

Online Reinforcement Learning in Average-Reward MDPs

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Online Average-Reward RL: Setting and Performance Metrics



Setting

Online Average-Reward RL. An **agent** interacts with an average-reward MDP $M = (\mathcal{S}, \mathcal{A}, P, R)$ for T rounds (potentially unbounded) **without any reset**.

At each time step $t = 1, 2, \dots$:

- The agent observes the current state s_t and takes an action $a_t \in \mathcal{A}$
- M decides a reward $r_t \sim R(s_t, a_t)$ and a next state $s_{t+1} \sim P(\cdot | s_t, a_t)$
- The agent receives r_t (any time in step t before start of $t + 1$)

M is unknown (beyond \mathcal{S} and \mathcal{A}), and the goal is to maximize $\sum_{t=1}^T r_t$ (in expectation) using collected experience (history):

$$h_t = (s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}, r_{t-1}, s_t)$$

Need for balancing **exploration** and **exploitation**.



Setup

The goal is to maximize T -step total reward

$$\sum_{t=1}^T r_t$$

Recall that under π^* ,

$$\sum_{t=1}^T r_t^* = Tg^* + \mathcal{O}\left(\text{sp}(b^*)\sqrt{T\log(T/\delta)}\right), \quad \text{w.p.} \geq 1 - \delta$$

Hence, the agent can resort to learning a gain-optimal policy $\pi^* \in \Pi^{\text{SD}}$ in M :

$$\pi^*(s) \in \arg \max_{\pi^* \in \Pi^{\text{SD}}} g^\pi(s)$$

Hence, in online average-reward RL, the goal can be set to learn π^* from collected experience.



Online RL: Performance Metrics

- Many offline algorithms can be made online with some tricks.
- But will they explore well?

For online RL, we need performance metrics to measure the quality of [exploration-exploitation](#) tradeoff.



Online RL: Performance Metrics

The performance of a learning algorithm \mathbb{A} can be measured through:

- **Convergence:** Whether \mathbb{A} converges to an optimal (or near-optimal) policy.
- **PAC Sample Complexity:** The number of steps where the value of the current policy output by \mathbb{A} is not near-optimal with high-probability.
- **Regret:** The amount of reward lost due to choosing sub-optimal actions by \mathbb{A} .

In fact these metrics measure how [exploration-exploitation](#) tradeoff is implemented.



Regret

In online average-reward RL, the **Regret** of an algorithm \mathbb{A} is the difference between cumulative reward of the optimal policy π^* (oracle) and that gathered by \mathbb{A} :

$$\mathfrak{R}(\mathbb{A}, T) := \sum_{t=1}^T r_t^* - \sum_{t=1}^T r_t$$

- Agent \mathbb{A} 's trajectory:

$$\forall t : \quad a_t = \mathbb{A}(h_t), \quad r_t \sim R(\mathbf{s}_t, a_t), \quad \mathbf{s}_{t+1} \sim P(\cdot | \mathbf{s}_t, a_t)$$

- Oracle's trajectory:

$$\forall t : \quad r_t^* \sim R(\mathbf{s}_t^*, \pi^*(\mathbf{s}_t^*)), \quad \mathbf{s}_{t+1}^* \sim P(\cdot | \mathbf{s}_t^*, \pi^*(\mathbf{s}_t^*))$$

Regret is directly connected to the agent's goal (maximizing $\sum_{t=1}^T r_t$).
Alternatively, the objective of the agent is to **minimize the regret**.



No-Regret Algorithm

- $\mathfrak{R}(\mathbb{A}, T)$ is a r.v., and we wish to control it **in expectation** or **with high probability**.
- An algorithm \mathbb{A} is a **no-regret learning algorithm** if either

$$\mathbb{E}[\mathfrak{R}(\mathbb{A}, T)] = o(T) \quad \text{or} \quad \mathfrak{R}(\mathbb{A}, T) = o(T) \quad \text{w.h.p.}$$

No-Regret Algorithm

An algorithm \mathbb{A} is called **no-regret** if there exists a deterministic function f with $\frac{f(T)}{T} \rightarrow_{T \rightarrow \infty} 0$, such that one of the following holds:

$$\mathbb{E}[\mathfrak{R}(\mathbb{A}, T)] \leq f(T)$$

$$\mathfrak{R}(\mathbb{A}, T) \leq f(T) \quad \text{with high probability.}$$

- f could be MDP-dependent but must be deterministic.
- Note that a high-probability bound on $\mathfrak{R}(\mathbb{A}, T)$ implies a bound on $\mathbb{E}[\mathfrak{R}(\mathbb{A}, T)]$, but not the other way around.



Exploration vs. Exploitation

The key difficulty to do so is to balance *exploration* and *exploitation*:

- Play the best action so far, ...
- ... or rather explore a different action?



Warm-up: A Simple Algorithm



Empirical MDP

For any $t \geq 1$, define

- $N_t(s, a, s')$: number of visits, up to t , to (s, a) followed by a visit to s'

$$N_t(s, a, s') = \sum_{i=1}^{t-1} \mathbb{I}\{s_i = s, a_i = a, s_{i+1} = s'\}$$

- $N_t(s, a) = \sum_{s' \in \mathcal{S}} N_t(s, a, s')$

Empirical Estimator for P :

$$\forall s' \in \mathcal{S} : \quad \hat{P}_t(s'|s, a) = \begin{cases} \frac{N_t(s, a, s')}{N_t(s, a)} & \text{if } N_t(s, a) > 0 \\ \frac{1}{S} & \text{otherwise} \end{cases}$$

Empirical Estimator for R :

$$\hat{R}_t(s, a) = \frac{1}{N_t(s, a)} \sum_{i=1}^{t-1} r_i \mathbb{I}\{s_i = s, a_i = a\}$$



Empirical MDP

The empirical MDP:

$$\widehat{M}_t = (\mathcal{S}, \mathcal{A}, \widehat{P}_t, \widehat{R}_t)$$

Why not only using \widehat{M}_t . I.e., finding the optimal policy in $\widehat{\pi}_t^*$ and taking $a_t = \widehat{\pi}_t^*(s_t)$ each step.

\Rightarrow No exploration-exploitation tradeoff. Will not lead to a no-regret algorithm.

A better proposal. At each time t ,

$$a_t = \begin{cases} \widehat{\pi}_t^*(s_t) & \text{w.p. } 1 - \varepsilon_t \\ \text{chosen uniformly at random over } \mathcal{A} & \text{w.p. } \varepsilon_t \end{cases}$$

Despite its simplicity, it will enjoy a sublinear regret for suitably chosen ε_t —E.g., $\varepsilon = \frac{1}{\sqrt{t}}$.



A Simple Algorithm

- **input:** $(\varepsilon_t)_{t \geq 1}$
- **initialization:** For all (s, a, s') , $N(s, a, s') = 0$
- **for** $t = 1, 2, \dots, T$
 - Compute estimates \hat{P}_t and \hat{R}_t
 - Find $\hat{\pi}_t^*$ using VI with accuracy $\frac{1}{\sqrt{t}}$
 - Take action

$$a_t = \begin{cases} \hat{\pi}_t^*(s_t) & \text{w.p. } 1 - \varepsilon_t \\ \text{chosen uniformly at random over } \mathcal{A} & \text{w.p. } \varepsilon_t \end{cases}$$

- Receive reward $r_t \sim R(s_t, a_t)$ and next-state $s_{t+1} \sim P(\cdot | s_t, a_t)$
- Update $N(s_t, a_t, s_{t+1})$



UCRL2: Upper Confidence Reinforcement Learning



The simple algorithm implements

the certainty equivalence principle + exploration.

- Intuitive design (+)
- For suitable ε_t , it becomes no-regret (+)
- Tuning ε_t is not easy, and may require prior knowledge to obtain sublinear regret (−)
- Weak empirical performance (−).

We need more powerful principle.



OFU Principle

Optimism in the Face of Uncertainty (OFU)

- A well-known principle in balancing **exploration-exploitation** in bandits and online RL dating back to (Lai & Robbins, 1985).
- Also known as the **Optimism** principle

The OFU Principle: In an uncertain world, suppose that the environment is the best possible (in terms of rewards)!

- If the chosen action is optimal \implies no penalty
- If sub-optimal \implies reducing uncertainty



Optimism in the Face of Uncertainty (OFU)

In bandits, OFU prescribes replacing unknown mean rewards by their corresponding high-probability UCBs. the most prominent example is the UCB algorithm.

In MDPs, different implementations exist depending on the approach

- Model-based: Select the best candidate environment (among all plausible models/MDPs), i.e. the one leading to the **highest possible value function**.
- Model-free: When updating the Q-function, be optimistic. Initialize all Q-values to their highest possible value and use “reward + exploration bonus” instead of “reward” alone.

This lecture: A no-regret algorithm (UCRL2) based on OFU.



OFU: Model-Based

UCRL2 (Jaksch et al., 2010):

- Stands for **Upper Confidence Reinforcement Learning**
- A **model-based** algorithm for average-reward designed based on OFU.

Model-based recipe for the optimism principle (OFU):

- **Step 1:** Maintains a set of **plausible MDPs (models)** (i.e., consistent with history h_t). This can be done by defining high-probability **confidence sets** for R and P , and forming a corresponding set of MDPs.
- **Step 2:** Choose an optimistic model (among all models) and an optimistic policy leading to the **highest gain**.



Step 1: Confidence Sets

$\delta \in (0, 1)$ is given.

Confidence Set for R :

- Define a confidence set for $R(s, a)$ as

$$C_{s,a} = \left\{ \lambda \in [0, 1] : |\hat{R}_t(s, a) - \lambda| \leq \beta_{N_t(s,a)} \right\}$$

for some suitable function $\beta_{N_t(s,a)}$.

- For example, using Hoeffding's inequality (combined with Laplace's methods):

$$\beta_n = \sqrt{\frac{1}{2n} \left(1 + \frac{1}{n}\right) \log \frac{SA\sqrt{n+1}}{\delta}}, \quad n \in \mathbb{N}.$$

$$\mathbb{P}\left(\forall t, \forall (s, a) : R(s, a) \in C_{s,a}\right) \geq 1 - \delta$$



Step 1: Confidence Sets

$\delta \in (0, 1)$ is given.

Confidence Set for P :

- Define a confidence set for $P(\cdot|s, a)$ as

$$C'_{s,a} = \left\{ q \in \Delta(\mathcal{S}) : \|\widehat{P}_t(\cdot|s, a) - q\|_1 \leq \beta'_{N_t(s,a)} \right\}$$

for some suitable function $\beta'_{N_t(s,a)}$.

- For example, using Weissman's inequality (combined with Laplace's methods):

$$\beta'_n = \sqrt{\frac{2}{n} \left(1 + \frac{1}{n}\right) \log \frac{SA(2^S - 2)\sqrt{n+1}}{\delta}}$$

$$\mathbb{P}\left(\forall t, \forall (s, a) : P(\cdot|s, a) \in C'_{s,a}\right) \geq 1 - \delta$$



Step 1: Set of Models

Confidence sets $\{C_{s,a}, C'_{s,a}\}_{s \in \mathcal{S}, a \in \mathcal{A}}$ yield a set of models consistent with h_t :

$$\mathcal{M}_t = \left\{ M' = (\mathcal{S}, \mathcal{A}, P', R') : \right. \\ \left. P'(\cdot | s, a) \in C'_{s,a} \text{ and } R'(s, a) \in C_{s,a}, \forall (s, a) \right\}$$

- \mathcal{M}_t collects *all MDPs* that could be a candidate for the true Model M (in view of h_t).
- M is trapped in \mathcal{M}_t with high probability, *simultaneously for all t* :

$$\mathbb{P}(\forall t : M \in \mathcal{M}_t) \geq 1 - 2\delta$$

- \mathcal{M}_t is called a *bounded parameter MDP*.



Step 2: Planning

Step 2: Planning. To implement OFU, we wish to find

$$\pi_t \in \arg \max_{\mathcal{M}_t} \max_{\pi \in \Pi^{\text{SD}}} g_{M'}^{\pi}$$

and then we choose $a_t = \pi_t(s_t)$.

Alternatively, by Bellman's optimality equation, we wish to find \tilde{g} and \tilde{b} satisfying:
For all s ,

$$\tilde{g} + \tilde{b}(s) = \max_{a \in \mathcal{A}} \left(\max_{R'(s,a) \in C_{s,a}} R'(s,a) + \max_{P'(\cdot|s,a) \in C'_{s,a}} \sum_x P'(x|s,a) \tilde{b}(s) \right)$$



Step 2: Planning

$$\tilde{g} + \tilde{b}(s) = \max_{a \in \mathcal{A}} \left(\max_{R'(s,a) \in C_{s,a}} R'(s,a) + \max_{P'(\cdot|s,a) \in C'_{s,a}} \sum_x P'(x|s,a) \tilde{b}(s) \right)$$

Compared to optimality equations for MDPs, we have two extra maximizations.

- The one in **blue** admits a closed-form solution:

$$\max_{R'(s,a) \in C_{s,a}} R'(s,a) = \hat{R}_t(s,a) + \beta_{N_t(s,a)}$$

- No closed-form solution to the second. However, for a fixed $u \in \mathbb{R}^S$, the problem

$$\max_{p \in C'(s,a)} \sum_x p(x) u(x)$$

can be solve using a simple procedure thanks to the shape of $C'_{s,a}$.

The second optimization problem can be efficiently solved using **Extended Value Iteration (EVI)**



Step 2: Planning

In summary,

- To implement OFU, we wish to find

$$\pi_t \in \arg \max_{M' \in \mathcal{M}_t} \max_{\pi \in \Pi^{\text{SD}}} g_{M'}^{\pi}$$

- It suffices to find a $\frac{1}{\sqrt{t}}$ -optimal policy π_t :

$$g^{\pi_t} \geq \arg \max_{M' \in \mathcal{M}_t} \max_{\pi \in \Pi^{\text{SD}}} g_{M'}^{\pi} - \frac{1}{\sqrt{t}}$$

- This can be done efficiently by Extended Value Iteration (EVI) —see next slides for the pseudo-code.



UCRL2-L: Planning

For technical issues with regret analysis, UCRL2-L does not update policy π_t at each step. Rather it proceeds in *internal epochs*:

- In each epoch, the policy will be kept unchanged.
- An epoch stops as soon as $N_t(s, a)$ for some (s, a) is doubled (compared to its number before the episode).

To implement this, UCRL2-L maintains two sets of counters:

- Global counters:
 $N(s, a, s')$ for each (s, a, s') , and $N(s, a) = \max 1, \sum_{s'} N(s, a, s')$
- Per-epoch counters: $\nu(s, a, s')$, which count the number of visits within an epoch. Further define:

$$\nu(s, a) = \sum_{s'} \nu(s, a, s')$$



UCRL2-L

- **input:** δ
- **initialization:** For all (s, a) ,
 - $N(s, a) = 0, v(s, a) = 0$
- **for** epochs $k = 1, 2, \dots$
 - $N(s, a, s') \leftarrow N(s, a, s') + \nu(s, a, s')$ for all (s, a)
 - Compute estimates \hat{P}_t and \hat{R}_t
 - Find π_k using EVI with accuracy $\frac{1}{\sqrt{t}}$
 - $\nu(s, a, s') = 0$ for (s, a, s')
 - **repeat**
 - Choose $a_t = \pi_k(s_t)$
 - Receive reward $r_t \sim R(s_t, a_t)$ and next-state $s_{t+1} \sim P(\cdot | s_t, a_t)$
 - Update $\nu(s_t, a_t, s_{t+1}) \leftarrow \nu(s_t, a_t, s_{t+1}) + 1$
 - Increment t
 - **until** $\nu(s, a) = N(s, a)$ for some (s, a)



UCRL2-L: EVI

- **input:** ε
- **initialization:** Select $u_0 \in \mathbb{R}^S$ arbitrarily. Set $n = -1$.
- **repeat:**
 - Increment n
 - Compute, for each (s, a) ,

$$R'(s, a) = \hat{R}_t(s, a) + \beta_{N(s, a)}$$

$$P'(\cdot|s, a) \in \operatorname{argmax} \left\{ \sum_{x \in \mathcal{S}} q(x) u_n(x) : q \in C'_{s, a} \right\}$$

- Update, for each (s, a) ,

$$u_{n+1}(s) = \max_a \left(R'(s, a) + \sum_{x \in \mathcal{S}} P'(x|s, a) u_n(x) \right)$$

until $\max_s (u_{n+1}(s) - u_n(s)) - \min_s (u_{n+1}(s) - u_n(s)) < \varepsilon$

- **output:** Policy π_k ,

$$\pi_k(s) \in \operatorname{argmax}_a \left(R'(s, a) + \sum_{x \in \mathcal{S}} P'(x|s, a) u_n(x) \right), \quad \forall s$$



UCRL2-L: Inner Maximization in EVI

Algorithm for solving

$$\max_{q \in C'_{s,a}} \sum_{x \in \mathcal{S}} q(x)u(x)$$

Index $\mathcal{S} = \{s_1, s_2, \dots, s_S\}$, and assume w.l.o.g. that

$$u(s_1) \geq u(s_2) \geq \dots \geq u(s_S)$$

- **initialization:** $q = \hat{P}_t(\cdot|s, a)$
- Set $q(s_1) = \hat{P}_t(s_1|s, a) + \frac{1}{2}\beta'_{N_t(s,a)}$
- $\ell = S$
- **while:** $\sum_{x \in \mathcal{S}} q(x) > 1$
 - Set $q(s_\ell) = \max\{0, 1 - \sum_{x \neq s_\ell} q(x)\}$
 - Decrement ℓ
- **output:** q



UCRL2-L: Regret Guarantee



UCRL2-L: Regret

Regret bound for UCRL2-L in **any communicating MDP** with S states, A actions, and diameter D :

Theorem (Regret of UCRL2-L)

Let $\delta \in (0, 1)$. The regret under UCRL2-L satisfies

$$\mathfrak{R}(T) \leq 24DS\sqrt{AT\log(T/\delta)},$$

with probability at least $1 - \delta$, and uniformly for all $T \geq 2$.

- Expected regret:

$$\mathbb{E}[\mathfrak{R}(T)] \leq (1 - \delta) \times 24DS\sqrt{AT\log(T/\delta)} + \delta T$$

Setting $\delta = 1/\sqrt{T}$ gives: $\mathbb{E}[\mathfrak{R}(T)] \leq 31DS\sqrt{AT\log(T)}$

- For rewards supported on $[a, b]$ (instead of $[0, 1]$), scale $\mathfrak{R}(T)$ by $(b - a)$.



UCRL2-L: Regret

The theorem tells us that UCRL2-L is **no-regret**. More precisely:

$$\mathbb{P}\left\{\sum_{t=1}^T r_t^* > \sum_{t=1}^T r_t + 24DS\sqrt{AT\log(T/\delta)}\right\} < \delta,$$

Note this is a worst-case bound:

$$\sup_{M'} \mathfrak{R}_{M'}(T) \leq 24DS\sqrt{AT\log(T/\delta)}, \quad \text{w.p.} \geq 1 - \delta$$

where M' is **any** communicating MDP with S states, A actions, and diameter D .
In particular, it holds for **hardest-to-learn** M' .



Regret: UCRL2-L vs. UCB

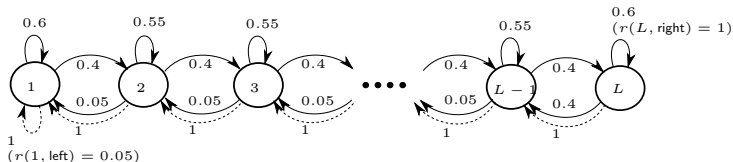
Regret bound in the theorem resembles that for UCB in K -armed bandits:

$$\underbrace{\mathcal{O}(DS\sqrt{AT\log(T)})}_{\text{UCRL2-L}} \quad \text{vs.} \quad \underbrace{\mathcal{O}(\sqrt{KT\log(T)})}_{\text{UCB}}$$

In K -armed bandits, we have K unknowns distributions, whereas in MDPs there are $2SA$ unknown distributions (one reward dist. and one transition dist. per state-action pair).

Why does regret must depend on diameter D ?

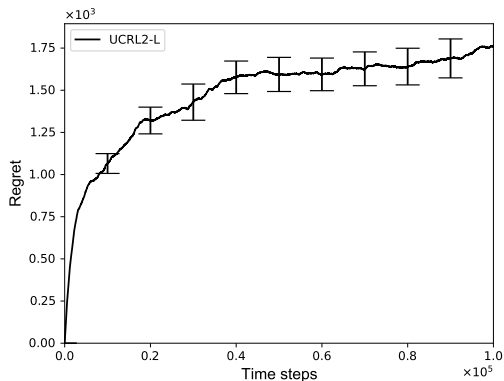
Intuitively, D captures the price to navigate in the MDP due to learning.



UCRL2-L: Empirical Performance



Numerical Experiments



UCRL2-L in 6-state RiverSwim: Average regret shown with 95% CIs for $n = 30$ experiments, $\delta = 0.01$.

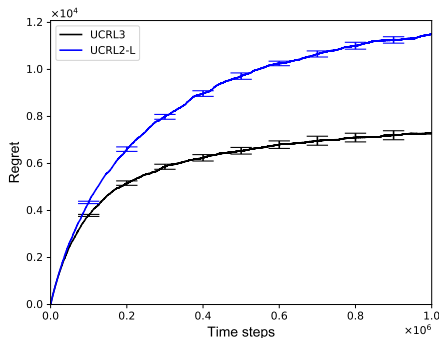


Numerical Experiments

- UCRL2-L brings massive improvement over UCRL2 in almost any MDP, yet achieving a smaller regret bound.
- This is due to using tighter confidence sets derived using a more advanced tool than vanilla union bounds.
- Can we do better empirically than UCRL2-L?



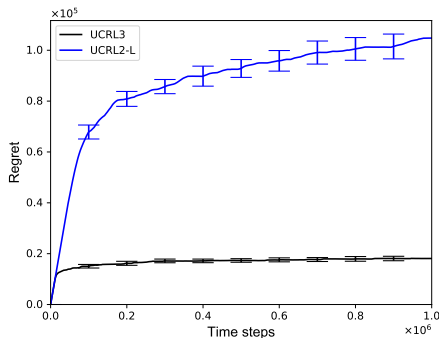
Numerical Experiments



- UCRL2-L in 4-room grid-world: Average regret shown with 95% CIs for $n = 30$ experiments).
- The black curve shows the result of UCRL3 (Bourel et al., 2020).



Numerical Experiments



- UCRL2-L in 25-state RiverSwim: Average regret shown with 95% CIs for $n = 30$ experiments)
- The black curve shows the result of UCRL3 (Bourel et al., 2020).



Worst-Case Regret Lower Bound



Worst-Case Lower Bound

How good is the regret bound of UCRL2? Could it be improved?

To answer these, we need to derive lower bounds on regret.

- Problem-dependent lower bound
- Worst-case lower bound



Worst-Case Lower Bound

Theorem (Worst-Case Regret Lower Bound)

Let $S, A \geq 5$, $D \geq 20 \log_A S$, and $T \geq DSA$. For *any learning algorithm* \mathbb{A} , there exists *an MDP* M with S states, A actions, and diameter D such that for any initial state, the T -step expected regret under \mathbb{A} satisfies

$$\mathbb{E}[\mathfrak{R}(\mathbb{A}, T)] \geq 0.015\sqrt{DSAT}$$

- A regret of $\Omega(\sqrt{DSAT})$ is a fundamental performance limit for communicating MDPs, which no algorithm can beat.
- Compare it to the minimax lower bound of $\Omega(\sqrt{KT})$ for stochastic K -armed bandits.



Worst-Case Lower Bound

$$\underbrace{\Omega\left(\sqrt{DSAT}\right)}_{\text{worst-case LB}} \quad \text{vs.} \quad \underbrace{\tilde{\mathcal{O}}\left(DS\sqrt{AT\log\frac{T}{\delta}}\right)}_{\text{UCRL2-L's regret}}$$

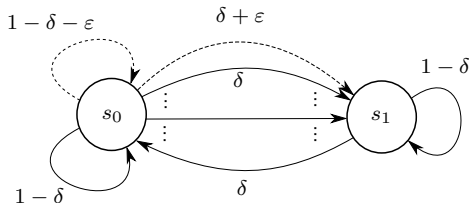
UCRL2 is **rate-optimal** (i.e., its regret has optimal dependence on T , up to logarithmic factors).

- There is a gap of $\sqrt{DS\log(T/\delta)}$ between the LB and the UB.
- The gap is reduced by improved variants of UCRL2 and UCRL2-L.



Worst-Case Lower Bound: Proof

A family of worst-case 2-state MDPs, parameterized by $\delta \in (0, \frac{1}{3})$ and $\varepsilon \leq \delta$:

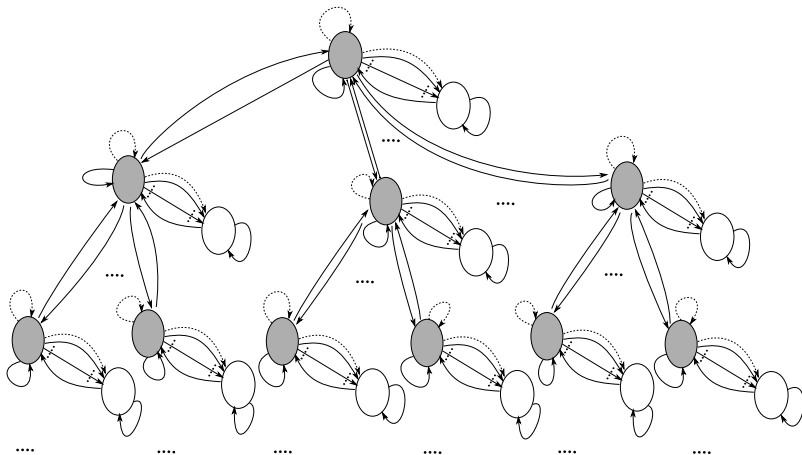


- A actions per state, all with identical rewards and transitions. Only one action in s_0 has a slightly different transition.
- For all actions, in s_0 the reward is 0, but in s_1 the reward is 1.
- Choosing $\varepsilon \propto \sqrt{\frac{A}{TD}}$ leads to a **worst-case MDP** with 2 states, A actions, and diameter D for any algorithm.



Worst-Case Lower Bound: Proof

A worst-case instance for $S > 2$ —for details, see (Jaksch et al., 2010).



Remarks

*UCRL2 is a model-based algorithm for **regret minimization** in average-reward MDPs.*

- Several variants of UCRL2 exist that improve upon its theoretical and/or empirical regret.
- State-of-the-art UCRL2-style algorithms achieve a regret bound *almost* matching the LB.
- These algorithms outperform model-free algorithms empirically, often by a large margin.
- Logarithmic regret bounds are mostly open.



Remarks

*UCRL2 is a model-based algorithm for **regret minimization** in average-reward MDPs.*

Key questions:

- Does it find a near-optimal policy?
- Does it output an accurate estimation \widehat{M} of the true MDP?
- Is it capable of doing generalization in MDPs (when possible)?



Remarks

UCRL2 is a model-based algorithm for regret minimization in average-reward MDPs.

Key questions:

- Does it find a near-optimal policy? It does not have a policy recommendation. In general, regret minimization is different than best policy identification.
- Does it output an accurate estimation \widehat{M} of the true MDP? Not necessarily. Some (rewarding) part of state-space could be visited much more than other parts.
- Is it capable of doing generalization in MDPs (when possible)? It doesn't. It assumes that various state-action pairs are unrelated in terms of p and r .



Remarks

(Near)-optimal behavior is easier to learn than the truth.

Minds & Machines (2016) 26:243–252
DOI 10.1007/s11023-016-9389-y



Optimal Behavior is Easier to Learn than the Truth

Ronald Ortner¹

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Abstract We consider a reinforcement learning setting where the learner is given a set of possible models containing the true model. While there are algorithms that



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