SEEM 5870: Computational Finance **2020-2021 Term2**

Homework 5 Solution

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Problem 1[30'] Prove the following statement: the value of an American call option equals the value of a European call option, assuming the underlying stock pays no dividend and both calls have the same strike price and expiration date.

Solution: The only difference between an American and European call option is that American call option can exercise on any business day. We will show that it is not profitable to exercise an Americal call option early, then the two options will have the same value.

We will show that the value of an American call option satisfies $C \geq S - PV(X)$, where S is the value of the asset and $PV(X)$ is the present value of the strike price.

Consider two portfolios: Portfolio 1 consists of buying an American call, Portfolio 2 consists of buying the asset and borrowing the present value of the strike price. The table below gives the values of these two portfolios at expiration:

	$S \leq X$	$S > X$
Portfolio 1 (buy C)	0	S-X
Portfolio 2 (buy S borrow PV(S))	S-X	S-X

Notice that regardless of the value of the asset at expiration Portfolio 1 has a value at least as great as Portfolio 2, which must be true at any time prior to expiration due to the arbitray free argument. So suppose that at some time t beofe expiration the holder of an Americal call option decides to exercise the option. Then he will realize a profit of $S - X$. But the value of the call option is at least $S - PV(X)$, which is greater than $S - X$. So rather than exercising the option, the holder of the call option will sell the call option and achieve a larger return. That is, it is not profitable to early exercise an American call option.

Problem 2[70'] Consider pricing an American put with strike $K = 12$ and maturity $T = 1$ year. The underlying stock pays no dividend, and the price process $S(\cdot)$ is modelled as a geometric Brownian motion with $S(0) = 10, r = 3\%, \sigma = 0.4$. For simplicity, assume the American put can only be exercised at time points $t = 1/3, 2/3, 1$. Also assume we only generate eight sample paths of the stock price process as given below. Use the Longstaff-Schwartz Method to compute the value of the Amercan put at time zero, and also find the optimal exercise strategy. (Hint: use the same method for the numerical example discussed in class)

Path	$t = 1/3$	$t = 2/3$	$t = 1$	$Y_3 = \max(K - S(1), 0)$
1	8.3826	9.9528	6.7581	5.2419
2	11.9899	13.8988	14.5060	0
3	13.1381	17.4061	13.4123	0
4	6.8064	7.8115	10.6520	1.3480
5	7.0508	9.1293	7.4551	4.5449
6	11.2214	8.3600	9.2896	2.7104
7	8.9672	8.7787	9.0822	2.9178
8	11.5336	10.9398	8.6958	3.3042

Solution:

Consider $t = 2/3$, the optionholder needs to decide whether to exercise the option immediately or continue the option when the option is in-the-money. We see that all the sample paths are in-the-money except path 2 and 3.

Path	$Y_3 e^{-r\Delta t}$	$S(2/3)$
1	5.1898	9.9528
2	—	13.8988
3	—	17.4061
4	1.3346	7.8115
5	4.4997	9.1293
6	2.6834	8.3600
7	2.8888	8.7787
8	3.2714	10.9398

Table 1: Regression at $t = 2/3$

We model the expected payoff from continuation at $t = 2/3$ as a quadratic polynomials of $S(2/3)$. The resulting formula is

$$E[Y_3 e^{-r\Delta t} | S(2/3)] = -82.5306 + 17.7780 \cdot S(2/3) - 0.9063 \cdot S(2/3)^2 := f_2(S(2/3))$$

Therefore, the optimal decision at $t = 2/3$

$$Y_2 = \begin{cases} K - S(2/3), & \text{if } K - S(2/3) \geq f_2(S(2/3)) \\ Y_3 e^{-r\Delta t}, & \text{otherwise} \end{cases}$$

Path	Exercise $K - S(2/3)$	Continuation $f_2(S(2/3))$	Y_2
1	2.0472	4.6379	5.1898
2	—	—	0
3	—	—	0
4	4.1885	1.0428	4.1885
5	2.8707	4.2388	4.4997
6	3.6400	2.7554	3.6400
7	3.2213	3.6958	2.8888
8	1.0602	3.4968	3.2714

Table 2: Optimal Decision at $t = 2/3$

Next, we repeat the procedure for $t = 1/3$. All the sample paths are in-the-money except path 3.

Path	$Y_2 e^{-r\Delta t}$	$S(1/3)$
1	5.1382	8.3826
2	0	11.9899
3	—	13.1381
4	4.1468	6.8064
5	4.4549	7.0508
6	3.6038	11.2214
7	2.8601	8.9672
8	3.2389	11.5336

Table 3: Regression at $t = 1/3$

Similaryly, we model the expected payoff from continuation at $t = 1/3$ as a quadratic polynomials of $S(1/3)$. The resulting formula is

$$E[Y_2 e^{-r\Delta t} | S(1/3)] = -8.9492 + 3.3105 \cdot S(1/3) - 0.2036 \cdot S(1/3)^2 := f_1(S(1/3))$$

Therefore, the optimal decision at $t = 1/3$

$$Y_1 = \begin{cases} K - S(1/3), & \text{if } K - S(1/3) \geq f_1(S(1/3)) \\ Y_2 e^{-r\Delta t}, & \text{otherwise} \end{cases}$$

Path	Exercise $K - S(1/3)$	Continuation $f_1(S(1/3))$	Y_1
1	3.6174	4.4921	5.1382
2	0.0101	1.4689	0
3	—	—	0
4	5.1936	4.1494	5.1936
5	4.9492	4.2688	4.9492
6	0.7786	2.5572	3.6038
7	3.0328	4.3620	2.8601
8	0.4664	2.1441	3.2389

Table 4: Optimal Decision at $t = 1/3$

Finally, the current price of the American option is estimated by the average of $Y_1 e^{-r\Delta t}$, that is 3.0919. And we summarize the exercise result below

Path	Exercise Time
1	t=1
2	—
3	—
4	t=1/3
5	t=1/3
6	t=2/3
7	t=1
8	t=1

Table 5: Optimal Decision for Each Sample Path