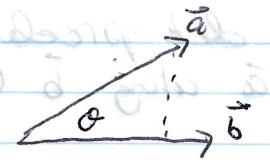
Projectors, Inner Products (Dot) and Exterior Product (as a projector)



Dot Product

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\text{comp}_{\vec{b}} \vec{a}$ ~ read component of $\vec{a}$ along $\vec{b}$

$$= |\vec{a}| \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|b|}$$

$\text{proj}_{\vec{b}} \vec{a}$ ~ read projection of $\vec{a}$ along $\vec{b}$

$$= (\text{comp}_{\vec{b}} \vec{a}) \underbrace{\hat{b}}_{\text{unit vector}} = \quad \hat{b} = b/|b|$$

A "Hat" (ie $\hat{a}$) means unit vector $\hat{a} = \vec{a}/|\vec{a}|$!!

$$= \left[\frac{|\vec{a}|}{|\vec{b}|} \cos\theta\right] \vec{b} = \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|^2} \vec{b}$$

I'm using a hat, $\hat{a}$, for an unit vector

ie. $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Let $\vec{a} = [a_1, a_2, \dots, a_n]^T$ and $b = [b_1, b_2, \dots, b_n]^T$ then the dot product can be written as

$\vec{a} \cdot \vec{b} = a^T b \rightarrow$ gives a scalar

Define the ~~exterior~~ ~~outer~~ Cartesian, Exterior, Outer, Tensor, product (these are all common names) as

$$A = \vec{a} \otimes \vec{b} = a b^T = \begin{bmatrix} a_1 b_1 & a_2 b_1 & \cdots & a_n b_1 \\ a_1 b_2 & a_2 b_2 & \cdots & a_n b_2 \\ \vdots & \vdots & \ddots & \\ a_1 b_n & a_2 b_n & & a_n b_n \end{bmatrix}$$

Just as we can use the dot product to find a projection of $\vec{a}$ along $\vec{b}$ (ie $proj_{\vec{b}}\,\vec{a}$) i.e.

$$proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$$

$$= \frac{a^T b}{b^T b}$$

We can use the exterior product to find the component of $\vec{a}$ along $\vec{b}$ ~~$comp_{\vec{b}}$~~ ($comp_{\vec{b}}\,\vec{a}$)

$$comp_{\vec{b}}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = (\vec{a}\cdot\hat{b})\,\hat{b}$$

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$$= \frac{\vec{b}\,(\vec{b}\cdot\vec{a})}{|\vec{b}|} = \hat{b}\,(\hat{b}\cdot\vec{a})$$

factor out $\vec{a}$ | factor out $\vec{a}$

$$= \left[\frac{\vec{b}\otimes\vec{b}}{|\vec{b}|^2}\right]\vec{a} = (\hat{b}\otimes\hat{b})\,\vec{a}$$

$$= \left[\frac{b\,b^T}{b^T b}\right]\vec{a} = \hat{b}\,\hat{b}^T a$$

Operator (ie matrix) of the form

$\hat{b}\otimes\hat{b}$

is called "projector". Because

$\vec{a} \longrightarrow comp_{\vec{b}}\,\vec{a}$

when it acts on it.

Projectors are important in quantum mechanics!!

Quantum Mechanics Digression

We can now think of the exterior product as a matrix operator that "hits" or "slaps" or "act" on the vector from the left side.

And in the process transforms the vector $\vec{a}$ into the component of $\vec{a}$ along $\vec{b}$. This happens to be very important in Quantum Mechanics. Say we have some state vector ~~....~~ $|\Psi\rangle$. Then we can write it as a super position of orthonormal vectors, typically we do this with "energy" eigenvector because the Hamiltonian ~~is~~ opperator happens to be hermition. This garentees that the "energy" eigen vectors ar orthonormal. I digress

$$\cancel{\vec{\Psi} = \sum_{a'} C_{a'} a}$$

$$\vec{\Psi} = \sum_n c_n \hat{a}_n \quad \text{where} \quad \hat{a}_n \cdot \hat{a}_m = \begin{cases} 1 & \text{is } m = n \\ 0 & \text{is } m \neq n \end{cases}$$

$\hat{a}_m$ are ortho-normal

[Sometimes written as $\hat{a}_n \cdot \hat{a}_m = \delta_{mn}$

$$\Psi = \sum c_n a_n$$

Then ~~slapps~~ splapping from the right with a_m

$$c_m = \hat{a}_m \cdot \vec{\Psi} = \sum_n c_n \underbrace{(\hat{a}_m \cdot \hat{a}_n)}_{=1 \text{ is } n=m}$$

$$= \hat{a}_m^T \Psi$$

We can now sub $c_m = \hat{a}_m \cdot \vec{\Psi}$ in

$$\cancel{\vec{\Psi} = \sum_n (\hat{a}_m \cdot \vec{\Psi})}$$

$$\vec{\Psi} = \sum_m c_n \hat{a}_m = \sum_m (\hat{a}_m \cdot \vec{\Psi}) \hat{a}_m$$

We can now write this in terms of an exterior product by factoring out $\vec{\Psi}$.

$$\vec{\Psi} = \sum_m \hat{a}_m (\hat{a}_m \cdot \vec{\Psi}) = \sum_m (\hat{a}_m \otimes \hat{a}_m) \vec{\Psi}$$

~~If we cancel~~

If we cancel out the $\vec{\Psi}$ then we arrive at the identity for orthonormal vectors

$$\mathbb{I} = \sum_m \hat{a}_m \otimes \hat{a}_m$$

where $\mathbb{I}$ is the identity matrix.

Let us define a projection opperator

$$P_m = \hat{a}_m \otimes a_m$$

then

$$P_m \vec{\Psi} = (\hat{a}_m \otimes a_m) \sum_n c_n \hat{a}_n$$

$$= \sum_n c_n \left[\hat{a}_m \underbrace{(\hat{a}_m \cdot \hat{a}_n)}_{= \delta_{mn}} \right] = c_m \hat{a}_m$$

In short projectors represent the action of "quantum measurement". In the words of Paul Dirac

"Measurement always causes a system to jump into the eigenstate of a dynamical variable that is being measured."

~~Before / Measurement~~

~~$\vec{\Psi}$~~

<u>Before</u>		<u>After</u>
$\vec{\Psi}$	$\xrightarrow{\text{measurement}}$	$P_m \vec{\Psi} = (\hat{a}_m \otimes \hat{a}_m) \vec{\Psi}$
		$= c_m \hat{a}_m$

Remember $c_m = \hat{a}_m \cdot \vec{\Psi}$. The probability of making this measurement is

$$\text{Prob}(m) = |\hat{a}_m \cdot \vec{\Psi}|$$