Let X_j be a set of identically distributed random variables. That is

$$E(X_j) = \mu \quad Var(X_j) = \sigma^2 \quad \text{for every } X_j.$$

However X_j's are not independent they are corelated.

$$Corr(X_i, X_j) = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_j)\, Var(X_i)}} = \frac{Cov(X_i, X_j)}{\sigma^2} = \rho$$

We want to simplify the expression, that is evaluate $Var(\bar{X})$ where $\bar{X} = \frac{1}{n} \sum_i X_i$

$$Var(\bar{X}) = \frac{1}{n^2} \underbrace{Var\left(\sum_i X_i\right)}$$

To simplify this expression we need to sum over all elements in the correlation matrix

Let us write down the correlation matrix

$$\begin{array}{cc} & \begin{array}{cccc} X_1 & X_2 & \cdots & X_n \end{array} \\ \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_n \end{array} & \begin{bmatrix} Var(X_1) & Cov(X_2, X_1) & \cdots & Cov(X_n, X_1) \\ Cov(X_1, X_2) & Var(X_2) & \cdots & Cov(X_n, X_2) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_1, X_n) & Cov(X_2, X_n) & \cdots & Var(X_n) \end{bmatrix} \end{array}$$

Not that this matrix is symetric because

$$Cov(X_i, X_j) = Cov(X_j, X_i) \quad \text{for ever } i, j$$

Our goal is to sum over every element in this matrix.

Define

$$I = Var\left(\sum_i X_i\right)$$

then $Var(\bar{X}) = \frac{1}{n^2} I$ and

$$I = Var\left(\sum_i X_i\right) = \sum_i Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

$$= \sum_i Var(X_i) + 2 \sum_{i<j} Cov(X_i, X_j)$$

From symetry.

Remember $Var(X_i) = \sigma^2$ and $Cov(X_i, X_j) = \rho\sigma^2$

$$I = n\sigma^2 + \rho\sigma^2 \underbrace{\sum_{i \neq j} (1)} = n\sigma^2 + \rho\sigma^2 (n^2 - n)$$

we need to count the number of off diagonal elements in the covarience matrix.

Therefore

$$Var(\bar{X}) = \frac{1}{n^2} \underbrace{\left[n\sigma^2 + \rho\sigma^2 n(n-1) \right]}_{= I} = \frac{\sigma^2}{n} + \rho\sigma^2 \left[\frac{n-1}{n}\right]$$

$$= \rho\sigma^2 + \left(\frac{1-\rho}{n}\right)\sigma^2$$

If there is no correlation that is $\rho = 0$ then $Var(\bar{X}) = \sigma/n$ and signal averaging works. If ~~they ar~~ X_i are perfecly correlated, $\rho = 1$, then ~~then $\rho=1$ and signal ave~~ $Var(\bar{X}) = \sigma$ and signal averaging does not work.