Faculty of Science

Policy Gradient Methods

Reinforcement Learning

WE WANT A GRADIENT FOR A RL PROBLEM

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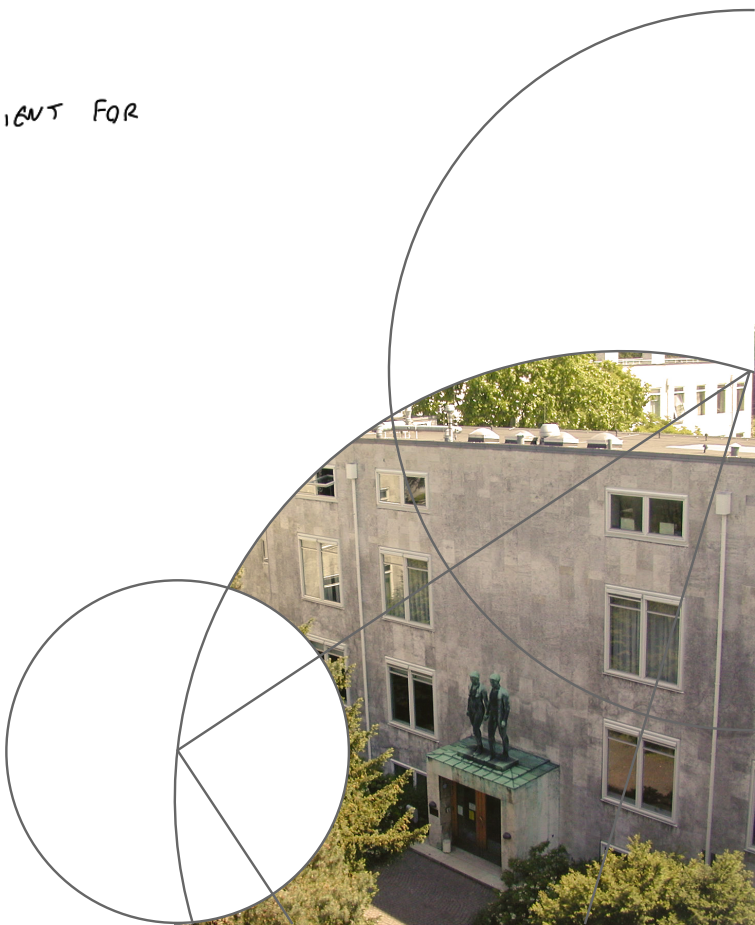

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Outline

Outline

Warmup: Monte Carlo methods

Crude Monte Carlo methods approximate an integral

$$\mathbb{E}_{p(z)}[f(z)] = \int f(z)p(z)\mathrm{d}z$$

by

$$\frac{1}{n}\sum_{i=1}^{n} f(z_i) \xrightarrow{n\to\infty} \mathbb{E}_{p(z)}[f(z)] \ ,$$

where

$$z_i \sim p(z)$$

for $i = 1, \ldots, n$.

Warmup: Importance sampling

For two distributions p and q with $p(z) > 0 \Rightarrow q(z) > 0$ we have

$$\mathbb{E}_{p(z)}[f(z)] = \int \frac{p(z)}{q(z)} f(z) q(z) \mathrm{d}z = \mathbb{E}_{q(z)}\left[\frac{p(z)}{q(z)} f(z)\right]$$

and thus

$$\frac{1}{n} \sum_{i=1}^{n} \frac{p(z_i)}{q(z_i)} f(z_i) \xrightarrow{n \to \infty} \mathbb{E}_{p(z)}[f(z)] ,$$

where $z_i \sim q(z)$ for $i = 1, \ldots, n$, that is, we approximate the expectation over p using samples from q.

Warmup: Control variates

Assume we want to estimate $\mathbb{E}[f]$ via Monte Carlo and are worried about the variance of our estimator.

We introduce a helper function (control variate) ϕ correlated with f for which we know $\overline{\phi} = \mathbb{E}[\phi]$ and write

$$\mathbb{E}[f] = \mathbb{E}[(f-\phi)] + \mathbb{E}[\phi] = \mathbb{E}[(f-\phi)] + \overline{\phi} = \mathbb{E}[(f-\phi) + \overline{\phi}] \ .$$

We have

$$\mathrm{Var}[(f-\phi+\overline{\phi}] = \mathrm{Var}[(f-\phi)] = \mathrm{Var}[f] - 2\,\mathrm{Cov}(f,\phi) + \mathrm{Var}[\phi] \ .$$

Thus, the control variate reduces the variance in the sense that $\mathrm{Var}[(f-\phi)] < \mathrm{Var}[(f)]$ if $2\,\mathrm{Cov}(f,\phi) > \mathrm{Var}[\phi]$.
Analogously, we have $\mathrm{Var}[(f+\phi)] < \mathrm{Var}[(f)]$ if $2\,\mathrm{Cov}(f,\phi) < \mathrm{Var}[\phi]$.

Warmup: "Log-derivative trick"

We have

PROBABILITY OF z WITH PARAMETERS θ

$$\nabla_\theta \log p(z;\theta) = \frac{\nabla_\theta p(z;\theta)}{p(z;\theta)} \Rightarrow \nabla_\theta p(z;\theta) = p(z;\theta)\nabla_\theta \log p(z;\theta)$$

which implies

$$\nabla_\theta \mathbb{E}_{p(z;\theta)}[f(z)] = \int f(z)\nabla_\theta p(z;\theta)\mathrm{d}z = \mathbb{E}_{p(z;\theta)}[f(z)\nabla_\theta \log p(z;\theta)] \ ,$$

where the RHS lends itself to Monte Carlo estimation via

$$\frac{1}{n}\sum_{i=1}^{n} f(z_i)\nabla_\theta \log p(z_i;\theta)$$

THERE'S NO DERIVATIVE OF f!

with $z_i \sim p(z)$ for $i = 1, \ldots, n$. $\nabla_\theta \log p(z;\theta)$ is called *score function*.

Outline

Notation

- Discrete time t, states S, actions A
- State-action-reward sequence $s_0, a_0, r_1, \ldots,$ with first time arriving in a terminal state in episodic tasks denoted by T (and for $t > T$: $r_t = 0$ and $s_t = s_T$), start state distribution p_{start}
- $P^a_{ss'} = \Pr\{s_{t+1} = s' \,|\, s_t = s, a_t = a\}$ and $R^a_{ss'} = \mathbb{E}\{r_{t+1} \,|\, s_t = s, s_{t+1} = s', a_t = a\}$
- $R^a_s = \mathbb{E}\{r_{t+1} \,|\, s_t = s, a_t = a\} = \sum_{s'} P^a_{ss'} R^a_{ss'}$
- $\pi(s, a, \boldsymbol{\theta}) = \pi(s, a) = \Pr\{a_t = a \,|\, s_t = s, \boldsymbol{\theta}\}$ → Parameters of the distribution
- $\Pr\{s \xrightarrow{k} x \,|\, \pi\}$: probability of going from state s to state x in k steps under policy π ($\Pr\{s \xrightarrow{0} s \,|\, \pi\} = 1$ and $\Pr\{s \xrightarrow{0} s' \,|\, \pi\} = 0$ for $s \neq s'$)

Stochastic vs. deterministic policy

- Toy Markov Decision Process (MDP) with $\mathcal{S} = \{s, s_T\}$, s_T is terminal state, s is start state, $\gamma = 1$, $\mathcal{A} = \{\text{left}, \text{right}\}$, all actions lead to T, $R_s^{\text{left}} = 1$, $R_s^{\text{right}} = 0$
- Deterministic policy:

$$\pi_\theta(s) = \begin{cases} \text{left} & \text{if } \theta \geq 0 \\ \text{right} & \text{otherwise} \end{cases}$$

- Stochastic policy:

$$\pi'_\theta(s) = \begin{cases} \text{left} & \text{with probability } \sigma(\theta) \\ \text{right} & \text{with probability } 1 - \sigma(\theta) \end{cases} \quad \text{with} \quad \sigma(\theta) = \frac{1}{1 + e^{-\theta}}$$

- $V^\pi(s) = \mathbb{I}[\theta \geq 0], \quad V^{\pi'}(s) = \sigma(\theta)$
- $\frac{\partial}{\partial\theta} V^{\pi'}(s) = (1 - \sigma(\theta))\sigma(\theta)$ ← This is differentiable
  $\frac{\partial}{\partial\theta} V^{\pi}(s) = \textbf{?}$ ← But what do I do here?

Introduction: Value function approaches to RL

- "Standard approach" to reinforcement learning (RL) is to
  - estimate a value function (V- or Q-function) and then
  - define a "greedy" policy on top of it.
- One may argue that this approach is
  - somehow "indirect" and
  - oriented towards deterministic policies.
- Problems:
  - "Strong causality" violated (small changes may have drastic effects)

Introduction: Policy gradient approaches to RL

- Model a stochastic policy by a function approximator (the "actor") with own parameters $\boldsymbol{\theta}$, for example for discrete action set

$$\pi(s, a \,|\, \boldsymbol{\theta}) = \frac{e^{h(s,a \,|\, \boldsymbol{\theta})}}{\sum_{a'} e^{h(s,a' \,|\, \boldsymbol{\theta})}}$$

  with *preferences* $h(s, a \,|\, \boldsymbol{\theta})$

- Adapt policy according to

$$\Delta\boldsymbol{\theta} \approx \alpha \nabla_{\boldsymbol{\theta}} J(\pi)$$

  where $J(\pi)$ is a performance measure of the policy π and α a positive step-size/learning-rate

- Subsumes known methods such as actor-critic approaches and the REINFORCE algorithms

Average reward formulation I

Expected reward per time step

$$J(\pi) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\{r_1 + \cdots + r_t \,|\, \pi\} = \sum_s \mu^\pi(s) \sum_a \pi(s,a) R_s^a$$

We assume that the stationary distribution μ^π of states under π exists and is independent of s_0 (e.g., the underlying Markov chain is finite and ergodic, i.e., $\exists k \in \mathbb{N}\, \forall s, s' \in S\, \exists k' \leq k : \Pr\{s \xrightarrow{k'} s' \,|\, \pi\} > 0$)

$$\mu^\pi(s) = \lim_{t\to\infty} \Pr\{s_t = s \,|\, s_0; \pi\} = \lim_{t\to\infty} \Pr\{s_t = s \,|\pi\}$$

$\mu^\pi(s)$ is also called the on-policy distribution and corresponds to the fraction of time spent in s under π.

Average reward formulation II

In general it holds

$$\Pr\{s_{t+1} = s' \,|\, \pi\} = \sum_s \Pr\{s_t = s \,|\, \pi\} \Pr\{s_{t+1} = s' \,|\, s_t = s, \pi\}$$

Stationarity

$$\mu^\pi(s') = \lim_{t\to\infty} \Pr\{s_t = s' \,|\, \pi\} = \lim_{t\to\infty} \Pr\{s_{t+1} = s' \,|\, \pi\}$$

implies for $t \to \infty$

$$\mu^\pi(s') = \sum_s \underbrace{\mu^\pi(s)}_{\Pr\{s_t=s\,|\pi\}} \underbrace{\sum_a \pi(s,a) P^a_{ss'}}_{\Pr\{s_{t+1}=s' \,|\, s_t=s,\pi\}}$$

Average reward formulation III

Let's redefine

$$Q^\pi(s,a) = \sum_{t=1}^{\infty} \mathbb{E}\{r_t - J(\pi) \,|\, s_0 = s, a_0 = a; \pi\}$$

and with

$$V^\pi(s) = \sum_a \pi(s,a) Q^\pi(s,a)$$

we have

$$\begin{aligned} Q^\pi(s,a) &= \sum_{t=1}^{\infty} \mathbb{E}\{r_t - J(\pi) \,|\, s_0 = s, a_0 = a, \pi\} \\ &= R_s^a - J(\pi) + \sum_{s'} P_{ss'}^a V^\pi(s') \,. \end{aligned}$$

This is actually better referred to as some type of advantage (A, see below) under stationary assumption.

Start-state formulation

Goal is to maximize the expected return

$$J(\pi) = \mathbb{E}\left\{ \sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_0, \pi \right\}$$

with $\gamma \in [0, 1]$, $\gamma = 1$ only for episodic tasks, and

$$Q^{\pi}(s, a) = \mathbb{E}\left\{ \sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k} \,\middle|\, s_t = s, a_t = a, \pi \right\} \quad .$$

On-policy distribution in episodic tasks

- In episodic tasks, on-policy distribution $\mu^\pi(s)$ depends on initial states, i.e., the start state distribution $p_{\text{start}}(s)$.
- Let $\eta^\pi(s)$ denote the number of time steps spent, on average, in s in a single episode:

$$\eta^\pi(s) = \mathbb{E}_{s_0 \sim p_{\text{start}}} \left[\sum_{k=0}^{\infty} \Pr\{ s_0 \xrightarrow{k} s \mid \pi \} \right]$$

- Time is spent in a state s if episodes start in s, or if transitions are made into s from a preceding state s' in which time is spent:

$$\eta^\pi(s) = p_{\text{start}}(s) + \sum_{s'} \eta^\pi(s') \sum_a P^a_{s's} \pi(a, s')$$

  This system of equations can be solved for $\eta(s)$.
- $\mu^\pi(s)$ is then the fraction of time spent in each state:

$$\mu^\pi(s) = \eta^\pi(s) / \sum_{s'} \eta^\pi(s')$$

Outline

Policy gradient theorem, average reward

Theorem

For any MDP, in average-reward formulation

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \sum_s \mu^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a) \ .$$

- No $\nabla_{\boldsymbol{\theta}} \mu^\pi(s)$ terms $\longrightarrow$ THAT'S IMPORTANT!
- If s is sampled following π, then

$$\sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a)$$

is an unbiased estimate of $\nabla_{\boldsymbol{\theta}} J(\pi)$

Policy gradient theorem, start-state formulation

Theorem

For any MDP, in start-state formulation

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \sum_s \eta_\gamma^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a)$$

$$\propto \mathbb{E}_{\substack{\text{state-action} \\ \text{sequences} \\ \text{following } \pi}} \left[\sum_{k=0}^{\infty} \gamma^k \frac{1}{\pi(s_k, a_k)} \nabla_{\boldsymbol{\theta}} \pi(s_k, a_k) Q^\pi(s_k, a_k) \right] .$$

- Here we define

$$\eta_\gamma^\pi(s) = \mathbb{E}_{s_0 \sim p_{\text{start}}} \left[\sum_{k=0}^{\infty} \gamma^k \Pr\{ s_0 \xrightarrow{k} s \,|\, \pi \} \right] .$$

Not normalized ($\sum_s \eta_\gamma^\pi(s) \neq 1$) for $\gamma < 1$, a factor $(1-\gamma)$ is missing.

REINFORCE algorithm

- If s is sampled from distribution following π and $\gamma = 1$, then

$$\sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a)$$

  is an unbiased estimate of $\nabla_{\boldsymbol{\theta}} J(\pi)$ – note equal weighting of actions (uniform distribution)

- If s and a are sampled from distribution following π, then

$$\frac{1}{\pi(s,a)}$$

  re-weights the samples ($\rightarrow$ importance weighting)

- $Q^\pi(s,a)$ is usually unknown; $Q^\pi(s_t,a_t)$ can be estimated using actual returns $R_t = \sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k}$ or $R_t = \sum_{k=1}^{T-t} r_{t+k}$ (for a finite episode of length T) as estimates

REINFORCE pseudocode

REINFORCE algorithm:

$$\Delta\boldsymbol{\theta}_t \propto \gamma^t \frac{1}{\pi(s_t, a_t)} \nabla_{\boldsymbol{\theta}} \pi(s_t, a_t) R_t \quad \overset{\text{“log derivative trick”}}{=} \quad \gamma^t \underbrace{\nabla_{\boldsymbol{\theta}} \ln \pi(s_t, a_t)}_{\text{“score function”}} R_t$$

Algorithm 1: REINFORCE

Input: differential policy π parameterized by $\boldsymbol{\theta}$, learning rate $\alpha > 0$, initial policy parameters $\boldsymbol{\theta}$

repeat
  Generate episode $s_0, a_0, r_1, \ldots, s_{T-1}, a_{T-1}, r_T$
  foreach $t = 1, \ldots, T-1$ **do**
    $R_t = \sum_{k=1}^{T-t} \gamma^{k-1} r_{t+k}$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha R_t \gamma^t \nabla_{\boldsymbol{\theta}} \ln \pi(s_t, a_t)$
until *stopping criterion is met*

Adding a baseline (Because it was really unstable)

The policy gradient theorem can be generalized, in either average-reward or start-state formulations, to include a baseline, e.g.,

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \sum_s \mu^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) \left(Q^\pi(s,a) - b(s)\right) \quad ,$$

where $b(s) : S \to \mathbb{R}$ is an arbitrary baseline function.

HA4?

$\phi(s) = \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) b(s)$ acts as a control variate. Note $\mathbb{E}[\phi] = 0$.

A possible choice of $b(s)$ would be some estimate of the value function $V^\pi(s)$. We call

$$A^\pi(s,a) = Q^\pi(s,a) - V^\pi(s)$$

the *advantage function*.

Proof policy gradient, start-state, undiscounted I

We start by assuming $\gamma = 1$.

$$\nabla_{\boldsymbol{\theta}} V^{\pi}(s) = \nabla_{\boldsymbol{\theta}} \sum_a \pi(s,a) Q^{\pi}(s,a)$$

$$\overset{\text{product rule}}{=} \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \nabla_{\boldsymbol{\theta}} Q^{\pi}(s,a) \right]$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \nabla_{\boldsymbol{\theta}} \left[R_s^a + \sum_{s'} P_{ss'}^a V^{\pi}(s') \right] \right]$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \sum_{s'} P_{ss'}^a \nabla_{\boldsymbol{\theta}} V^{\pi}(s') \right]$$

Proof policy gradient, start-state, undiscounted II

$$\nabla_{\boldsymbol{\theta}} V^{\pi}(s) = \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \sum_{s'} P^a_{ss'} \nabla_{\boldsymbol{\theta}} V^{\pi}(s') \right]$$

$$= \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \nabla_{\boldsymbol{\theta}} V^{\pi}(s')$$

$$= \Pr\{s \xrightarrow{0} s \,|\, \pi\} \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \sum_{s'} \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \nabla_{\boldsymbol{\theta}} V^{\pi}(s')$$

$$= \sum_{s'} \left[\Pr\{s \xrightarrow{0} s' \,|\, \pi\} \sum_a \nabla_{\boldsymbol{\theta}} \pi(s',a) Q^{\pi}(s',a) + \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \nabla_{\boldsymbol{\theta}} V^{\pi}(s') \right]$$

$$= \sum_{s'} \sum_{k=0}^{\infty} \Pr\{s \xrightarrow{k} s' \,|\, \pi\} \sum_a \nabla_{\boldsymbol{\theta}} \pi(s',a) Q^{\pi}(s',a)$$

Proof policy gradient, start-state: Unrolling

A STEP APPLIED ONCE CAN ACTUALLY BE APPLIED n TIMES TO INFINITY →

Closer look at last step on previous slide:

$$\sum_{s'} \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \nabla_{\boldsymbol{\theta}} V^{\pi}(s')$$

$$= \sum_{s'} \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \left[\sum_{a} \nabla_{\boldsymbol{\theta}} \pi(s', a) Q^{\pi}(s', a) + \sum_{s''} \Pr\{s' \xrightarrow{1} s'' \,|\, \pi\} \nabla_{\boldsymbol{\theta}} V^{\pi}(s'') \right]$$

$$= \sum_{s'} \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \left[\sum_{a} \nabla_{\boldsymbol{\theta}} \pi(s', a) Q^{\pi}(s', a) \right]$$

$$+ \sum_{s'} \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \left[\sum_{s''} \Pr\{s' \xrightarrow{1} s'' \,|\, \pi\} \nabla_{\boldsymbol{\theta}} V^{\pi}(s'') \right]$$

$$= \sum_{s'} \Pr\{s \xrightarrow{1} s' \,|\, \pi\} \left[\sum_{a} \nabla_{\boldsymbol{\theta}} \pi(s', a) Q^{\pi}(s', a) \right] + \sum_{s''} \Pr\{s \xrightarrow{2} s'' \,|\, \pi\} \nabla_{\boldsymbol{\theta}} V^{\pi}(s'')$$

etc.

Proof policy gradient, start-state, undiscounted III

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \nabla_{\boldsymbol{\theta}} \mathbb{E}\left\{ \sum_{t=1}^{\infty} r_t \,\middle|\, \pi \right\} = \mathbb{E}_{s_0 \sim p_{\text{start}}} \left[\nabla_{\boldsymbol{\theta}} V^{\pi}(s_0) \right]$$

$$\nabla_{\boldsymbol{\theta}} V^{\pi}(s_0) \overset{\text{slide 25}}{=} \sum_s \underbrace{\sum_{k=0}^{\infty} \Pr\{s_0 \xrightarrow{k} s \,|\, \pi\}}_{\eta^{\pi}(s), \text{ see slide 17}} \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a)$$

Thus we have (note equal weighting of actions in first line):

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \mathbb{E}_{s_0 \sim p_{\text{start}}} \left[\sum_s \sum_{k=0}^{\infty} \Pr\{s_0 \xrightarrow{k} s \,|\, \pi\} \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) \right]$$

$$\propto \mathbb{E}_{\substack{\text{state-action} \\ \text{sequences} \\ \text{following } \pi}} \left[\sum_{k=0}^{\infty} \frac{1}{\pi(s_k, a_k)} \nabla_{\boldsymbol{\theta}} \pi(s_k, a_k) Q^{\pi}(s_k, a_k) \right]$$

Adding discounting: Time and ensemble average

- Consider an MDP $\mathcal{M}_\gamma$ with discount factor $\gamma \in]0,1[$ leading to the expected return for some policy:

$$J_{\mathcal{M}_\gamma}(\pi) = \mathbb{E}_{\mathcal{M}_\gamma}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t\right]$$

- Let $\mathcal{M}_1$ be the undiscounted version of that MDP.
- Now consider the MDP $\mathcal{M}_{1,\gamma}$, which is the same as $\mathcal{M}_1$ except that after each action (and receiving the reward) the process is terminated with probability of $1-\gamma$ by going to a new terminal state s_{dummy}:

$$J_{\mathcal{M}_{1,\gamma}}(\pi) = \sum_{l=0}^{\infty} (1-\gamma)\gamma^l \mathbb{E}_{\mathcal{M}_1}[r_1 + \cdots + r_{l+1}]$$

$(1-\gamma)\gamma^l$ is the probability that the transition to s_{dummy} happens after l steps.

Time and ensemble average II

$$
\begin{aligned}
J_{\mathcal{M}_{1,\gamma}}(\pi) &= \sum_{l=0}^{\infty}(1-\gamma)\gamma^l \mathbb{E}_{\mathcal{M}_1}[r_1 + \cdots + r_{l+1}] \\
&= \mathbb{E}_{\mathcal{M}_1}\left[\sum_{l=0}^{\infty}(1-\gamma)\gamma^l [r_1 + \cdots + r_{l+1}]\right] \\
&= \mathbb{E}_{\mathcal{M}_1}\left[\sum_{t=1}^{\infty}(1-\gamma)\sum_{i=t}^{\infty}\gamma^{i-1} r_t\right]
\end{aligned}
$$

First sum picks the time step t, second sum accumulates r_t terms:

$$
\begin{aligned}
&(1-\gamma)\gamma^0[r_1] \\
&(1-\gamma)\gamma^1[r_1 + r_2] \\
&(1-\gamma)\gamma^2[r_1 + r_2 + r_3] \\
&\vdots
\end{aligned}
$$

Time and ensemble average III

We have ($\rightarrow$ geometric series):

$$(1-\gamma)\sum_{t=1}^{\infty}\sum_{i=t}^{\infty}\gamma^{i-1}r_t = \sum_{t=1}^{\infty}\gamma^{t-1}r_t(1-\gamma)\sum_{i=0}^{\infty}\gamma^{i} = \sum_{t=1}^{\infty}\gamma^{t-1}r_t$$

and thus:

$$J_{\mathcal{M}_{1,\gamma}}(\pi) = J_{\mathcal{M}_{\gamma}}(\pi)$$

As this holds for all start-state distributions, this implies

$$V^{\pi}_{\mathcal{M}_{1,\gamma}}(s) = V^{\pi}_{\mathcal{M}_{\gamma}}(s)$$

for all states s. As $\mathcal{M}_{1,\gamma}$ differs from $\mathcal{M}_1$ only *after* taking each action and receiving the corresponding reward, this implies $Q^{\pi}_{\mathcal{M}_{1,\gamma}}(s,a) = Q^{\pi}_{\mathcal{M}_{\gamma}}(s,a)$ for all state-action pairs.

Policy gradient theorem discounted case I

We derive the policy gradient theorem for $\mathcal{M}_\gamma$ using $\mathcal{M}_{1,\gamma}$ and $\mathcal{M}_1$ and apply the undiscounted start-state version to $\mathcal{M}_{1,\gamma}$.

If $\Pr\{s_0 \xrightarrow{k} s \,|\, \pi\}$ is the probability of going from s_0 to s in k steps in $\mathcal{M}_1$, then the corresponding probability in $\mathcal{M}_{1,\gamma}$ is $\gamma^k \Pr\{s_0 \xrightarrow{k} s \,|\, \pi\}$.

Thus last step of the proof on slide 27 for $\mathcal{M}_{1,\gamma}$ becomes:

$$\mathbb{E}_{s_0 \sim p_{\text{start}}} [\nabla_{\boldsymbol{\theta}} V^\pi(s_0)] =$$

$$\mathbb{E}_{s_0 \sim p_{\text{start}}} \Big[\sum_s \sum_{k=0}^{\infty} \underbrace{\gamma^k \underbrace{\Pr\{s_0 \xrightarrow{k} s \,|\, \pi\}}_{\text{probability under } \mathcal{M}_1}}_{\text{probability under } \mathcal{M}_{1,\gamma}} \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a) \Big] =$$

$$\mathbb{E}_{s_0 \sim p_{\text{start}}} \left[\sum_s \eta_\gamma^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a) \right]$$

Policy gradient theorem discounted case II

And in the same way:

$$\mathbb{E}_{s_0 \sim p_{\text{start}}} [\nabla_{\boldsymbol{\theta}} V^{\pi}(s_0)] =$$

$$\mathbb{E}_{s_0 \sim p_{\text{start}}} \Big[\sum_{s} \sum_{k=0}^{\infty} \underbrace{\gamma^k \underbrace{\Pr\{s_0 \xrightarrow{k} s \,|\, \pi\}}_{\text{probability under } \mathcal{M}_1}}_{\text{probability under } \mathcal{M}_{1,\gamma}} \sum_{a} \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) \Big] \propto$$

$$\mathbb{E}_{\substack{\text{state-action} \\ \text{sequences} \\ \text{following } \pi \text{ on } \mathcal{M}_\gamma}} \left[\sum_{k=0}^{\infty} \gamma^k \frac{1}{\pi(s_k, a_k)} \nabla_{\boldsymbol{\theta}} \pi(s_k, a_k) Q^{\pi}(s_k, a_k) \right]$$

Note: $\Pr\{s_0 \xrightarrow{k} s \,|\, \pi\}$ is the same for $\mathcal{M}_\gamma$ and $\mathcal{M}_1$ and Q^{π} is the same for $\mathcal{M}_\gamma$ and $\mathcal{M}_1$ as can be shown using the "time and ensemble average" approach as before

Proof policy gradient, average reward I

$$\nabla_{\boldsymbol{\theta}} V^{\pi}(s) = \nabla_{\boldsymbol{\theta}} \sum_a \pi(s,a) Q^{\pi}(s,a)$$

$$\overset{\text{product rule}}{=} \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \nabla_{\boldsymbol{\theta}} Q^{\pi}(s,a)\right]$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \nabla_{\boldsymbol{\theta}} \left[R_s^a - J(\pi) + \sum_{s'} P_{ss'}^a V^{\pi}(s')\right]\right]$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \left[-\nabla_{\boldsymbol{\theta}} J(\pi) + \sum_{s'} P_{ss'}^a \nabla_{\boldsymbol{\theta}} V^{\pi}(s')\right]\right]$$

Proof policy gradient, average reward II

$$\nabla_{\boldsymbol{\theta}} V^{\pi}(s) = \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \left[-\nabla_{\boldsymbol{\theta}} J(\pi) + \sum_{s'} P^a_{ss'} \nabla_{\boldsymbol{\theta}} V^{\pi}(s') \right] \right] \Rightarrow$$

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a) + \pi(s,a) \sum_{s'} P^a_{ss'} \nabla_{\boldsymbol{\theta}} V^{\pi}(s') \right] - \nabla_{\boldsymbol{\theta}} V^{\pi}(s) \Rightarrow$$

$$\sum_s \mu^{\pi}(s) \nabla_{\boldsymbol{\theta}} J(\pi) = \sum_s \mu^{\pi}(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^{\pi}(s,a)$$
$$+ \sum_s \mu^{\pi}(s) \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \nabla_{\boldsymbol{\theta}} V^{\pi}(s') - \sum_s \mu^{\pi}(s) \nabla_{\boldsymbol{\theta}} V^{\pi}(s)$$

Proof policy gradient, average reward III

$$\sum_s \mu^\pi(s) \nabla_{\boldsymbol{\theta}} J(\pi) = \sum_s \mu^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a)$$

$$+ \underbrace{\sum_s \mu^\pi(s) \sum_a \pi(s,a) \sum_{s'} P^a_{ss'}}_{\mu^\pi(s), \text{ see slide 14}} \nabla_{\boldsymbol{\theta}} V^\pi(s') - \sum_s \mu^\pi(s) \nabla_{\boldsymbol{\theta}} V^\pi(s)$$

$$= \sum_s \mu^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a)$$

$$+ \sum_s \mu^\pi(s) \nabla_{\boldsymbol{\theta}} V^\pi(s) - \sum_s \mu^\pi(s) \nabla_{\boldsymbol{\theta}} V^\pi(s)$$

$$\nabla_{\boldsymbol{\theta}} J(\pi) = \sum_s \mu^\pi(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi(s,a) Q^\pi(s,a)$$

Example: Softmax policy

Consider vector of features ($\rightarrow$ neural network) $\boldsymbol{\phi}(s,a), \forall a \in A, s \in S$; policy is a Gibbs distribution in a linear combination of the features

$$\pi(s,a) = \frac{e^{\boldsymbol{\theta}^\mathsf{T}\boldsymbol{\phi}(s,a)}}{\sum_b e^{\boldsymbol{\theta}^\mathsf{T}\boldsymbol{\phi}(s,b)}} .$$

Alternatively, for discrete actions, consider a feature map $\boldsymbol{\phi}$ for the features and

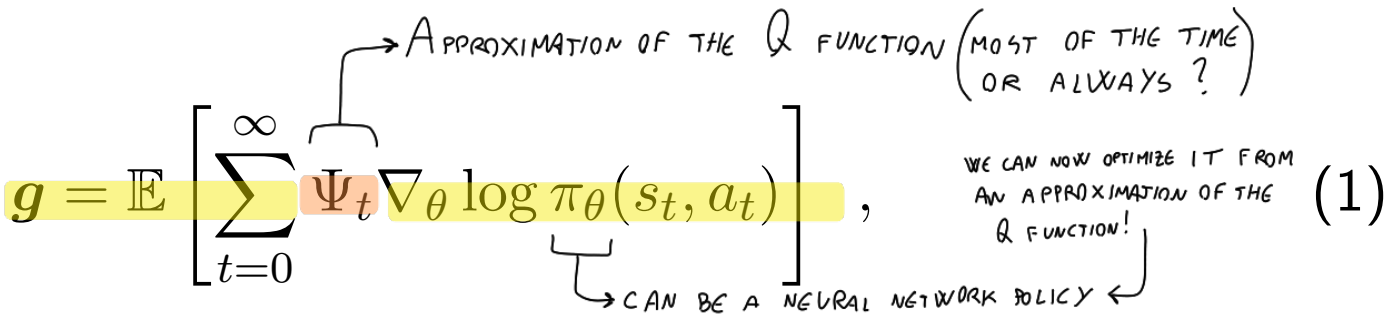
$$\pi(s,a) = \frac{e^{\boldsymbol{\theta}_a^\mathsf{T}\boldsymbol{\phi}(s)}}{\sum_b e^{\boldsymbol{\theta}_b^\mathsf{T}\boldsymbol{\phi}(s)}}$$

with parameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{|A|})$.

Outline

Policy gradients in the wild

In the wild, you will find several expressions for the policy gradient, which have the form



where Ψ_t may be one of the following (Schulman et al., 2016):

1. $\sum_{t=0}^{\infty} r_t$: total reward of the trajectory
2. $\sum_{t'=t}^{\infty} r_{t'}$: reward following action a_t
3. $\sum_{t'=t}^{\infty} r_{t'} - b(s_t)$: baselined version of previous formula
4. $Q^\pi(s_t, a_t)$: state-action value function
5. $A^\pi(s_t, a_t)$: advantage function
6. $r_t + V^\pi(s_{t+1}) - V^\pi(s_t)$: temporal difference (TD) residual

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