Stochastic Bandits
The UCB algorithm

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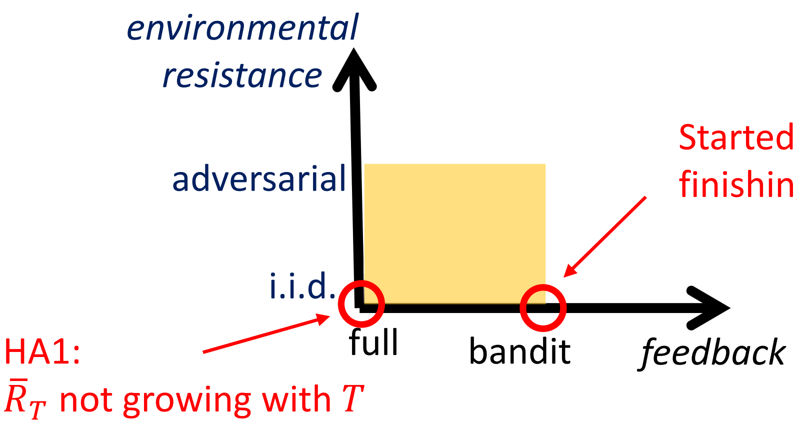
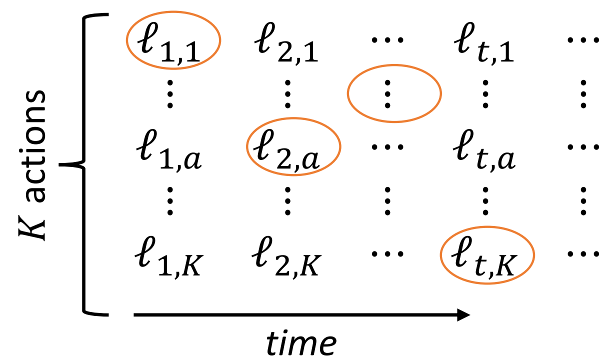
Quick recap of the last lecture





- Regret: $R_T = \sum_{t=1}^T \ell_{t,A_t} - \min_a \sum_{t=1}^T \ell_{t,a}$
- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^T \ell_{t,a}\right]$
- Pseudo-regret: $\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^T \ell_{t,a}\right] = \mathbb{E}\left[\sum_{t=1}^T \ell_{t,A_t}\right] - T\mu^*$

$$= \sum_{a=1}^K \Delta(a)\mathbb{E}[N_T(a)]$$

Lower Confidence Bound (LCB) algorithm for losses
(Originally Upper Confidence Bound (UCB) for rewards)
("Optimism in the face of uncertainty" approach)

- Define $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{3\ln t}{2N_{t-1}(a)}}$ lower confidence bound
  - (We will show that with high probability $L_t^{CB}(a) \le \mu(a)$ for all t)

- LCB Algorithm:
  - Play each arm once
  - For $t = K+1, K+2, \ldots$:
    - Play $A_t = \arg\min_a L_t^{CB}(a)$

- No knowledge of T
- No knowledge of Δ
- Works for any K

Rewards ↔ Losses

$\ell_{t,a} = 1 - r_{t,a}$

$r_{t,a} = 1 - \ell_{t,a}$

- Theorem:

$$\bar{R}_T \le 6 \sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)} + \left(1 + \frac{\pi^2}{3}\right)\sum_a \Delta(a)$$

DEPENDS ON:

- TIMESTAMP t
- EMPIRICAL MEAN $\hat{\mu}_{t-1}(a)$
- NUMBER OF TIMES THAT a WAS PICKED $N_{t-1}(a)$

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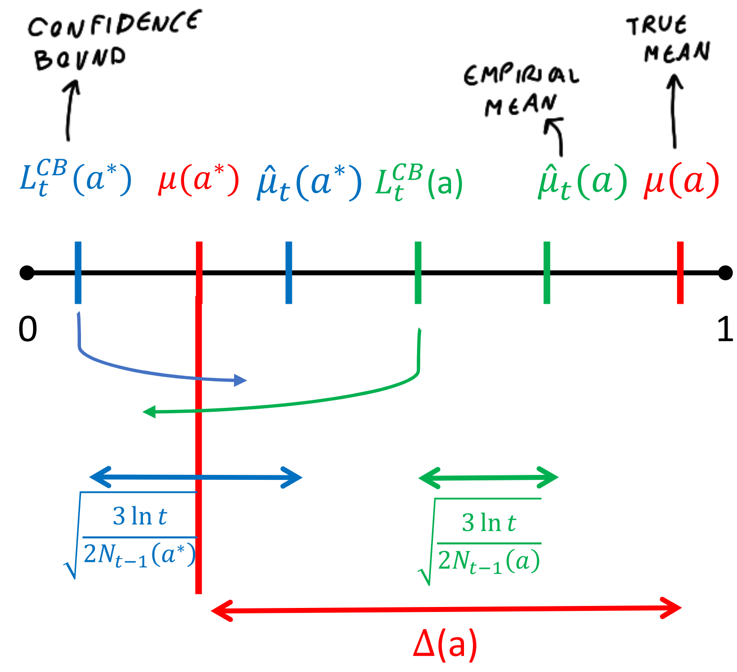
- Theorem:

$$\bar{R}_T \leq 6 \sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)} + \left(1 + \frac{\pi^2}{3}\right) \sum_a \Delta(a)$$

- Proof:
  - $\bar{R}_T = \sum_{a=1}^K \Delta(a) \mathbb{E}[N_T(a)]$
  - When can we play $a \neq a^*$?
  - Bound the expected number of times $L_t^{CB}(a) \leq L_t^{CB}(a^*)$



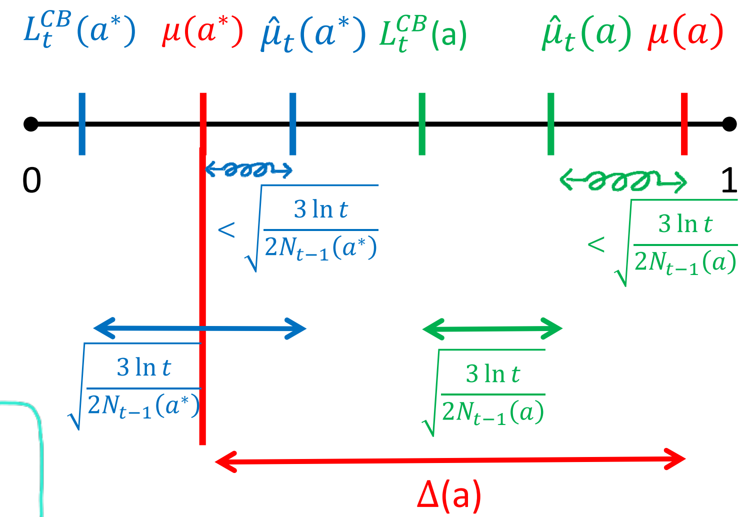
Proof

- $\bar{R}_t(a) = \sum_a \Delta(a)\mathbb{E}[N_T(a)]$
- $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{3\ln t}{2N_{t-1}(a)}}$

- Bound the expected number of times $L_t^{CB}(a) \le L_t^{CB}(a^*)$

- The expected number of times $L_t^{CB}(a) \le L_t^{CB}(a^*)$ is bounded by
    1. The expected number of times $L_t^{CB}(a^*) \ge \mu(a^*)$
    2. Plus expected the number of times $L_t^{CB}(a) \le \mu(a^*)$

Proof continued



1. The expected number of times $L_t^{CB}(a^*) \geq \mu(a^*)$ is bounded by

The expected number of times $\hat{\mu}_t(a^*) \geq \mu(a^*) + \sqrt{\frac{3 \ln t}{2N_{t-1}(a^*)}}$

2. The expected the number of times $L_t^{CB}(a) \leq \mu(a^*)$ is bounded by

2.1 The expected number of times $\hat{\mu}_t(a) \leq \mu(a) - \sqrt{\frac{3 \ln t}{2N_{t-1}(a)}}$

"THE CONCENTRATION IS "UNDER CONTROL" " (?)

2.2 If $\hat{\mu}_t(a) > \mu(a) - \sqrt{\frac{3 \ln t}{2N_{t-1}(a^*)}}$ then

$$L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{3 \ln t}{2N_{t-1}(a)}} > \mu(a) - 2\sqrt{\frac{3 \ln t}{2N_{t-1}(a)}} = \mu(a^*) + \Delta(a) - \sqrt{\frac{6 \ln t}{N_{t-1}(a)}}$$

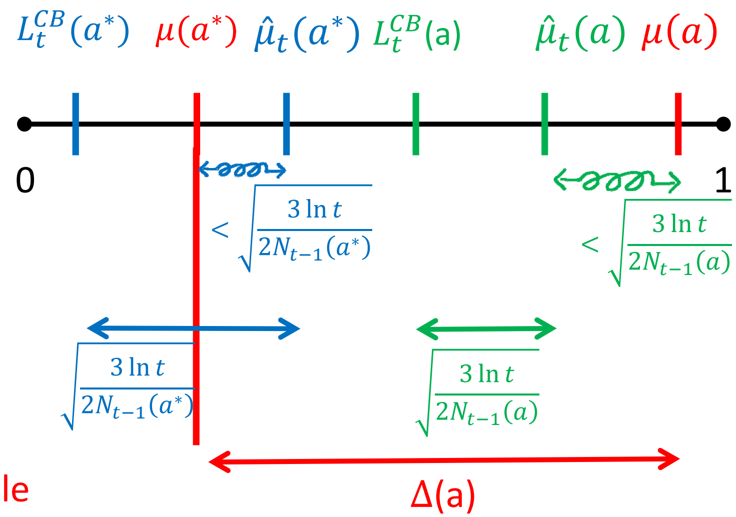
and so we may have $L_t^{CB}(a) \leq \mu(a^*)$ if $\sqrt{\frac{6 \ln t}{N_{t-1}(a)}} > \Delta(a)$

$$\Rightarrow \quad N_t(a) \leq \frac{6 \ln t}{\Delta(a)^2} \leq \frac{6 \ln T}{\Delta(a)^2}$$

- Mid-summary: $\mathbb{E}[N_T(a)] \leq \left\lceil \frac{6 \ln T}{\Delta(a)^2} \right\rceil + \mathbb{E}[1.] + \mathbb{E}[2.1]$

EXPECTED AMOUNT OF TIMES THAT WE PLAY THE WRONG MOVE

Proof continued



- $\mathbb{E}[N_T(a)] \le \left\lceil \frac{6\ln T}{\Delta(a)^2} \right\rceil + \mathbb{E}[\ ] + \mathbb{E}[\ ]$
- Let $F(a^*)$ be the expected number of times $\hat{\mu}_t(a^*) \ge \mu(a^*) + \sqrt{\frac{3\ln t}{2N_{t-1}(a^*)}}$
- Bound $\mathbb{P}\left(\hat{\mu}_{t-1}(a^*) - \mu(a^*) \ge \sqrt{\frac{3\ln t}{2N_{t-1}(a^*)}}\right)$ ← $N_{t-1}(a^*)$ is a random variable dependent on $\hat{\mu}_t(a^*)$!

USING HOEFFDING — BUT THERE'S A PROBLEM!

- Idea: break dependent events into independent events and take a union bound
- Introduce $X_1, \dots, X_T$ r.v. with the same distribution as ℓ_{t,a^*}
- Let $\bar{\mu}_s = \frac{1}{s}\sum_{i=1}^{s} X_i$
- $\mathbb{P}\left(\hat{\mu}_{t-1}(a^*) - \mu(a^*) \ge \sqrt{\frac{3\ln t}{2N_{t-1}(a^*)}}\right) \le \mathbb{P}\left(\exists s: \bar{\mu}_s - \mu(a^*) \ge \sqrt{\frac{\ln t^3}{2s}}\right)$

$$\underset{\text{union BOUND}}{\le} \sum_{s=1}^{t} \mathbb{P}\left(\bar{\mu}_s - \mu(a^*) \ge \sqrt{\frac{\ln t^3}{2s}}\right)$$

$$\underset{\text{Hoeffding}}{\le} \sum_{s=1}^{t} \frac{1}{t^3} = \frac{1}{t^2}$$

- $\mathbb{E}[F(a^*)] = \sum_{t=1}^{\infty} \mathbb{P}\left(L_t^{CB}(a^*) \ge \mu(a^*)\right) \le \sum_{t=1}^{\infty} \frac{1}{t^2} \le \frac{\pi^2}{6}$

Hoeffding:

$$\mathbb{P}\left(\frac{1}{n}\sum_{i}^{n} Z_i - \mu \ge \sqrt{\frac{\ln\frac{1}{\delta}}{2n}}\right) \le \delta$$

$\ell_{1,a^*}, \ell_{2,a^*}, \ell_{3,a^*}, \dots$

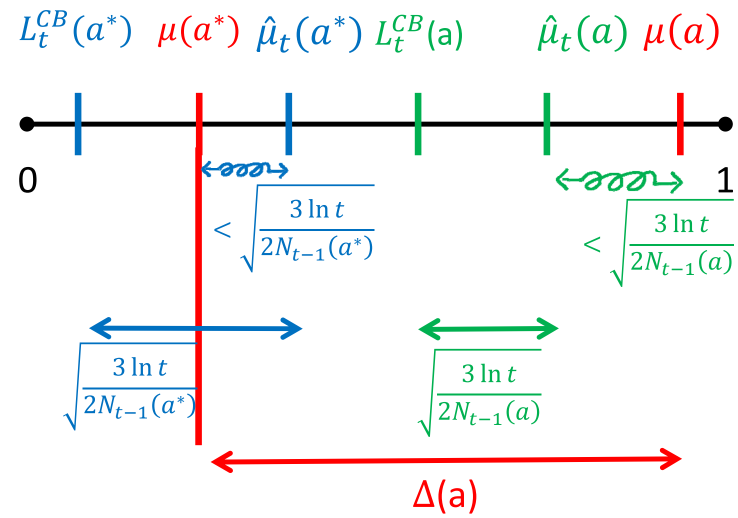
$X_1, X_2, X_3, \dots$

Proof summary

- $\bar{R}_t(a) = \sum_a \overbrace{\Delta(a)}^{\text{DISTANCE BETWEEN OPT. AND SUBOPT.}} \overbrace{\mathbb{E}[N_T(a)]}^{\text{EXPECTED NUMBER OF TIMES TO TAKE THE WRONG ACTION}}$

- $\mathbb{E}[N_T(a)] \leq \underbrace{\left\lceil \frac{6 \ln T}{\Delta(a)^2} \right\rceil}_{\text{The time it takes for confidence intervals to start working}} + \underbrace{\frac{\pi^2}{6} + \frac{\pi^2}{6}}_{\text{The expected number of times confidence intervals fail}}$

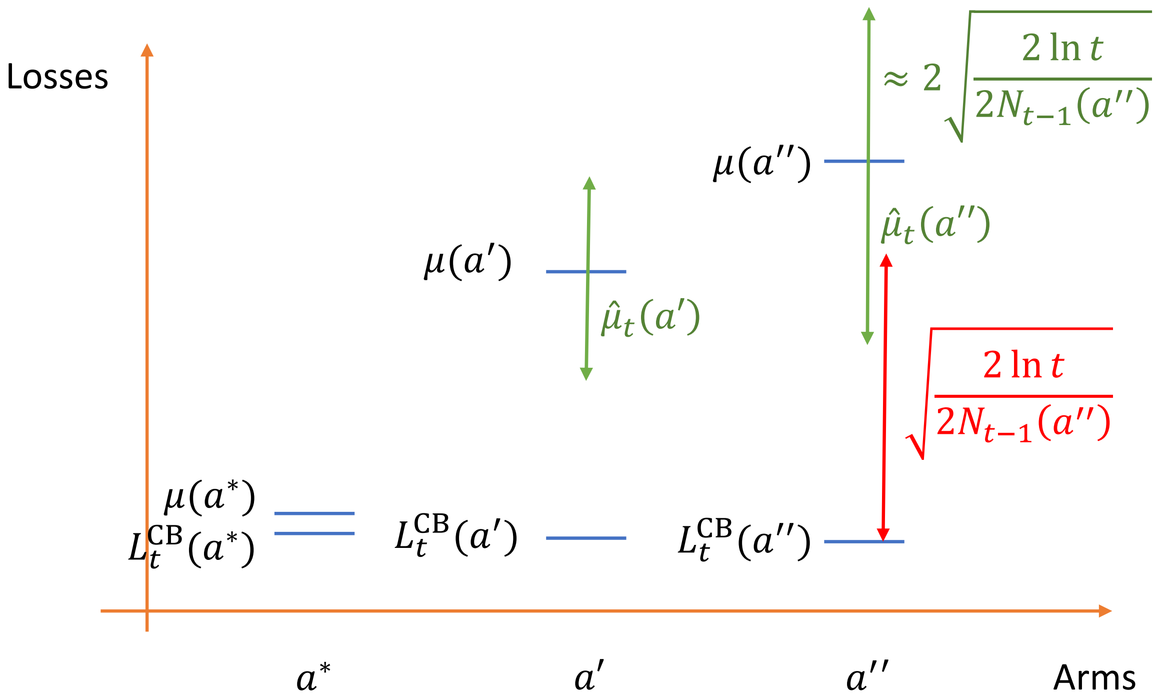
- $\bar{R}_T \leq 6 \sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)} + \left(1 + \frac{\pi^2}{3}\right) \sum_a \Delta(a)$



- Home assignment:
  - Take $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{2 \ln t}{2N_{t-1}(a)}}$ (instead of $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{3 \ln t}{2N_{t-1}(a)}}$; i.e. confidence $\frac{1}{t^2}$ instead $\frac{1}{t^3}$)
  - Show $\bar{R}_T \leq 4 \sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)} + (2 \ln T + 3) \sum_a \Delta(a)$

At time t we pick the one with the lowest $L_t^{CB}(a)$
(We roughly want to follow the empirical means but we have to account other factors and make sure to pick apparently suboptimal actions sometimes to make sure that on the long time our empirical means are equally valid and meaningful)

LCB algorithm dynamics (with $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{2\ln t}{2N_{t-1}(a)}}$)



- Confidence interval of the played arm shrinks ($N_{t-1}(a)$ grows)
- Confidence intervals of all other arms grow ($\ln t$ grows)
- $\Rightarrow$ all LCBs are roughly at the same level
- Most of the time $L_t^{CB}(a^*) \leq \mu(a^*)$
- a^* is played a lot, so $L_t^{CB}(a^*)$ is very close to $\mu(a^*)$
- All other arms are played just enough to keep $\sqrt{\frac{2\ln t}{2N_{t-1}(a)}} = \theta(\Delta(a))$, i.e. $N_t(a) = \theta\left(\frac{\ln t}{\Delta(a)^2}\right)$