# Theory of Discounted Markov Decision Processes

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# Markov Decision Process

NO REAL PROBLEMS IS ACTUALLY "INFINITE", BUT IT'S A GOOD APPROXIMATION FOR TASKS THAT ARE LONG ENOUGH

An infinite-horizon discounted MDP is a tuple $M = (\mathcal{S}, \mathcal{A}, P, R, \gamma)$:

- State-space $\mathcal{S}$ (finite, countably infinite, or continuous)
- Action-space $\mathcal{A} = \cup_{s \in \mathcal{S}} \mathcal{A}_s$ (finite, countably infinite, or continuous)
  - $\mathcal{A}_s$ is the set of actions available in state $s$
- Transition function $P$: Selecting $a \in \mathcal{A}_s$ in $s \in \mathcal{S}$ leads to a transition to $s'$ with probability $P(s'|s, a)$. $P(\cdot|s, a)$ is a probability distribution over $\mathcal{S}$, i.e.,

$$\sum_{s' \in \mathcal{S}} P(s'|s, a) = 1$$

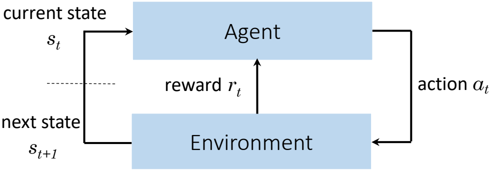
- Reward function $R$: Selecting $a \in \mathcal{A}_s$ in $s \in \mathcal{S}$ yields a reward $r \sim R(s, a)$.
- Discount factor $\gamma$: Future rewards are discounted geometrically with a rate $0 < \gamma < 1$.

# Recap: Interaction with MDP

An **agent** interacts with the MDP for $N$ rounds.

At each time step $t$:

- The agent observes the current state $s_t$ and takes an action $a_t \in \mathcal{A}_{s_t}$
- The environment (MDP) decides a reward $r_t := r(s_t, a_t) \sim R(s_t, a_t)$ and a next state $s_{t+1} \sim P(\cdot|s_t, a_t)$
- The agent receives $r_t$ (any time in step $t$ before start of $t+1$)




This interaction produces a trajectory (or history)

$$h_t = (s_1, a_1, r_1, s_2, a_2, r_2, \ldots, s_{t-1}, a_{t-1}, r_{t-1}, s_t)$$

# Objective Function

**Infinite-Horizon Discounted MDPs:** $N = \infty$, and the goal is to maximize the total expected sum of discounted rewards

$$\max_{\text{all strategies}} \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t)\right]$$

Two views on discounting with a discount factor $\gamma \in [0, 1)$:

- Earlier rewards are more important. A unit reward at present will worth $\gamma$ in the next slot.
- Problems with random horizon $N$ and absorbing states ← CHESS!

→ RANDOM $N$ = YOU DON'T KNOW WHEN GAME END

→ ABSORBING STATE = POSITION WHERE GAME IS FINISHED

# Reward Function: Some Comments

- **Bounded Rewards Assumption**: We assume

$$R_{\max} := \sup_{s,a} \left|\mathbb{E}_{r\sim R(s,a)}[r]\right| < \infty$$

- For simplicity, we assume *deterministic rewards*
  - Hence, $r \sim R(s,a)$ means $r = R(s,a)$.
  - Hence, we may use $r(s,a)$ and $R(s,a)$ interchangeably, but tend to keep $r(s,a)$ for generality.
  - The results in this lecture will hold for stochastic rewards under mild assumptions (and often by replacing $R(s,a)$ or $r(s,a)$ with its mean).

**This lecture:** We consider deterministic and bounded rewards.

NOTE: IN RIVERSWIM REWARDS ARE DETERMINISTIC BUT TRANSITIONS AREN'T

# Value Function

# Recap: Policy

When interacting with an MDP, actions are taken according to some **policy**:

| | deterministic | randomized |
|---|---|---|
| stationary | $\pi : \mathcal{S} \to \mathcal{A}, \quad \Pi^{\mathsf{SD}}$ | $\pi : \mathcal{S} \to \Delta(\mathcal{A}), \quad \Pi^{\cancel{\mathsf{SD}}\,\mathsf{SR}}$ |
| history-dependent | $\pi : \mathcal{H} \to \mathcal{A}, \quad \Pi^{\cancel{\mathsf{SD}}\,\mathsf{HD}}$ | $\pi : \mathcal{H} \to \Delta(\mathcal{A}), \quad \Pi^{\cancel{\mathsf{SD}}\,\mathsf{HR}}$ |

- $\Delta(\mathcal{A})$ denotes the simplex of probability distributions over $\mathcal{A}$.
- $\mathcal{H}$ the set of all possible histories (trajectories).

For $\pi \in \Pi^{\mathsf{SR}}$, we write $a \sim \pi(\cdot|s)$ or $a \sim \pi(s)$ . Also, given $f : \mathcal{A}_s \to \mathbb{R}$,

$$\mathbb{E}_{a\sim\pi(s)}[f(a)] = \sum\nolimits_{a\in\mathcal{A}_s} f(a)\pi(a|s)$$

# Value Function

The value function of policy $\pi$ (or simply, value of $\pi$) is a mapping $V^\pi : \mathcal{S} \to \mathbb{R}$ defined as

$$V^\pi(s) := \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \middle| s_1 = s\right].$$

where $\mathbb{E}^\pi$ indicates expectation over trajectories generated by $\pi$.

- Intuitively, $V^\pi(s)$ measures the sum of future discounted rewards (in expectation) when the agent starts in $s$ and follows $\pi$.
- We have

$$|V^\pi(s)| \le \frac{R_{\max}}{1-\gamma}, \quad \forall s \in \mathcal{S}$$

# Action-Value Function

The action-value function of policy $\pi$ (or simply, Q-value of $\pi$) is a mapping $Q^\pi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ defined as (Under the bounded reward assumption)

$$Q^\pi(s,a) := \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \middle| s_1 = s, a_1 = a\right].$$

- Intuitively, $Q^\pi(s,a)$ measures the sum of future discounted rewards (in expectation) when the agent starts in $s$ and takes action $a$ in the first step (possibly $a \neq \pi(s)$), and then follows $\pi$ afterwards.
- We have
$$|Q^\pi(s,a)| \le \frac{R_{\max}}{1-\gamma}, \quad \forall s \in \mathcal{S}, \forall a \in \mathcal{A}$$
- For all $s \in \mathcal{S}$, $Q^\pi(s,\pi(s)) = V^\pi(s)$.

# Policy Evaluation

# Recap: Induced Markov Chains

- Every $\pi \in \Pi^{\mathsf{SR}}$ induces a Markov chain on $M$, with transition probability matrix $P^\pi$ given by:

$$P^\pi_{s,s'} = \sum_{a \in \mathcal{A}_s} P(s'|s,a)\pi(a|s)\,, \quad s, s' \in \mathcal{S}.$$

- Every $\pi \in \Pi^{\mathsf{SR}}$ induces a reward vector $r^\pi \in \mathbb{R}^S$ on $M$ defined by:

$$r^\pi(s) = \sum_{a \in \mathcal{A}_s} R(s,a)\pi(a|s)\,, \quad s \in \mathcal{S}.$$

- If $\pi \in \Pi^{\mathsf{SD}}$, then $P^\pi_{s,s'} = P(s'|s,\pi(s))$ and $r^\pi(s) = R(s,\pi(s))$.

Every policy $\pi \in \Pi^{\mathsf{SR}}$ induces a **Markov Reward Process (MRP)** on $M$, specified by $r^\pi$ and $P^\pi$.

# Bellman Equation for $\pi$

## Theorem (Bellman Equation for $\pi$)

*Let* $\pi \in \Pi^{\mathsf{SR}}$. *For all* $s \in \mathcal{S}$,

$$V^\pi(s) = \mathbb{E}_{a\sim\pi(s)}[r(s,a)] + \gamma \mathbb{E}_{a\sim\pi(s)}\Big[\sum\nolimits_{x\in\mathcal{S}} P(x|s,a)V^\pi(x)\Big]$$
$$= \sum\nolimits_{a\in\mathcal{A}_s} \pi(a|s) r(s,a) + \gamma \sum\nolimits_{a\in\mathcal{A}_s} \pi(a|s) \sum\nolimits_{x\in\mathcal{S}} P(x|s,a)V^\pi(x)$$

*Equivalently,* $V^\pi = r^\pi + \gamma P^\pi V^\pi$.

- These relations are called the Bellman equation.
- The theorem tells us that for $\pi \in \Pi^{\mathsf{SR}}$, $V^\pi$ satisfies the Bellman equation.
- For a deterministic policy $\pi \in \Pi^{\mathsf{SD}}$, the Bellman equation becomes:

$$V^\pi(s) = r(s,\pi(s)) + \gamma \sum_{x\in\mathcal{S}} P(x|s,\pi(s))V^\pi(x)\,, \quad s \in \mathcal{S}.$$

# Bellman Operator for $\pi$

The Bellman operator associated to $\pi \in \Pi^{\text{SR}}$ is a mapping $\mathcal{T}^\pi : \mathbb{R}^S \to \mathbb{R}^S$, such that for any function $f : \mathcal{S} \to \mathbb{R}$,

$$\mathcal{T}^\pi f := r^\pi + \gamma P^\pi f\,.$$

- Intuitively, $\mathcal{T}^\pi$ is the value of $\pi$ for the same one-stage problem.
- $\mathcal{T}^\pi$ applies to (or *operates on*) a function defined on $\mathcal{S}$ and returns another function defined on $\mathcal{S}$.
- The Bellman equation $V^\pi = r^\pi + \gamma P^\pi V^\pi$ reads

$$V^\pi = \mathcal{T}^\pi V^\pi$$

In other words, $V^\pi$ is the *unique* fixed-point of the operator $\mathcal{T}^\pi$.

# Bellman Equation for $\pi$

We prove the theorem for $\pi \in \Pi^{\text{SD}}$. (See Lecture Notes for $\pi \in \Pi^{\text{SR}}$.)

**Proof.** Let $\pi \in \Pi^{\text{SD}}$ and $s \in \mathcal{S}$. We have

$$\begin{aligned} V^\pi(s) &= \mathbb{E}^\pi\Big[\sum_{t=1}^\infty \gamma^{t-1} r(s_t, \pi(s_t))\Big| s_1 = s\Big] \\ &= r(s,\pi(s)) + \mathbb{E}^\pi\Big[\sum_{t=2}^\infty \gamma^{t-1} r(s_t, \pi(s_t))\Big| s_1 = s\Big] \\ &= r(s,\pi(s)) + \gamma \sum_{x\in\mathcal{S}} \mathbb{P}(s_2 = x | s_1 = s, a_1 = \pi(s_1))\, \underbrace{\mathbb{E}^\pi\Big[\sum_{t=2}^\infty \gamma^{t-2} r(s_t, \pi(s_t))\Big| s_2 = x\Big]}_{=V^\pi(x)} \\ &= r(s,\pi(s)) + \gamma \sum_{x\in\mathcal{S}} \mathbb{P}(s_2 = x | s_1 = s, a_1 = \pi(s_1)) V^\pi(x) \\ &= r(s,\pi(s)) + \gamma \sum_{x\in\mathcal{S}} P(x|s,\pi(s)) V^\pi(x)\,. \end{aligned}$$

# Bellman Equation for $\pi$

The notion of the Bellman operator can be extended to Q-value functions.

The Bellman operator for Q-value of $\pi$ is a mapping $\mathcal{T}^\pi : \mathbb{R}^{S\times A} \to \mathbb{R}^{S\times A}$ defined as follows: For any function $f : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$,

$$(\mathcal{T}^\pi f)(s,a) = r(s,a) + \gamma \mathbb{E}_{a' \sim \pi(s)}\Big[\sum_y P(y|s,a') f(y,a')\Big], \qquad (s,a) \in \mathcal{S} \times \mathcal{A}$$

- Hence, we have $Q^\pi = \mathcal{T}^\pi Q^\pi$
- In other words, $Q^\pi$ is the fixed point of the operator $\mathcal{T}^\pi$ (for Q-function).

# Policy Evaluation

**Policy Evaluation:** Computing $V^\pi$ for a given $\pi$

- **Direct Computation:** Using Bellman equation,

$$V^\pi = r^\pi + \gamma P^\pi V^\pi \quad \Longrightarrow_{I-\gamma P^\pi \text{ is invertible}} \quad V^\pi = (I - \gamma P^\pi)^{-1} r^\pi$$

- **Iterative Policy Evaluation:** Using $V^\pi = \mathcal{T}^\pi V^\pi$, the sequence

$$V_{n+1} = \mathcal{T}^\pi V_n = \underbrace{\mathcal{T}^\pi \cdots \mathcal{T}^\pi}_{n+1 \text{ times}} V_0$$

  converges to $V^\pi$ starting from any $V_0$.

- **Monte-Carlo Method:** Generate a number of trajectories of $\pi$ and use the sample mean as an estimator to $V^\pi$.

So far:

- We defined policies and the value function.
- We characterized the value of stationary policies (via Bellman equations and operator).
- We developed ways to compute the value of a *fixed* stationary policy.

*How to find an optimal strategy/policy? Alternatively, how to find policies with good values?*

# Optimization in Discounted MDPs: Optimal Policy and Value

# Optimal Value and Policy

Solving a discounted MDP $M$ amounts to solving the following optimization problem:

$$V^\star(s) = \sup_{\pi \in \Pi^{\mathsf{HR}}} V^\pi(s) , \qquad \forall s \in \mathcal{S}.$$

(i) $V^\star : \mathcal{S} \to \mathbb{R}$ is called the optimal value function.

(ii) If there exists $\pi^\star$ such that $V^{\pi^\star}(s) = V^\star(s)$ for all $s \in \mathcal{S}$, then $\pi^\star$ is called an optimal policy.

(iii) $\pi$ is $\varepsilon$-optimal for $\varepsilon > 0$ if

$$V^\pi(s) \geq V^\star(s) - \varepsilon, \quad \forall s \in \mathcal{S}$$

# Bellman Optimality Equation

**Theorem**

$V^\star$ *satisfies the optimal Bellman equation:*

$$V^\star(s) = \max_{a \in \mathcal{A}_s} \Big( r(s,a) + \gamma \sum_{x \in \mathcal{S}} P(x|s,a) V^\star(x) \Big), \quad s \in \mathcal{S}$$

The optimal Bellman operator is a mapping $\mathcal{T} : \mathbb{R}^S \to \mathbb{R}^S$, such that for any function $f : \mathcal{S} \to \mathbb{R}$,

$$(\mathcal{T} f)(s) := \max_{a \in \mathcal{A}_s} \Big( r(s,a) + \gamma \sum_{x \in \mathcal{S}} P(x|s,a) f(x) \Big), \quad s \in \mathcal{S}$$

- $V^\star$ satisfies $\mathcal{T} V^\star = V^\star$.
- We can define $\mathcal{T}$ and optimal Bellman equation for the optimal Q function.

# Optimality Theorems

**Theorem**

*Suppose the state space $\mathcal{S}$ is finite. Then there exists a policy $\pi^\star \in \Pi^{SD}$.*

- Thus, when seeking $\pi^\star$ in a discounted MDP with a finite $\mathcal{S}$, we can restrict our attention to $\Pi^{\mathsf{SD}}$.
- In other words, for finite $\mathcal{S}$,

$$\sup_{\pi \in \Pi^{\mathsf{HR}}} V^\pi = \sup_{\pi \in \Pi^{\mathsf{SD}}} V^\pi = \max_{\pi \in \Pi^{\mathsf{SD}}} V^\pi$$

# Optimality Theorems

A fundamental result in the theory of discounted MDPs:

**Theorem**

*A stationary deterministic policy $\pi$ is optimal if and only if*

$$\mathcal{T}^\pi V^\star = \mathcal{T} V^\star$$

*Equivalently, $\pi$ is optimal if and only if it attains the maximum in the Bellman optimality equations: For all $s \in \mathcal{S}$,*

$$\pi(s) \in \operatorname*{arg\,max}_{a \in \mathcal{A}_s} \left( r(s,a) + \sum_{x \in \mathcal{S}} P(x|s,a) V^\star(x) \right).$$

So far:

- We defined policies and the value function.
- We characterized the value of stationary policies (via Bellman equations and operator).
- We developed ways to compute the value of a *fixed* stationary policy.
- We defined the notion of optimality and showed that there exists $\pi^\star \in \Pi^{\mathsf{SD}}$ when $\mathcal{S}$ is finite.
- We characterized the optimal value function $V^\star$ (via optimal Bellman equation).

*How to actually compute $\pi^\star$?*

# Algorithms for Solving Discounted MDPs

# Major Solution Methods

Three major classes of algorithms for solving discounted MDPs:

- Value Iteration
- Policy Iteration
- Linear Programming

# Value Iteration

Value Iteration (`VI`)

- The most well-known, and perhaps the simplest, algorithm for solving discounted MDPs
- Around since the early days of MDPs
- Also known as successive approximation, backward induction, etc.

**Idea:** The optimal Bellman operator $\mathcal{T}$ is *contracting*. Iterate $\mathcal{T}$ until convergence:

$$V_{n+1} = \mathcal{T} V_n, \quad n = 0, 1, 2, \ldots$$

Indeed, `VI` is an algorithm for approximating the fixed point of $\mathcal{T}$.

# Value Iteration (VI)

**input:** $\varepsilon$

- **initialization:** Select a value function $V_0 \in \mathbb{R}^S$, $V_1 = R_{\max}/(1-\gamma)\mathbf{1}$, and set $n = 0$
- **while** $\left(\|V_{n+1} - V_n\|_\infty \geq \frac{\varepsilon(1-\gamma)}{2\gamma}\right)$
  (i) Update, for each $s \in \mathcal{S}$,
  $$V_{n+1}(s) = \max_{a \in \mathcal{A}_s} \left( r(s,a) + \gamma \sum_{x \in \mathcal{S}} P(x|s,a) V_n(x) \right)$$
  (ii) Increment $n$.

**output:**

$$\pi^{\mathtt{VI}}(s) \in \arg\max_{a \in \mathcal{A}_s} \left( r(s,a) + \gamma \sum_{x \in \mathcal{S}} P(x|s,a) V_n(x) \right), \quad s \in \mathcal{S}$$

# Why VI works?

*Why does VI work?*

$\Longrightarrow$ Because of contraction properties of Bellman operators.

# Contraction Mapping

An operator (or mapping) $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$ is called a *$\kappa$-contraction mapping* (with respect to $\|\cdot\|$) if there exists $\kappa \in [0, 1)$ such that for all $v, v' \in \mathbb{R}^n$,

$$\|\mathcal{L}v - \mathcal{L}v'\| \le \kappa \|v - v'\| .$$

**Theorem (Banach Fixed-Point Theorem)**

*Suppose $\mathcal{L}$ is a contraction mapping. Then*

(i) *there exists a unique $v^\star \in \mathbb{R}^n$ such that $\mathcal{L}v^\star = v^\star$;*

(ii) *for any $v_0 \in \mathbb{R}^n$, the sequence $(v_n)_{n\ge 0}$ with $v_{n+1} = \mathcal{L}v_n = \mathcal{L}^{n+1}v_0$ for $n \ge 0$ converges to $v^\star$.*

# $\mathcal{T}^\pi$ and $\mathcal{T}$ Are Contraction Mapping

**Lemma**

*For any $v, v' \in \mathbb{R}^S$, and any $\pi$,*

$$\|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty \leq \gamma \|v - v'\|_\infty ,$$
$$\|\mathcal{T} v - \mathcal{T} v'\|_\infty \leq \gamma \|v - v'\|_\infty .$$

*Hence, $\mathcal{T}^\pi$ and $\mathcal{T}$ are $\gamma$-contraction mappings w.r.t. $\|\cdot\|_\infty$.*

**Proof.** First statement is easy to prove. For the second, we have:

$$\begin{aligned}
&\|\mathcal{T} v - \mathcal{T} v'\|_\infty \\
&\quad = \max_s \left| \max_{a \in \mathcal{A}_s} \Big( r(s,a) + \gamma \sum_j P(j|s,a) v(j) \Big) - \max_{a \in \mathcal{A}_s} \Big( r(s,a) + \gamma \sum_j P(j|s,a) v'(j) \Big) \right| \\
&\quad \leq \max_s \max_{a \in \mathcal{A}_s} \left| \gamma \sum_j P(j|s,a) \big( v(j) - v'(j) \big) \right| \\
&\qquad\qquad \text{(Using inequality } |\max_x f(x) - \max_x g(x)| \leq \max_x |f(x) - g(x)| \text{)} \\
&\quad \leq \gamma \max_s \max_{a \in \mathcal{A}_s} \max_j |v(j) - v'(j)| \sum_j P(j|s,a) = \gamma \|v - v'\|_\infty
\end{aligned}$$

# VI: Convergence

VI is a *globally convergent* method for finding an $\varepsilon$-optimal policy. Formally:

**Theorem**

*Let $(V_n)_{n\geq 0}$ a sequence of value functions generated by VI with some $\varepsilon > 0$ starting from an arbitrary initial point $V_0 \in \mathbb{R}^S$. Then,*

(i) *$V_n$ converges to $V^\star$ in norm;*

(ii) *the algorithm stops after finitely many iterations;*

(iii) *$\pi^{VI}$ is $\varepsilon$-optimal;*

(iv) *when convergence criterion is satisfied, $\|V_{n+1} - V^\star\|_\infty < \varepsilon/2$.*

- Each iteration of VI involves $O(S^2 A)$ arithmetic calculations.
- The iteration complexity of VI depends on both $\varepsilon$ and $\gamma$. The larger the $\gamma$, the more iteration until the algorithm finds an $\varepsilon$-optimal policy.

# Policy Iteration

Policy Iteration (PI)

- A popular algorithm for solving discounted MDPs
- Around since early days of MDPs
- Like VI, it is an iterative algorithm but directly searches in the space of policies.

**Idea:** Starting from an initial policy, at each iterate $n$,

(i) Find $V^{\pi_n}$ (policy evaluation)

(ii) Improve $\pi_n$ to $\pi_{n+1}$ using $V^{\pi_n}$ (policy improvement)

# Policy Iteration (PI)

- **initialization:** Select $\pi_0$ and $\pi_1$ arbitrarily ($\pi_0 \neq \pi_1$), and set $n = 0$
- **while**$\left(\pi_{n+1} \neq \pi_n\right)$
  (i) *Policy Evaluation:* Find $V_n$, the value of $\pi_n$ by solving
  $$(I - \gamma P^{\pi_n})V_n = r^{\pi_n}$$
  (ii) *Policy Improvement:* Choose $\pi_{n+1}$ such that
  $$\pi_{n+1}(s) \in \arg\max_{a \in \mathcal{A}_s} \left(r(s,a) + \gamma \sum_{x \in \mathcal{S}} P(x|s,a)V_n(x)\right)$$
  and if possible, set $\pi_{n+1} = \pi_n$.
  (iii) Increment $n$.
- **output:** $\pi^{\mathtt{PI}} = \pi_n$

# PI: Convergence

**Theorem**

*Suppose $M$ has a finite state-action space. Then,*

(i) *PI terminates in at most*

$$O\Big(\max\Big\{\frac{SA}{1-\gamma}\log\frac{1}{1-\gamma}, \frac{A^S}{S}\Big\}\Big) \qquad \textit{iterations;}$$

(ii) $\pi^{PI} = \pi^\star$.

- Under PI, $V_{n+1} \geq V_n$ for any $n$. Further, the number of policies is finite $A^S$.
- Each iteration in PI involves solving a linear system with $S$ equations and $S$ unknowns. Hence, per iteration complexity of PI is $O(S^3 + S^2 A)$.
- In practice, PI converges within, at most, a few tens of iterations.