# Mini Projects on Option Pricing

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## Common Modeling Setup

Under the risk-neutral measure, the underlying follows geometric Brownian motion (GBM):

$$dS_t = (r - q)\, S_t\, dt + \sigma S_t\, dW_t,$$

where $r$ is the risk-free rate, $q$ the dividend yield (or convenience yield), and $\sigma$ the volatility. I report prices in present value (discounted by $e^{-rT}$).

## Mini Project 1: European vs. American Options

- **Methods.** European options are priced with *Black–Scholes* (BS). Both European and American options are priced on a *Cox–Ross–Rubinstein (CRR)* binomial tree.
- **Validation.** As the number of steps $N$ grows, the *European* CRR price converges to BS. This checks numerical correctness.
- **Early Exercise.** For non-dividend-paying underlyings, an American call has no early-exercise premium (so American $\approx$ European). American puts can have early-exercise value, hence $V_{\text{Am put}} \geq V_{\text{Eu put}}$. I visualize the optimal-exercise region on the tree.

## Mini Project 2: Options on Futures (Black–76)

- **Cost-of-Carry.** The theoretical futures price is $F_0 = S_0 e^{(r-q)T}$.
- **Pricing.** European *options on futures* are priced with *Black–76*.
- **Implied Volatility.** Given a market price $V^{\text{mkt}}$, I solve for $\sigma$ in $V^{\text{model}}(\sigma) = V^{\text{mkt}}$ via a robust bisection routine.
- **Hedging Sketch.** A one-step P&L example illustrates how a futures position offsets option exposure.

## Mini Project 3: Asian Options (Arithmetic vs. Geometric)

- **Payoffs.** I consider options on the average price over $N$ observation dates. The *geometric-average* Asian option admits a closed form; the *arithmetic-average* one typically requires Monte Carlo (MC).
- **Control Variate.** Using the geometric Asian as control for the arithmetic Asian greatly reduces variance. The estimator is

$$\hat{V}_{\text{CV}} \;=\; \overline{X} \;-\; \beta\big(\overline{Y} - \mu_Y\big), \qquad \beta^\star \approx \frac{\text{Cov}(X,Y)}{\text{Var}(Y)},$$

where $X$ is the arithmetic payoff, $Y$ the geometric payoff simulated on the same paths, and $\mu_Y$ the analytic geometric price.

- **Illustrative Numbers (Call, $N = 12$, $M = 10{,}000$).**

| Method | Price | 95% CI |
|---|---|---|
| Geometric Asian (continuous) | 5.0865 | – |
| Geometric Asian (discrete, $N = 12$) | 5.4353 | – |
| Arithmetic Asian (plain MC) | 5.5276 | $\pm\,0.1608$ |
| Arithmetic Asian (control variate) | 5.4658 | $\pm\,0.0043$ |

This corresponds to roughly a $38\times$ tighter confidence interval (i.e., over $10^3\times$ lower variance) for the same number of paths.

## Mini Project 4: Barrier Options

- **Contracts.** I analyze knock-out and knock-in structures (up/down), with optional rebates.
- **Methods.** Where applicable, I reference *Reiner–Rubinstein* closed-form formulas as baselines. For discrete monitoring MC, I use *Brownian-bridge* corrections to account for barrier crossings between time steps; antithetic and control-variates are available to stabilize estimates.
- **Takeaway.** Barrier features materially alter pricing relative to vanilla options; careful treatment of monitoring frequency and crossing probabilities is essential.

## Key Takeaways

- **Validation**: CRR $\to$ BS for Europeans; geometric-Asian closed forms validate MC.
- **Efficiency**: Control variates (and antithetics) deliver large variance reductions.
- **Design Matters**: Early exercise, averaging, and barrier logic reshape price and risk.

*Note.* Parameters in the demos typically use $S_0$=100, $K$=100, $r\approx3\%$, $q\approx0$, $T$ around 0.5–1 year, and $\sigma$ set within the notebooks. Adapt as needed for your PDF.