Phys514 Fall 2013: HW6 Solution

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1 Foot 10.1 - Magnetic trapping (20 pts)

Numerical values: $g_F m_F = 1/2$, $M = 23\ (6.02 \cdot 10^{26})^{-1}$ kg, $B_0 = 3 \cdot 10^{-4}$ T, $b' = 3$ T m^{-1}, $b'' = 300$ T m^{-2}

The radial trap frequency is given by Foot 10.12, $\omega_r = \sqrt{\frac{g_F m_F \mu_B}{M B_0}} \cdot b' = \boxed{(2\pi)\ 304\ \text{Hz}}$. On the other hand, the oscillation frequency in the axial direction can be derived from Foot 10.13, $\omega_z = \sqrt{\frac{g_F m_F \mu_B b''}{M}} = \boxed{(2\pi)\ 30\ \text{Hz}}$

2 Foot 10.2 - Loading a trap (20 pts)

Numerical values: $N = 10^{10}$, $n = 10^{16}$ m^{-3}, $T = 2.4 \cdot 10^{-4}$ K, $M = 23\ (6.02 \cdot 10^{26})^{-1}$ kg

Part a. The radius of the cloud can be approximated by $r = (\frac{3N}{4\pi n})^{1/3} = 6.2$ mm. From Foot 10.46, the trapping frequency is $\omega = \sqrt{\frac{k_B T}{M r^2}} = 47.5$ rad/s $= \boxed{(2\pi)\ 7.5\ \text{Hz}}$. The volume of the cloud in terms of the trap frequency is $V = \frac{4\pi r^3}{3} = \frac{4\pi}{3\omega^3}\left(\frac{k_B T}{M}\right)^{3/2}$. We conclude that the gas size is smaller for a stiffer trap and larger for a weaker trap.

Part b. The de Broglie wavelength is $\lambda_{dB} = \frac{h}{\sqrt{2\pi k_B T M}} = 23$ nm, and the phase-space density is $\frac{n\lambda_{dB}^3}{2.6} = \boxed{4.9 \cdot 10^{-8}}$

Part c. The phase-space density during the adiabatic process is constant and scales as $n\lambda_{dB}^3 \propto \frac{1}{VT^{3/2}}$, therefore $TV^{2/3}$ is constant. In the part a we found that the radius of the cloud is related to the trap frequency as $r = \frac{1}{\omega}\sqrt{\frac{k_B T'}{M}}$, from which we obtain the volume $V' = \frac{4\pi}{3} r_r^2 r_z = \frac{4\pi}{3\omega_r^2 \omega_z}\left(\frac{k_B T'}{M}\right)^{3/2}$. By conserving the product $TV^{2/3}$ we find that the temperature after compression is $T' = \sqrt{\frac{MT}{k_B}}\left(\frac{3V\omega_r^2\omega_z}{4\pi}\right)^{1/3} = \boxed{3.2\ \text{mK}}$. The new radii of the gas are $r_r = 0.7$ mm and $r_z = 10.7$ mm; the new volume is $V' = 2.1 \cdot 10^{-8}$ m^3; and the new density is $n' = \frac{N}{V'} = \boxed{4.8 \cdot 10^{17}\ \text{m}^{-3}}$

3 Foot 10.4 - Evaporative cooling (20 pts)

$N_{tot} = A\int_0^\infty \frac{E^2}{2(\hbar\bar{\omega})^3} e^{-\beta E} dE = \frac{A}{(\beta\hbar\bar{\omega})^3}$, then $A = N_{tot}(\beta\hbar\bar{\omega})^3$

$E_{tot} = \frac{N_{tot}\beta^3}{2}\int_0^\infty E^3 e^{-\beta E} dE = \boxed{3N_{tot}k_B T}$ and therefore $\bar{E} \equiv E_{tot}/N_{tot} = \boxed{3\beta^{-1}}$

Part a. $\frac{\Delta N}{N_{tot}} = \frac{\beta^3}{2}\int_\epsilon^\infty E^2 e^{-\beta E} dE = \boxed{(1 + \beta\epsilon + \frac{1}{2}\beta^2\epsilon^2)e^{-\beta\epsilon}}$

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Part b. The fractional change in the mean energy is

$$\frac{\Delta\bar{E}}{\bar{E}} \equiv \frac{\bar{E}-\bar{E}'}{\bar{E}} \qquad \text{; where} \quad \bar{E}' \equiv E'_{tot}/N'_{tot}$$

$$= 1 - \frac{1}{3\beta^{-1}} \frac{\frac{N_{tot}\beta^3}{2}\int_0^\epsilon E^3 e^{-\beta E} dE}{\frac{N_{tot}\beta^3}{2}\int_0^\epsilon E^2 e^{-\beta E} dE}$$

$$= 1 - \frac{1-(1+\beta\epsilon+\frac{1}{2}\beta^2\epsilon^2+\frac{1}{6}\beta^3\epsilon^3)e^{-\beta\epsilon}}{1-(1+\beta\epsilon+\frac{1}{2}\beta^2\epsilon^2)e^{-\beta\epsilon}}$$

$$= \boxed{\frac{\frac{1}{6}\beta^3\epsilon^3 e^{-\beta\epsilon}}{1-(1+\beta\epsilon+\frac{1}{2}\beta^2\epsilon^2)e^{-\beta\epsilon}}}$$

Part c.

$\beta\epsilon$	$\Delta N/N_{tot}$	$\Delta\bar{E}/\bar{E}$	$\frac{\Delta\bar{E}/\bar{E}}{\Delta N/N_{tot}}$
3	0.42	0.39	0.92
6	0.06	0.09	1.54

A deeper cut ($\beta\epsilon = 3$) removes more energy per atom than a shallower cut ($\beta\epsilon = 6$); however, it also removes more atoms. A shallower cut is more efficient in evaporative cooling.

Part d. The peak density scales as $n_{pk} \propto n \propto \frac{N_{tot}}{r^3} \propto N_{tot}\ \beta^{3/2}$. On the other hand, the mean velocity scales as $\bar{v} \propto \beta^{-1/2}$. In consequence, the collision rate scales as $R_{coll} \propto n_{pk}\bar{v}\sigma \propto N_{tot}\beta \propto \boxed{N_{tot}/\bar{E}}$

During the evaporation process, $R_{coll} \propto \frac{N_{tot}-\Delta N}{\bar{E}-\Delta\bar{E}} \approx 1 - \frac{\Delta N}{N_{tot}} + \frac{\Delta\bar{E}}{\bar{E}}$. The collision rate increases if $\frac{\Delta\bar{E}/\bar{E}}{\Delta N/N_{tot}} > 1$

4 Foot 10.5 - The properties at the phase transition (20 pts)

Numerical values: $M = 87\ (6.02\cdot 10^{26})^{-1}$ kg, $N = 10^6$ atoms, $\omega_z = (2\pi)$ 16 Hz, $\omega_r = (2\pi)$ 250 Hz

The mean trap frequency is $\bar{\omega} = (\omega_z\omega_r^2)^{1/3} = (2\pi)$ 100 Hz. From Foot 10.19, the critical temperature of a gas with harmonic confinement is $T_c = \frac{\hbar\bar{\omega}N^{1/3}}{k_B} = \boxed{478\text{ nK}}$. From Foot 10.14, the de Broglie wavelength is $\lambda_{dB} = \frac{h}{\sqrt{2\pi k_B T M}} = 26.9\ \mu$m and, therefore, the phase-space density is $n = \frac{2.6}{\lambda_{dB}^3} = \boxed{1.3\cdot 10^{20}\text{ m}^{-3}}$

5 Foot 10.8 - Expansion of a non-interacting condensate (20 pts)

Numerical values: $\omega_z = (2\pi)$ 16 Hz, $\omega_r = (2\pi)$ 250 Hz.

The width of the gas in the transverse and axial directions are respectively $\Delta r = \sqrt{\frac{\hbar}{M\omega_r}}$ and $\Delta z = \sqrt{\frac{\hbar}{M\omega_z}}$. From the uncertainty principle, $\Delta x \cdot \Delta p \sim \hbar/2$, we can estimate the momentum spread in each direction: $\Delta p_r \sim \frac{1}{2}\sqrt{M\hbar\omega_r}$ and $\Delta p_z \sim \frac{1}{2}\sqrt{M\hbar\omega_z}$. Matching the gas size in both directions after a expansion time t gives

$$\sqrt{\frac{\hbar}{M\omega_r}} + \frac{1}{2}\sqrt{\frac{\hbar\omega_r}{M}}t = \sqrt{\frac{\hbar}{M\omega_z}} + \frac{1}{2}\sqrt{\frac{\hbar\omega_z}{M}}t$$

whose solution is $t = \frac{2}{\sqrt{\omega_r\omega_z}} = 5$ ms

In a more rigorous way, the width of a gaussian packet expands as[1] $\sigma(t) = a\sqrt{1+\omega^2 t^2}$, where $a = \sqrt{\frac{\hbar}{M\omega}}$ is the initial width. The cloud becomes spherical when $a_r\sqrt{1+\omega_r^2 t^2} = a_z\sqrt{1+\omega_z^2 t^2}$, from where we obtain the time of flight $t = \frac{1}{\sqrt{\omega_r\omega_z}} = \boxed{2.5\text{ ms}}$.

[1] Griffith, Introduction to Quantum Mechanics