Compartmental epidemic models with nonlinear recovery

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1 The model

Assume the recovery rate, $\mathcal{K}(I)$, is a smooth, non-negative function on $[0, 1]$.

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \varepsilon\big(1 - S\big) - \rho SI \tag{1a}$$ {E:SIR;S}

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = \rho SI - \mathcal{K}(I)I \tag{1b}$$ {E:SIR;I}

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = I - \varepsilon(1 - S) \tag{1c}$$ {E:SIR;R}

where $S + I + R = 1$. In terms of the usual SIR parameters, the dimensionless parameters above are

$$\varepsilon = \frac{\mu}{\gamma + \mu}\,, \tag{2a}$$ {E:eps}

$$\rho = \frac{\beta N}{\gamma + \mu}\,. \tag{2b}$$ {E:rho}

If $\mathcal{K}(I) \equiv 1$ then ε is the mean time in the infectious class as a fraction of the of the mean lifetime and ρ is $\mathcal{R}_0$. Note that $0 \le \varepsilon < 1$ necessarily. I will write initial conditions as $(S_{\rm i}, I_{\rm i}, R_{\rm i})$.

2 Equilibria

There are typically two equilibria:

2.1 Disease free equilibria (DFE)

As for the standard KM SIR model:
Unique if $\varepsilon > 0$: $(S, I) = (1, 0)$.
Continuum of DFEs if $\varepsilon = 0$: $(S_0, 0)$ is a DFE for any $S_0 \in [0, 1]$.

2.2 Endemic equilibria (EE)

$(S, I) = (\hat{S}, \hat{I})$.

$$0 = \varepsilon\big(1 - \hat{S}\big) - \rho\hat{S}\hat{I} \tag{3}$$ {E:}

$$0 = \rho\hat{S} - \mathcal{K}(\hat{I}) \tag{4}$$

Hence

$$\hat{S} = \frac{\mathcal{K}(\hat{I})}{\rho} \tag{5}$$ {E:Shat}

$$\mathcal{K}(\hat{I}) = \frac{\varepsilon}{\hat{I} + (\varepsilon/\rho)} \tag{6}$$

To solve for $\hat{I}$ we need a specific form for $\mathcal{K}(I)$. In principle there can be multiple EEs.
For the standard KM SIR model,

$$\hat{I} = \varepsilon\Big(1 - \frac{1}{\rho}\Big)\,, \qquad \mathcal{K}(I) \equiv 1. \tag{7}$$ {E:IhatKM}

If the recovery rate is density-dependent, it is most natural to assume $\mathcal{K}(I)$ is decreasing, since this means that higher prevalence reduces recovery rates and hence lengthens infectious periods. Here are some examples:

$$\hat{I} = \begin{cases} \frac{\varepsilon}{\rho} \cdot \frac{a\rho - 1}{1 - b\varepsilon} & \mathcal{K}(I) = \frac{1}{a+bI}, \qquad \rho > \frac{1}{a}, \quad \varepsilon < \frac{1}{b} \\ \frac{a}{2} - \frac{\varepsilon}{2\rho} - \sqrt{\left(\frac{a}{2} - \frac{\varepsilon}{2\rho}\right)^2 - \varepsilon\left(1 - \frac{a}{\rho}\right)}\,, & \mathcal{K}(I) = a - I, \quad a \geq 1 \\ -\frac{1}{b} W(-b\varepsilon e^{-b\varepsilon/\rho}) - \frac{\varepsilon}{\rho} & \mathcal{K}(I) = e^{-bI}, \quad b > 0 \end{cases} \tag{8}$$ {E:Ihat}

Here, W is Lambert's W function. For the above cases, **Figure 1** shows the dependence of the EE on ρ for $\varepsilon = 0.1$.

The (singular) recovery function $\mathcal{K}(I) = I^{-1/2}$ has been used a fair bit in the literature even though it blows up at $I = 0$, so I'm mentioning it. This has two EEs if $\rho > 4/\varepsilon$ and none if $\rho < 4/\varepsilon$:

$$\hat{I} = \frac{\varepsilon}{2\rho}\left(\rho\varepsilon - 2 \pm \sqrt{\rho\varepsilon(\rho\varepsilon - 4)}\right), \qquad \mathcal{K}(I) = I^{-1/2}, \qquad \rho > 4/\varepsilon \tag{9}$$ {E:Ihat-1/2

This is quite peculiar and surely of mathematical interest only.

In case it is somehow helpful, here's an example with $\mathcal{K}(I)$ increasing:

$$\hat{I} = \varepsilon\left(\sqrt{\frac{1}{\varepsilon} + \frac{1}{4\rho^2}} - 1\right), \qquad \mathcal{K}(I) = I \tag{10}$$ {E:Ihatinc}

3 Infectious period distribution at EE

At the EE, prevalence is constant, so there is no change in the recovery rate over time. Consequently, individuals recover according to

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = -\mathcal{K}(\hat{I})\, I\,, \tag{11}$$ {E:}

i.e., the infectious period distribution is exponential with mean $1/\mathcal{K}(\hat{I})$ rather than mean 1.

4 Infectious period distribution near DFE (invasion)

In this limit, we can use the linearized system to obtain an approximation $I_{\mathrm{lin}}(\tau)$ initially. Then, for as long as this approximation is valid, the recovery rate is $\mathcal{K}(I_{\mathrm{lin}}(\tau))$, so we can approximate the initial distribution of infectious by solving

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = -\mathcal{K}\big(I_{\mathrm{lin}}(\tau)\big)\, I\,, \tag{12}$$ {E:invlim}

5 Infectious period distribution at any time

More generally, between the limits of invasion and endemic equilibrium, we could plug in the exact numerically computed $I(\tau)$ rather than $I_{\mathrm{lin}}(\tau)$ into Equation (12) and compute the distribution of infectious periods at any time. However, this is a pain, not just because it is numerical but because we really need to integrate to $\tau = \infty$. Still, this is an alternative to agent-based simulations.

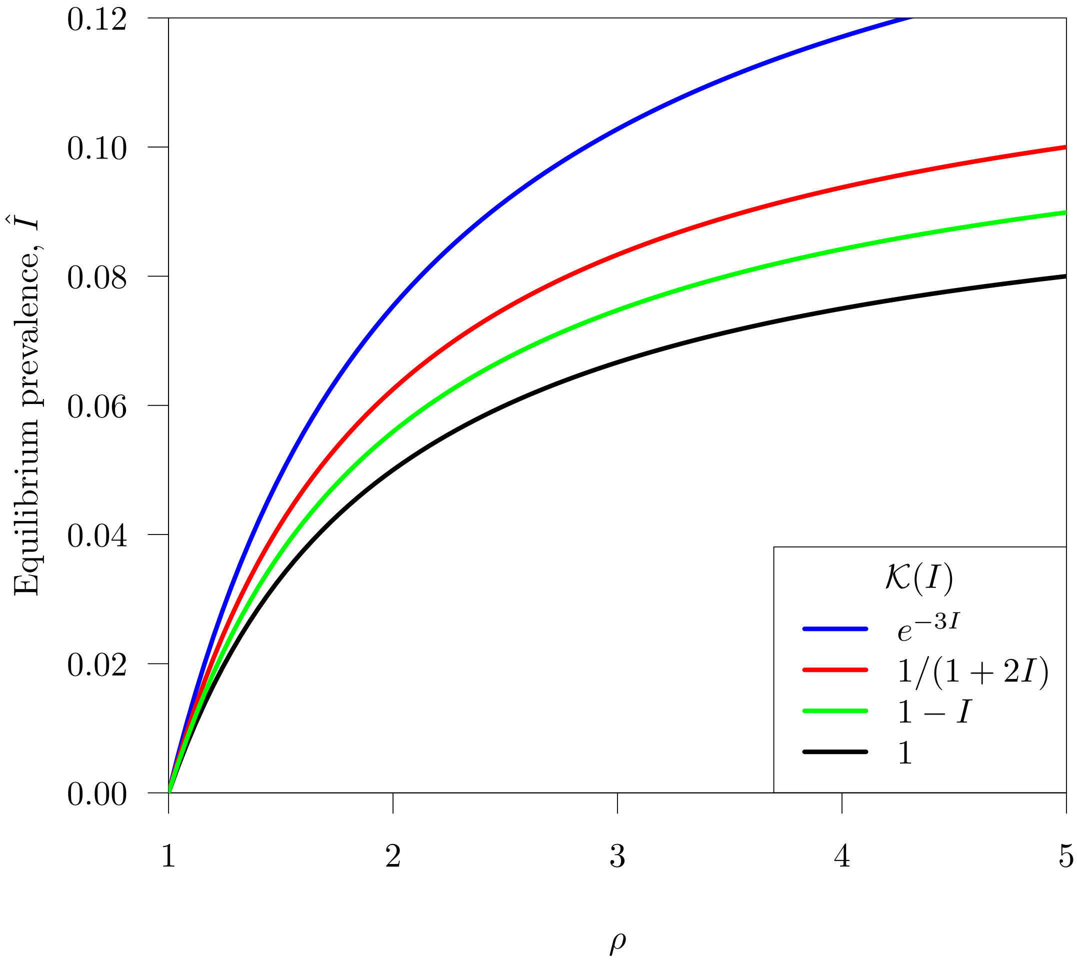

Figure 1: Equilibrium prevalence $\hat{I}$ as a function of ρ for several recovery functions $\mathcal{K}(I)$, with $\varepsilon = 0.1$.

References