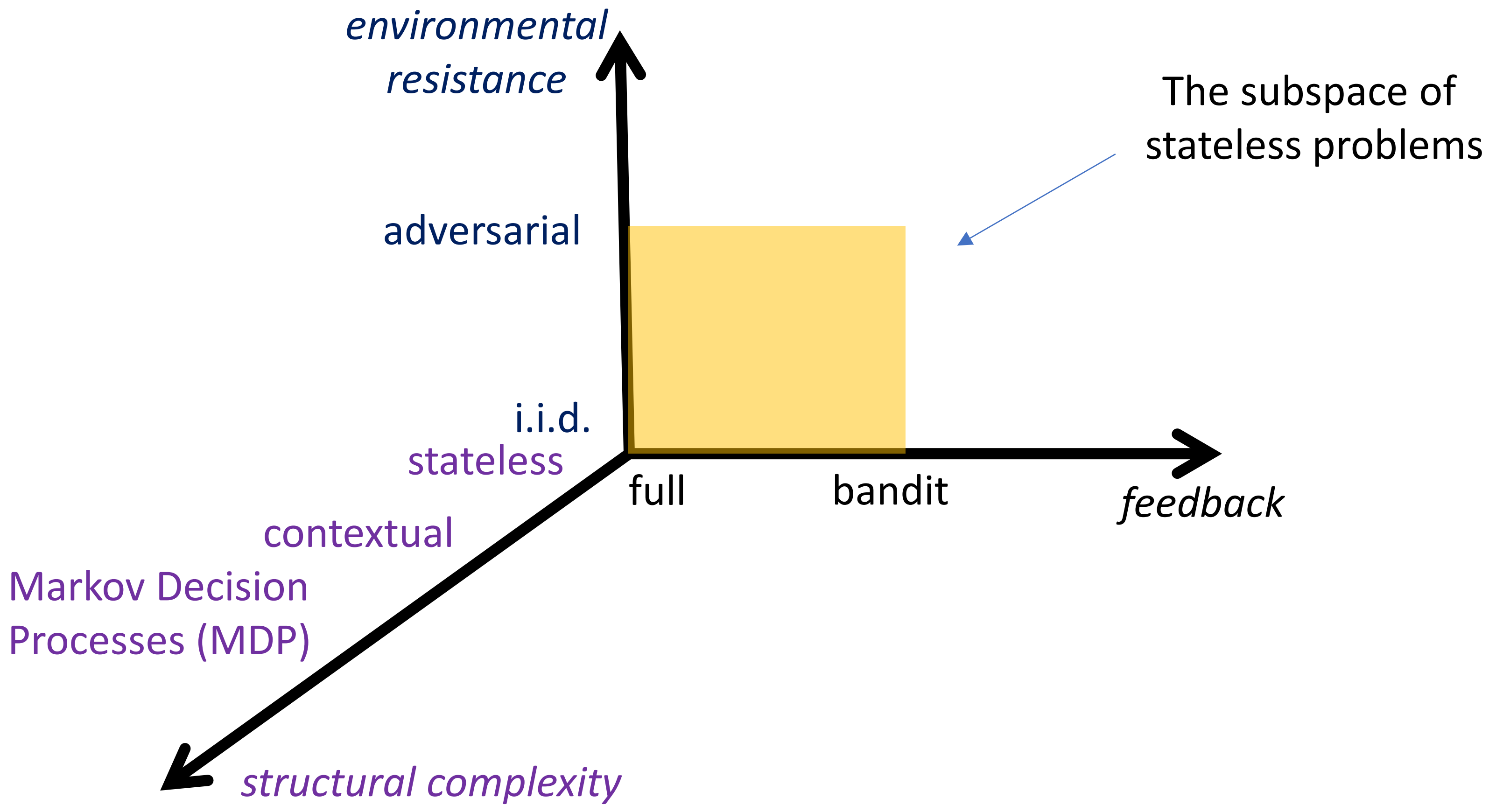
Online Learning Setup and Stochastic Bandits

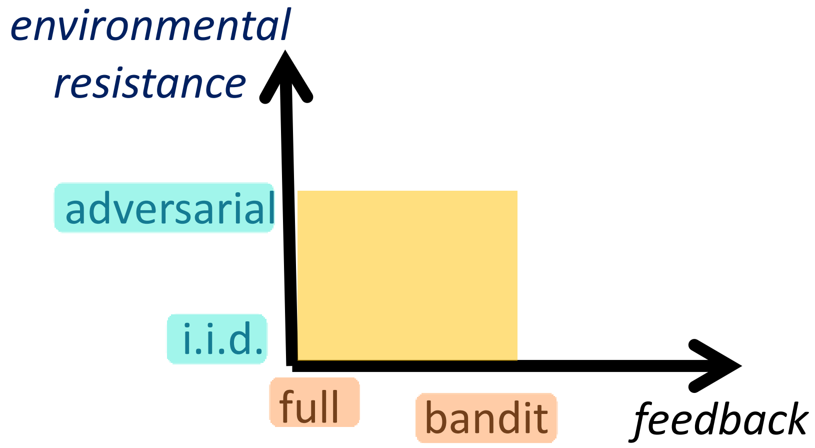
Yevgeny Seldin

Online Learning Setup

The space of online learning problems



The stateless setting



$$
K \text{ actions} \left\{ \begin{array}{ccccc}
\ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots
\end{array} \right.
$$

$\xrightarrow{\quad\quad\quad\quad}$ *time*

$\ell_{t,a} \in [0,1]$

<u>Game protocol:</u>

For $t = 1,2, \ldots$:

1. Pick a row A_t
2. Suffer the loss ℓ_{t,A_t}
3. Observe …

YOU KNOW THE LOSS OF THE ACTION YOU HAVE TAKEN AND THE ONES YOU HAVE NOT

YOU ONLY KNOW THE LOSS OF THE ACTION YOU HAVE CHOSE

GIVEN AT PRIORY

Observations / Generation of $\ell_{t,a}$	Full: $\ell_{t,1}, \ldots, \ell_{t,K}$	Bandit: ℓ_{t,A_t}
Adversarial: $\ell_{t,a}$ arbitrary		
I.I.D.: $\ell_{t,a}$ sampled i.i.d., such that $\mathbb{E}[\ell_{t,a}] = \mu(a)$		

WE HAVE A WAY TO GENERATE THE LOSS (CALCULATE)

Performance measure

REFERS TO THE LOSS OF ONE SINGLE ROW (THE BEST)

$$
\begin{array}{c}
K \text{ actions}
\end{array}
\begin{array}{cccccc}
\ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots
\end{array}
$$

$\xrightarrow{\textit{time}}$

- Regret: $R_T = \underbrace{\sum_{t=1}^{T} \ell_{t,A_t}}_{\text{Loss of the algorithm}} - \underbrace{\min_a \sum_{t=1}^{T} \ell_{t,a}}_{\text{Loss of the best action in hindsight}}$

$$
\begin{array}{ccccc}
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{array}
\longrightarrow \frac{K-1}{K} T = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right]
$$

ASSUME THAT THERE IS ONLY 1 ZERO AND IS PLACED RANDOMLY

$\downarrow$

THE "BEST ACTION IN HINDSIGHT" IS A ROW, AND THE EXPECTED LOSS IS $\frac{K-1}{K} T$

$\ell_{t,a} \in [0,1]$

- Regret of order T means no learning
  - The loss of A_t stays at the same distance from the loss of the optimal action as the game proceeds
- The aim is to achieve sublinear regret $\rightarrow$ IF OUR REGRET INCREASE LINEARLY WITH T IT MEANS THAT WE ARE NOT LEARNING WHICH ONE IS THE BEST OPTION TO PICK IN THE LONG RUN
- Why do we compare to the best fixed action in hindsight and not to the best path in hindsight?
  - "The best path in hindsight" is an overly strong competitor – we cannot guarantee sublinear regret
  - Show that the regret relative to the best path in hindsight can be as large as $\frac{K-1}{K} T$

Performance measures

$$K \text{ actions} \left\{ \begin{array}{ccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array} \right.$$

$\xrightarrow{\hspace{6em}}$ *time*

$\ell_{t,a} \in [0,1]$

- Regret: $R_T = \underbrace{\sum_{t=1}^{T} \ell_{t,A_t}}_{\text{Loss of the algorithm}} - \underbrace{\min_a \sum_{t=1}^{T} \ell_{t,a}}_{\text{Loss of the best action in hindsight}}$

- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$

$$\underbrace{=}_{\text{oblivious adversary}} \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \sum_{t=1}^{T} \ell_{t,a}$$

↳ IS NOT REACTING TO YOUR ACTIONS

- Oblivious adversary:
  - $\ell_{t,a}$ is independent of $A_1, \ldots, A_{t-1}$
  - The losses can be written down before the game starts
- Adaptive adversary:
  - $\ell_{t,a}$ may depend on $A_1, \ldots, A_{t-1}$

Performance measures

$$\text{K actions}\left\{\begin{array}{cccccc} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,a} & \ell_{2,a} & \cdots & \ell_{t,a} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ell_{1,K} & \ell_{2,K} & \cdots & \ell_{t,K} & \cdots \end{array}\right.$$

$\xrightarrow{\hspace{3em}}$ *time*

Regret: $R_T = \underbrace{\sum_{t=1}^{T} \ell_{t,A_t}}_{\text{Loss of the algorithm}} - \underbrace{\min_a \sum_{t=1}^{T} \ell_{t,a}}_{\text{Loss of the best action in hindsight}}$

- Pseudo-regret (stochastic setting):

$$\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right]$$

I AM MAKING THE DIFFERENCE BETWEEN THE EXPECTED LOSS OF MY CHOISES AND THE EXPECTED LOSS OF THE BEST POSSIBLE ACTION (ROW) (NOT SURE?)

THE BEST ACTION IS THE ONE WITH THE LOWEST EXPECTED LOSS

$$= \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - T \underbrace{\min_a \mu(a)}_{\mu^*}$$

$\min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \min_a \sum_{t=1}^{T} \mathbb{E}[\ell_{t,a}] = \min_a \sum_{t=1}^{T} \mu^* = \min_a T\mu^* = T \min_a \mu^*$

$$= \mathbb{E}\left[\sum_{t=1}^{T} (\ell_{t,A_t} - \mu^*)\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \Delta(A_t)\right]$$

$$= \mathbb{E}\left[\sum_{a=1}^{K} \Delta(a) \underbrace{N_T(a)}_{\text{NUMBER OF TIMES THE AGORITHM CHOOSE ACTION } a}\right]$$

$$= \sum_{a=1}^{K} \Delta(a) \mathbb{E}[N_T(a)]$$

- $N_T(a) = \sum_{t=1}^{T} \mathbb{1}(A_t = a)$
- $\mathbb{E}[\ell_{t,a}] = \mu(a)$
- $\mu^* = \min_a \mu(a)$ ← IF THE BEST ACTION IN HINDSIGHT GIVES 0 LOSS 70% OF THE TIME AND 1 LOSS 30% OF THE TIME THEN THIS NUMBER IS GOING TO BE 0.3
- $a^* \in \arg\min_a \mu(a)$
  - An optimal arm (may be multiple optimal arms with the same μ^*)
- $\Delta(a) = \mu(a) - \mu^*$ suboptimality gap
- $\mathbb{E}[\ell_{t,A_t} - \mu^*] = \mathbb{E}\left[\mathbb{E}[\ell_{t,A_t} - \mu^* | A_1, \dots, A_t]\right] = \mathbb{E}[\mu(A_t) - \mu^*] = \mathbb{E}[\Delta(A_t)]$

Expected regret:

$$\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$$

Expected regret vs. Pseudo regret

$$
K \text{ actions} \left\{ \begin{array}{ccccc}
\ell_{1,1} & \ell_{2,1} & \cdots & \ell_{t,1} & \cdots \\
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$\xrightarrow{\hspace{3em}}$ *time*

- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$
- Pseudo-regret: $\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - T\mu^*$

- $\mathbb{E}\left[\min_a f(a,B)\right] \leq \min_a \mathbb{E}[f(a,B)] \Rightarrow \bar{R}_T \leq \mathbb{E}[R_T]$

HERE THE BEST ACTION IN HINDSIGHT IS ONE AND FIXED

HERE THE BEST ACTION IN HINDSIGHT CAN BE DIFFERENT (LIKE SAYING: TO MAKE THE EXPECTATION I RUN THE EXPERIMENT MULTIPLE TIMES, AND THE BEST ACTION IS THE BEST EVERY TIME (THERE IS NOT A FIXED ONE FOR THE WHOLE EXPERIMENTATION))

- Oblivious adversarial setting:
  - $\ell_{t,a}$ are deterministic and the two notions of regret coincide
  - $\mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right] = \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \min_a \sum_{t=1}^{T} \ell_{t,a}$

Expected regret vs. Pseudo regret

$$
\begin{array}{c}
K \text{ actions} \left\{ \begin{array}{ccccc}
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\end{array} \right. \\
\xrightarrow{\hspace{6em}} \\
\textit{time}
\end{array}
$$

- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$
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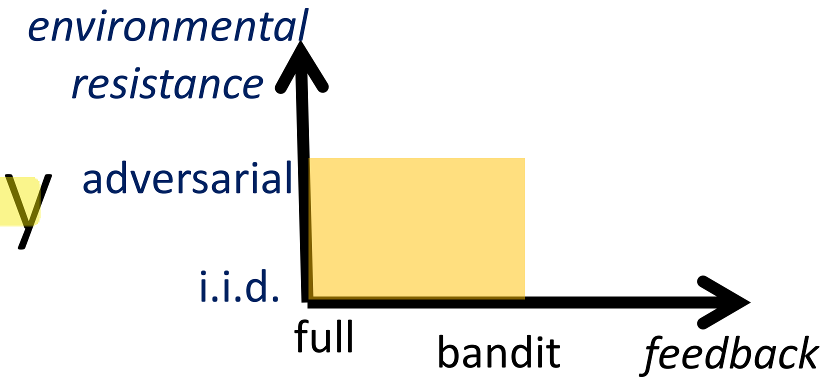
- $\mathbb{E}\left[\min_a f(a,B)\right] \le \min_a \mathbb{E}[f(a,B)] \;\Rightarrow\; \bar{R}_T \le \mathbb{E}[R_T]$

- Stochastic setting: imagine that $\mu(a) = \frac{1}{2}$ for all a. Then
  - $\mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] = \frac{1}{2}T$
  - $\mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \frac{1}{2}T$ for all a
  - $\bar{R}_T = 0$
  - $\mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right] \approx T\mu^* - \sqrt{\frac{1}{2}\left(\frac{1}{2}T\right)\ln K}$ → If the competitor is allowed to select out of the K action will have an advantage
  - $\mathbb{E}[R_T] \approx \sqrt{\frac{1}{2}\left(\frac{1}{2}T\right)\ln K}$
  - Pseudo-regret is a more reasonable quantity to look at
  - Expected regret provides an artificial advantage to the competitor due to their ability to select out of K trials

Online Learning Setup - Summary



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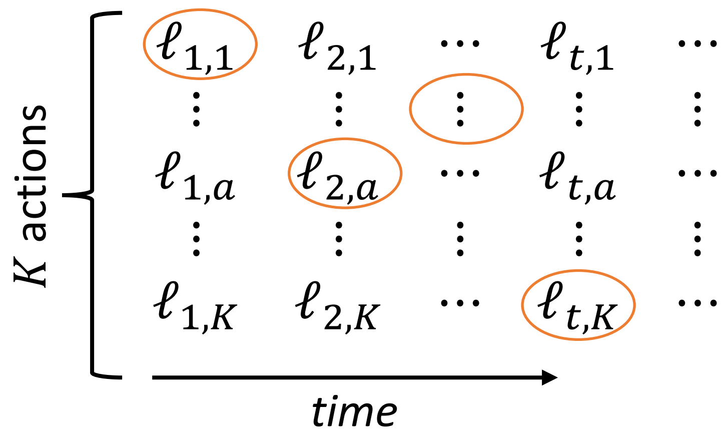
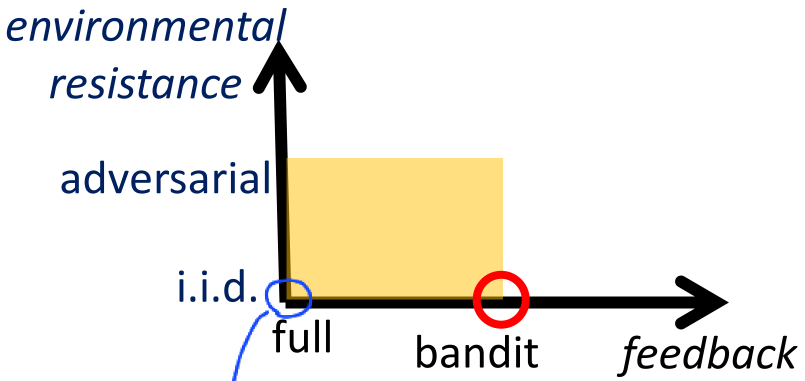


- Regret: $R_T = \sum_{t=1}^{T} \ell_{t,A_t} - \min_a \sum_{t=1}^{T} \ell_{t,a}$
- Expected regret: $\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \mathbb{E}\left[\min_a \sum_{t=1}^{T} \ell_{t,a}\right]$
- Pseudo-regret: $\bar{R}_T = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - \min_a \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,a}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,A_t}\right] - T\mu^*$



THIS ONE PROBABLY IS SOLVED
JUST PICKING THE ACTIO THAT WE
SEE IS PERFORMING BETTER

Stochastic (i.i.d.) bandits

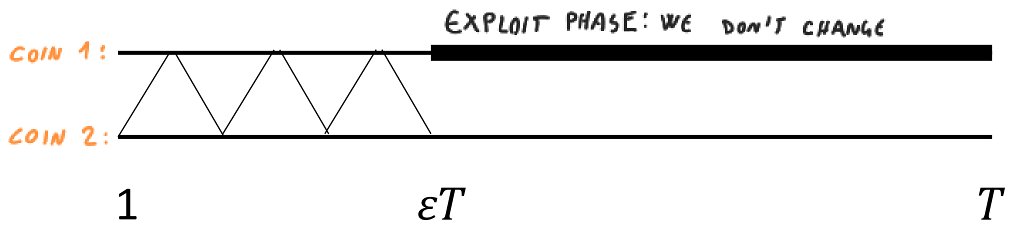
Exploration-Exploitation trade-off: a simple approach

I HAVE TWO COINS:
- 1 HAS EXPECTATION μ
- 1 HAS EXPECTATION $\mu + \Delta$

WE DON'T KNOW WHICH IS WHICH, SO WE HAVE TO FIND A WAY TO UNDERSTAN WHICH IS THE BEST

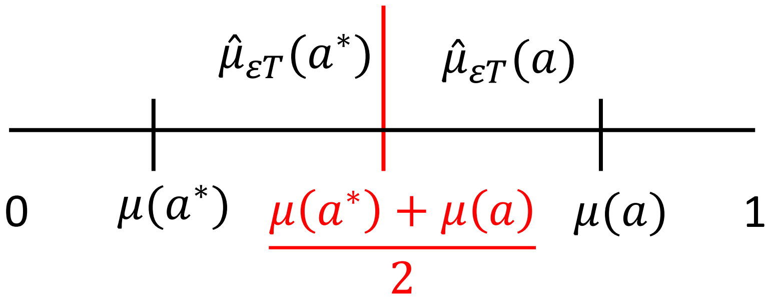
- Setting:
  - Two actions
  - Bandit feedback
  - T is known
  - Δ is known

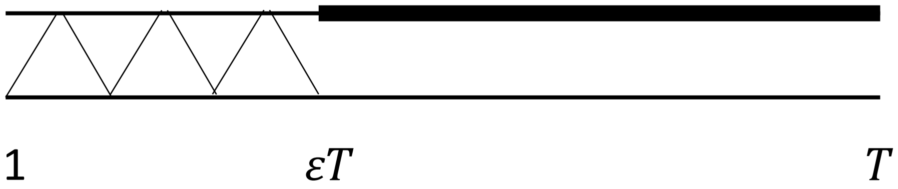
HOW MUCH IS ε? THAT'S THE HARD QUESTION!



- Approach:
  - Explore 50/50 for εT rounds
  - Exploit for the remaining rounds

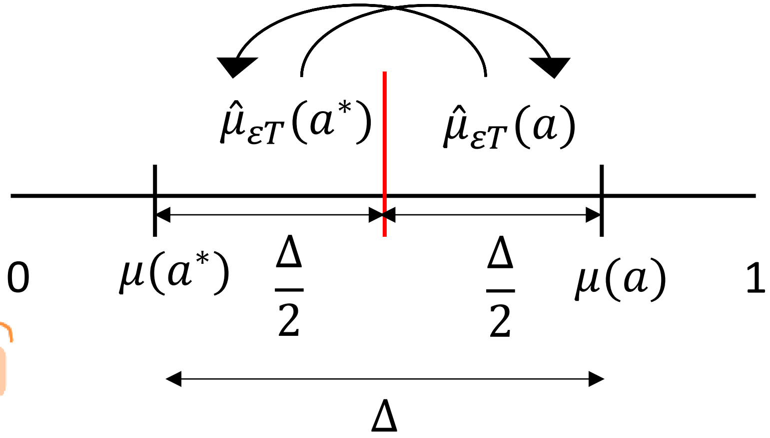
- Analysis approach:
  - Take a separation line at $\frac{\mu(a^*)+\mu(a)}{2}$
  - If at time εT the empirical means are on the "correct" side of the separation line, the arm selection for exploitation will be correct
  - Bound the probability that at εT the empirical means are estimated incorrectly





Analysis

PROBABILITY THAT WE CHOOSE THE WRONG COIN AFTER THE EXPLORATION OF ε STEPS



- Let $\delta(\varepsilon) = \mathbb{P}\big(\hat{\mu}_{\varepsilon T}(a) \le \hat{\mu}_{\varepsilon T}(a^*)\big)$ be the prob. of confusion

OUR LOSS

UPPER BOUND OF THE LOSS (WE CHOOSE ε TO MINIMIZE THIS)

- $\bar{R}_T = \sum_{t=1}^{T} \Delta(A_t) \le \underbrace{\frac{1}{2}\varepsilon T\Delta}_{\text{Exploration}} + \underbrace{\delta(\varepsilon)(1-\varepsilon)T\Delta}_{\text{Exploitation}} \le \left(\frac{\varepsilon}{2} + \delta(\varepsilon)\right)T\Delta$

- $$\delta(\varepsilon) = \mathbb{P}\big(\hat{\mu}_{\varepsilon T}(a) \le \hat{\mu}_{\varepsilon T}(a^*)\big)$$
$$\le \mathbb{P}\left(\hat{\mu}_{\varepsilon T}(a^*) \ge \mu(a^*) + \frac{1}{2}\Delta\right) + \mathbb{P}\left(\hat{\mu}_{\varepsilon T}(a) \le \mu(a) - \frac{1}{2}\Delta\right)$$
$$\le 2e^{-2\frac{\varepsilon T}{2}\left(\frac{1}{2}\Delta\right)^2} = 2e^{-\varepsilon T\Delta^2/4}$$

FROM ML A COURSE:

$$\mathbb{P}\Big(\underbrace{\frac{1}{n}\sum_{i=1}^{n} z_i}_{\hat{\mu}_{\varepsilon T}(a^*)} - \underbrace{\mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^{n} z_i\Big]}_{\mu(a^*)} \ge \underbrace{\varepsilon}_{\frac{1}{2}\Delta}\Big) \le e^{-2n\varepsilon^2}$$

- Minimization of $\frac{\varepsilon}{2} + 2e^{-\varepsilon T\Delta^2/4}$ with respect to ε gives $\varepsilon^* = \frac{4\ln(T\Delta^2)}{T\Delta^2}$

- With exploration phase of length $\varepsilon^* T$, we get $\bar{R}_T \le \frac{2(\ln(T\Delta^2)+1)}{\Delta}$

Reflection

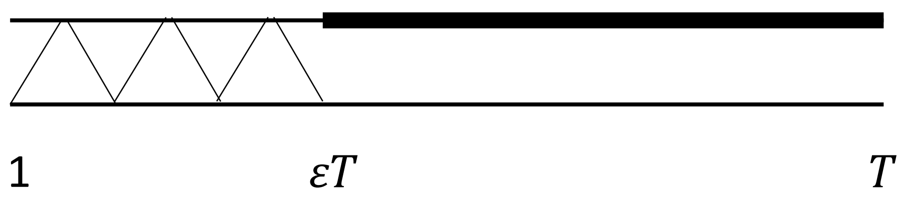
- $\varepsilon^* = \frac{4\ln(T\Delta^2)}{T\Delta^2}$
- Exploration phase: $\varepsilon^* T = \frac{4\ln(T\Delta^2)}{\Delta^2}$
- $\delta(\varepsilon^*) = 2e^{-\varepsilon^* T\Delta^2/4} = \frac{1}{T\Delta^2}$
- $\bar{R}_T \leq \frac{2(\ln(T\Delta^2)+1)}{\Delta}$

- It takes $\sim \frac{\ln T}{\Delta^2}$ rounds to identify the best arm with confidence $\frac{1}{T\Delta^2}$
- On each exploration round we pay Δ
- Total regret order:

$$\bar{R}_T \approx \underbrace{\frac{\ln T}{\Delta^2}\Delta}_{\text{Exploration}} + \underbrace{\frac{1}{T\Delta^2}T\Delta}_{\text{Exploitation}} \approx \frac{\ln T}{\Delta}$$

- Small $\Delta \Longrightarrow$ **Harder** problem (**Larger** $\bar{R}_T$)

Limitations



- Assumes knowledge of T
- Assumes knowledge of Δ
- Generalization to more than two actions is not straightforward

Lower Confidence Bound (LCB) algorithm for losses (Originally Upper Confidence Bound (UCB) for rewards) ("Optimism in the face of uncertainty" approach)

- Define $L_t^{CB}(a) = \hat{\mu}_{t-1}(a) - \sqrt{\frac{3 \ln t}{2N_{t-1}(a)}}$ lower confidence bound
  - (We will show that with high probability $L_t^{CB}(a) \leq \mu(a)$ for all t)
- LCB Algorithm:
  - Play each arm once
  - For $t = K + 1, K + 2, \ldots$:
    - Play $A_t = \arg\min_a L_t^{CB}(a)$
- Theorem:

$$\bar{R}_T \leq 6 \sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)} + \left(1 + \frac{\pi^2}{3}\right) \sum_a \Delta(a)$$

- Proof: next time

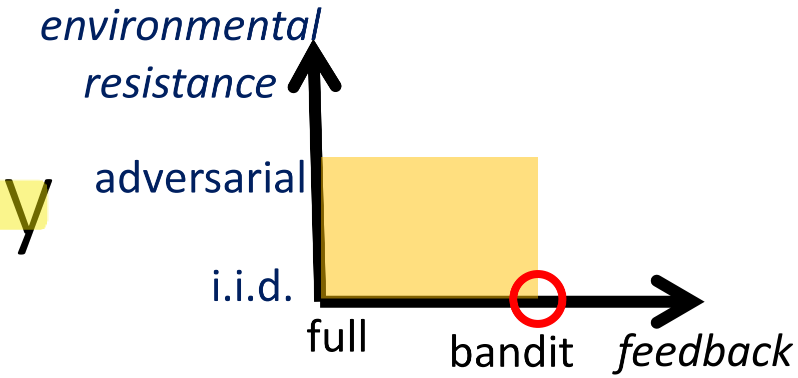
- No knowledge of T
- No knowledge of Δ
- Works for any K

Rewards ↔ Losses

$\ell_{t,a} = 1 - r_{t,a}$

$r_{t,a} = 1 - \ell_{t,a}$

Stochastic bandits – mid-summary



- It takes $\sim \frac{\ln T}{\Delta^2}$ rounds to identify the best arm with confidence $\frac{1}{T\Delta^2}$

- Each exploration round costs Δ, but their number grows as $\frac{1}{\Delta^2}$!

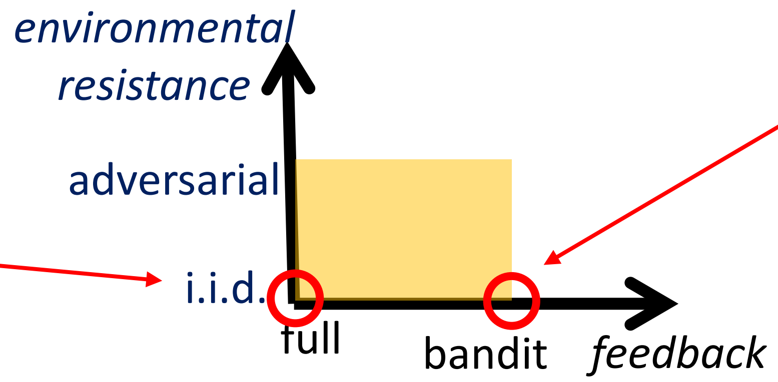
- $\overline{\mathrm{R}}_T \approx \underbrace{\frac{\ln T}{\Delta^2}\Delta}_{\text{Exploration}} + \underbrace{\frac{1}{T\Delta^2}\mathrm{T}\Delta}_{\text{Exploitation}} \approx \frac{\ln T}{\Delta}$

- Problems with small Δ are **harder** than problems with large Δ!

Home Assignment

TO DO "MATHEMATICALLY", SO USING FORMULAS SUCH AS IN THE SLIDES

$$\bar{R}_T = O\left(\sum_{a:\Delta(a)>0} \frac{1}{\Delta(a)}\right)$$



$$\bar{R}_T = O\left(\sum_{a:\Delta(a)>0} \frac{\ln T}{\Delta(a)}\right)$$

- In full information there is no need for exploration

YOU JUST PICK THE BRANCH WITH THE HIGHEST MEAN

- $\ln T$ factor is the cost of exploration (the cost of bandit feedback) in i.i.d.