SEEM 5870: Computational Finance **2020-2021 Term2**

Homework 1 Solution

Instructor: Professor Xuefeng Gao TA: Yi Xiong

Problem 1[10'] Suppose you have an algorithm to generate $U \sim \text{Unif}[0,1]$, the uniform distribution on $[0,1]$. Explain how to use the inverse transform method to generate samples from an exponential distribution with density function $f(x) = \lambda e^{-\lambda x}$ for given $\lambda > 0$. You do not need to write codes for this exercise.

Solution: The cumulative density function (CDF) of exponential distribution is $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Then generate $U \sim \text{Unif}[0,1]$, then $X = F^{-1}(U) = -\frac{1}{\lambda}\log(1-U)$ follows exponential distribution. Moerover, it can also be implemented by $X = -\frac{1}{\lambda}\log(U)$ because U and $1-U$ follow the same distribution.

Problem 2[30'] One way to generate samples from univariate normal distributions is the Box-Muller algorithm.

Algorithm 1 [1] Box-Muller algorithm

Generate $U_1, U_2 \sim \text{Unif}[0,1]$
$R \leftarrow -2\log(U_1)$
$V \leftarrow 2\pi U_2$
$Z_1 \leftarrow \sqrt{R}\cos(V)$, $Z_2 \leftarrow \sqrt{R}\sin(V)$
return Z_1, Z_2

One can prove that Z_1 and Z_2 are independent and both follow $N(0,1)$ (no need to prove it here). Please generate 100,000 samples from the standard normal distribution using the Box-Muller algorithm. Plot the histogram of the simulated samples and use the Q-Q plot (Quantile-Quantile plot) to assess its normality.

Solution:

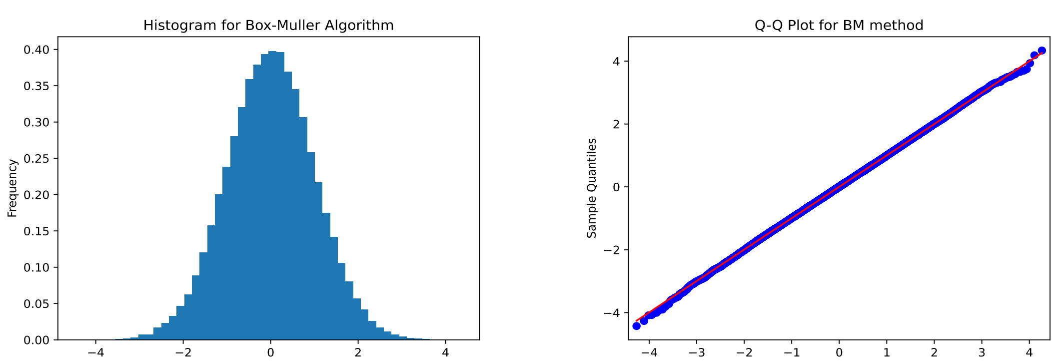

Figure 1: (a) Histogram of Z_1; (b) Q-Q Plot of Z_1.

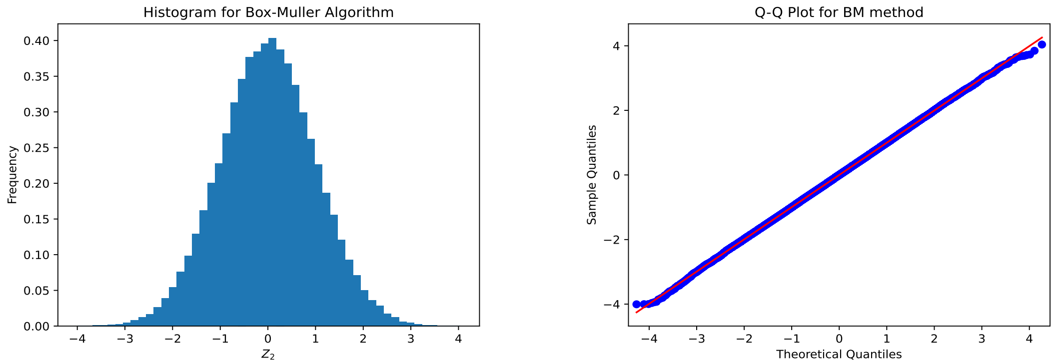

Figure 2: (a) Histogram of Z_2; (b) Q-Q Plot of Z_2.

We can see that the historgrams of generated samples Z_1 and Z_2 have the shape of normal distirubion, and the Q-Q plots both appear linear, showing the distributions of data generated are normal.

Problem 3[60'] Suppose the risk-neutral dynamics of underlying asset price can be modeled by the geometric brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Suppose $S_0 = 50, T = 0.25, r = 0.05, \sigma = 0.3$.

(1) [10'] Please simulate 100 samples of S_T and plot the histogram.

(2) [30'] Please use Monte Carlo simulation with sample size $M = 10^5$ to estimate the price of the European put option $e^{-rT}\mathbb{E}[(K - S_T)^+]$ for $K = 40, 50, 60$ at time 0 and provide confidence intervals with confidence level 95%.

(3) [20'] Also vary M from 1 to 10^6 to plot the Monte Carlo error as a function of M, by comparing the estimated values from simulation with the true value obtained from the Black-Scholes formula.

Solution:

(1) Applying Ito's formula, for $u < t$, we can obtain

$$\begin{aligned} S_T &= S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma(W_T - W_0)\right] \\ &= S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right], \quad Z \sim N(0,1). \end{aligned}$$

Then we plot the histogram of 100 samples of S_T.

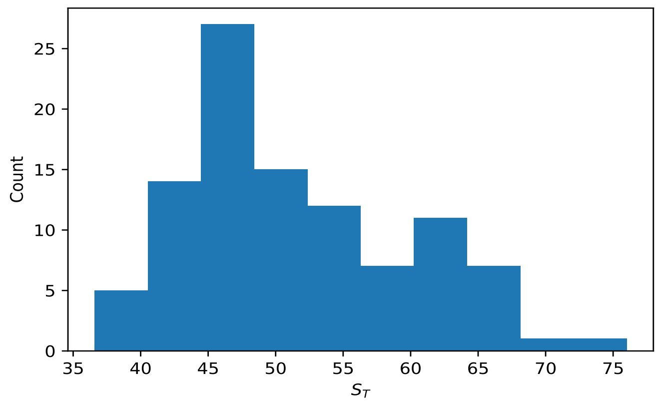

Figure 3: Histogram of S_T.

(2) The payoff of the European put option at time T is $V(T) = (K - S_T)^+$, thus the price of the option at time t is

$$V(t) = \mathbb{E}[e^{-r(T-t)}V(T)] = \mathbb{E}[e^{-r(T-t)}(K - S_T)^+].$$

Then let $\tau = T - t$, we can get the Black-Scholes-Merton value of the price:

$$P(\tau, S_t) = Ke^{-r\tau}N\left(-d_-(\tau, S_t)\right) - S_tN\left(-d_+(\tau, S_t)\right),$$

where $d_\pm(\tau, S_t) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})\tau\right]$.
Therefore, the price of the European put option at time 0 is

$$P(T, S_0) = Ke^{-rT}N\left(-d_-(T, S_0)\right) - S_0N\left(-d_+(T, S_0)\right).$$

Given sample size $M = 10^5$, the 95% confidence level of the Monte Carlo estimator $\hat{P}$ is in the form

$$\left(\hat{P} - Z_{0.025} * \frac{\hat{\sigma}}{\sqrt{M}}, \hat{P} + Z_{0.025} * \frac{\hat{\sigma}}{\sqrt{M}}\right), \tag{1}$$

where $\hat{\sigma}$ is the sample standard deviation of samples $\hat{P}_1, \cdots, \hat{P}_M$, and $Z_{0.025} = 1.96$. Then we have following results table.

K	40	50	60
BSM value $P(T, S_0)$	0.1655	2.6704	9.7792
MC value $\hat{P}$	0.1676	2.6786	9.7927
Sample Variance	0.6797	14.4048	42.4660
Confidence Interval	[0.1625, 0.1727]	[2.6550,2.7021]	[9.7524, 9.8331]

We can see that the trun BSM values are contained in the confidence intervals of the MC value.

(3) Monte Carlo error can be computed by $\hat{P} - P(T, S_0)$, then we plot the estimated error as a functioin of the sample size M from 1 to 10^6.

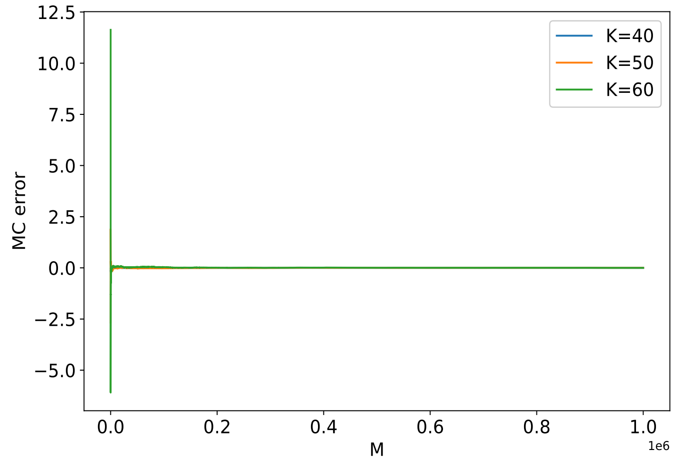

Figure 4: Monte Carlo error as a function of sample size (M varying from 1 to 10^6)

Moreover, we can see the more detailed figure when M varying from 1 to 1000 as below:

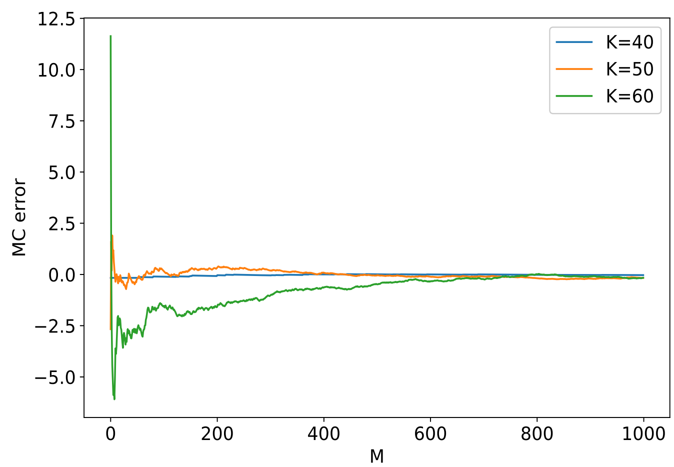

Figure 5: Monte Carlo error as a function of sample size (M varying from 1 to 10^3)

We can see that when sample size is large enough, the Monte Carlo error closes to 0.

References

[1] Glasserman, P. (2013). Monte Carlo methods in financial engineering (Vol. 53). Springer Science & Business Media.