ECE-GY 6143
Homework 1

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Problem 1

Let $\{x_1, x_2, ..., x_n\}$ be a set of points in d-dimensional space. Suppose we wish to produce a single point estimate $\mu \in \mathbb{R}^d$ that minimizes the mean squared-error:

$$\frac{1}{n}(\|(x_1 - \mu)\|_2^2 + \|(x_2 - \mu)\|_2^2 + ... + \|(x_n - \mu)\|_2^2)$$

Find a closed form expression for μ and prove that your answer is correct.

Solution

Problem 2

Not all norms behave the same; for instance, the ℓ_1-norm of a vector can be dramatically different from the ℓ_2-norm, especially in high dimensions. Prove the following norm inequalities for d-dimensional vectors, starting from the definitions provided in class and lecture notes. (Use any algebraic technique/result you like, as long as you cite it.)

$$a. \|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\,\|x\|_\infty \tag{1}$$
$$b. \|x\|_\infty \leq \|x\|_1 \leq d\,\|x\|_\infty \tag{2}$$

Solution
Proof of $\|x\|_2 \leq \|x\|_1$

$$\|x\|_1^2 = (\sum_{i=1}^{d} x_i)^2 = \sum_{i=1}^{d} x_i^2 + 2\sum_{i<j}^{d} x_i x_j \tag{3}$$
$$\|x\|_1^2 \geq \sum_{i=1}^{d} x_i^2 = \|x\|_2^2 \tag{4}$$

Proof of $\|x\|_\infty \leq \|x\|_2$

$$\|x\|_2 = (\sum_{i=1}^{d} x_i^2)^{\frac{1}{2}} \geq (\max_{i=1\ldots d} (x_i)^2)^{\frac{1}{2}} \tag{5}$$
$$\|x\|_2 \geq \max_{i=1\ldots d} (x_i) \tag{6}$$
$$\|x\|_2 \geq \|x\|_\infty \tag{7}$$

Hence $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$

Now, proof of $\|x\|_2 \leq \sqrt{d}\,\|x\|_\infty$

$$\|x\|_2^2 = \sum_{i=1}^{d} |x_i|^2 \leq d \max_{i=1\ldots d}(|x_i|^2) = d\,\|x\|_\infty^2 \tag{8}$$
$$\|x\|_2 = (\sum_{i=1}^{d} |x_i|)^{\frac{1}{2}} \leq \sqrt{d}\,\|x\|_\infty \tag{9}$$

Finally, proof of $\|x\|_1 \leq d\,\|x\|_\infty$

$$\|x\|_1 = \sum_{i=1}^{d} x_i \tag{10}$$
$$\|x\|_\infty = \max_{i=1\ldots d} (x_i) \tag{11}$$
$$\sum_{i=1}^{d} x_i \leq \sum_{i=1}^{d} \max_{i=1\ldots d} (x_i) \tag{12}$$
$$\|x\|_1 = \sum_{i=1}^{d} x_i \leq d \max_{i=1\ldots d} (x_i) = d\,\|x\|_\infty \tag{13}$$

Problem 3

In this problem, you will practice using Python for exploratory data analysis.
Solution
See pdf or google colab.