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Trading Strategies Generated by Path-dependent Functionals of Market Weights

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Abstract

Almost twenty years ago, E.R. Fernholz introduced portfolio generating functions which can be used to construct a variety of portfolios, solely in the terms of the individual companies' market weights. I. Karatzas and J. Ruf recently developed another methodology for the functional construction of portfolios, which leads to very simple conditions for strong relative arbitrage with respect to the market. In this paper, both of these notions of functional portfolio generation are generalized in a pathwise, probability-free setting; portfolio generating functions are substituted by path-dependent functionals, which involve the current market weights, as well as additional bounded-variation functions of past and present market weights. This generalization leads to a wider class of functionally-generated portfolios than was heretofore possible, and yields improved conditions for outperforming the market portfolio over suitable time-horizons.

Keywords and Phrases: Stochastic portfolios, pathwise functional Itô formula, trading strategies, functional generation, regular functionals, strong relative arbitrage.

1 Introduction

The concept of 'functionally generated portfolios' was introduced by Fernholz (1999, 2002) and has been one of the essential components of stochastic portfolio theory; see Fernholz and Karatzas (2009) for an overview. Portfolios generated by appropriate functions of the individual companies' market weights have wealth dynamics which can be expressed solely in terms of these weights, and do not involve any stochastic integration. Constructing such portfolios does not require any statistical estimation of parameters, or any optimization. Completely observable quantities such as the current values of 'market weights', whose temporal evolution is modeled in terms of continuous semimartingales, are the only ingredients for building these portfolios. Once this structure has been discerned, the mathematics underpinning its construction involves just a simple application of Itô's rule. Then the goal is to find such portfolios that outperform a reference portfolio, for example, the market portfolio.

Karatzas and Ruf (2017) recently found a new way for the functional generation of trading strategies, which they call 'additive generation' as opposed to Fernholz's 'multiplicative generation' of portfolios. This new methodology weakens the assumptions on the market model: asset prices and market weights are continuous semimartingales, and trading strategies are constructed from 'regular' functions of the

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semimartingales without the help of stochastic calculus. Trading strategies generated in this additive way require simpler conditions for strong relative arbitrage with respect to the market over appropriate time horizons; see also Fernholz et al. (2018).

Along a different, but related, development, Föllmer (1981) has shown that Itô calculus can be developed 'path by path', without any probability structure. Then Dupire (2009), and Cont and Fournié (2010, 2013) introduced an associated functional Itô calculus. This new type of stochastic calculus extends the classical Itô formula to functionals depending on the entire history of the path, not only on its current value. Versions of this pathwise Itô calculus have recently been applied to various fields in mathematical finance, in particular, to stochastic portfolio theory. Schied et al. (2016) developed path-dependent functional version of Fernholz's 'master formula' for portfolios which are generated multiplicatively by functionals of the entire past evolution of the market weights.

In this paper, we generalize both additive and multiplicative functional generation of trading strategies in two ways. First, we use pathwise functional Itô calculus to show that one can construct trading strategies additively and multiplicatively from a functional depending on the entire history of market weights, in a manner completely devoid of probability considerations. The only analytic structure we impose is that the market weights admit continuous covariations in a pathwise sense. The usage of this advanced Itô calculus enables us to construct trading strategies which depend not only on the current market weights, but also on the past market weights. Secondly, we admit generating functionals that depend on an additional argument of finite variation. Introducing a new argument other than the market weights gives extra flexibility in constructing portfolios; see Strong (2014), Schied et al. (2016), Ruf and Xie (2018). We present various types of additional argument, to the effect that a variety of new trading strategies can be generated from a functional depending on it; these strategies yield new sufficient conditions for strong relative arbitrage with respect to the market portfolio.

We also present a new sufficient condition for strong relative arbitrage via additively generated trading strategies. The existing sufficient condition in Karatzas and Ruf (2017) requires the generating function to be 'Lyapunov', or the corresponding 'Gamma functional' to be nondecreasing. In contrast, the new sufficient condition in this paper depends on the intrinsic nondecreasing structure of the generating function itself. This new condition shows that trading strategies outperforming the market portfolio can be generated from a much richer collection of functions, or of functionals depending on the market weights and on an additional argument of finite variation. We give some interesting examples of such additively generated trading strategies and empirical analysis of them.

This paper is organized as follows: Section 2 presents the pathwise functional Itô calculus that will be needed for our purposes. Section 3 first defines trading strategies and regular functionals, then discusses how to generate trading strategies from regular functionals in ways both additive and multiplicative. Section 4 gives sufficient conditions for such trading strategies to generate strong relative arbitrage with respect to the market. Section 5 builds trading strategies depending on current and past values of market weights, in contexts where the use of the pathwise functional Itô calculus is essential. Section 6 gives some examples of trading strategies additively generated from entropic functionals and corresponding sufficient conditions for strong arbitrage. Section 7 contains empirical results of portfolios discussed in Section 6. Finally, Section 8 concludes.

2 Pathwise functional Itô calculus

In what follows, we let $X = (X_1, X_2, \cdots, X_d)'$ be a $[0, \infty)^d$-valued continuous function, representing a d-dimensional vector of assets whose values change over time, and $X_i(t)$ is the value of the i^{th} asset at time $t \geq 0$. As in Schied et al. (2016), we require that the components of X admit continuous covariations in the pathwise sense with respect to a given, refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of $[0,\infty)$. The sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is such that each partition $\mathbb{T}_n = \{t_0^{(n)} < t_1^{(n)} < \cdots\}$ satisfies $\lim_{k\to\infty} t_k^{(n)} = \infty$, for each $n \in \mathbb{N}$, as well as $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \cdots$, and the mesh size of $\mathbb{T}_n$ decreases to zero on each compact interval as $n \to \infty$. We fix such a sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions for the remainder of this paper. We denote by t'_n the successor of a given $t \in [0,\infty)$ in $\mathbb{T}_n$, i.e.,

$$t'_n = \min\{u \in \mathbb{T}_n | u > t\}. \tag{2.1}$$

With this notation, we define the pathwise covariation of X_i and X_j on the interval $[0,t]$, denoted by $\langle X_i, X_j\rangle(t)$ for any $1 \leq i,j \leq d$, as the limit of the sequence

$$\sum_{\substack{s\in\mathbb{T}_n\\ s\leq t}} (X_i(s'_n) - X_i(s))(X_j(s'_n) - X_j(s)), \quad n \in \mathbb{N}.$$

We assume that this limit is finite, and that the resulting mapping $t \mapsto \langle X_i, X_j\rangle(t)$ is continuous. We also define the pathwise quadratic variation of X_i by $\langle X_i\rangle := \langle X_i, X_i\rangle$ as usual.

Here, we need to emphasize that the existence of the pathwise covariations and quadratic variations for the components of X depends heavily on the choice of the sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions. Example 5.3.2. in Cont (2016), and the arguments following this example, illustrate this fact. Also, we note that the existence of the pathwise covariations and quadratic variations is required for Itô's formula to hold in a pathwise sense.

We will consider only the finite time-horizon case, so we fix $T \in (0,\infty)$ for the remainder of this paper. For an open subset V of a Euclidean space, we denote by $D([0,T],V)$ the space of right continuous V-valued functions with left limits on $[0,T]$. As usual, we denote by $C([0,T],V)$ the space of continuous V-valued functions; whereas $BV([0,T],V)$ stands for the space of those functions in $D([0,T],V)$ which are of bounded variation, and $CBV([0,T],V)$ stands for the space of continuous functions in $BV([0,T],V)$.

Let $U \subset \mathbb{R}^d_+$ be an open set of $\mathbb{R}^d_+ := (0,\infty)^d$, and W an open subset of $\mathbb{R}^m$. Here, d is the dimension of our vector function $X = (X_1,\cdots,X_d)'$ mentioned earlier, and m is the dimension for some additional vector function $A = (A_1,\cdots,A_m)'$ of finite variation on compact time-intervals. For $X, \tilde{X}$ in $D([0,T],U)$, and for $A, \tilde{A}$ in $D([0,T],W)$, we denote the distance

$$d_\infty\big((X,A),(\tilde{X},\tilde{A})\big) := \sup_{u\in[0,T]} |X(u) - \tilde{X}(u)| + \sup_{u\in[0,T]} |A(u) - \tilde{A}(u)|.$$

Next, we state a pathwise functional Itô formula, as well as some definitions and notation based on Appendix of Schied et al. (2016).

2.1 Non-anticipative functional

A functional $F : [0,T] \times D([0,T],U) \times BV([0,T],W) \to \mathbb{R}$ is called *non-anticipative*, if $F(t,X,A) = F(t,X^t,A^t)$ holds for each $t \in [0,\infty)$. Here we denote by $Y^t(\cdot) := Y(\cdot \wedge t)$ the function, or "path", stopped at $t \in [0,T]$, for any given path Y defined on $[0,T]$. We use the term "path" instead of function, to emphasize that it represents an evolution of value which changes over time. We present some concepts regarding a non-anticipative functional F, acting on a space of such paths, as follows:

1. F is called *left-continuous* if, for any given $\epsilon > 0$ and for any given $(t, X, A) \in [0, T] \times D([0, T], V) \times BV([0, T], W)$, there exists $\eta > 0$ such that, for all $(\tilde{X}, \tilde{A}) \in D([0, T], V) \times BV([0, T], W)$ and $\tilde{t} < t$ with $d_\infty\big((X, A), (\tilde{X}, \tilde{A})\big) + (t - \tilde{t}) < \eta$, we have

$$|F(t, X, A) - F(\tilde{t}, \tilde{X}, \tilde{A})| < \epsilon.$$

2. F is called *boundedness-preserving* if, for any given compact sets $K \subset U$ and $L \subset W$, there exists a constant $C_{K,L}$ such that

$$|F(t, X, A)| \leq C_{K,L}$$

holds for all $(t, X, A) \in [0, T] \times D([0, T], K) \times BV([0, T], L)$.

2.2 Horizonal and vertical derivatives

We present now two notions of differentiability for a given non-anticipative functional F. Intuitively, the first of these notions, horizontal differentiation, concerns the arguments $A_0, A_1, \cdots, A_m$ that correspond to functions of bounded variation; we set here $A_0(t) \equiv t$. The second notion, vertical differentiation, concerns the arguments $X_1, \cdots, X_d$ that correspond to general, right continuous functions with left limits.

Definition 2.1 (Horizontal derivative)**.** A non-anticipative functional F is called *horizontally differentiable* if, for each $(t, X, A) \in [0, T] \times D([0, t], U) \times BV([0, T], W)$, the limits

$$D_0 F(t, X, A) := \lim_{h \downarrow 0} \frac{F(t, X^{t-h}, A^{t-h}) - F(t-h, X^{t-h}, A^{t-h})}{h}$$

$$D_k F(t, X, A) := \lim_{h \downarrow 0} \frac{F(t, X^{t-h}, A_1^{t-h}, \cdots, A_k^t, \cdots, A_m^{t-h}) - F(t, X^{t-h}, A^{t-h})}{A_k(t) - A_k(t-h)}, \qquad k = 1, \cdots, m,$$

exist and are finite. We use here the convention $0/0 = 0$. Then, the *horizontal derivative* of F at (t, X, A) is given by the $(m+1)$-dimensional vector

$$DF(t, X, A) = \big(D_0 F(t, X, A), D_1 F(t, X, A), \cdots, D_m F(t, X, A)\big)',$$

and the map

$$\begin{aligned} DF : [0, T] \times D([0, T], U) \times BV([0, T], W) &\to \mathbb{R}^{m+1} \\ (t, X, A) &\mapsto DF(t, X, A) \end{aligned}$$

defines a non-anticipative vector-valued functional called the *horizontal derivative* of F, with the convention $DF(0, X, A) := 0$.

We note that our definition of horizontal derivative is based on the left-hand limit as in Schied et al. (2016), and is different from that of Dupire (2009) and Cont and Fournié (2010). Thus, only the past evolution of the underlying path is relevant for this definition, while no assumptions on future values are imposed.

Now let us fix $X \in D([0, T], U)$, $t \in [0, \infty)$ and $h \in \mathbb{R}^d$, and define the vertical perturbation $X^{t,h}$ of the stopped path X^t, as the right-continuous path obtained by shifting the value of the path X^t stopped at t by the amount h on the interval $[t, T]$, i.e.,

$$X^{t,h}(u) := X^t(u) + h \mathbb{1}_{[t,T]}(u), \qquad 0 \leq u \leq T.$$

Definition 2.2 (Vertical derivative)**.** A non-anticipative functional F is called *vertically differentiable* at $(t, X, A) \in [0, T] \times D([0, T], U) \times BV([0, T], W)$ if the map $\mathbb{R}^d \ni v \mapsto F(t, X^{t,v}, A)$ is differentiable at $v = 0$. In this case, the *i*th vertical partial derivative of the functional F at (t, X, A) is defined as

$$\begin{aligned}\partial_i F(t, X, A) &:= \lim_{\theta \to 0} \frac{F(t, X^{t,\theta e_i}, A) - F(t, X^t, A)}{\theta} \\ &= \lim_{\theta \to 0} \frac{F(t, X^{t,\theta e_i}, A) - F(t, X, A)}{\theta}, \qquad i = 1, \cdots, d.\end{aligned}$$

Here e_i is the *i*th unit vector in $\mathbb{R}^d$, and the last equality holds because F is non-anticipative. The corresponding gradient is denoted by

$$\nabla_X F(t, X, A) = \big(\partial_1 F(t, X, A), \cdots, \partial_d F(t, X, A)\big)'$$

and is called the *vertical derivative* of F at (t, X, A). If F is vertically differentiable at every triple (t, X, A) in $[0, T] \times D([0, T], U) \times BV([0, T], W)$, then the mapping

$$\begin{aligned}\nabla_X F : [0, T] \times D([0, T], U) \times BV([0, T], W) &\to \mathbb{R}^d \\ (t, X, A) &\mapsto \nabla_X F(t, X, A)\end{aligned}$$

defines a non-anticipative functional ∇_X with values in $\mathbb{R}^d$, called the *vertical derivative* of F.

We can iterate the operations of horizontal (DF) and vertical $(\nabla_X F)$ differentiation, to define higher-order horizontal and vertical derivatives as long as the functional F admits horizontal and vertical derivatives. In particular, we define the mixed vertical derivatives

$$\partial_{i,j} F := \partial_i(\partial_j F), \qquad 1 \le i, j \le d.$$

Definition 2.3. We denote by $\mathbb{C}^{j,k}[0, T]$ the set of all non-anticipative functionals F which satisfy the following conditions.

(i) F is continuous at fixed times t, uniformly in t over compact intervals. That is, for all $\epsilon > 0$ and $(t, X, A) \in [0, T] \times D([0, T], U) \times BV([0, T], W)$, there exists $\eta > 0$ such that for all $(\tilde{t}, \tilde{X}, \tilde{A})$ with $d_\infty\big((X, A), (\tilde{X}, \tilde{A})\big) + |t - \tilde{t}| < \eta$, we have

$$|F(t, X, A) - F(t, \tilde{X}, \tilde{A})| < \epsilon.$$

(ii) F is *j*-times horizontally differentiable and *k*-times vertically differentiable.

(iii) F and all its horizontal and vertical derivatives are left-continuous and boundedness-preserving.

2.3 Pathwise functional Itô formula

We are in a position now, to state the celebrated functional change-of-variable formula, in the form of Schied et al. (2016) or Schied and Voloshchenko (2016). The proof can be found in Schied and Voloshchenko (2016). As before, we let $(\mathbb{T}_n)$ be a fixed refining sequence of partitions of $[0, T]$, and suppose a continuous function X admits finite covariations $\langle X_i, X_j \rangle$, $1 \le i, j \le d$ along the sequence of partitions $(\mathbb{T}_n)$.

Theorem 2.4 (Pathwise functional Itô formula)**.** *Let $X \in C([0,T],U)$ and $A \in CBV([0,T],W)$ be given functions, recall the notation of* (2.1)*, define the n-th piecewise-constant approximation $X_n \in D([0,t],U)$ of X by*

$$X_n(t) := \sum_{s \in \mathbb{T}_n} X(s'_n)\mathbb{1}_{[s,s'_n)}(t) + X(T)\mathbb{1}_{\{T\}}(t), \quad 0 \le t \le T,$$

and let $X_n^{s-} := \lim_{r \uparrow s}(X_n)^r$. Then, for any functional $F \in \mathbb{C}^{1,2}[0,T]$, the pathwise Itô integral along $(\mathbb{T}_n)$, namely

$$\int_0^T \nabla_X F(s,X,A)dX(s) := \lim_{n\to\infty} \sum_{s \in \mathbb{T}_n} \nabla_X F(s, X_n^{s-}, A) \cdot (X(s'_n) - X(s)),$$

exists; and with $A_0(t) := t$, we have the expansion

$$\begin{aligned} F(T,X,A) - F(0,X,A) = \int_0^T \nabla_X F(s,X,A)dX(s) + \sum_{k=0}^m \int_0^T D_k F(s,X,A)dA_k(s) \\ + \frac{1}{2}\sum_{i,j=1}^d \int_0^T \partial^2_{i,j} F(s,X,A)d\langle X_i, X_j\rangle(s). \end{aligned}$$

3 Trading strategies generated by path-dependent functionals

As in the previous section, let $X = (X_1, X_2, \cdots, X_d)'$ be a $[0,\infty)^d$-valued continuous function which admits continuous covariations with respect to a refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of $[0,T]$ and $A = (A_1, A_2, \cdots, A_m)'$ be an additional vector function of finite variation. For the purpose of this section, the components of X will denote the value processes of d tradable assets, and eventually stand for the vector of market weights in an equity market. At the same time, the components of A will model the evolution of an observable, but non-tradable quantity related to the market weights. We have the following definition of trading strategy with respect to the pair (X,A) in the manner of Karatzas and Ruf (2017).

Definition 3.1 (Trading strategies)**.** For the pair (X,A) of a d-dimensional function $X \in C([0,T],U)$ and an m-dimensional function $A \in CBV([0,T],W)$, suppose that $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_d)'$ is a d-dimensional function with a representation

$$\vartheta_i(\cdot) = \Theta_i(\cdot, X, A), \qquad i = 1, \cdots, d,$$

for a vector $\Theta = (\Theta_1, \cdots, \Theta_d)'$ of non-anticipative functionals, for which we can define an integral $\int_0^\cdot \vartheta(t)dX(t) \equiv \int_0^\cdot \sum_{i=1}^d \vartheta_i(t)dX_i(t)$ with respect to X; we write $\vartheta \in \mathcal{L}(X,A)$, to express this. We shall say that $\vartheta \in \mathcal{L}(X,A)$ is a *trading strategy with respect to* X if it is 'self-financed', in the sense that

$$V^\vartheta(\cdot, X) - V^\vartheta(0,X) = \int_0^\cdot \sum_{i=1}^d \vartheta_i(t)dX_i(t) \tag{3.1}$$

holds. Here and in what follows,

$$V^{\vartheta}(t, X) := \sum_{i=1}^{d} \vartheta_i(t) X_i(t), \qquad 0 \le t \le T \tag{3.2}$$

denotes the value process of the strategy ϑ at time t.

The interpretation here, is that $\vartheta_i(t)$ stands for the "number of shares" invested in asset i at time t. If $X_i(t)$ is the price of this asset, then $\vartheta_i(t)X_i(t)$ is the dollar amount invested in asset i at time t, and $V^{\vartheta}(t, X)$ is the total value of investment across all assets.

The preceding Itô formula in Theorem 2.4 suggests that integrands $\vartheta \in \mathcal{L}(X, A)$ of the special form $\vartheta(t) = \nabla_X F(t, X, A)$, for some non-anticipative functional $F \in \mathbb{C}^{1,2}[0, T]$, play a very important role for integrators $X \in C([0, T], U)$ that admit finite covariations $\langle X_i, X_j \rangle$, $1 \le i, j \le d$ along an appropriate nested sequence of partitions. This gives rise to the following definition.

Definition 3.2 (Admissible trading strategy)**.** Let X be a d-dimensional function in $C([0, T], U)$, and A be an m-dimensional function in $CBV([0, T], W)$. A d-dimensional trading strategy $\vartheta : [0, \infty) \to \mathbb{R}^d$ in $\mathcal{L}(X, A)$ is called an *admissible trading strategy for the pair* (X, A), if there exists a non-anticipative functional $F : [0, T] \times D([0, T], U) \times BV([0, T], W) \to \mathbb{R}$ in the space $\mathbb{C}^{1,2}[0, T]$, such that

$$\vartheta(t) = \nabla_X F(t, X, A), \quad 0 \le t \le T. \tag{3.3}$$

If ϑ is an admissible trading strategy for (X, A), the last integral of (3.1) above can be viewed as either the usual vector Itô integral (when X is a continuous vector semimartingale in a probabilistic setting), or as the pathwise Itô integral (in the context of our Theorem 2.4).

In the following, we will define a regular functional for the d-dimensional continuous function X and the m-dimensional function A in $CBV([0, T], W)$ in a manner similar to that of Karatzas and Ruf (2017).

Definition 3.3 (Regular functional)**.** We say that a non-anticipative functional $G : [0, T] \times D([0, T], U) \times BV([0, T], W) \to \mathbb{R}$ in $\mathbb{C}^{1,2}[0, T]$ is *regular* for the pair (X, A) of a d-dimensional continuous function X and a function $A \in CBV([0, T], W)$, if the continuous function

$$\Gamma^G(t) := G(0, X, A) - G(t, X, A) + \int_0^t \sum_{i=1}^{d} \vartheta_i(s) dX_i(s), \quad 0 \le t \le T \tag{3.4}$$

has finite variation on compact intervals of $[0, T]$. Here, ϑ is the function of (3.3) right above, with components

$$\vartheta_i(t) := \partial_i G(t, X, A), \quad i = 1, \cdots, d, \quad 0 \le t \le T. \tag{3.5}$$

Remark 3.4. In order to define a pathwise functional Itô integral and be able to use pathwise functional Itô formulas, we need a sufficiently smooth (in general, at least $\mathbb{C}^{1,2}[0, T]$) functional, and an integrand which can be cast in the form of a vertical derivative of this functional. Thus, thanks to the above definition, we can always apply the pathwise functional Itô formula (Theorem 2.4) to the functional G as in Definition 3.3 above, and get another expression for the so-called "Gamma functional" $\Gamma^G(\cdot)$ in (3.4); namely,

$$\Gamma^G(t) = -\sum_{k=0}^{m} \int_0^t D_k G(s, X, A) dA_k(s) - \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial^2_{i,j} G(s, X, A) d\langle X_i, X_j \rangle(s). \tag{3.6}$$

Here $D_k G(s, X, A)$ and $\partial^2_{i,j} G(s, X, A)$ are, respectively, the horizontal derivative and the second-order vertical derivative of G at (s, X, A).

The difference in Definition 3.3 here, with Definition 3.1 of Karatzas and Ruf (2017), should be noted and stressed. In Karatzas and Ruf (2017), the integrand ϑ_i need not be the form of 'gradient' of a regular function G. Here, we need the special structure of (3.5) for the integrand; this is the "price to pay" for being able to work in a pathwise, probability-free setting, without having to invoke the theory of rough paths.

3.1 Trading strategies depending on market weights

We place ourselves from now onward in a frictionless equity market with a fixed number $d \geq 2$ of companies. For an open set U with $[0, \infty)^d \subset U \subset \mathbb{R}^d$, we also consider a vector of continuous functions $S = (S_1, \cdots, S_d)' \in C([0, T], U)$ where $S_i(t)$ represents the capitalization of the i^{th} company at time $t \in [0, T]$. Here we take $S_i(0) > 0$ and allow $S_i(t)$ to vanish at some time $t > 0$ for all $i = 1, \cdots, d$, but assume that the total capitalization $\Sigma(t) := S_1(t) + \cdots + S_d(t)$ does not vanish at any time $t \in [0, T]$. Then we define another vector of continuous functions $\mu = (\mu_1, \cdots, \mu_d)'$ that consists of the companies' relative market weights

$$\mu_i(t) := \frac{S_i(t)}{\Sigma(t)} = \frac{S_i(t)}{S_1(t) + \cdots + S_d(t)}, \qquad t \in [0, T], \quad i = 1, \cdots, d. \tag{3.7}$$

We also assume that the components of μ admit finite covariations $\langle \mu_i, \mu_j \rangle$, $1 \leq i, j \leq d$ along a nested sequence of partitions $\mathbb{T}_n$ of $[0, T]$. In the following, we will consider only regular functionals of the form $G(\cdot, \mu, A)$ which depend on the vector of market weights μ and on some additional function $A \in CBV([0, T], W)$. Examples of such functions A appear in (4.3), (4.4).

Remark 3.5. In order to simplify the expression of (3.6) and to do concave analysis in a manner analogous to that of Karatzas and Ruf (2017), we can make $G(t, \mu, A)$ depend only on the function μ. For example, if we consider the Gibbs entropy function $H(x) := -\sum_{i=1}^d x_i \log x_i$ and set $G(t, \mu, A) = H\big(\mu(t)\big)$, elementary computations show that the first term on the right-hand side of (3.6) vanishes and we obtain $\Gamma^G(t) = \frac{1}{2} \sum_{i=1}^d \int_0^t \frac{d\langle \mu_i \rangle(s)}{\mu_i(s)}$; this coincides with Example 5.3 of Karatzas and Ruf (2017).

3.2 Additively generated trading strategies

We would like now to introduce an additively-generated trading strategy, starting from a regular functional. For this, we will need a result from Karatzas and Ruf (2017). For any given functional G which is regular for the pair (μ, A), where μ is the vector of market weights and A an appropriate function in $CBV([0, T], W)$, we consider the vector ϑ with components

$$\vartheta_i := \partial_i G(\cdot, \mu, A), \quad i = 1, \cdots, d, \tag{3.8}$$

as in (3.5) of the Definition 3.3, and the vector φ with components

$$\varphi_i(t) := \vartheta_i(t) - Q^\vartheta(t) - C(0), \quad i = 1, \cdots, d, \quad 0 \leq t \leq T. \tag{3.9}$$

Here,

$$Q^\vartheta(t) := V^\vartheta(t) - V^\vartheta(0) - \int_0^t \sum_{i=1}^d \vartheta_i(s) d\mu_i(s) \tag{3.10}$$

is the "defect of self-financibility" at time $t \in [0,T]$ of the integrand ϑ in (3.8), and

$$C(0) := \sum_{i=1}^{d} \partial_i G(0,\mu,A)\mu_i(0) - G(0,\mu,A) \tag{3.11}$$

is the "defect of balance" at time $t=0$ of the regular functional G. By analogy with Proposition 2.3 of Karatzas and Ruf (2017), the vector $\varphi = (\varphi_1,\cdots,\varphi_d)'$ of (3.9), (3.8) defines a trading strategy with respect to μ.

Definition 3.6 (Additive generation)**.** We say that the trading strategy φ of the form (3.9), (3.8) is *additively generated* by the functional $G : [0,T] \times D([0,T],U) \times BV([0,T],W) \to \mathbb{R}$, which is assumed to be regular for the vector $\mu = (\mu_1,\cdots,\mu_d)'$ of market weights.

Proposition 3.7. *The trading strategy φ, generated additively as in* (3.9) *by a regular functional G for the pair (μ, A), where $\mu = (\mu_1,\cdots,\mu_d)'$ is the vector of market weights and $A \in CBV([0,T],W)$, has value*

$$V^{\varphi}(t) = G(t,\mu,A) + \Gamma^G(t), \quad 0 \le t \le T \tag{3.12}$$

as in Definitions 3.1 and 3.3, and its components can be represented, for $i = 1,\cdots,d$, in the form

$$\begin{aligned} \varphi_i(t) &= \partial_i G(t,\mu,A) + \Gamma^G(t) + G(t,\mu,A) - \sum_{j=1}^{d} \mu_j(t)\partial_j G(t,\mu,A) \qquad (3.13)\\ &= V^{\varphi}(t) + \partial_i G(t,\mu,A) - \sum_{j=1}^{d} \mu_j(t)\partial_j G(t,\mu,A). \end{aligned}$$

We can think of $\Gamma^G(\cdot)$ in (3.4), (3.6) and (3.12), as expressing the "cumulative earnings" of the strategy φ around the "baseline" $G(\cdot,\mu,A)$.

Proof. The proof is exactly the same as the proof of Proposition 4.3 of Karatzas and Ruf (2017), if we change $G(\mu(t))$, $D_j G(\mu(t))$ there, into $G(t,\mu,A)$, $\partial_j G(t,\mu,A)$. ☐

Remark 3.8.

(i) When the functional G in Proposition 3.7 satisfies the 'balance' condition,

$$G(\cdot,\mu,A) = \sum_{j=1}^{d} \mu_j(\cdot)\partial_j G(\cdot,\mu,A), \tag{3.14}$$

the additively generated trading strategy φ in (3.13) takes the considerably simpler form

$$\varphi_i(t) = \partial_i G(t,\mu,A) + \Gamma^G(t), \qquad i = 1,\cdots,d. \tag{3.15}$$

(ii) For an additively generated trading strategy φ with strictly positive value process $V^{\varphi} > 0$, the corresponding portfolio weights are defined as

$$\pi_i := \frac{\varphi_i\mu_i}{V^{\varphi}} = \frac{\varphi_i\mu_i}{\sum_{i=1}^{d}\varphi_i\mu_i}, \qquad i = 1,\cdots,d,$$

or with the help of (3.12) and (3.13), as

$$\pi_i(t) = \mu_i(t)\left(1 + \frac{1}{G(t,\mu,A) + \Gamma^G(t)}\Big(\partial_i G(t,\mu,A) - \sum_{j=1}^{d} \mu_j(t)\partial_j G(t,\mu,A)\Big)\right), \tag{3.16}$$

for $i = 1, \cdots, d$.

3.3 Multiplicatively generated trading strategies

Next, we introduce the notion of multiplicatively generated trading strategy. We suppose that a functional $G : [0,T] \times D([0,T],U) \times BV([0,T],W) \to [0,\infty)$ is regular as in Definition 3.3 for the pair (μ, A), where μ is the vector of market weights and A is some additional function in $CBV([0,T],W)$, and that $1/G(\cdot,\mu,A)$ is locally bounded. This holds, for example, if G is bounded away from zero. We consider the vector function $\eta = (\eta_1, \cdots, \eta_d)'$ defined by

$$\eta_i := \vartheta_i \times \exp\Big(\int_0^{\cdot} \frac{d\Gamma^G(t)}{G(t,\mu,A)}\Big) = \partial_i G(\cdot,\mu,A) \times \exp\Big(\int_0^{\cdot} \frac{d\Gamma^G(t)}{G(t,\mu,A)}\Big) \tag{3.17}$$

in the notation of (3.4), (3.8) for $i = 1, \cdots, d$. The integral here is well-defined, as $1/G(\cdot,\mu,A)$ is assumed to be locally bounded. Moreover, we have $\eta \in \mathcal{L}(\mu)$, since $\vartheta \in \mathcal{L}(\mu)$ from Definition 3.1, and the exponential process is again locally bounded. As before, we turn this η into a trading strategy $\psi = (\psi_1, \cdots, \psi_d)'$ by setting

$$\psi_i := \eta_i - Q^\eta - C(0), \quad i = 1, \cdots, d \tag{3.18}$$

in the manner of (3.9), and with Q^η, $C(0)$ defined as in (3.10) and (3.11).

Definition 3.9 (Multiplicative generation)**.** The trading strategy $\psi = (\psi_1, \cdots, \psi_d)'$ of (3.18), (3.17) is said to be *multiplicatively generated* by the functional $G : [0,T] \times D([0,T],U) \times BV([0,T],W) \to [0,\infty)$.

Proposition 3.10. *The trading strategy $\psi = (\psi_1, \cdots, \psi_d)'$, generated as in* (3.18) *by a given functional $G : [0,T] \times D([0,T],U) \times BV([0,T],W) \to (0,\infty)$ which is regular for the pair (μ, A) with vector $\mu = (\mu_1, \cdots \mu_d)'$ of market weights and a suitable $A \in CBV([0,T],W)$ such that $1/G(\cdot,\mu,A)$ is locally bounded, has value process*

$$V^\psi = G(\cdot,\mu,A)\exp\Big(\int_0^{\cdot} \frac{d\Gamma^G(t)}{G(t,\mu,A)}\Big) > 0 \tag{3.19}$$

in the notation of (3.4)*. This strategy ψ can be represented for $i = 1, \cdots, d$ in the form*

$$\psi_i(t) = V^\psi(t)\Big(1 + \frac{1}{G(t,\mu,A)}\Big(\partial_i G(t,\mu,A) - \sum_{j=1}^{d} \mu_j(t)\partial_j G(t,\mu,A)\Big)\Big). \tag{3.20}$$

Proof. We follow the proof of Proposition 4.8 in Karatzas and Ruf (2017), using the pathwise functional Itô formula instead of the standard Itô formula. If we denote the exponential $\exp\Big(\int_0^t \frac{d\Gamma^G(s)}{G(s,\mu,A)}\Big)$ in (3.19) by $K(t)$, the pathwise functional Itô formula (Theorem 2.4) yields

$$
\begin{aligned}
G(t,\mu,A)K(t) = G(0,\mu,A)K(0) &+ \int_0^t \sum_{i=1}^d \partial_i G(s,\mu,A)K(s)d\mu_i(s) + \int_0^t K(s)d\Gamma^G(s) \\
&+ \int_0^t \sum_{i=0}^m D_i G(s,\mu,A)K(s)dA_i(s) \\
&+ \frac{1}{2}\int_0^t \sum_{i=1}^d \sum_{j=1}^d \partial^2_{i,j} G(s,\mu,A)K(s)d\langle \mu_i,\mu_j\rangle(s) \\
= G(0,\mu,A)K(0) &+ \int_0^t \sum_{i=1}^d \partial_i G(s,\mu,A)K(s)d\mu_i(s) \\
= G(0,\mu,A)K(0) &+ \int_0^t \sum_{i=1}^d \eta_i(s)d\mu_i(s) \\
= G(0,\mu,A)K(0) &+ \int_0^t \sum_{i=1}^d \psi_i(s)d\mu_i(s), \qquad 0 \le t \le T.
\end{aligned}
$$

Here, the second equality uses the expression in (3.6), and the last equality relies on Proposition 2.3 of Karatzas and Ruf (2017). Since (3.19) holds at time zero, it follows that (3.19) holds at any time t between 0 and T. The justification of (3.20) is exactly the same with that of Proposition 4.8 in Karatzas and Ruf (2017). □

Remark 3.11.

(i) The multiplicatively generated trading strategy ψ in (3.20) takes the simpler form

$$
\psi_i(t) = \partial_i G(t,\mu,A)\exp\Big(\int_0^t \frac{d\Gamma^G(s)}{G(s,\mu,A)}\Big), \qquad i = 1,\cdots,d \tag{3.21}
$$

when the functional G in Proposition 3.10 is 'balanced' as in (3.14).

(ii) The portfolio weights corresponding to the multiplicatively generated trading strategy ψ, are similarly defined as

$$
\Pi_i := \frac{\psi_i\mu_i}{V^\psi} = \frac{\psi_i\mu_i}{\sum_{i=1}^d \psi_i\mu_i}, \qquad i = 1,\cdots,d,
$$

and with the help of (3.19) and (3.20), takes the form

$$
\Pi_i(t) = \mu_i(t)\Big(1 + \frac{1}{G(t,\mu,A)}\big(\partial_i G(t,\mu,A) - \sum_{j=1}^d \mu_j(t)\partial_j G(t,\mu,A)\big)\Big), \qquad i = 1,\cdots,d.
$$

For a functional G that satisfies the "balance" condition (3.14), this simplifies to

$$
\Pi_i(t) = \mu_i(t)\,\frac{\partial_i G(t,\mu,A)}{G(t,\mu,A)}, \qquad i = 1,\cdots,d.
$$

4 Sufficient conditions for strong relative arbitrage

We consider the vector $\mu = (\mu_1, \cdots, \mu_d)'$ of market weights as in (3.7). For a given trading strategy φ with respect to the market weights μ, let us recall the value process $V^\varphi(\cdot, \mu) = \sum_{i=1}^d \varphi_i \mu_i$ from Definition 3.1. As we will always consider trading strategies with respect to market weights, we write $V^\varphi(\cdot)$ instead of $V^\varphi(\cdot, \mu)$ from now on. For some fixed $T_* \in (0, T]$, we say that φ is *strong relative arbitrage with respect to the market* over the time-horizon $[0, T_*]$, if we have

$$V^\varphi(t) \geq 0, \quad \forall\, t \in [0, T_*], \tag{4.1}$$

along with

$$V^\varphi(T_*) > V^\varphi(0). \tag{4.2}$$

Remark 4.1. The notion of strong relative arbitrage defined above does not depend on any probability measure and is slightly more strict than the existing definition of strong relative arbitrage. The classical definition involves an underlying filtered probability space, and posits that the market weights $\mu_1, \cdots, \mu_d$ should be continuous, adapted stochastic processes on this space. Also, there are two types of classical arbitrage; relative arbitrage and 'strong' relative arbitrage as in Definition 4.1 of Fernholz et al. (2018). In this old definition, an underlying probability measure is essential in defining this 'weak' version of relative arbitrage. However, if we say that φ is strong relative arbitrage when (4.2) holds for 'every' realization of μ, instead of 'almost sure' realization of μ, the notion of strong relative arbitrage can be established without referring to any probability structure. Since we constructed trading strategies in a pathwise, probability-free setting, the 'strong' version of relative arbitrage is a more appropriate concept of arbitrage for this paper, and we adopt the above strict definition of strong relative arbitrage from now on.

The value process of a trading strategy generated functionally, either additively or multiplicatively, admits a quite simple representation in terms of the generating functional G and the derived functional Γ^G as in (3.12) and (3.19). This simple representation provides in turn nice sufficient conditions for strong relative arbitrage with respect to the market; for example, as in Theorem 5.1 and Theorem 5.2 of Karatzas and Ruf (2017). In this section, we find such conditions on trading strategies generated by a pathwise functional $G(\cdot, \mu, A)$ which depends not only on the vector of market weights μ, but also on an additional finite-variation process A related to μ. We also give a new sufficient condition leading to strong relative arbitrage for additively generated trading strategies, which is different from Theorem 5.1 of Karatzas and Ruf (2017).

Until now, we have not specified the m-dimensional function $A \in CBV([0, T], W)$, so it is time to consider some plausible candidates for this finite variation function. A first suitable candidate would be the d-dimensional vector

$$A = \langle \mu \rangle = \big(\langle \mu_1 \rangle, \langle \mu_2 \rangle, \cdots, \langle \mu_d \rangle\big)' \tag{4.3}$$

of quadratic variation of market weights. We can also think about a more general candidate; namely, the S_d^+-valued covariation process of market weights. Here, S_d^+ is the notation for symmetric positive $d \times d$ matrices, and we will use double bracket $\langle\langle \quad \rangle\rangle$ to distinguish this d^2-dimensional vector from (4.3): namely,

$$A = \langle\langle \mu \rangle\rangle, \quad (A)_{i,j} = \langle \mu_i, \mu_j \rangle \qquad 1 \leq i, j \leq d. \tag{4.4}$$

The advantage of choosing A as in (4.4), is that we can match the integrators of the two integrals in (3.6), and the resulting expression for $\Gamma^G(\cdot)$ can then be cast as one integral.

There are many other functions of finite variation which can be candidates for the process A. We list some examples below:

1. The moving average $\bar{\mu}$ of μ defined by

$$\bar{\mu}_i(t) := \begin{cases} \frac{1}{\delta}\int_0^t \mu_i(s)ds + \frac{1}{\delta}\int_{t-\delta}^0 \mu_i(0)ds, & t \in [0, \delta), \\ \frac{1}{\delta}\int_{t-\delta}^t \mu_i(s)ds, & t \in [\delta, T], \end{cases} \qquad i = 1, \cdots, d.$$

2. The running maximum μ^* of the market weights with the components $\mu_i^*(t) := \max_{0 \le s \le t} \mu_i(s)$, and the running minimum μ_* of the market weights with the components $\mu_{*i}(t) := \min_{0 \le s \le t} \mu_i(s)$ for $i = 1, \cdots, d$.

3. The local time process $\Lambda^{(k,\ell)}$ of the continuous semimartingale $\mu_{(k)} - \mu_{(\ell)} \ge 0$ at the origin, for $1 \le k < \ell \le d$. We call this process the "collision local time of order $\ell - k + 1$" for the ranked market weights
$$\mu_{(1)} := \max_i \mu_i \ge \mu_{(2)} \ge \cdots \ge \mu_{(d)} =: \min_i \mu_i.$$

Since the vectors $\bar{\mu}$, μ^*, and μ_*, defined above, are d-dimensional vectors, $m = d$ holds for these choices of A. For the choice of $\frac{1}{2}d(d-1)$-dimensional vector Λ with the components $(\Lambda)_{k,\ell} := \Lambda^{k,\ell}$, the dimension m of A is $\frac{1}{2}n(n-1)$. Empirical results using the moving average $\bar{\mu}$ can be found in Section 3 of Schied et al. (2016). The collision local time processes $\Lambda^{(k,\ell)}$ are from Example 3.9 of Karatzas and Ruf (2017).

We first consider conditions leading to strong relative arbitrage with respect to the market with general A as the third input of generating functional G. Then we present some examples of G with specific finite variation process A chosen from among the above candidates, and continue with empirical results regarding these examples.

4.1 Additively generated strong relative arbitrage

We start with a condition leading to additively generated strong arbitrage, which is similar to Theorem 5.1 of Karatzas and Ruf (2017).

Theorem 4.2 (Additively generated strong relative arbitrage when Γ^G is nondecreasing)**.** *For open sets $U \subset \mathbb{R}_+^d$ and $W \in \mathbb{R}^m$, fix a regular functional $G : [0,T] \times D([0,T], U) \times BV([0,T], W) \to [0, \infty)$ for the pair (μ, A) such that the process $\Gamma^G(\cdot)$ in* (3.4) *or* (3.6) *is nondecreasing. Here, μ is the vector of market weights and A is some m-dimensional function in $CBV([0,T], W)$, as before. For some real number $T_* > 0$, suppose that*
$$\Gamma^G(T_*) > G(0, \mu, A) \tag{4.5}$$
holds. Then the additively generated strategy φ of Definition 3.6 is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \le t \le T$.

Proof. Since $\Gamma^G(\cdot)$ is nondecreasing, we obtain $V^\varphi(t) = G(t, \mu, A) + \Gamma^G(t) \ge \Gamma^G(0) = 0$ for $\forall t \in [0, T_*]$ from (3.12). We also have $V^\varphi(t) = G(t, \mu, A) + \Gamma^G(t) \ge \Gamma^G(T_*) > G(0, \mu, A) = V^\varphi(0)$ for all $T_* \le t \le T$. The last equality holds because $\Gamma^G(0) = 0$. ☐

Remark 4.3. If we choose $A = \langle\langle \mu \rangle\rangle$ as in (4.4), then from (3.6), the process $\Gamma^G(\cdot)$ is nondecreasing when
$$-\int_0^\cdot D_0 G(s, \mu, \langle\langle \mu \rangle\rangle)ds - \sum_{i,j=1}^d \int_0^\cdot \Big(D^1_{(i,j)} + \frac{1}{2}\partial^2_{i,j}\Big) G(s, \mu, \langle\langle \mu \rangle\rangle) d\langle \mu_i, \mu_j \rangle(s)$$

is nondecreasing. Here, $D^1_{(i,j)}$ denotes the first-order horizontal derivative operator with respect to the (i,j)th entry of $\langle\langle\mu\rangle\rangle$. Also, we substitute from (4.4), (3.6) into (4.5) to obtain the more explicit form

$$\begin{aligned}\Gamma^G(T_*) = &-\int_0^{T_*} D_0 G(s,\mu,\langle\langle\mu\rangle\rangle)ds \\ &- \sum_{i,j=1}^{d}\int_0^{T_*}\Big\{D^1_{(i,j)}G(s,\mu,\langle\langle\mu\rangle\rangle) + \frac{1}{2}\partial^2_{i,j}G(s,\mu,\langle\langle\mu\rangle\rangle)\Big\}d\langle\mu_i,\mu_j\rangle(s) \\ &> G(0,\mu,\langle\langle\mu\rangle\rangle),\end{aligned} \tag{4.6}$$

of the condition for strong relative arbitrage. Thus, unlike the situation of Theorem 3.7 in Karatzas and Ruf (2017), we can have a nondecreasing Γ^G and a chance of strong relative arbitrage, even if we lose the 'concavity' of G in μ.

Remark 4.4. Let us assume that μ and A are 'separated' in G, in the sense that there exist two regular functionals K and H, with the property that K depends only on μ^t and H depends on t and A, and such that

$$G(t,\mu^t,A^t) = K(\mu^t) + H(t,A^t), \quad \forall\, t\in[0,T] \tag{4.7}$$

holds. Then, we easily get $\partial^2_{i,j}G(t,\mu^t,A^t) = \partial^2_{i,j}K(\mu^t)$ and $D_kG(t,\mu^t,A^t) = D_kH(t,A^t)$. Substituting these expressions into (3.6), we have

$$\Gamma^G(T_*) = -\sum_{k=0}^{m}\int_0^{T_*} D_kH(s,A^s)dA_k(s) - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^{T_*}\partial^2_{i,j}K(\mu^s)d\langle\mu_i,\mu_j\rangle(s), \tag{4.8}$$

and from (3.12) of Proposition 3.7, the relative value process of the additively generated trading strategy φ by G can be expressed as

$$V^\varphi(T_*) = K(\mu^{T_*}) + H(T_*,A^{T_*}) + \Gamma^G(T_*), \tag{4.9}$$

and

$$V^\varphi(0) = K(\mu^0) + H(0,A^0). \tag{4.10}$$

After substituting (4.8), (4.9), and (4.10) into (4.2) and rearranging terms in such a manner, that the left-hand side contains only terms involving G_1, the strong arbitrage condition (4.2) takes the form

$$K(\mu^{T_*}) - K(\mu^0) - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^{T_*}\partial^2_{i,j}K(\mu^s)d\langle\mu_i,\mu_j\rangle(s) > B_H(T_*,A^{T_*}), \tag{4.11}$$

where

$$B_H(T_*,A^{T_*}) := -H(T_*,A^{T_*}) + H(0,A^0) + \sum_{k=0}^{m}\int_0^{T_*} D_kH(s,A^s)dA_k(s).$$

When we apply the pathwise functional Itô formula of Theorem 2.4 to the functional $H(t,\langle\langle\mu\rangle\rangle^t)$, $\quad 0\le t\le T$, the right-hand side of the above expression vanishes. Hence, the requirement (4.11) becomes

$$K(\mu^{T_*}) - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^{T_*}\partial^2_{i,j}K(\mu^s)d\langle\mu_i,\mu_j\rangle(s) > K(\mu^0)$$

and we are in very similar situation as in Theorem 5.1 of Karatzas and Ruf (2017).

To be more precise, if K takes non-negative values and is a 'Lyapunov functional' for the vector μ of market weights, in the sense that $\Gamma^K(t) := -\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{i,j}^2 K(\mu^s)d\langle\mu_i, \mu_j\rangle(s)$ is nondecreasing, then the requirement $\Gamma^K(T_*) > K(\mu^0)$ ensures strong relative arbitrage condition over every time-horizon $[0, t]$ with $T_* \le t \le T$ as in Theorem 5.1 of Karatzas and Ruf (2017). Thus, in this 'separated' case, we cannot achieve more than the result in Theorem 5.1 of Karatzas and Ruf (2017), as all terms on the right-hand side of (4.11) that involve H vanish. This is because when we additively generate the trading strategy φ in (3.9) from a regular functional G, only vertical derivatives of G in (3.8) are involved in φ, and this makes the H term in (4.7) meaningless in generating φ. Therefore, to be able to find new sufficient conditions for strong relative arbitrage, we need forms of G more complicated than (4.7). All the following examples of G we develop in this paper are of those more complicated forms.

From (3.12), the value V^φ at time t of the additively generated trading strategy with respect to the market, has two additive components, $G(t, \mu, A)$ and $\Gamma^G(t)$. In Theorem 4.2, we derived the strong arbitrage condition from the "nondecreasing property" of $\Gamma^G(\cdot)$, but there is no reason to discriminate between $G(t, \mu, A)$ and $\Gamma^G(t)$. If the mapping $t \mapsto G(t, \mu, A)$ is nondecreasing, we may derive a strong arbitrage condition like Theorem 4.2, switching the role of $G(t, \mu, A)$ and $\Gamma^G(t)$. However, it is difficult to find functionals $G(t, \mu, A)$ which are monotone in t, because G must depend on the market weights $\mu(\cdot)$ and these fluctuate all the time. Thus, we have to 'extract a nondecreasing structure' from the generating functional $G(\cdot, \mu, A)$, and use this nondecreasing structure instead of G to derive a new strong arbitrage condition.

Theorem 4.5 (Additively generated strong relative arbitrage when Γ^G admits a lower bound)**.** *For open sets $U \subset \mathbb{R}_+^d$ and $W \in \mathbb{R}^m$, fix a regular functional $G : [0, T] \times D([0, T], U) \times BV([0, T], W) \to [0, \infty)$ for the pair (μ, A), where μ is the vector of market weights and A is a m-dimensional function in $CBV([0, T], W)$, such that the following conditions are satisfied;*

- *(i) $V^\varphi(\cdot) = G(\cdot, \mu, A) + \Gamma^G(\cdot) \ge 0$, where the process $\Gamma^G(\cdot)$ is from* (3.4) *or* (3.6)*;*
- *(ii) there exists a functional $F(t, \mu, A)$ satisfying $G(t, \mu, A) \ge F(t, \mu, A)$ for all $t \in [0, T]$ and the mapping $t \mapsto F(t, \mu, A)$ is nondecreasing;*
- *(iii) $\Gamma^G(\cdot) \ge \kappa$ holds by some constant κ.*

For some real number $T_* > 0$, suppose that

$$F(T_*, \mu, A) > G(0, \mu, A) - \kappa \tag{4.12}$$

holds. Then the additively generated strategy φ of Definition 3.6 is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \le t \le T$.

Proof. The inequality (4.1) is satisfied by the first condition above. From the last two conditions with (3.12) and (4.12), we obtain $V^\varphi(t) = G(t, \mu, A) + \Gamma^G(t) \ge F(t, \mu, A) + \kappa \ge F(T_*, \mu, A) + \kappa > G(0, \mu, A) = V^\varphi(0)$ for every $t \in [T_*, T]$. □

In Theorem 4.5, the functional $F(t, \mu, A)$ can be seen as the 'extracted nondecreasing structure' of G. This theorem states that the generating functional G need not be 'Lyapunov' to yield a strong arbitrage relative to the market as in Theorem 5.1 of Karatzas and Ruf (2017). There can be a strong relative arbitrage even if $\Gamma^G(\cdot)$ is nonincreasing. This is intuitively plausible from the equation (3.12) when $G(\cdot, \mu, A)$ grows faster than $\Gamma^G(\cdot)$ decays. Some applications of Theorem 4.5 will appear in Section 6 (Example 6.4 and Example 6.6).

4.2 Multiplicatively generated strong relative arbitrage

In this subsection, in order to make the arguments simpler, we assume that the regular functional G takes only nonnegative values and satisfies $G(0, \mu, A) = 1$. This normalization can be achieved by replacing G by $G/G(0, \mu, A)$ if $G(0, \mu, A) > 0$, or by $G + 1$ if $G(0, \mu, A) = 0$. The following result is similar to Theorem 5.2 of Karatzas and Ruf (2017), but requires somewhat more stringent 'boundedness' conditions.

Theorem 4.6 (Multiplicatively generated strong relative arbitrage)**.** *For open sets $U \subset \mathbb{R}^d_+$ and $W \in \mathbb{R}^m$, let us fix a regular functional $G : [0,T] \times D([0,T],U) \times BV([0,T],W) \to [0,\infty)$ for the pair (μ, A) with the market weights μ and some m-dimensional function $A \in CBV([0,T],W)$, satisfying one of the following conditions:*

(i) G is bounded from above;

(ii) $A \in BV([0,T],L)$ for some compact set $L \in \mathbb{R}^{d\times d}$.

For some real number $T_* > 0$, suppose that there exists an $\epsilon = \epsilon(T_*) > 0$ such that

$$\Gamma^G(T_*) > 1 + \epsilon. \tag{4.13}$$

Then, there exists a constant $c = c(T_*, \epsilon) > 0$ such that the trading strategy $\psi^{(c)} = (\psi^{(c)}_1, \cdots, \psi^{(c)}_d)'$, multiplicatively generated by the regular functional

$$G^{(c)} := \frac{G + c}{1 + c}$$

as in Definition 3.9, is strong arbitrage relative to the market over the time-horizon $[0, T_*]$; as well as over every time-horizon $[0, t]$ with $T_* \le t \le T$, if $\Gamma^G(t)$ is nondecreasing in t.

Proof. Since $G \in \mathbb{C}^{1,2}[0,T]$, and because G is boundedness-preserving in the sense of Definition 2.3, there exists on account of assumption (ii) a constant C_L, such that

$$|G(t, \mu, A)| \le C_L \quad \forall\, t \in [0,T], \quad \mu \in D([0,T], \Delta^d), \quad A \in BV([0,T], L)$$

holds. Here, Δ^d denotes the unit simplex in $\mathbb{R}^d$. Thus, in both cases of (i) and (ii), G admits an upper bound κ. For $c > 0$, (3.19) yields $V^{\psi^{(c)}}(0) = 1$, $V^{\psi^{(c)}} > 0$, and

$$V^{\psi^{(c)}}(T_*) \ge \frac{c}{1+c} \times \exp\Big(\int_0^{T_*} \frac{d\Gamma^G(t)}{G(t,\mu,A) + c}\Big) > \frac{c}{1+c} \times \exp\Big(\frac{1+\epsilon}{\kappa + c}\Big).$$

In the first inequality, we used $G \ge 0$ and the identity $\Gamma^{G^{(c)}} = \Gamma^G/(1 + c)$. The rest of the proof is the same as that of Theorem 5.2 in Karatzas and Ruf (2017). □

The following example provides a condition for strong relative arbitrage more general than Example 5.5 of Karatzas and Ruf (2017), using for $A(t)$ the running maximum of the market weights

$$A(t) \equiv \mu^*_i(t) := \max_{0 \le s \le t} \mu_i(s), \qquad i = 1, \cdots, d.$$

Example 4.7 (Quadratic functional). For fixed constant $c \in \mathbb{R}$ and $p > 0$, consider the following functional

$$
\begin{aligned}
G^{(c,p)}(t, \mu, \mu^*) &:= c - \sum_{i=1}^{d} \big(\mu_i(t)\big)^2 - p \sum_{i=1}^{d} \mu_i(t)\mu_i^*(t) \\
&= c - \sum_{i=1}^{d} \big(\mu_i(t)\big)^2 - p \sum_{i=1}^{d} \mu_i(t)\big\{ \max_{0 \le s \le t} \mu_i(s)\big\},
\end{aligned}
$$

which is the same as $Q^{(c)}$ in Example 5.5 of Karatzas and Ruf (2017) except for the last term. Note that $G^{(c,p)}$ takes values in the interval $\big[c - (1+p),\ c - \frac{1}{d}(1+p)\big]$. After some straightforward computation, we have

$$
\begin{aligned}
D_i G^{(c,p)}(t, \mu, \mu^*) &= -p\mu_i(t), \\
\partial_i G^{(c,p)}(t, \mu, \mu^*) &= -2\mu_i(t) - p\mu_i^*(t), \\
\partial^2_{i,i} G^{(c,p)}(t, \mu, \mu^*) &= -2,
\end{aligned}
$$

for $i = 1, \cdots, d$, and using these equations along with (3.6), we obtain

$$
\Gamma^{G^{(c,p)}}(t) = \sum_{i=1}^{d} \int_0^t p\mu_i(s) d\mu_i^*(s) + \sum_{i=1}^{d} \langle \mu_i \rangle(t).
$$

As $\mu_i^*(\cdot)$ is nondecreasing and $p\mu_i(\cdot) \ge 0$, the integral term is always non-negative and nondecreasing in t, which makes $\Gamma^{G^{(c,p)}}(\cdot)$ nondecreasing and non-negative. Also, using the property that the increment $d\mu_i^*(s)$ is positive only when $\mu_i(s) = \mu_i^*(s)$, we have

$$
\int_0^t \mu_i(s) d\mu_i^*(s) = \int_0^t \mu_i^*(s) d\mu_i^*(s) = \frac{1}{2}\big\{ \big(\mu_i^*(t)\big)^2 - \big(\mu_i^*(0)\big)^2 \big\}.
$$

Then,

$$
\Gamma^{G^{(c,p)}}(t) = \frac{p}{2} \sum_{i=1}^{d} \big\{ \big(\mu_i^*(t)\big)^2 - \big(\mu_i^*(0)\big)^2 \big\} + \sum_{i=1}^{d} \langle \mu_i \rangle(t).
$$

Since $G^{(1+p,p)} \ge 0$, let us consider the case $c = 1 + p$ from now on. Using the same argument in the proof of Theorem 4.2, the condition

$$
\frac{p}{2} \sum_{i=1}^{d} \big\{ \big(\mu_i^*(T)\big)^2 - \big(\mu_i^*(0)\big)^2 \big\} + \sum_{i=1}^{d} \langle \mu_i \rangle(T) > G^{(1+p,p)}(0, \mu, \mu^*), \tag{4.14}
$$

where

$$
G^{(1+p,p)}(0, \mu, \mu^*) = (1+p)\Big\{ 1 - \sum_{i=1}^{d} \big(\mu_i(0)\big)^2 \Big\} > 0,
$$

yields a strategy which is strong relative arbitrage with respect to the market on $[0, T]$. This strategy is additively generated by the functional $G^{(1+p,p)}/G^{(1+p,p)}(0, \mu, \mu^*)$. If we compare the condition (4.14) with the condition

$$
\sum_{i=1}^{d} \langle \mu_i \rangle(T) > 1 - \sum_{i=1}^{d} \big(\mu_i(0)\big)^2, \tag{4.15}
$$

that is, (5.4) of Example 5.5 in Karatzas and Ruf (2017), there is a trade-off between the left- and the right-hand sides. The existence of the extra nondecreasing term $(p/2)\sum_{i=1}^{d}\big\{\big(\mu_i^*(T)\big)^2-\big(\mu_i^*(0)\big)^2\big\}$ in (4.14), guarantees that its left-hand side grows faster than the left-hand side of (4.15), as T increases; but we also have a bigger constant on the right-hand side of (4.14), namely,

$$(1+p)\Big\{1-\sum_{i=1}^{d}\big(\mu_i(0)\big)^2\Big\} > 1-\sum_{i=1}^{d}\big(\mu_i(0)\big)^2.$$

Thus, if we choose the value of p wisely, we may obtain bounds for the times T, for which there is strong relative arbitrage with respect to the market over $[0,T]$, better than those of Example 5.5 in Karatzas and Ruf (2017).

5 Trading strategies depending on the values of market weights at multiple time points

In the previous sections, we used the pathwise functional Itô formula instead of the usual Itô formula to construct trading strategies, but we did not illustrate much of the superiority of the 'pathwise version'. In this section, we will see situations where the use of this pathwise functional formula is essential.

Suppose we want to construct a trading strategy which depends not only on the current market weights but also on past market weights. To be specific, we want our trading strategy to depend on the values of $\mu(t)$ and $\mu(t-\delta)$ for some fixed $\delta>0$. In order to do this, first we fix the time interval $\delta>0$ and enlarge the domain of each μ_i from $[0,T]$ to $[-\delta,T]$. This extension of domain can be done easily because even before we start investing to our trading strategy at time $t=0$, there must be past stock prices and past market weights. We simply attach these past data to the left of the timeline, so as to extend its domain.

We will develop our theory by demonstrating one example of a functional G which depends on $\mu(\cdot)$ and $\mu(\cdot-\delta)$; namely,

$$G(t,\mu)=2-\sum_{i=1}^{d}\big(\mu_i(t)-\mu_i(t-\delta)\big)^2. \tag{5.1}$$

Note that this functional G is non-anticipative and satisfies $G\geq 0$. We assume that $\mu_i(-\delta)<1$ for all $i=1,\cdots,d$ so that $G(0,\mu)>0$. For simplicity, as G does not depend on some additional process A as in the previous section, we removed A from the third input of G in (5.1). When we use results and quote equations from Section 3 or 4, we can just treat $A\equiv 0$.

Showing the regularity (Definition 3.3) of the functional G in (5.1) is straightforward. Using the Definition 2.2 of the vertical derivative, we obtain

$$\begin{aligned}\vartheta_i(t):=\partial_i G(t,\mu)&=\lim_{h\to 0}\frac{G(t,\mu^{t,he_i})-G(t,\mu)}{h}\\&=\lim_{h\to 0}\frac{-\big(\mu_i(t)+h-\mu_i(t-\delta)\big)^2+\big(\mu_i(t)-\mu_i(t-\delta)\big)^2}{h}\\&=-2\big(\mu_i(t)-\mu_i(t-\delta)\big),\end{aligned} \tag{5.2}$$

and also

$$\partial^2_{i,i} G(t, \mu) = -2$$

for $i = 1, \cdots, d$. From the equation (3.6), which is derived from (3.4) with the help of the pathwise functional Itô formula, we obtain

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t d\langle \mu_i \rangle(s) = \sum_{i=1}^{d} \big\{ \langle \mu_i \rangle(t) - \langle \mu_i \rangle(0) \big\}. \tag{5.3}$$

Note here that $\langle \mu_i \rangle(\cdot) = \int_{-\delta}^{\cdot} d\langle \mu_i \rangle(s)$ as we extended the domain of each $\mu_i(\cdot)$ to $[-\delta, T]$ and thus $\langle \mu_i \rangle(0)$ is not zero anymore. Equations (5.2)-(5.3) show that G in (5.1) is indeed regular and we can additively generate the trading strategy φ via the recipe (5.2)-(5.3) and (3.9)-(3.13); namely,

$$\begin{aligned}\varphi_i(t) = & -2\big(\mu_i(t) - \mu_i(t-\delta)\big) + \sum_{j=1}^{d} \big\{ \langle \mu_j \rangle(t) - \langle \mu_j \rangle(0) \big\} \\ & + 2 - \sum_{j=1}^{d} \big(\mu_j(t) - \mu_j(t-\delta)\big)^2 + 2\sum_{j=1}^{d} \mu_j(t)\big(\mu_j(t) - \mu_j(t-\delta)\big)\end{aligned} \tag{5.4}$$

for $i = 1, \cdots, d$, with relative value process

$$V^\varphi(t) = 2 - \sum_{i=1}^{d} \big(\mu_i(t) - \mu_i(t-\delta)\big)^2 + \sum_{i=1}^{d} \big\{ \langle \mu_i \rangle(t) - \langle \mu_i \rangle(0) \big\}.$$

Remark 5.1 (Illustration of difficulties). We were able to generate the additive trading strategy φ as in (5.4), by using the pathwise functional version of Itô formula. However, what happens if we try to generate such strategy from a 'function' of current and past market weights in the 'primitive' way with the standard Itô formula? Let us set aside the above functional for the moment, and try to set the right-hand side of (5.1) as a function G_1 of $2d$-dimensional argument, defined as

$$G_1\big(\mu(t), \mu(t-\delta)\big) := 2 - \sum_{j=1}^{d} \big(\mu_j(t) - \mu_j(t-\delta)\big)^2,$$

instead of the functional G defined as in (5.1). Then we will eventually find out that the function G_1 is not regular in the sense of Definition 3.1 in Karatzas and Ruf (2017).

Indeed, by applying the standard Itô formula to G_1, we have

$$\begin{aligned}G_1\big(\mu(t), \mu(t-\delta)\big) = G_1\big(\mu(0), \mu(-\delta)\big) &+ B_1(t) \\ &+ \int_0^t \sum_{j=1}^{d} \partial_j G_1\big(\mu(s), \mu(s-\delta)\big) d\mu_j(s) \\ &+ \int_0^t \sum_{j=d+1}^{2d} \partial_j G_1\big(\mu(s), \mu(s-\delta)\big) d\mu_j(s-\delta),\end{aligned}$$

for some finite variation process $B_1(\cdot)$. Observe that the last term in the above equation integrates the value $\partial_j G_1\big(\mu(s), \mu(s-\delta)\big)$ which is observable at the 'current' time s, with respect to the 'past' increment $d\mu_j(s-\delta)$ of the path $\mu_j(\cdot)$, a major difficulty! Notice also that there are two stochastic integral terms with different integrators in the last equation. This is because we have to treat $\mu(\cdot)$ and $\mu(\cdot - \delta)$ as two different semimartingales, even though they correspond to values from the same path μ. Then, from (3.2) of Karatzas and Ruf (2017), the gamma functional of G_1 is obtained as

$$\Gamma^{G_1}(t) = -\int_0^t \sum_{j=d+1}^{2d} \partial_j G_1\big(\mu(s), \mu(s-\delta)\big) d\mu_j(s-\delta) - B_1(t);$$

and because of the appearance of stochastic integral terms in this expression, G_1 is indeed not regular. Hence, we cannot construct trading strategies as in Definition 4.1 or Definition 4.7 of Karatzas and Ruf (2017), with a function that depends on both past and current market weights.

Remark 5.2. The power of the pathwise functional Itô formula, which overcomes the difficulties discussed in the previous remark, comes from the definition of the vertical derivative in Definition 2.2. The vertical derivative is calculated via the vertical perturbation $\mu^{t,h}$, which only shifts the current value at t by h and does not alter the past values of μ. In order words, when taking a vertical derivative of G with respect to the market weights μ, only the current value of μ matters and we can treat past values of μ as constants. This means that applying the pathwise functional Itô formula to G results in the presence of a 'single' stochastic integral term, which in turn guarantees the absence of a stochastic integral term in $\Gamma^G(\cdot)$, as is the case in (5.3).

Remark 5.3. On the right-hand side of (5.4), only the first term depends on i, i.e., we can rewrite (5.4) as

$$\varphi_i(t) = p(t) - 2\big(\mu_i(t) - \mu_i(t-\delta)\big), \qquad i = 1, \cdots, d,$$

where

$$p(t) = 2 + \sum_{j=1}^{d} \Big\{ \langle \mu_j \rangle(t) - \langle \mu_j \rangle(0) - \big(\mu_j(t) - \mu_j(t-\delta)\big)^2 + 2\mu_j(t)\big(\mu_j(t) - \mu_j(t-\delta)\big) \Big\}.$$

This trading strategy can be adopted by investors who believe in the mean-reverting property of stock prices. If the value of $\mu_i(t) - \mu_i(t-\delta)$ is significantly positive, then such investors might think that the i^{th} stock is overpriced now, and expect that the price of this stock will drop in the near future. Thus, they 'penalize' $\varphi_i(t)$, the number of shares they invest in i^{th} stock, by the amount of $-2\big(\mu_i(t) - \mu_i(t-\delta)\big)$ from the 'universal' amount of $p(t)$ across all i's. At the same time, if $\mu_j(t) - \mu_j(t-\delta)$ is negative, investors expect jth stock will rise soon and put more money in jth stock by adding $-2\big(\mu_j(t) - \mu_j(t-\delta)\big)$ to $p(t)$.

From Theorem 4.2 and (5.3), the additively generated strategy φ in (5.4) is strong arbitrage relative to the market over $[0, t]$ with $T_* \leq t \leq T$, where T_* is the smallest value satisfying

$$\sum_{i=1}^{d} \big\{ \langle \mu_i \rangle(T_*) - \langle \mu_i \rangle(0) \big\} > 2 - \sum_{i=1}^{d} \big(\mu_i(0) - \mu_i(-\delta)\big)^2.$$

We can also construct the multiplicatively generated trading strategy ψ from G in (5.1) using (5.2)-(5.3), and (3.17)-(3.20). Here we further assume $\max_{1 \leq i \leq d} \mu_i(\cdot) < 1 - \zeta$ or $\min_{1 \leq i \leq d} \mu_i(\cdot) > \zeta$ for some $\zeta > 0$ to make sure that $G(\cdot, \mu)$ is bounded away from 0 which in turn implies $1/G(\cdot, \mu)$ is locally bounded.[1] Then, from (5.2), (5.3), and (3.19), (3.20), after some computations, the trading strategy ψ can be represented in the form

$$\psi_i(t) = K(t)\Big\{2 - 2\big(\mu_i(t) - \mu_i(t-\delta)\big) + \sum_{j=1}^{d} \big(\mu_j^2(t) - \mu_j^2(t-\delta)\big)\Big\},$$

for $i = 1, \cdots, d$, and ψ has value process

$$V^{\psi}(t) = K(t)\Big\{2 - \sum_{j=1}^{d} \big(\mu_j(t) - \mu_j(t-\delta)\big)^2\Big\},$$

where

$$K(t) := \exp\left(\sum_{i=1}^{d} \int_0^t \frac{d\langle\mu_i\rangle(s)}{2 - \sum_{j=1}^{d} \big(\mu_j(s) - \mu_j(s-\delta)\big)^2}\right).$$

From Theorem 4.6, if we suppose that the function $\Gamma^G(\cdot)$ in (5.3) satisfies the condition (4.13) for some ϵ and $T_* > 0$, i.e., $\sum_{i=1}^{d} \big(\langle\mu_i\rangle(T_*) - \langle\mu_i\rangle(0)\big) > 1 + \epsilon$, then there is a trading strategy $\psi^{(c)}$, multiplicatively generated by the regular functional $G^{(c)} = ((G/G(0,\mu)) + c)/(1 + c)$ for an appropriate choice of c, and this strategy $\psi^{(c)}$ is strong arbitrage relative to the market over every time-horizon $[0, t]$ with $T_* \le t \le T$.

Now we present a trading strategy generated from a functional G which depends on the 'ratio' between current market weights and past market weights.

Example 5.4 (Relative entropy). Let us consider now the relative entropy

$$\sum_{i=1}^{d} \mu_i(t) \log\Big(\frac{\mu_i(t)}{\mu_i(t-\delta)}\Big)$$

of current market weights $\{\mu_i(t)\}_{i=1}^d$ with respect to past market weights $\{\mu_i(t-\delta)\}_{i=1}^d$. If the value of $\mu_i(t-\delta)$ is close to 0 while the value of $\mu_i(t)$ is close to 1, the ratio $\mu_i(t)/\mu_i(t-\delta)$ may have very large value. Thus, we impose a condition that restricts the growth of the market weight $\mu_i(\cdot)$ over the time span δ; namely, we suppose that there exists a constant $\zeta > 1$ such that

$$\max_{1 \le i \le d} \Big(\frac{\mu_i(t)}{\mu_i(t-\delta)}\Big) < \zeta \tag{5.5}$$

holds for all $t \in [0, T]$. Then, we define a functional G as

$$G(t, \mu) = \log\zeta - \sum_{i=1}^{d} \mu_i(t) \log\Big(\frac{\mu_i(t)}{\mu_i(t-\delta)}\Big). \tag{5.6}$$

[1] If we want to avoid imposing such a condition on μ, we can simply change the definition of G into

$$G(t, \mu) = 2 + \beta - \sum_{i=1}^{d} \big(\mu_i(t) - \mu_i(t-\delta)\big)^2$$

in (5.1) by adding some positive constant β to ensure that $1/G$ is locally bounded.

The condition (5.5) guarantees that this G is positive. From the Definition 2.2 of the vertical derivative, we can compute $\vartheta_i(t) := \partial_i G(t, \mu)$ as

$$\begin{aligned}
\partial_i G(t,\mu) &= \lim_{\theta \to 0} \frac{-(\mu_i(t)+\theta)\log\left(\frac{\mu_i(t)+\theta}{\mu_i(t-\delta)}\right) + \mu_i(t)\log\left(\frac{\mu_i(t)}{\mu_i(t-\delta)}\right)}{\theta} \\
&= \lim_{\theta \to 0} \frac{-(\mu_i(t)+\theta)\log\left(\frac{\mu_i(t)+\theta}{\mu_i(t-\delta)}\right) + \mu_i(t)\log\left(\frac{\mu_i(t)+\theta}{\mu_i(t-\delta)}\right)}{\theta} \\
&\quad + \lim_{\theta \to 0} \frac{-\mu_i(t)\log\left(\frac{\mu_i(t)+\theta}{\mu_i(t-\delta)}\right) + \mu_i(t)\log\left(\frac{\mu_i(t)}{\mu_i(t-\delta)}\right)}{\theta} \\
&= -\log\left(\frac{\mu_i(t)}{\mu_i(t-\delta)}\right) - \mu_i(t) \cdot \lim_{\theta \to 0} \frac{\log(\mu_i(t)+\theta) - \log \mu_i(t)}{\theta} \\
&= -\log\left(\frac{\mu_i(t)}{\mu_i(t-\delta)}\right) - 1,
\end{aligned}$$

where the last equality is obtained by applying L'Hôpital's rule. Similar computation leads us to

$$\partial^2_{i,i} G(t,\mu) = -\frac{1}{\mu_i(t)}, \qquad \Gamma^G(t) = \sum_{i=1}^d \int_0^t \frac{d\langle \mu_i \rangle(s)}{2\mu_i(s)}$$

from (3.6). Then, we again use (3.9)-(3.13) to obtain a representation

$$\varphi_i(t) = \log\zeta - \log\left(\frac{\mu_i(t)}{\mu_i(t-\delta)}\right) + \sum_{j=1}^d \int_0^t \frac{d\langle \mu_j \rangle(s)}{2\mu_j(s)}, \qquad i = 1, \cdots, d \tag{5.7}$$

of the additively generated trading strategy φ from G of (5.6) with corresponding value process

$$V^\varphi(t) = \log\zeta - \sum_{i=1}^d \mu_i(t)\log\left(\frac{\mu_i(t)}{\mu_i(t-\delta)}\right) + \sum_{i=1}^d \int_0^t \frac{d\langle \mu_i \rangle(s)}{2\mu_i(s)}.$$

From Theorem 4.2, φ is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \leq t \leq T$, where T_* is the smallest positive number satisfying the condition

$$\sum_{i=1}^d \int_0^{T_*} \frac{d\langle \mu_i \rangle(s)}{2\mu_i(s)} \geq \log\zeta - \sum_{i=1}^d \mu_i(0)\log\left(\frac{\mu_i(0)}{\mu_i(-\delta)}\right).$$

Remark 5.5. This trading strategy φ of (5.7) has a similar characteristic as the trading strategy of (5.4) regarding the mean-reversion property of stock prices. If the market weight $\mu_i(t)$ of i^{th} stock at time t is bigger than the same market weight $\mu_i(t-\delta)$ at earlier time $t-\delta$, the log value of ratio $\mu_i(t)/\mu_i(t-\delta)$ would be positive. Then, the investor must reduce the number of shares invested in this i^{th} stock by subtracting the amount of this log-value from the universal amount $\log\zeta + \sum_{j=1}^d \int_0^t \frac{1}{2\mu_j(s)} d\langle \mu_j \rangle(s)$ in (5.7).

The trading strategy ψ multiplicatively generated from G as in (5.6), and the corresponding sufficient condition for strong relative arbitrage, can be obtained in a similar manner, using equations (3.17)-(3.20) and Theorem 4.6.

The same methodology of this section constructs various trading strategies from other forms of functionals which depend on current and past market weights; this is not possible using the primitive form of the Itô formula. Also, we can build up trading strategies depending on N different values of past market weights and current market weights by setting $\delta_1 > \cdots > \delta_N > 0$, enlarging the domain of each μ_i from $[0, T]$ to $[-\delta_1, T]$, and applying the pathwise functional Itô formula as before.

6 Examples of entropic functionals

In this section, we present some examples of trading strategies additively generated from variants of the 'entropy function', and the corresponding conditions for strong relative arbitrage. Empirical results regarding these examples will be presented in the next section. We focus only on additively generated strong relative arbitrage, as the condition from Theorem 4.2 is much simpler to derive than that of Theorem 4.6. We also present some applications of Theorem 4.5.

Consider the Gibbs entropy function

$$H(x) = -\sum_{i=1}^{d} x_i \log(x_i), \qquad x \in (0,1)^d, \tag{6.1}$$

with values in (0, $\log d$). Being nonnegative, twice-differentiable and concave, this function is one of the most frequently used functions in stochastic portfolio theory. See Fernholz (2002); Fernholz and Karatzas (2009); Karatzas and Ruf (2017) for its usage in generating portfolios, and also Ruf and Xie (2018), Schied et al. (2016) for some variants of portfolios generated by this function.

Example 6.1 (Entropy function). In order to compare the trading strategy generated from the original entropy function with those generated from variants of functionals related to this function, we first derive and summarize the trading strategy additively generated from the original entropy function. Consider the "shifted entropy"

$$G(\mu(t)) := -\sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_i(t)\big) = -\log p - \sum_{i=1}^{d} \mu_i(t)\log(\mu_i(t)), \tag{6.2}$$

for some given real constant $p \geq 1$, where the last equality uses the fact $\sum_{i=1}^{d} \mu_i(t) = 1$. This function coincides with the original entropy function $H(\mu(t))$ in (6.1) when $p = 1$; the reason for inserting the additive constant will be explained in the following remark. From (3.6), (3.12), and (3.13), the trading strategy φ additively generated by this entropy function can be represented as

$$\varphi_i(t) = -\log\big(p\mu_i(t)\big) + \Gamma^G(t), \qquad i = 1, \cdots, d, \tag{6.3}$$

where

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \frac{d\langle \mu_i \rangle(s)}{2\mu_i(s)}, \tag{6.4}$$

and the value of this trading strategy is given as

$$V^{\varphi}(t) = -\sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_i(t)\big) + \sum_{i=1}^{d} \int_0^t \frac{d\langle \mu_i \rangle(s)}{2\mu_i(s)}. \tag{6.5}$$

Note that φ in (6.3) has a relatively simple form, because G in (6.2) is 'almost balanced', in the sense that

$$G(\mu(\cdot)) - 1 = \sum_{j=1}^{d} \mu_j(\cdot)\partial_j G(\mu(\cdot))$$

holds; compare this equation with (3.14), and (6.3) with (3.15). Then, the condition (4.5) for strong arbitrage in Theorem 4.2 is given as

$$\sum_{i=1}^{d} \int_0^{T_*} \frac{d\langle \mu_i \rangle(s)}{2\mu_i(s)} > -\sum_{i=1}^{d} \mu_i(0) \log\big(p\mu_i(0)\big). \tag{6.6}$$

Remark 6.2. The construction of trading strategies described in the previous sections does not require any optimization or statistical estimation of parameters. However, we can improve relative performance of trading strategies with respect to the market by introducing a parameter or a set of parameters in the generating functional G. Though the original entropy function is the equation (6.2) with $p = 1$, we purposely inserted a constant p inside the logarithm. To achieve strong relative arbitrage faster, or to find the smaller such T_* satisfying (6.6), or more generally (4.5), making the 'threshold' value $G(0, \mu, A)$ in the right-hand side of the inequality smaller, while keeping the 'growth rate' of $\Gamma^G(\cdot)$ fixed, is very helpful. In this sense, we placed the parameter p to the function in (6.2); inserting such constant $p > 1$ inside the $\log$ would make the initial value $G(\mu(0))$ smaller by the amount $\log p$, compared to the case $p = 1$, while it does not affect the $\Gamma^G(\cdot)$ as subtracting a constant $\log p$ from G does not change any derivatives (both vertical and horizontal) of G with respect to the market weights. However, if p is too large such that $-\sum_{i=1}^{d} \mu_i(t) \log \mu_i(t) < \log p$ holds at some time t, then $G(t, \mu, A)$ has a negative value. Theoretically, $-\sum_{i=1}^{d} \mu_i(t) \log \mu_i(t)$ has the minimum value of 0 only when one of the market weights, say $\mu_1(t)$, is equal to 1 and all the others $\mu_i(t)$ for $i = 2, \cdots, d$ vanish, which does not happen in the real world. Empirically, the value of $-\sum_{i=1}^{d} \mu_i(t) \log \mu_i(t)$ is always bounded away from zero, and we can guarantee this condition theoretically by imposing a weak condition on the market weights. For example, restricting the maximum value of the market weights, say $\max_i \mu_i(\cdot) \le 0.5$, makes the value of $-\sum_{i=1}^{d} \mu_i(t) \log \mu_i(t)$ strictly positive and bounded away from 0 all the time. Finding a suitable value of $p > 1$ while maintaining G nonnegative should be statistically done and it depends on d, the number of stocks. Empirical estimation of such p can be found in the next section.

Making the initial value of $G(0, \mu, A)$ small while keeping the growth rate of $\Gamma^G(\cdot)$ is also beneficial for calculating the 'excess return rate' of trading strategies with respect to the market. The excess return rate of the trading strategy φ at time $t \in (0, T]$ can be defined as

$$R^{\varphi}(t) := \frac{V^{\varphi}(t) - V^{\varphi}(0)}{V^{\varphi}(0)}, \tag{6.7}$$

and from (3.12), this can be represented as

$$R^{\varphi}(t) = \frac{G(t, \mu, A) + \Gamma^G(t) - G(0, \mu, A)}{G(0, \mu, A)}.$$

Thus, if we somehow make the value $G(0, \mu, A)$, the denominator of above fraction, smaller while keeping the value of $\Gamma^G(t)$ in the numerator, we can obtain larger excess return rate of trading strategy φ. From now on, we use this trick to decrease the initial value $G(0, \mu, A)$ of generating functional whenever possible.

The following two examples use for the component A two opposite functions of finite variation; the running maximum $\mu_i^*(t) := \max_{0\le s\le t}\mu_i(s)$, and the running minimum $\mu_{*i}(t) := \min_{0\le s\le t}\mu_i(s)$ of the market weights.

Example 6.3 (Entropy function with running maximum). Consider an entropic functional of the type

$$G(t,\mu,A) \equiv G(t,\mu,\mu^*) := -\log p - \sum_{i=1}^d \mu_i(t)\log\mu_i^*(t), \tag{6.8}$$

with the notation of the vector function $A \equiv \mu^* = (\mu_1^*,\cdots,\mu_d^*)'$. As before, $p \ge 1$ is a constant as in Remark 6.2, and the initial value $G(0,\mu,\mu^*) = -\log p - \sum_{i=1}^d \mu_i(0)\log\mu_i(0)$ is the same as in Example 6.1. We then easily obtain

$$\partial_i G(t,\mu,\mu^*) = -\log\mu_i^*(t), \qquad \partial^2_{i,j}G(t,\mu,\mu^*) = 0, \qquad D_iG(t,\mu,\mu^*) = -\frac{\mu_i(t)}{\mu_i^*(t)},$$

for $1 \le i,\ j \le d$. From (3.6), we also have

$$\begin{aligned}\Gamma^G(t) &= \sum_{i=1}^d \int_0^t \frac{\mu_i(s)}{\mu_i^*(s)}d\mu_i^*(s) = \sum_{i=1}^d \int_0^t \frac{\mu_i^*(s)}{\mu_i^*(s)}d\mu_i^*(s),\\ &= \sum_{i=1}^d \big(\mu_i^*(t) - \mu_i(0)\big) = \sum_{i=1}^d \mu_i^*(t) - 1,\end{aligned} \tag{6.9}$$

where we used the fact that the increment $d\mu_i^*(s)$ is positive only when $\mu_i(s) = \mu_i^*(s)$. As the functional G of (6.8) is linear in $\mu_i(\cdot)$, the second order vertical derivatives of G vanish, and the nondecreasing structure of $\Gamma^G(\cdot)$ comes solely from $\mu_i^*(\cdot)$. Also from (3.12), and (3.13), the trading strategy φ generated additively from this functional, is expressed as

$$\varphi_i(t) = -\log\big(p\mu_i^*(t)\big) + \sum_{j=1}^d \mu_j^*(t) - 1, \qquad i = 1,\cdots,d; \tag{6.10}$$

and the value of this trading strategy is given as

$$V^\varphi(t) = -\sum_{i=1}^d \mu_i(t)\log\big(p\mu_i^*(t)\big) + \sum_{i=1}^d \mu_i^*(t) - 1. \tag{6.11}$$

The strong arbitrage condition (4.5) in Theorem 4.2 is

$$\sum_{i=1}^d \mu_i^*(T_*) > 1 - \sum_{i=1}^d \mu_i(0)\log\big(p\mu_i(0)\big). \tag{6.12}$$

Empirical results regarding this example can be found in the next section.

The Gamma functional $\Gamma^G(\cdot)$ which represents the "cumulative earnings" of the additively generated trading strategy φ of the next example is nonincreasing, but surprisingly, the empirical value $V^\varphi(\cdot)$ of trading strategy grows asymptotically in the long run as the value of G grows, as indicated in the next section. Thus, in this case, it is more appropriate to apply Theorem 4.5 regarding the strong arbitrage condition.

Example 6.4 (Entropy function with running minimum). Consider a functional

$$G(t, \mu, A) \equiv G(t, \mu, \mu_*) := -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t), \tag{6.13}$$

with the notation of the vector function $A \equiv \mu_* = (\mu_{*1}, \cdots, \mu_{*d})'$. As before, p is a constant and the initial value $G(0, \mu, \mu_*)$ is the same as previous examples. Then, similarly as before, we have

$$\partial_i G(t, \mu, \mu_*) = -\log \mu_{*i}(t), \qquad \partial^2_{i,j} G(t, \mu, \mu_*) = 0, \qquad D_i G(t, \mu, \mu_*) = -\frac{\mu_i(t)}{\mu_{*i}(t)},$$

for $1 \leq i, j \leq d$. Also from (3.6), we obtain

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \frac{\mu_i(s)}{\mu_{*i}(s)} d\mu_{*i}(s) = \sum_{i=1}^{d} \int_0^t 1 \; d\mu_{*i}(s) = \sum_{i=1}^{d} \mu_{*i}(t) - 1, \tag{6.14}$$

which is nonpositive and nonincreasing function of t. By (3.13), the trading strategy φ additively generated from this functional is expressed as

$$\varphi_i(t) = -\log \big(p\mu_{*i}(t)\big) + \sum_{j=1}^{d} \mu_{*j}(t) - 1, \qquad i = 1, \cdots, d. \tag{6.15}$$

Note that $\varphi_i(t)$ admits the lower bound

$$\varphi_i(t) = -\log p - \log \mu_{*i}(t) + \mu_{*i}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{d} \mu_{*j}(t) - 1 \geq -\log p - \log \mu_i(0) + \mu_i(0) - 1, \tag{6.16}$$

because the function $x \mapsto -\log x + x$ is decreasing in the interval $x \in (0, 1)$ and, thus, $\varphi_i(t)$ is positive provided that

$$p < e^{-\log \mu_i(0) + \mu_i(0) - 1}.$$

holds. By (3.12), the value of this trading strategy is given as

$$V^{\varphi}(t) = -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) + \Big(\sum_{i=1}^{d} \mu_{*i}(t) - 1\Big). \tag{6.17}$$

While $\Gamma^G(t) = \sum_{i=1}^{d} \mu_{*i}(t) - 1$, the last terms in the right-hand side of (6.17), is nonincreasing, the second term $-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t)$ asymptotically increase as the mapping $t \mapsto -\log \mu_{*i}(t)$ is nondecreasing. Actually, as we can see in the next section, the value of this trading strategy grows in the long run. We can apply Theorem 4.5, rather than Theorem 4.2, to find a strong arbitrage condition, because $\Gamma^G(\cdot)$ in this example is not nondecreasing.

In order to apply Theorem 4.5, we first need to show that $V^{\varphi}(\cdot) \geq 0$ holds. From (6.16), we obtain

$$\begin{aligned} -\log \mu_{*i}(t) &\geq -\sum_{j=1}^{d} \mu_{*i}(t) - \log \mu_i(0) + \mu_i(0) \\ &\geq -1 - \log \mu_i(0) + \mu_i(0) \\ &\geq -1 - \log \big(\max_{j=1,\cdots,d} \mu_j(0)\big) + \max_{j=1,\cdots,d} \mu_j(0) \end{aligned}$$

holds for all $i = 1, \cdots, d$. The last inequality follows from the fact that the function $x \mapsto -\log x + x$ is decreasing in the interval $x \in [0, 1]$. Then, we also obtain

$$-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) \geq -1 - \log \big(\max_{j=1,\cdots,d} \mu_j(0) \big) + \max_{j=1,\cdots,d} \mu_j(0),$$

because $-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t)$ is just the weighted arithmetic average of $\{-\log \mu_{*i}(t)\}_{i=1,\cdots,d}$ with weights $\mu_i(t)$ with $\sum_{i=1}^{d} \mu_i(t) = 1$. Thus, $V^{\varphi}(t)$ in (6.17) admits the lower bound

$$V^{\varphi}(t) \geq -\log p - 2 - \log \big(\max_j \mu_j(0) \big) + \max_j \mu_j(0)$$

for any $t \in [0, T]$, and $V^{\varphi}(\cdot) \geq 0$ is guaranteed when

$$p \leq e^{-2 - \log \big(\max_j \mu_j(0) \big) + \max_j \mu_j(0)} \tag{6.18}$$

holds. Regarding the second condition of Theorem 4.5, we have

$$\begin{aligned} G(t, \mu, \mu_*) &= -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) \\ &\geq -\log p - \sum_{i=1}^{d} \mu_i(t) \log \big(\max_{i=1,\cdots,d} (\mu_{*i}(t)) \big) \\ &= -\log p - \max_{i=1,\cdots,d} \big\{ \log \mu_{*i}(t) \big\} \\ &:= F(t, \mu, \mu_*), \end{aligned} \tag{6.19}$$

where we used the fact $\sum_{i=1}^{d} \mu_i(t) = 1$; now the mapping $t \mapsto \mu_{*i}(t)$ is nonincreasing, so $F(t, \mu, \mu_*)$ is nondecreasing in t. Finally, the last condition of Theorem 4.5 follows easily from (6.14), as

$$\Gamma^G(t) \geq -1 := \kappa. \tag{6.20}$$

Thus, Theorem 4.5 shows that the additively generated strategy φ in (6.15) is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \leq t \leq T$, satisfying the condition

$$\sum_{i=1}^{d} \mu_i(0) \log \mu_i(0) - \max_{i=1,\cdots,d} \big\{ \log \mu_{*i}(T_*) \big\} > 1.$$

Remark 6.5. In Remark 6.2, we need to find a suitable value for p satisfying an inequality, for instance, $-\sum_{i=1}^{d} \mu_i(t) \log(\mu_i(t)) \geq \log p$ for all $t \in [0, T]$ in Example 6.1, to make the functional G nonnegative. This inequality usually depends on the values $\mu_i(t)$, $t \in [0, T]$ which are not observable at time 0. Thus, we need to impose some condition on the market weights or statistically analyze historical market data to find an appropriate value for p before we construct the trading strategy.

However, in Example 6.4, due to its unique structure, we can analytically find a suitable value of p without any statistical estimation at time $t = 0$. Indeed, from (6.19), we have that

$$\begin{aligned} G(t, \mu, \mu_*) &\geq -\log p - \max_{i=1,\cdots,d} \big\{ \log \mu_{*i}(t) \big\} \\ &\geq -\log p - \max_{i=1,\cdots,d} \big\{ \log \mu_i(0) \big\} \end{aligned}$$

holds; and setting

$$p = \frac{1}{\max_{i=1,\cdots,d} \mu_i(0)} \tag{6.21}$$

guarantees the condition $G(t, \mu, \mu_*) \geq 0$ for all $t \in [0, T]$. Note that this p can be calculated from absolutely observable values at time 0. Actually, p satisfying (6.18) also guarantees the nonnegativity condition of G because $G(\cdot, \mu, \mu_*) \geq V^{\varphi}(\cdot) = G(\cdot, \mu, \mu_*) + \Gamma^G(\cdot) \geq 0$ holds due to the nonpositivity of $\Gamma^G(\cdot)$. Of course, one can perform a statistical estimation of p using past market data, to obtain a better value of p while satisfying both $G(\cdot, \mu, \mu_*) \geq 0$ and $V^{\varphi}(\cdot) \geq 0$.

The next example provides yet another application of Theorem 4.5.

Example 6.6 (Iterated entropy function with running minimum). In this example, we first fix a positive constant r such that the following condition on the initial market weights holds;

$$\mu_i(0) \leq \frac{1}{re}, \qquad i = 1, \cdots, d. \tag{6.22}$$

Here, e is the exponential constant. As the initial market weights are observable before we construct a trading strategy, we can find and fix such value of r at the moment we start investing in our trading strategy. For example, if no single stock takes more than 12% of total capitalization at time 0, we can set $r = 3$, as $\frac{1}{3e} \approx 0.123$. Then, we consider a functional

$$G(t, \mu, A) \equiv G(t, \mu, \mu_*) := -p - \sum_{i=1}^{d} \mu_i(t) \log\big\{ -r\mu_{*i}(t) \log\big(r\mu_{*i}(t)\big)\big\}, \tag{6.23}$$

with the notation of the vector function $A \equiv \mu_* = (\mu_{*1}, \cdots, \mu_{*d})'$. As in Remark 6.5, we can predetermine the value of the constant p, without any statistical estimation, because of the series of inequalities

$$\begin{aligned}
G(t, \mu, \mu_*) &\geq -p - \sum_{i=1}^{d} \mu_i(t) \log\Big[\max_{j=1,\cdots,d} \big\{ -r\mu_{*j}(t) \log\big(r\mu_{*j}(t)\big)\big\}\Big] \\
&\geq -p - \log\Big[\max_{j=1,\cdots,d} \big\{ -r\mu_{*j}(t) \log\big(r\mu_{*j}(t)\big)\big\}\Big] =: F(t, \mu, \mu_*) \\
&\geq -p - \log\Big[\max_{j=1,\cdots,d} \big\{ -r\mu_{j}(0) \log\big(r\mu_{j}(0)\big)\big\}\Big], \qquad \forall\, t \in [0, T].
\end{aligned} \tag{6.24}$$

The first inequality uses the fact that $x \mapsto -\log x$ is decreasing function and the second inequality is from the equation $\sum_{i=1}^{d} \mu_i(t) = 1$. The last inequality holds because $x \mapsto -rx \log(rx)$ is increasing in the interval $[0, \frac{1}{re}]$ and

$$0 \leq \mu_{*i}(\cdot) \leq \mu_i(0) \leq \frac{1}{re}, \tag{6.25}$$

holds from the assumption (6.22). Note that $F(t, \mu, \mu_*)$ defined in (6.24) is a nondecreasing in t as the mappings $t \mapsto \mu_{*i}(t)$ and $t \mapsto -r\mu_{*i}(t) \log\big(r\mu_{*i}(t)\big)$ are nonincreasing. Then, the choice

$$p \leq -\log\Big[\max_{i=1,\cdots,d} \big\{ -r\mu_i(0) \log\big(r\mu_i(0)\big)\big\}\Big], \tag{6.26}$$

which is completely observable value at time 0, guarantees that $G(\cdot, \mu, \mu_*)$ is always nonnegative. Next, after some computation, we obtain

$$\begin{aligned}
&\partial_i G(t, \mu, \mu_*) = -\log\big\{ -r\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)\big\} \geq 1, \\
&\partial^2_{i,j} G(t, \mu, \mu_*) = 0, \\
&D_i G(t, \mu, \mu_*) = -\frac{\mu_i(t)\log\big(r\mu_{*i}(t)\big) + \mu_i(t)}{\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)},
\end{aligned} \tag{6.27}$$

for $1 \leq i, j \leq d$. We note that $\partial_i G(t, \mu, \mu_*) \geq 1$ holds again because the mapping $x \mapsto -rx\log(rx)$ is increasing from 0 to $\frac{1}{e}$ in the interval $[0, \frac{1}{re}]$. From (3.6) and the fact that the increment $d\mu_{*i}(s)$ is positive only when $\mu_i(s) = \mu_{*i}(s)$, we obtain

$$\Gamma^G(t) = \sum_{i=1}^d \int_0^t \Big(1 + \frac{1}{\log\big(r\mu_{*i}(s)\big)}\Big) d\mu_{*i}(s) \tag{6.28}$$

which is nonincreasing function of t, because $0 \leq 1 + \frac{1}{\log\big(r\mu_{*i}(\cdot)\big)} \leq 1$ holds by the equation (6.25). This function admits the lower bound

$$\begin{aligned}
\Gamma^G(t) &= \sum_{i=1}^d \int_0^t 1\, d\mu_{*i}(s) + \sum_{i=1}^d \int_0^t \frac{1}{\log\big(r\mu_{*i}(s)\big)} d\mu_{*i}(s) \\
&= \sum_{i=1}^d \mu_{*i}(t) - \sum_{i=1}^d \mu_{*i}(0) + \sum_{i=1}^d li_r(\mu_{*i}(t)) - \sum_{i=1}^d li_r(\mu_i(0)), \\
&\geq -1 - \sum_{i=1}^d li_r(\mu_i(0)) =: \kappa,
\end{aligned} \tag{6.29}$$

with the notation

$$li_r(x) := \int_0^x \frac{du}{\log(ru)} = \frac{1}{r}\int_0^{rx} \frac{dv}{\log v} = \frac{1}{r} li(rx).$$

Here, $li(x) = \int_0^x \frac{du}{\log u}$ represents the logarithmic integral function. Note that the function $li_r(x)$ has negative value and is decreasing from 0 to $-\infty$ in the interval $x \in [0, \frac{1}{r})$. The last inequality holds because the inequality

$$\mu_{*i}(\cdot) + li_r(\mu_{*i}(\cdot)) \geq 0 \tag{6.30}$$

is satisfied for all $\mu_{*i}(\cdot)$ with the condition (6.25). We also note that κ defined in (6.29) satisfies $-1 < \kappa \leq -1 + \sum_{i=1}^d \mu_i(0) = 0$ from the same inequality (6.30). On the other hand, by (3.13), the trading strategy φ additively generated from this functional is expressed as

$$\varphi_i(t) = -p - \log\big\{ -r\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)\big\} + \sum_{i=1}^d \int_0^t \Big(1 + \frac{1}{\log\big(r\mu_{*i}(s)\big)}\Big) d\mu_{*i}(s). \tag{6.31}$$

Finally, by (3.12), the value of this trading strategy is given as

$$V^\varphi(t) = -p - \sum_{i=1}^d \mu_i(t)\log\big\{ -r\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)\big\} + \sum_{i=1}^d \int_0^t \Big(1 + \frac{1}{\log\big(r\mu_{*i}(s)\big)}\Big) d\mu_{*i}(s), \tag{6.32}$$

and is estimated as

$$V^{\varphi}(t) \geq -p - \log\Big[\max_{i=1,\cdots,d}\big\{-r\mu_i(0)\log\big(r\mu_i(0)\big)\big\}\Big] + \kappa,$$

from (6.24) and (6.29). Thus, the choice

$$p = -\log\Big[\max_{i=1,\cdots,d}\big\{-r\mu_i(0)\log\big(r\mu_i(0)\big)\big\}\Big] + \kappa \tag{6.33}$$

guarantees $V^{\varphi}(\cdot) \geq 0$ and also satisfies (6.26). We emphasize here again that p defined as in (6.33) depends only on the initial market weights $\mu_i(0)$, thus no statistical estimation of p is required. Using the same technique as in (6.24), $\varphi_i(t)$ in (6.31) is greater or equal to

$$-p - \log\Big[\max_{i}\big\{-r\mu_i(0)\log\big(r\mu_i(0)\big)\big\}\Big] + \kappa,$$

which is 0 by (6.33). Thus, this trading strategy is 'long-only', i.e., $\varphi_i(\cdot) \geq 0$ for all $i = 1, \cdots, d$.

As we showed above that all conditions of Theorem 4.5 are satisfied, the additively generated strategy φ in (6.31) is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \leq t \leq T$, satisfying the condition

$$-\log\Big[\max_{i=1,\cdots,d}\big\{-r\mu_i(T_*)\log\big(r\mu_i(T_*)\big)\big\}\Big] > -\sum_{i=1}^{d}\mu_i(0)\log\big\{-r\mu_i(0)\log\big(r\mu_i(0)\big)\big\} - \kappa,$$

with κ in (6.29).

7 Empirical results

We present some empirical results regarding the behavior of additively-generated portfolios as in the previous section, using historical market data. We first analyze the value function $V^{\varphi}(\cdot)$ of these portfolios with respect to the market by decomposing it with generating functional G and corresponding Gamma functional Γ^G in (3.12). Especially, we show that all value functions of portfolios in Section 6 outperform the market portfolio. Then, we present that the different choice of the parameter p, explained in Remark 6.2, indeed significantly influences the performance of portfolios.

7.1 Data description and notations

In order to simulate a perfect 'closed market', we construct a "universe" with $d = 1085$ stocks which had been continuously traded during 4528 consecutive trading days between 2000 January 1st and 2017 December 31st. These 1085 stocks were chosen from those listed at least once among the constituents of the S&P 1500 index in this period, and did not undergo mergers, acquisitions, bankruptcies, etc.

Remark 7.1. This selection of 1085 stocks is somewhat biased, in the sense that we are looking ahead into the future at time $t = 0$ by blocking out those stocks which will go bankrupt in the future. However, the reason for this biased selection is to keep the number of stocks d constant all the time which is the essential assumption of our 'closed' market model. If we compose our portfolio from $d = 1500$ stocks included in S&P 1500 index at the beginning, remove one stock whenever it goes bankrupt, or take in a new stock whenever it is newly added to the index, the number d of stocks in our portfolio fluctuates over time and the generating functional G would be discontinuous whenever d changes.

One possible solution to this problem is to consider an 'open market'. We first fix the value of d, say $d = 1500$ at the beginning, keep track of price dynamics of all stocks in the market (which should be composed of more than d stocks, say D stocks with $D > d$), rank them by the order of their market capitalization, and construct our portfolio using the top $d = 1500$ stocks among D stocks. In this way we can keep the same number d of companies all the time, but considering ranked market weights always involves a 'leakage' issue. As explained in Chapter 4.2, 4.3 of Fernholz (2002) and Example 6.2 of Karatzas and Ruf (2017), this refers to the loss incurred when we have to sell a stock that has been relegated from top d capitalization index to the lower capitalization index. Even worse, as we want to invest only in the top d companies among D companies in this open market, our trading strategy $\varphi = (\varphi_1, \cdots, \varphi_D)$ should satisfy the equations $\varphi_i(t) = 0$ for $i = 1, \cdots, D$ whenever the i-th company fails to be included in the top d companies at time t. However, we do not know how to construct such trading strategy yet.

Thus, it is not easy to make a perfect empirical model, and we decided to select $d = 1085$ stocks in a biased manner which fits better to our theoretic model described in the previous sections.

We obtained daily closing prices and total number of shares of these stocks from the CRSP and Compustat data sets. The data can be found here; https://wrds-web.wharton.upenn.edu/wrds/. We used ***R*** and ***C++*** to program our portfolios.

As we used daily data for $N = 4528$ days, we discretized the time horizon as $0 = t_0 < t_1 < \cdots, < t_{N-1} = T$. For $\ell \in \{1, 2, \cdots, N\}$, we summarize our notations here;

1. $S_i(t_\ell)$: the capitalization (daily closing price multiplied by total number of shares) of i^{th} stock at the end of day t_ℓ.

2. $\Sigma(t_\ell) := \sum_{i=1}^{d} S_i(t_\ell)$: the total capitalization of d stocks at the end of day t_ℓ. This quantity also represents dollar value of the market portfolio at the end of day t_ℓ with the initial wealth $\Sigma(0)$.

3. $\mu_i(t_\ell) := \frac{S_i(t_\ell)}{\Sigma(t_\ell)}$: the i^{th} market weight at the end of day t_ℓ.

4. $\pi_i(t_\ell)$: the additively generated portfolio weight of the i^{th} stock at the end of day t_ℓ which can be computed using the equation (3.16). Note that $\sum_{i=1}^{d} \pi_i(t_\ell) = 1$ holds.

5. $W(t_\ell)$: the total value of the portfolio at the end of day t_ℓ. Then, $W(t_\ell)\pi_i(t_\ell)$ represents the amount of money invested by our portfolio in i^{th} stock at the end of day t_ℓ.

As the capitalization of i^{th} stock at the beginning of day t_ℓ should be equal to $S_i(t_{\ell-1})$, the capitalization of the same stock at the end of the last trading day $t_{\ell-1}$, we also deduce that $\Sigma(t_{\ell-1})$, $\mu_i(t_{\ell-1})$, $\pi_i(t_{\ell-1})$, and $W(t_{\ell-1})$ represent the total capitalization, i^{th} market weight, i^{th} additively generated portfolio weight, and the money value of portfolio at the beginning of day t_ℓ, respectively.

The transaction, or rebalancing, of our portfolio on day t_ℓ, is made at the beginning of the day t_ℓ, using the market weights $\mu_i(t_{\ell-1})$ at the end of the last trading day. We compute $\pi_i(t_{\ell-1})$ from $\mu_i(t_{\ell-1})$

via (3.16), and re-distribute the generated value $W(t_{\ell-1})$ according to the these weights $\pi_i(t_{\ell-1})$. Then, the monetary value of portfolio $W(t_\ell)$ at the end of day t_ℓ can be calculated as

$$W(t_\ell) = \sum_{i=1}^{d} W(t_{\ell-1})\pi_i(t_{\ell-1})\frac{S_i(t_\ell)}{S_i(t_{\ell-1})}.$$

In order to compare the performance of our portfolios with the market portfolio, we set our initial wealth as $W(0) = \Sigma(0)$ and compare the evolutions of $\Sigma(\cdot)$ and $W(\cdot)$. Once the initial amount $W(0)$ invested in our portfolio is determined, the monetary value of the portfolio can be obtained recursively by the above equation. However, $W(\cdot)$ can be defined with the trading strategy $\varphi_i(\cdot)$ in (3.9) or (3.13);

$$W(\cdot) = \sum_{i=1}^{d} S_i(\cdot)\varphi_i(\cdot). \tag{7.1}$$

Then, the value $V^\varphi(\cdot, \mu)$ with respect to the market, defined as in (3.2) or represented as in (3.12), has another representation as the ratio between the money value of our portfolio and total market capitalization;

$$V^\varphi(\cdot) = \sum_{i=1}^{d} \varphi_i(\cdot)\mu_i(\cdot) = \sum_{i=1}^{d} \varphi_i(\cdot)\frac{S_i(\cdot)}{\Sigma(\cdot)} = \frac{W(\cdot)}{\Sigma(\cdot)},$$

and the expression 'value of trading strategy (or portfolio) with respect to the market' makes sense. Furthermore, the excess return rate $R^\varphi(\cdot)$ of the portfolio defined in (6.7) can be represented as

$$R^\varphi(\cdot) = \frac{V^\varphi(\cdot) - V^\varphi(0)}{V^\varphi(0)} = \frac{\frac{W(\cdot)}{\Sigma(\cdot)} - 1}{1} = \frac{W(\cdot) - \Sigma(\cdot)}{\Sigma(\cdot)} \; \big(= V^\varphi(\cdot) - 1\big),$$

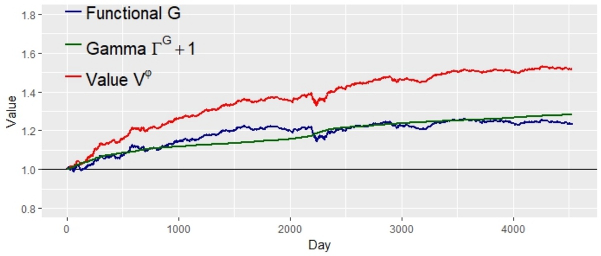
and the expression 'excess return rate with respect to the market' also makes sense. Here, $V^\varphi(0) = 1$ because we set $W(0) = \Sigma(0)$. In the last part of following subsection, we show the evolutions of $W(\cdot)$ of several portfolios to compare their performance.

7.2 Empirical results

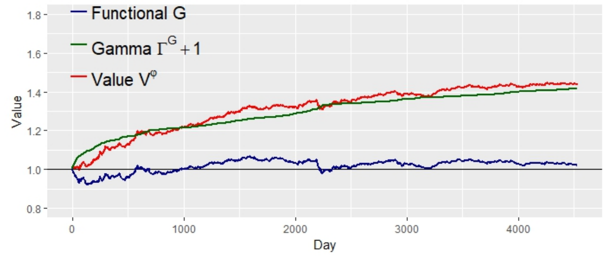
We first decompose the value functions $V^\varphi(\cdot)$ of trading strategies additively generated from the functionals G in entropic examples (Example 6.1, 6.3, 6.4, and 6.6) into the generating functional $G(\cdot, \mu, A)$ and the corresponding Gamma functional $\Gamma^G(\cdot)$. For easy comparison, we normalized all generating functionals so that $G(0, \mu, A) = 1$ holds, and shifted up the Gamma functionals by 1 in Figure 1.

Figure 1 confirms that all trading strategies additively generated in Section 6 outperform the market as the values V^φ (red lines in the figure) gradually increase. In sub-figures (a) and (b), the growth of the value V^φ comes from the growth of the Gamma functional. In contrast, even though the Gamma functional decreases, the value of trading strategy grows as the functional G increases substantially in sub-figures (c) and (d). In the sub-figure (d), we set the parameter $r = 5$ as it is the largest integer satisfying the equation (6.22); initial market weights data give us $\max_i \mu_i(0) = 0.065$ and $0.065 < 1/(5e)$ holds. We chose the same parameter $p = 9$ (See Remark 6.2) in all sub-figures for fair comparison, but this is a very sloppy choice of the parameter p for (a), (b), and (c). If we chose the value of p using

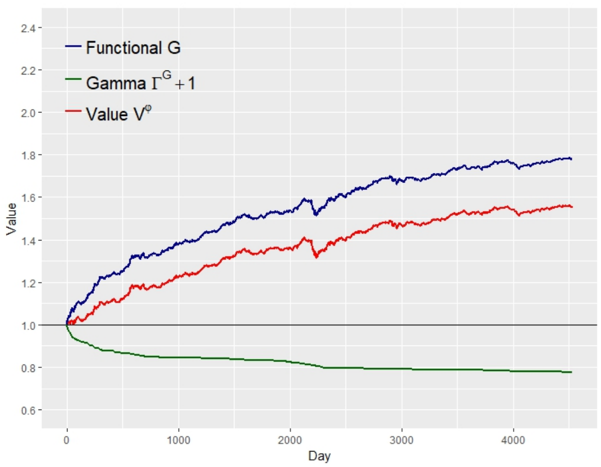
Figure 1: Decomposition of value function of additively generated trading strategies



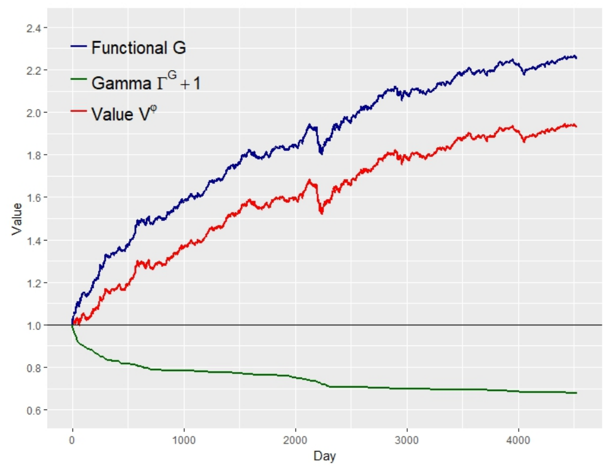
(a) Example 6.1
Original entropy, $p = 9$



(b) Example 6.3
Entropy with running maximum, $p = 9$

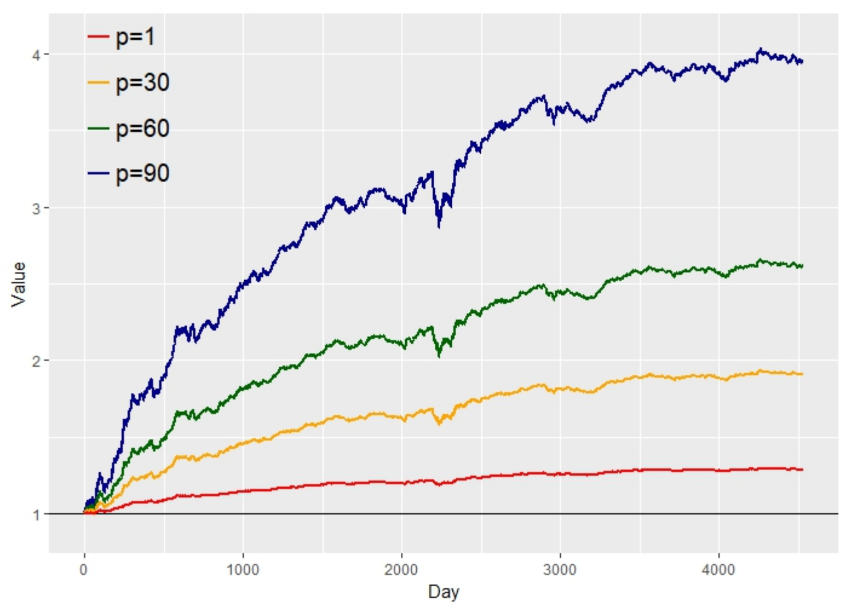


(c) Example 6.4
Entropy with running minimum, $p = 9$



(d) Example 6.6
Iterated entropy with running minimum, $p = 9$, $r = 5$

Figure 2: Value of additively generated trading strategies from Example 6.1 with different p values

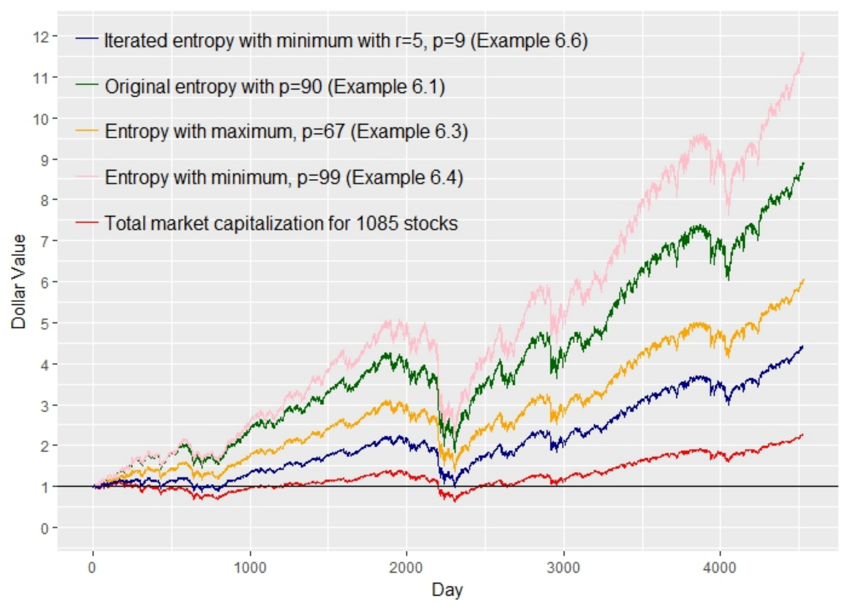


statistical estimation elaborately in each examples, the performance of portfolio would be improved, as Figure 2 represents in the case of Example 6.1.

Figure 2 shows the values of additively generated portfolios in Example 6.1 with different choices of the parameter p. We can verify that trading strategy with bigger value of p performs better as described in Remark 6.2. From the data, the Gibbs entropy $-\sum_{i=1}^{1085} \mu_i(t) \log \mu_i(t)$ of market weights ranged from 4.954 to 5.726 during 4528 days. Thus, $p = 90$ is a safe estimation of the parameter p which guarantees the non-negativity of the functional G in (6.2) as $\log 90 < 4.5 < 4.954$ holds.

Finally, 'dollar values' $W(\cdot)$ of portfolios from four examples of Section 6, which are defined as in (7.1), along with the total market value $\Sigma(\cdot)$ of $d = 1085$ stocks from the start of 2000 to the end of 2017, are illustrated in Figure 3. Dollar values are normalized by replacing $W(\cdot)$ by $W(\cdot)/W(0)$. In Figure 3, while the market capitalization had been approximately doubled during 18 years, the dollar values of all other portfolios had been grown more than 4.5 times. Parameters are appropriately chosen using statistical estimation in each portfolio.

Figure 3: (Normalized) Dollar values of portfolios over 18 years



8 Conclusion

Karatzas and Ruf (2017) introduced an alternative "additive" way of functional generation of trading strategies and compared it to the original "multiplicative" way of E.R. Fernholz. This new approach weakens the assumption on the asset prices from Itô processes to continuous semimartingales, characterizes the class of functions called "Lyapunov functions" which generate trading strategies leading to strong arbitrage with respect to the market, and gives a very simple sufficient condition for strong arbitrage. The present paper takes more generalized approaches to these two ways of functional generation. The results of this paper can be summarized as follows:

1. We show how to generate both additively and multiplicatively, trading strategies without any probabilistic assumptions on the market model. This is done by using the celebrated pathwise functional Itô calculus, and the only analytic assumption we impose is that the market weights admit continuous covariations in a pathwise sense.

2. We significantly extend the class of functionals which generate trading strategies. These functionals take the entire history of market weights and an additional non-anticipative argument of finite variation as the input. As this generating functional contains the whole information of past evolution, we can build up the trading strategies which not only depend on the current market weights, but also depend on the past of the market weights. Introducing an additional argument in the generating functional gives another extra flexibility in generating portfolios.

3. We also extend the class of functionals which generate additive strong relative arbitrage by giving a new sufficient condition leading to this arbitrage. The new condition allows the functional not be "Lyapunov", or concave with respect to the market weights, in order to generate strong relative arbitrage or to outperform the market portfolio in the long run. We also present empirical results of portfolios which indeed outperform the market.

While this paper generalizes the functional generation of portfolios in several aspects, we suggest some new questions. First, this paper assumes a 'closed market', in other words, the number of stocks d is fixed. In this respect, it fails to represent or resemble the real market. As explained in Remark 7.1, an 'open market' models the real world better, but nothing seems to be known on how to construct trading strategies in this open market. Secondly, the market weights in this paper should have finite second variation along a sequence of time partitions; can something be said, along the lives of Cont and Perkowski (2018), regarding price dynamics, or market weights, with finite p-th variation for $p > 2$?

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