SEEM 5870: Computational Finance	**2020-2021 Term2**

Homework 2 Solution

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Suppose the risk-neutral dynamics of underlying asset price can be modeled by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Suppose $S_0 = 50, T = 0.25, r = 0.05, \sigma = 0.3$.

Problem 1[40'] Please use Monte Carlo simulation with sample size $M = 10^5$ to estimate the price of the European put option $e^{-rT}\mathbb{E}[(K - S_T)^+]$ for $K = 40, 50, 60$ at time T using two different control variates: S_T and W_T. Compare the variances of the control-variate estimators with the variance of the naive Monte Carlo estimator.

Solution: We we want to estimate $\mathbb{E}[Y]$ using the Monte Carlo average $\bar{Y} = \frac{Y_1+\cdots+Y_n}{n}$, we can use the control-variate method to reduce the variance. It can be conducted by defining a new estimator

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i(b) = \frac{1}{n}\sum_{i=1}^{n} \left(Y_i - b(X_i - \mathbb{E}[X_i])\right),$$

where $b = \frac{Cov(X,Y)}{Var(X)} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$

The price of the European put option obtained from the Black-Scholes-Merton formula at time 0 is

$$P(T, S_0) = Ke^{-rT}N\left(-d_-(T, S_0)\right) - S_0 N\left(-d_+(T, S_0)\right).$$

where $d_\pm(T, S_0) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K} + (r \pm \frac{\sigma^2}{2})T\right]$. Then the Black-Scholes-Merton values and their corresponding variances under different strike prices K:

K	40	50	60
BSM value $P(T, S_0)$	0.16034497	2.68403827	9.74933184
MC value $\hat{P}$	0.169127	2.66955	9.76195
Sample Variance	0.64288313	14.43622487	42.3745417
Estimator Variance	6.42883129e-06	1.44362249e-04	4.23745417e-04

Here, I want to remark that the estimator variance equals sample variance divided by sample size.

Choosing three different control variates, the simulation results are showed below under different strike prices K:

when $X = S_T$

K	40	50	60
$\hat{P}$	0.16034497	2.68403827	9.74933184
$\lvert\rho_{XY}\rvert$	0.39096313	0.81144152	0.96764948
Sample Variance	0.54461705	4.93087535	2.69732926
Estimator Variance	5.44617050e-06	4.93087535e-05	2.69732926e-05

when $X = W_T$

K	40	50	60
$\hat{P}$	0.16034497	2.68403827	9.74933184
$\lvert\rho_{XY}\rvert$	0.45401635	0.8645061	0.98337253
Sample Variance	0.51036509	3.64701195	1.39744738
Estimator Variance	5.10365086e-06	3.64701195e-05	1.39744738e-05

when $X = W_T^2$

K	40	50	60
$\hat{P}$	0.16034497	2.68403827	9.74933184
$\lvert\rho_{XY}\rvert$	0.60992609	0.4629888	0.07079784
Sample Variance	0.40372428	11.34169548	42.16214633
Estimator Variance	4.03724280e-06	1.13416955e-04	4.21621463e-04

We can see that the MC estimators under three different control variates are the same, while their variance are quite different. We can obtain that a good control variate (large $|\rho_{XY}|$ value) can decrease the estimator variance, while inappropriate control variates cannot achieve the goal.

Problem 2[30'] Partition the real line into ten disjoint sets $A_1, \ldots, A_{10}$ so that $P(Z \in A_i) = 0.1$ for each i, where $Z \sim N(0,1)$. Use stratified sampling method (by stratifying $N(0,1)$) to estimate the price of the European put option $e^{-rT}\mathbb{E}[(K - S_T)^+]$ for $K = 40, 50, 60$. Use total sample size $M = 10^5$ and proportional sampling from each strata. Report your estimates and the standard errors. (Hint: You may need to use the inverse transform method or other methods to generate samples from the conditional distribution of Z given $Z \in A_i$)

Solution: We want to simulate $Z \sim N(0,1)$, let $\Phi(z)$ is the cumulative density function of Z. Given the equal probabilities $p_i = 0.1$, for $i = 1, \cdots, 10$. Define $a_0 = \infty$, $a_1 = \Phi^{-1}(0.1), a_2 = \Phi^{-1}(0.2), \cdots, a_{10} = \Phi^{-1}(1)$. We can define the following 10 strata: $A_1 = (a_0, a_1], A_i = (a_{i-1}, a_i], \cdots, A_{10} = (a_9, a_{10}]$. Then we have $\mathbb{P}(Z \in A_i) = \Phi(a_i) - \Phi(a_{i-1}) = 0.1$. Let $U \sim \text{Unif}[0,1]$, then $V = a_{i-1} + U(a_i - a_{i-1})$ is uniformly distributed between a_{i-1}, then $\Phi^{-1}(V)$ has the distribution of Z conditional on $Z \in A_i$.

K	40	50	60
MC value $\hat{P}$	0.168251	2.67325	9.62348
Standard Error	0.00254122	0.0120233	0.0206119

(The definition of standard error given in page 29, Lecture 1)

Moreover, you can read the Example 4.3.2 in the textbook by Paul Glasserman for reference.

Problem 3[30'] (Exercise in Page 38 of Lecture Notes Topic 2) Fix the sample size $M = 10^4$. Use both Plain Monte Carlo and importance sampling methods to estimate $\mathbb{P}(S_T \leq K)$ for $K = 10, 20, 60$. For each method, report your estimate, the standard error, and the relative error (standard error divided by the estimate).

Solution: S_T is a geometrica Brownian motion with

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W_T\right)$$

Then we let $X = \log(S_T) = \log(S_0) + (r - \frac{1}{2}\sigma^2)T + \sigma W_T$, which follows $N(\log(S_0) + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$

So the naive MC estimator can be estimated by generate i.i.d. samples $X_1, \cdots, X_M$ following $N(\log(S_0) + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$, then calculate the estimator

$$\frac{1}{M}\sum_{i=1}^{M} 1_{\{X_i \le \log(K)\}}.$$

The results are shown below:

K	10	20	60
MC value $\hat{P}$	0	0	0.8829
Standard Error	0	0	0.00321555
Relative Error	nan	nan	0.00364204

Then, for important sampling method, we generate i.i.d. samples $X_1, \cdots, X_M$ following $N(\log(K), \sigma^2 T)$, then calculate the estimator

$$\frac{1}{M}\sum_{i=1}^{M} 1_{\{X_i \le \log(K)\}} * \exp\left(\frac{(X_i - \mu_1/2 - \mu_2/2)(\mu_1 - \mu_2)}{\sigma^2 T}\right),$$

where $\mu_1 = \log(S_0) + (r - \frac{1}{2}\sigma^2)T, \mu_2 = \log(K)$.

The results are shown below:

K	10	20	60
MC value $\hat{P}$	3.36155403e-27	4.79930712e-10	8.93935734e-01
Standard Error	1.19752990e-28	1.24819341e-11	1.92906155e-02
Relative Error	0.03562429	0.02600778	0.02157942