Cohort model to solve for the duration of infectiousness distribution

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Consider an SIR model with non-linear recovery:

$$\begin{aligned} S' &= \mu S - \beta S I && (1) \\ I' &= \beta S I - \gamma I \mathcal{K}(I) && (2) \end{aligned}$$

Note that equation (2) can be rewritten as a per-capita rate of prevalence change

$$\frac{I(t)'}{I(t)} = \beta S(t) - \gamma \mathcal{K}(I(t)) \quad (3)$$

where the time dependence ($t > 0$) has been explicitly made.

In order to find the expression of the duration of infectiousness distribution, we consider the cohort of individuals who acquired the disease at exactly time $\alpha > 0$. Let's label the size of this cohort (as a proportion of the whole population) c_α. This cohort is depleted at the same per-capita rate as the one given in equation (3). Hence, if $c_\alpha(\tau)$ is the size of this cohort τ units of time after disease acquisition, we have:

$$\frac{c_\alpha(\tau)'}{c_\alpha(\tau)} = -\gamma \mathcal{K}(I(\alpha + \tau)) \quad (4)$$

The initial condition for c_α is theoretically arbitrary, and if we choose $c_\alpha(0) = 1$, we can interpret $c_\alpha(\tau)$ as the probability of still being infectious τ units of time after having acquired the disease at time α, in other words, the duration of infectiousness distribution.

Let's express (4) with the original time variable $t = \alpha + \tau$:

$$\frac{c_\alpha(t-\alpha)'}{c_\alpha(t-\alpha)} = -\gamma \mathcal{K}(I(t)) \quad (5)$$

and for convenience, let's introduce p as

$$p_\alpha(t) = c_\alpha(t - \alpha)$$

The duration of infectiousness distribution conditional on disease acquisition at time α, p_α, can be determined numerically by solving the following system:

$$\begin{aligned} S' &= \mu S - \beta S I && (6) \\ I' &= \beta S I - \gamma I \mathcal{K}(I) && (7) \\ p_\alpha' &= -\gamma p_\alpha \mathcal{K}(I) && (8) \end{aligned}$$

with the initial conditions $I(0) = i_0$, $S(0) = 1 - i_0$ and $p_\alpha(\alpha) = 1$. Note that p_α is not epidemiologically defined for $t < \alpha$, but it is nonetheless possible to arbitrary set $p_\alpha(t) = 1$ for $t \leq \alpha$.