The design of the estimator required a direct correspondence towards the system it tries to model and predict. With a Bayesian framework, we obtain a belief in order to form an effective estimation. Furthermore, there are several assumptions made to allow the estimator to be simple yet robust so that it may lead to useful estimator.

Figure XXX describes the overall computation workflow of the estimator.

3.1 The Bayesian Model

There are a series of fundamental assumptions on how NMR T2 relaxation data typically behaves to support a Bayesian estimation framework. Setting the general constraints that the model may operate on are essential so that the estimator may deliver usable results. Additionally, translating the model from previous literature into a compatible Bayesian context is allows for comparability.

3.1.1 Modelling the Measurement

The time domain T2 relaxation measurement is the starting point for a Bayesian estimator model. We assume the noise is additive, white (same magnitude for all frequencies), zero mean, and Gaussian. In addition, each time domain measurement is independent of each other – the noise from one measurement does not `leak’ to the next measurement. This means that the m vector has a diagonal covariance matrix to describe how certain we are of our m vector.

To borrow from the previous literature, we can hypothesize that the measurement is the result of a mapping of the true density function of relaxation times to the time domain by a kernel matrix. The kernel matrix maps a time constant to a time domain exponential decay. Therefore, we obtain a measurement vector from a density function that we are trying to find. This results in our final expression of the measurement noise in equation XXX that we use as part of our quotient in the Bayesian framework.

3.1.2 Density Function Model

The aim of the estimator is to obtain the density function of T2 relaxation conditioned on the measurement data. This is the posterior in this Bayesian framework. Obtaining this density function requires finding p(f) and p(m) in the framework as well.

**Modelling the prior:**

The prior’s model is a multivariate Gaussian. There are two components: a vector of magnitudes at different bins of T2 relaxation times; and a covariance matrix of the variance of each density value and cross-correlation between different relaxation bins. This portable model allows for a straightforward and computable framework. This also forms a significant design point of the system discussed in sections 3.1.3 and 3.2.

An important flaw of the Gaussian assumption is that a Gaussian distribution allows for negative relaxation times. This is not valid as it violates the non-negative assumption that is required for decaying exponentials in the measurement data.

**Modelling the quotient:**

In order for a directly computable density function estimation, all parts Bayes’ Theorem must be computable, including the probability of unconditioned measurements p(m). This can be achieved by marginalizing p(f)p(m|f) over all density functions. The resultant distribution from this marginalization results in equation XX.

**Analytical Expression:**

With the multiplication of the different components of Bayes’ Theorem, we obtain the following distribution of the posterior.

As the result is a Gaussian distribution, the most likely value density function is the mean itself. This immediately computable analytical expression has a lower computation load than the optimization frameworks in section XXX as well as combining different possible uncertainties in a measurement.

3.2 Construction of the Prior

The most essential component of the estimator is the integration of prior high quality experimental data into the estimator framework. The specific point of development is the Gaussian distribution prior of p(f). The strength of this technique lies in the analytical expression given in equation XXX.

The multivariate Gaussian of the density function has two design aspects:

1. The mean of potential density functions
2. The covariance between each T2 relaxation bin to model dependence and uncertainty.

High quality experimental data forms the basis for a prior in the Bayesian framework. Thirty NMR relaxation experimental density functions from Schlumberger Doll Research form the prior. These high quality measurements reflect true rock data that make the technique’s performance representative of typical application and use.

**One Dimensional Interpolation:**

For all of the estimates of the prior they must be compatible with the rest of the Bayesian framework. For example, there are 100 T2 relaxation bins for the prior but only 30 different divided T2 relaxation bins for the actual density function computation. Interpolation of the prior to the actual framework’s dimensionality is used to bridge between these two domains. In particular, Shape-preserving piecewise cubic interpolation (PCHIP) interpolation is used. The benefits of this technique over other candidates are:

1. Smooth interpolation between points – unlike linear interpolation
2. No oscillation between data points that may violate the non-negativity constraint of the density function – unlike splines
3. It is compatible with the non-uniform spacing the T2 relaxation bins – unlike cubic convolution
4. There is not constraint on computation time as interpolation is not required during time sensitive processes where the estimator is used.

3.2.1 Estimation of the mean of the prior

Computation of the mean of the prior involves taking the mean all of all of the experimental density functions over each T2 relaxation bin. This is the estimation of the expectation of all of the T2 density functions of porous media that are of interest for Schlumberger

3.2.2 Uniform Independent Estimation Covariance

The first iteration of the estimation of the covariance assumed that the uncertainty for each T2 relaxation bin was uniform and independent. This made the covariance equivalent to a scalar multiplied with an identity matrix. This simple assumption’s main flaw came no intrinsic indicator of the best estimation of the uniform covariance.

3.2.3 Density Function Independent Estimation Covariance

This estimation of the covariance of the prior density function takes into account the varying uncertainty for different T2 relaxation bins. The assumption of independence is carries so the covariance is a positive definite diagonal matrix (all of the non-zero values are positive and on the diagonal). Though more flexible than the method described bed in Section 3.2.2, there is no intrinsic value to minimize the error in the framework.

3.2.4 Density Function Dependent Estimation Covariance

Estimating the covariance of all of the experimental density functions yielded the most robust covariance for the prior. This modelled the dependence between different T2 relaxation bins in the density function. This extensive model also meant that the full scope of variance in the density function made it representative of it. Therefore, it was the best intrinsic indicator of density.

3.3 Integral Transform

The goal of the estimator is to estimate the bound fluid fraction of porous media sample, not the density function. To obtain this value requires the computation of an integral transforms of the estimated density function for the bound fluid volume and the porosity.

3.3.1 Sharp Bound Fluid Volume

The bound fluid volume is defined as the integration of the density function from T2=0 to T2=Tc. In the discretized version, this is equivalent to the dot product of the density function with a vector with ones for values below Tc and zeros for values above. There is no smooth transition from the bound fluid to the free fluid; hence, it is the sharp bound fluid volume.

The strength of using an integral transform like this is that it is explicit with what it considers bound fluid volume. The weakness of this however is that it does not have any tolerance for ‘leaking’ from one T2 relaxation bin to another, something that a noisy measurement can inflict onto an estimation of the density function. This brings into the picture another candidate – the tapered integral transform.

3.3.2 Tapered Bound Fluid Volume

Tapered integral transforms have a looser tolerance towards bound fluid volume. A looser tolerance allows for a more flexible estimator for noisy environments that may cause ‘leakage’ between T2 relaxation bins. The candidate integral transform for comparison with its sharp counterpart is the exponential Haar transform (EHT) proposed in Gruber et al 2013 []. The discretized version multiplies a discretized and scaled EHT with the density function to give the bound fluid volume.

3.3.3 Porosity

Calculating the porosity of the discretized density function is more straightforward. It is by definition the sum of all the T2 relaxation bins, an approximation of the integration of the density function over the entire T2 domain. There is no decision between what relaxation times to consider so it is more straightforward.

3.4 Metrics

A vital aspect of estimation is to have defined metrics to evaluate and compare it. The goal is for the evaluation of viability and effectiveness in a fair manner.

3.4.1 Error

The primary format of error evaluation root mean square error (RMSE). This is the optimal form of evaluation as it expectation of the bias, and the expectation of the variance. The RMSE combines these two in a balanced manner. The additional benefit of this metric is that the error is comparable with different cut off times so there can be a comparison for different cut off times – allowing for sensitivity analysis.

3.4.2 Computational Effort

Though not exhaustive, the computation time of the algorithm delivers a useful comparative perspective on algorithmic feasibility. For an algorithm to be feasible in the field, it should have a comparatively smaller computation time. The computer and its computational load are held constant and averaged over several computations to strengthen the validity of comparison.