: Introduction to Graphics and Image Processing

A little more on matrices as coordinate transformations…

Determinant

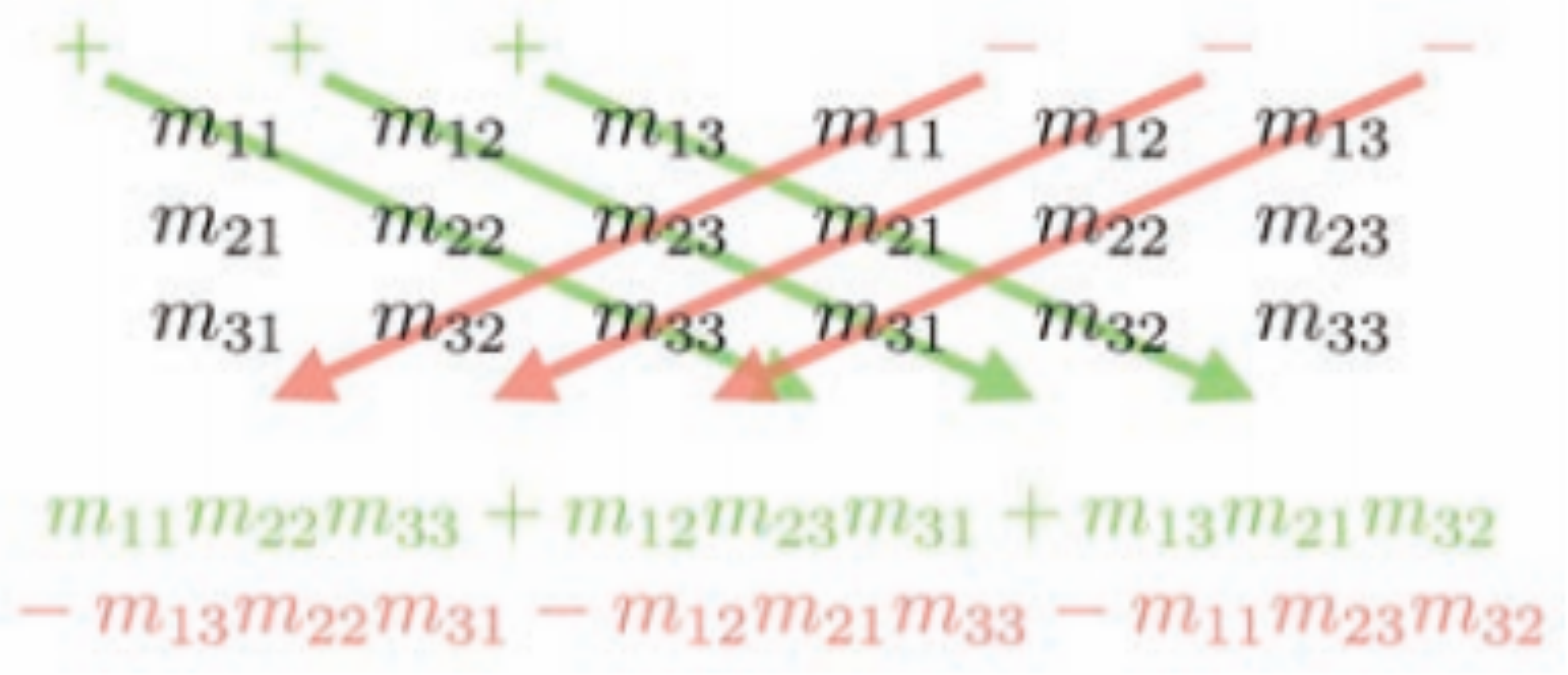
$$|\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}$$

Determinant

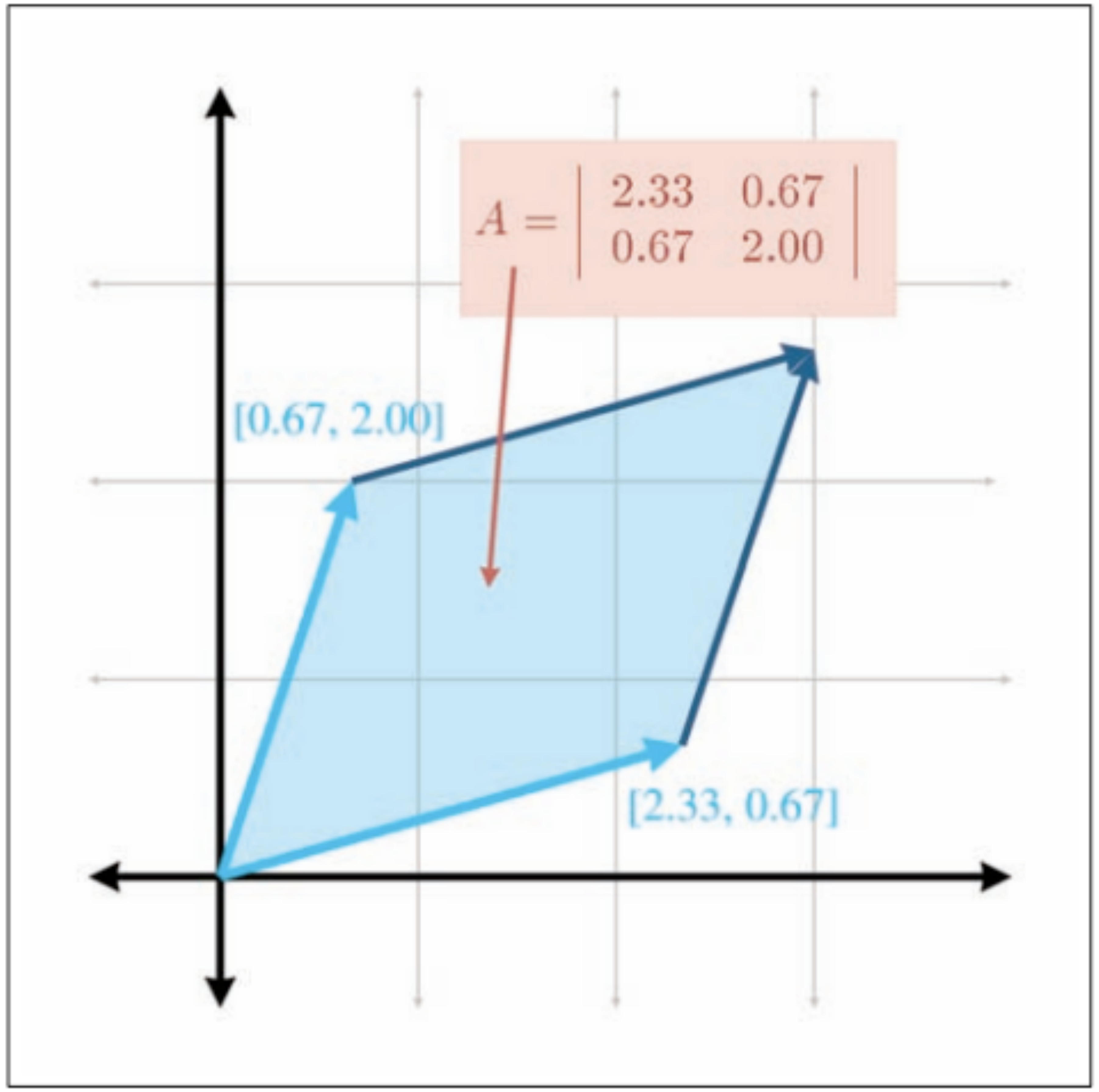
$$|\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = {\color{green} m_{11}m_{22}} - {\color{red} m_{12}m_{21}}$$

Determinant

$$|\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}$$



Geometric Interpretation



Properties of Determinants

$$|\mathbf{I}| = 1$$

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$$

$$\left|\mathbf{M}^T\right| = |\mathbf{M}|$$

$$\left|\mathbf{M}^{-1}\right| = \frac{1}{|\mathbf{M}|}$$

Linear Independence

A set of vectors is said to be *linearly dependent* if at least one of them can be expressed as a linear combination (weighted sum) of the others:

$$\mathbf{v}_j = \sum_{i \neq j} w_i \ \mathbf{v}_i$$

If not linearly *dependent*, then linearly *independent*

Singular Matrices

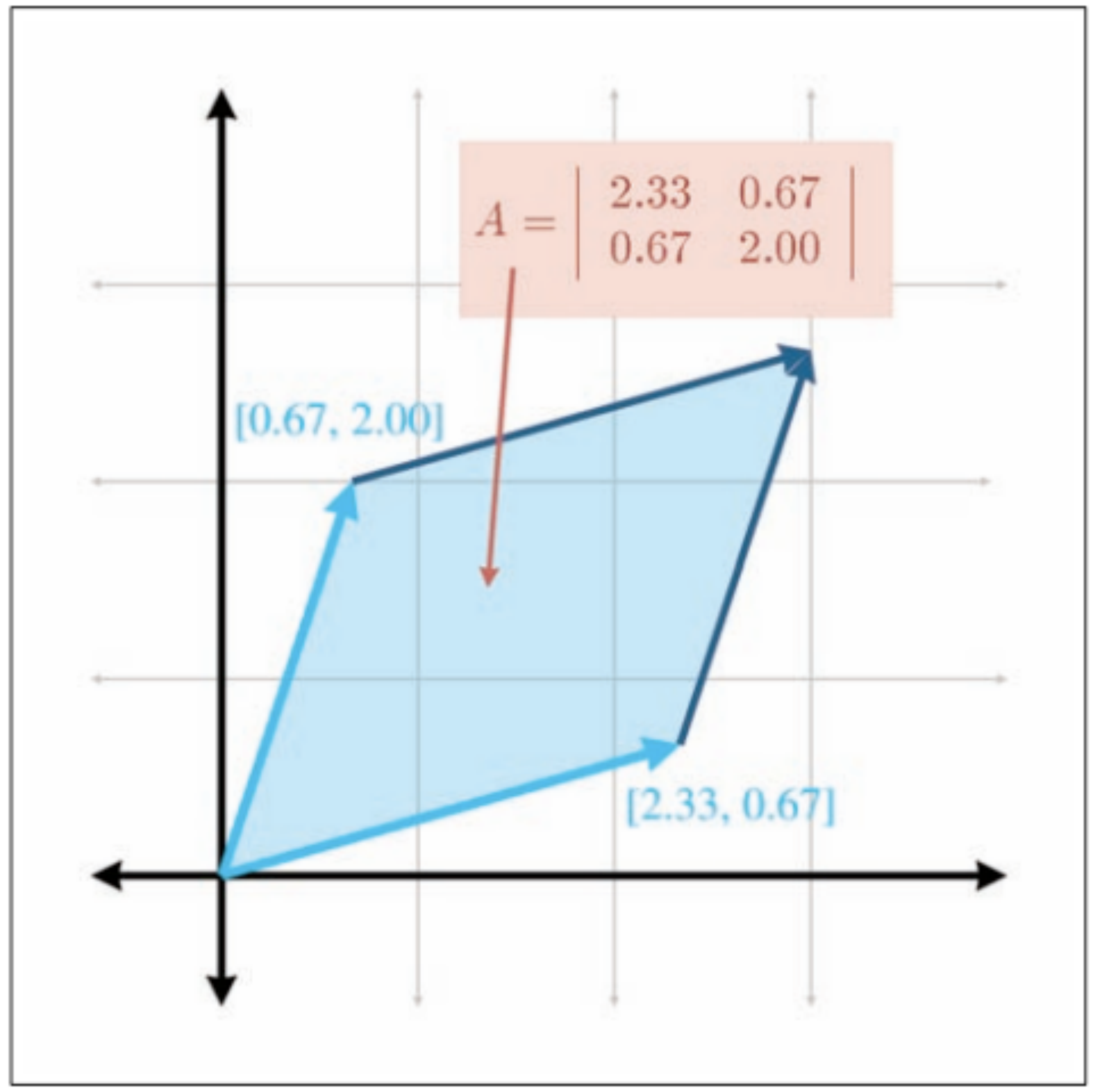
$$\left|\mathbf{M}^{-1}\right| = \frac{1}{|\mathbf{M}|}$$

But what if $|\mathbf{M}| = 0$?

A matrix whose determinant is zero
has no inverse and is said to be *singular*

Singular Matrices

- What does a singular matrix mean geometrically?
- *The rows are linearly dependent*



Matrix Rank

- The *rank* of a matrix is the number of linearly independent rows
- When used as transforms, matrices with *full rank* transform to the full space
- Singular matrices have *insufficient rank* and collapse to a corresponding subspace
- Geometric interpretation: the rank of a matrix is the dimensionality of the (sub)space that matrix maps to

Orthogonal Matrices

- Two (square) matrices are said to be orthogonal iff

$$\mathbf{M}\mathbf{M}^T = \mathbf{I}$$

- Implies rows are orthonormal vectors

Orthogonal Matrices

- Orthogonal matrices are also easily invertible:

$$\mathbf{M}^{-1} = \mathbf{M}^T$$

- Implies

$$|\mathbf{M}| = \left|\mathbf{M}^{-1}\right| = 1$$

Orthogonal Matrices

- All rotation matrices are orthogonal

AND

- All orthogonal matrices are rotations!

Coming up…

- 3D Graphics!