SEEM 5870: Computational Finance	**2020-2021 Term2**

Homework 3 Solution

Instructor: Professor Xuefeng Gao — TA: Yi Xiong

Problem 1[30'] Suppose the dynamics of two asset prices (S_1, S_2) are modeled as a two-dimensional geometric Brownian motion (GBM) under the risk neutral measure where

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t),$$

where W_i is one-dimensional standard Brownian motion for $i = 1, 2$, and $W_1(t)$ and $W_2(t)$ have correlation $\rho_{12} \in [-1, 1]$. Suppose $r = 0.01, \sigma_1 = 0.3, S_1(0) = 50, \sigma_2 = 0.2, S_2(0) = 30, \rho_{12} = 0.1, T = 0.5, K = 15$. Using the exact simulation of two dimensional GBM to price spread options with payoff $(S_1(T) - S_2(T) - K)^+$.

Solution: Let $W_1(t) = \sqrt{t}Z_1, W_2(t) = \sqrt{t}Z_2$, then

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\mathbf{0}, \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}\right).$$

Then one can directly use some programming pacakage to simulate Z_1, Z_2 or follow the linear property $Z = AY$ where $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N(\mathbf{0}, I)$, and A is a lower triangular matrix with $AA^T = \Sigma$. Using the Cholesky factorization, we can get $A = \begin{pmatrix} 1 & 0 \\ 0.1 & \sqrt{0.99} \end{pmatrix}$, then $Z_1 = Y_1, Z_2 = 0.1Y_1 + \sqrt{0.99}Y_2$, where $Y_1 \sim N(0, 1), Y_2 \sim N(0, 1)$. Then,

$$S_1(t) = S_1(0) \exp\left(\left(r - \frac{1}{2}\sigma_1^2\right)t + \sigma_1\sqrt{t}Z_1\right),$$
$$S_2(t) = S_2(0) \exp\left(\left(r - \frac{1}{2}\sigma_2^2\right)t + \sigma_2\sqrt{t}Z_2\right).$$

Simulate $M = 10^5$ samples of value $(S_1(T)-S_2(T)-K)^+$. , we can get the naive MC estimator $\frac{1}{M}\sum_{i=1}^{M} e^{-rT}(S_{1,i}(T)-S_{2,i}(T) - K)^+ = 7.2430$.

Problem 2[30'] Merton's jump diffusion model can be described through

$$\frac{dS(t)}{S(t-)} = rdt + \sigma dW(t) + dJ(t) \tag{1}$$

and

$$J(t) = \sum_{i=1}^{N(t)} (Y_i - 1),$$

where $\log Y_j \sim N(a, b^2)$ and $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Suppose $T = 0.25, S(0) = 52, K = 50, r = 0.05, \sigma = 0.3, a = 0, b = 1, \lambda = 0.25$. Please simulate $\{S(t) : t \geq 0\}$ to estimate the price of European call option $e^{-rT}\mathbb{E}[(S(T) - K)^+]$. Compare your simulation results with the analytical formula obtained by Merton.

Solution: Learning in the lecture 3, the analytical formula for the European call option is

$$e^{-rT}\mathbb{E}[(S(T)-K)^+] = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!} e^{-\lambda' T} BS(S(0), \sigma_n, T, r_n, K),$$

where $\lambda' = \lambda\mathbb{E}[Y_j] = \lambda e^{a+b^2/2}, \sigma_n^2 = \sigma^2 + b^2 n/T, r_n = r - \lambda(\mathbb{E}[Y_j]-1) + n\log(\mathbb{E}[Y_j])/T$, and $BS(\cdot)$ denotes the Black-Scholes formula for a call option.

The simulation schemes of Merton's jump diffusion model can be founded in lecture 3 notes, page 39-42.

Then we can get that the approximate value under the analytical formula is 5.8794, and the MC simulation value is 7.1675. The MC error is 7.1675-5.8794=1.2881, which seems not good. When we choose another pair of parameters, $T = 0.25, S(0) = 52, K = 50, r = 0.05, \sigma = 0.3, a = 0, b = 0.01, \lambda = 2$, you can find that the approximate value under the analytical formula is 4.5221 and the MC simulation value is 4.5290, which is desired.

Problem 3[40'] Consider the CIR model for interest rate movements:

$$dX_t = a(b - X_t)dt + \sigma\sqrt{X_t}dW_t.$$

Suppose $a = 0.2, b = 0.05, \sigma = 0.1, X_0 = 4\%, T = 0.5$. Please implement the exact simulation method and the Euler scheme to simulate X_T. Quantify the discretization error of the Euler Scheme based on your simulation results.

Solution:

Exact simulation method (simulation result is denoted by X_T^1): We find that $d = 4ba/\sigma^2 = 4 > 1$, then applying the algorithm given in case 1, page 124, [1].

Euler scheme (simulation result is denoted by X_T^2):

$$\begin{aligned}\hat{X}_{t_{i+1}} &= \hat{X}_{t_i} + a(\hat{X}_{t_i})(t_{i+1} - t_i) + b(\hat{X}_{t_i})\sqrt{t_{i+1} - t_i}Z_{i+1} \\ &= \hat{X}_{t_i} + a(b - \hat{X}_{t_i})(t_{i+1} - t_i) + \sigma\sqrt{\hat{X}_{t_i}} \cdot \sqrt{t_{t+1} - t_i}Z_{i+1},\end{aligned}$$

where Z_i is the independent standard normal random variables sequence.

Choose steps size $n = 100$ and sample size $M = 10^5$ to get the above two simulation results: $X_T^1 = 0.04096, X_T^2 = 0.04090$. The discretization error is $\mathbb{E}X_T^2 - X_T^1] = -4.7547 \times 10^{-5}$.

References

[1] Glasserman, P. (2013). Monte Carlo methods in financial engineering (Vol. 53). Springer Science & Business Media.