Notes on MAT157: Analysis 1

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Chapter 1

The real numbers

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Section 1. Definition and Notations

1.1. Definition: The real number is a **complete, ordered, field.**

1.2. Definition: A **field** is a set F with two binary operations, $+$, and $\cdot$, such that,

(1). $a + (b + c) = (a + b) + c$

(2). $\exists 0 : a + 0 = 0 + a = a$

(3). $a + (-a) = (-a) + a = 0$

(4). $a + b = b + a$

(5). $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(6). $\exists 1 \neq 0 : a \cdot 1 = 1 \cdot a = a$

(7). $a \cdot a^(- 1) = a^(- 1) \cdot a = 1$ for $a \neq 0$

(8). $a \cdot b = b \cdot a$

(9). $a \cdot (b + c) = a \cdot b + a \cdot c$

1.3. Remark:

- 0 is unique. $(0 = 0 + 0\prime = 0\prime)$
- $-a$ is unique.
- *For all* a, b, $\quad (-a) \cdot b = -(ab)$
- *For all* a, b, $\quad (-a) \cdot (-b) = a \cdot b$
- *For all* a, b, $\quad a - b = b - a \iff a \cdot (1 + 1) = b \cdot (1 + 1)$

1.4. Theorem (Trasitivity of $\leq$): *For every $a, b, c \in \mathbb{R}$, if $a \leq b$ and $b \leq c$, then $a \leq c$.*

1.5. Definition: Given F is a field. F is an ordered field if there exist a subset $P \in F$ such that it is closed under addition and multiplcation and for every $a \in F$:

$$a = 0, a \in P, -a \in P$$

1.6. Remark: if $P \in F$ is an ordered field, then $1 \in P$

1.7. Definition: The **absolute value** of a is

$$|a| := \begin{cases} a & \text{if } a >= 0 \\ -a & \text{if } a < 0 \end{cases}$$

1.8. Theorem: *The triangle inequality:* $|x+y| \leq |x| + |y|$

1.9. Definition: Let A be a subset of an ordered field F. A is bounded from above if there is an element $b \in F$, called the **upper bound**, such that $b \geq a$ for all $a \in A$. A **least upper bound (supremum)** of A is an element $b_0 \in F$ such that it is an upper bound of A and if b is any upper bound, $b_0 \leq b$. Similarly, A is bounded from below if there exist $c \in A$, called the **lower bound**, such that $c \leq a$ for every $a \in A$. If c_0 is a lower bound of A such that $c_0 \geq c$ for every lower bound c of A, then c_0 is called the **greatest lower bound (infimum)**.

1.10. Proposition: *The supremum and infimum of a set is unique if it exists. inf* $(A) \leq$ *sup* (A) *if they both exist*

1.11. Definition: F is a complete ordered field if and only if for every subset $A \subset F, A \neq \emptyset$ such that A has an upper bound, then there exist $\sup(A)$ for $A \in F$

1.12. Theorem:
A: A complete ordered field exist
B: Complete ordered field is unique up to isomorphism

1.13. Corollary:
1) For every real number x*, there is an integer* k *such that* $k > x$
2)For any $\epsilon > 0$*, there is an* $n > 0$ *such that* $0 < \frac{1}{n} < \epsilon$
3) Let $x, y \in \mathbb{R}$*, if* $y - x > 1$*, then there is an* $k \in \mathbb{Z}$ *with* $x < k < y$*.*
4) $x < y \in \mathbb{R}$*, then there is a* $r \in \mathbb{Q}$ *such that* $x < r < y$

Section 2. Induction

2.1. Definition (Principal of mathematical induction):
Suppose $A \subset \mathbb{N}$ such that

(1). $1 \in A$

(2). $k \in A \Rightarrow k + 1 \in A$

then $A = \mathbb{N}$

2.2. Theorem (Well ordering principal): *Since* $A \in \mathbb{N}, A \neq \emptyset$*, then* A *has a least element.*

2.3. Proposition: *If m is any integer and n is a positive integer, there exist unique q and r such that $m = qn + r$ and $0 \leq r < n$*

Proof. Let $m, q, r \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, $A = \{m - qn | q \in \mathbb{Z}\} \cap \mathbb{N}$
Claim: A is non-empty.
If $m >= 0$, then $m \in A$ when $q = 0$.
If $m < 0$, then the element $m - qn$ such that q is the smallest integer such that $m - qn > 0$ is in the set.
Then by WO, there exist a smallest element $m - qn = r \in A$, s.t. $r \geq 0$.
Now all that's left is to prove $r < n$.
We prove by contradiction, assume $r \geq n$, then $m - qn \geq n$, $m - n(q + 1) \geq 0$, since this also satisfies the conditions to be in A and is smaller than r, a contradiction occurs as we claimed r to be the smallest element in A. □

2.4. Theorem: *Mathmatical induction $\Leftrightarrow$ Complete Mathmatical induction$\Leftrightarrow$ Well Ordering*

Proof. CMI $\Rightarrow$ MI: Suppose $C \in \mathbb{N}$, $1 \in C$, and $1, ..., k \in C \Rightarrow k+1 \in C$, let $A = \{k \in \mathbb{N} | 1, 2, ..., k \in C\}$. Assume CMI, so $1 \in A$, we want to show $k \in A \Rightarrow k+1 \in A$. Suppose $k \in A$, thus $1, ..., k \in C$, by definition of C, this implies $k + 1 \in C$, so $k + 1 \in A$.
MI $\Rightarrow$ CMI: Suppose we have a set $A = \{k \in \mathbb{N} | k \in A \Rightarrow k+1 \in A\}$, $1 \in A$, and $C = \{k \in \mathbb{N} | k \in A\}$, we want to show $C = \mathbb{N}$. Suppose $1, ..., k \in C$, thus $1, ..., k \in A \Rightarrow k + 1 \in A$, thus $k + 1 \in C$, and $C = \mathbb{N}$
CMI $\Rightarrow$ WO: Suppose we have $A = \{a \in \mathbb{N}\}, A \neq \emptyset, B = \{n \in \mathbb{N} | n \notin A\}$, we want to show $1 \in A$
Let $1, ..., k \in B$, so $1, ..., k \notin A$, then $k + 1 \notin A$, otherwise it is the least element of A. Then by strong induction, $\mathbb{N} \in B$ and $A = \emptyset$, which is a contradiction. Thus $MI \Rightarrow WO$
WO $\Leftarrow$ MI: Suppose we have a set P such that $1 \in P$ and $n \in P \Rightarrow n+1 \in P$, and S a non-empty set s.t. $S = \{n \in \mathbb{N} | n \notin P\}$. We want to show $P = \mathbb{N}$.
Suppse not, by WO, there exist a least element in S but it cannot be 1. Let k be its least element, then $k - 1 \notin S$, $k - 1 \in P$. But by definition if $k - 1 \in P \Rightarrow k \in P$, which is a contradiction. Thus $P = \mathbb{N}$ and $WO \Rightarrow MI$ □

2.5. Theorem (Fundamental Theorem of Arithmetic): *Every positive integer except 1 can be represented in one way up to isomorphism as a product of one or more primes.*

Proof. Base case: $2 = 2$, $3 = 3$, $4 = 2 \cdot 2$, $5 = 5$, clearly, first few numbers can be factored into primes.
Inductive step: Suppose every number $n \leq k$ can be factored in to product numbers. We consider $k + 1$, it is either a prime in which case we are done, or a composite number, thus it can be written as the product of 2 factors, so $k + 1 = n_1 n_2$ s.t. $n_1, n_2 \in \mathbb{Z}$ and $2 <= n_1, n_2 < k + 1$. By induction hypothesis, n_1 can be written in the form of $p_1 p_2 ... p_k$ and n_2 can be written in the form of $q_1 q_2 ... q_r$. Multiplying them, we get $p_1 q_1 p_2 q_2 ... p_k q_r$. Therefore, since $k + 1$ is a product of prime numbers, by strong induction, all $n \in \mathbb{Z}, n > 1$ can be written uniquely as a prime numbers.

□

Section 3. Functions

3.1. Definition: A function $f : A \to B$ is a subset $S \subseteq A \times B$ ($A \times B = \{(a, b)|a \in A, b \in B\}$), we write $f(a) = b$ if $(a, b) \in S$.
Such that

(1). we write $f(a) = b$ if $(a, b) \in S$

(2). $\forall a \in A, \exists (a, b) \in S$

(3). if $(a_1, b_1), (a_2, b_2) \in S$, then $a_1 = a_2 \Rightarrow b_1 = b_2$

3.2. Remark:
Domain: dom $f = \{a \in A|\exists b \in B, (a, b) \in S\}$
Range: ran $f = \{b \in B|\exists (a, b) \in S\}$

3.3. Theorem (Formula for an ellipse): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Proof. An ellipse is defined as the set of points whose distance from each of two "focus" adds up to the same value. For convinience, let them be $(-c, 0), (c, 0)$, and the sum of distances to be $2a$
Using the distance formula:

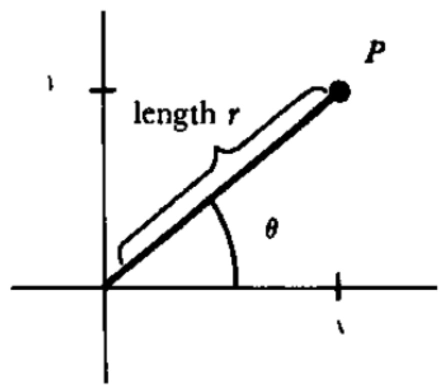
$$\sqrt{(x-(-c))^2+y^2}+\sqrt{(x-c)^2+y^2}=2a \tag{1.1}$$
$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2} \tag{1.2}$$
$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \tag{1.3}$$
$$4(cx-a^2)=-4a\sqrt{(x-c)^2+y^2} \tag{1.4}$$
$$c^2x^2-2cxa^2+a^4=a^2(x^2-2cx+c^2+y^2) \tag{1.5}$$
$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2) \tag{1.6}$$
$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \tag{1.7}$$
$$\text{we usually let } b=\sqrt{a^2-c^2} \text{ so the equation becomes} \tag{1.8}$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1.9}$$

□

3.4. Remark: The **hyperbola** is defined analogously, except we require the *difference* of the two distances to be constant
$\sqrt{(x-(-c))^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a, \implies \frac{x^2}{a^2}-\frac{y^2}{a^2-c^2}=1$
However, in this case, we must choose $c > a$, so $a^2 < c^2 = 0$, otherwise its a ellipse. So let $b = \sqrt{c^2-a^2}$, and get $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

<u>Polar Coordinate</u>

3.5. Remark:



(x, y) can be written as $(rcos\theta, rsin\theta)$
To convert from cartesian to polar coordinate, simply plug in $x = rcos\theta$ and $y = rsin\theta$. The polar coordinates.
Lets try to find an equation of an ellipse in polar coordinates. Let one of the focus be the origin, which makes our equation much nicer. We have our other focus at $(-2\epsilon a, 0)$. Since the total distance from a point (x, y) is $2a$. Let the distance from (x, y) to O be r, and to $(-2\epsilon a, 0)$ be $2a - r$.
The distance r from (x, y) to O is given by $r^2 = x^2 + y^2$. And