SIR Model For COVID

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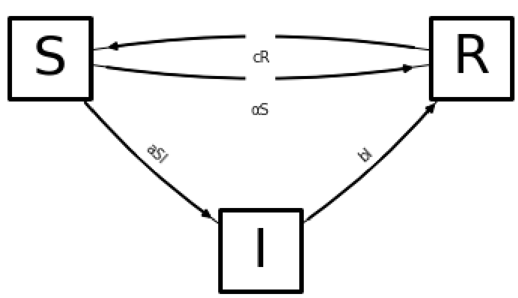
1 Introducing Variables and Parameters

SIR models provide a theoretical framework for the time rates of change of three populations in an outbreak of a contagious disease. The populations in the models are given the shorthand

- S for the number of people in the population that are susceptible to getting infected
- I for the number of people that are infected
- R for the people that are recovered from the disease (and are therefore immune, possibly only temporarily)

As time goes on, the three populations exchange members as shown in the diagram below. For example, susceptible people become infected. The following directed graph shows the rates exchanges in the model with a, b, c, and α as arbitrary numbers.

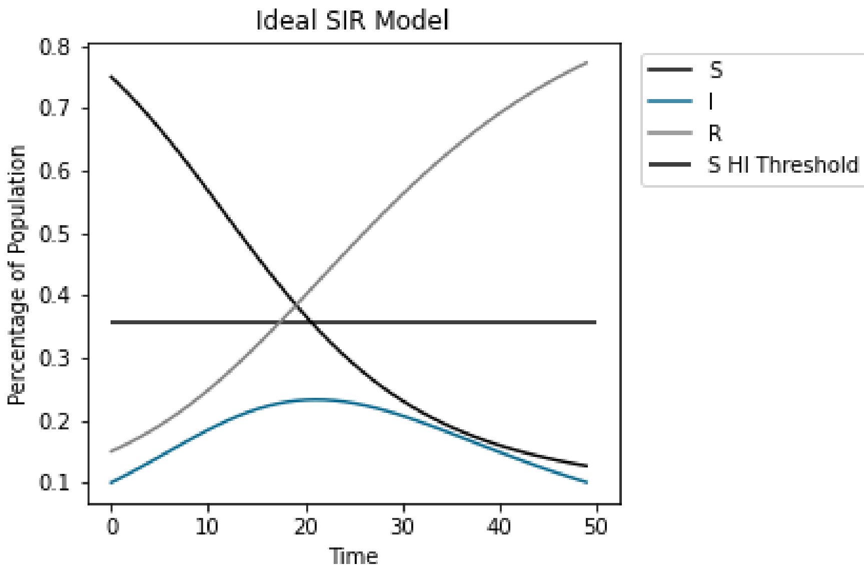
- a is called **transmissability**
- b is called **recovery rate**
- c is called **deimunization rate**
- α is called **vaccination rate**



If a vaccine is administered to α percent of the susceptible population each unit of time (e.g. ten percent every week) the the differential equation describing the populations becomes

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ R \end{pmatrix} = \begin{pmatrix} -\alpha S - aSI + cR \\ aSI - bI \\ bI - cR \end{pmatrix}.$$

The image below shows an ideal SIR evolution of an epidemic; the number of infection grows, removing people from the susceptible population, until the number of susceptible people is below a threshold called *herd immunity*. After that, the infection dies out.



2 Herd Immunity

The condition for a decrease in the number of infections in time, $\frac{dI}{dt} < 0$, a condition called **herd immunity**, is mathematically

$$\frac{d}{dt}I < 0 \Leftrightarrow S < \frac{b}{a}.$$

Administering a vaccine hastens the approach to herd immunity by removing people from S.

We now turn to finding reasonable values of the parameters of the SIR model.

3 Realistic Values of Parameters

3.1 Toward Realistic b

In SIR models, the term $-bI$ in the differential equation

$$\frac{dI}{dt} = aSI - bI,$$

describes the rate at which infected people move from the infected into the recovered compartment. It is common knowledge that a case of COVID lasts about two weeks, or 14 days. For this reason we use

$$b = \frac{1}{14\text{days}}.$$

As an example of why this is an intuitive value, if $b = \frac{1}{14\text{days}}$ and I $= 14\,$people then the number of people who recover in a day is $bI = 1\frac{\text{person}}{\text{day}}$.

3.2 Toward realistic a

3.2.1 Introducing Reproductive Number Through Exponential Models

SIR models are improvements upon exponential models in that

- exponential models treat the susceptible population as infinitely large,
- SIR models take into account that the susceptible population is finite and changes size.

In an exponential growth model of disease transmission, each infected person infects some number r_0 of susceptible people over the duration $1/b$ of their infection. This number r_0 is called **the reproductive number** of the disease. If the reproductive number of a disease is less than 1 then the disease will die out. To make this idea quantitative, let b be the recovery rate (so $1/b$ is the duration of disease). The differential equation below is then interpreted as "each day each infected person infects $r_0 b$ other people."

$$\frac{dI}{dt} = r_0 bI.$$

This is called an exponential model because the solutions (I as a function of time t) are the exponential functions of the form

$$I(t) = I_0(r_0 b)^t$$

with $I(0)$ the number of infections when $t = 0$. Changing base to Euler's number (e) the functions are

$$I(t) = I_0 e^{\ln(r_0 b)t}.$$

The doubling time for the infection is t_d such that $e^{\log(r_0 b)t_d} = 2 \Rightarrow t_d = \ln(2)/\ln(r_0 b)$.

3.2.2 Reproductive Number in SIR Models

SIR models have a similar feature, the reproductive number, but the value changes in time. Thus, SIR models have "effective reproductive number" that changes from an ideal initial time. We spell this out below.

Recall that in SIR models

$$\frac{dI}{dt} = aSI - bI$$

The first term on the right hand side of the equation is the number of new infections per time. Compare that term to the analogous term, new infections per time, in exponential growth models;

$$aSI \sim r_0 bI.$$

The dynamics of the two terms are different; the product aS changes in time while $r_0 b$ does not change in time. As a result, the exponential model grows exponentially without end, but the SIR models infections peak and then decay to 0. In the intermediate times, the **effective reproductive number** of the disease is $r = \frac{a}{b}S$. Intuitively, when almost all of the population is susceptible $S \approx p$, the growth of I is approximately exponential. The similarity then takes on the form

$$apI \sim r_0 bI \Leftrightarrow r_0 = p\frac{a}{b}$$

specifying the ideal initial **reproductive number** of a disease in an SIR model.

This is, the reproductive number r_0 of an SIR model may be expressed in terms of the transmission rate a, recovery rate b, and population p. The measured value of r_0 for COVID is approximately 2.8. This demands that the transmission rate of covid be

$$a = \frac{2.8b}{p}.$$

3.2.3 Effective Reproductive Rate and Herd Immunity From Another Argument

When S changes to below p as the infectionstarts to spread, the reproductive rate becomes an effective reproductive rate

$$r_{\text{eff}} = S\frac{a}{b}.$$

The condition that the disease starts to die out

$$r_{\text{eff}} < 1 \Leftrightarrow S < \frac{b}{a}$$

as presented through different means in section 1 above.

3.2.4 Conclusion: Realistic Values of Parameters a , b , and p

We conclude that

$$b = \frac{1}{14}\frac{\text{ppl}}{\text{day}}, \qquad a = \frac{2.8b}{p} = \frac{0.2}{p},$$

and that p will vary between the populations we model, state, county, nation, or world.

3.3 Toward Realistic c

Estimates of how long immunity to COVID lasts have been the subject of considerable research, debate, and disagreement. We use the order of magnitude estimate of immunity lasting for about 100 days, inline with the estimate of three months. That is, we use

$$c = \frac{1}{100\,\text{days}}.$$

3.4 Toward Realistic p

We use the the populations of each US state and Washington DC obtained from April 1st, 2020 census column in the wikipedia page for US states listed by population.

4 Conclusion: Realistic Values of Parameters a , b, c , and p

We conclude that, with units included,

$$b = \frac{1}{14}\frac{\text{ppl}}{\text{day}}, \quad a = \frac{0.2}{p\,\text{days}}, \quad c = \frac{1}{100\,\text{days}}$$

and that p will vary between the state populations we model.