Lab Session 1: State estimation

Laboratory on Least Mean Square position and bias estimation | **LAB BRIEFING**

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Least Mean Square Single Point Positioning (SPP)

- In the most general case, the **Position, Velocity and Time** (PVT) solution is estimated by linearizing a multilateration problem through a **Least Mean Square** (LMS) algorithm (a.k.a. LS)

$$\boxed{\Delta\mathbf{x} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\Delta\boldsymbol{\rho}}$$

- One possible way to solve for the PVT is to recursively apply the LMS algorithm by updating the linearization point to obtain a more accurate solution
- At each iteration, the estimated PVT is expected to be closer to the actual user location
- In this lab we will process pseudorange to solve for position and clock bias only.
  Velocity estimation requires Doppler measurements (not available).

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Recursive Least Mean Square

- For each time instant n, a number of iterations $k = 1, \dots, K$ is performed to process the same measurements vector $\boldsymbol{\rho}_n$

- Initial position/linearization point
- Geometrical ranges from $\hat{\boldsymbol{x}}_n^0$ and satellite positions
- Measured/Observed pseudorange measurements

Initialization	$\hat{\mathbf{x}}_n^0$
Procedure for k-th iteration	$\Delta\hat{\boldsymbol{\rho}}_n^k = \hat{\boldsymbol{\rho}}_n^k - \boldsymbol{\rho}_n$
	$\Delta\hat{\mathbf{x}}_n^k = \left((\mathbf{H}_n^k)^{\mathrm{T}}\mathbf{H}_n^k\right)^{-1}(\mathbf{H}_n^k)^{\mathrm{T}}\Delta\hat{\boldsymbol{\rho}}_n^k$
	$\hat{\mathbf{x}}_n^k = \hat{\mathbf{x}}_n^{k-1} + \Delta\hat{\mathbf{x}}_n^k$

- $\mathbf{H}$ is updated at each iteration k
- The initial linearization point, $\hat{\mathbf{x}}_n^0$, could be the same at each time n
- K has to be chosen in order to obtain a “stable” solution (generally $K < 10$)

Recursive Least Mean Square

- $\hat{\mathbf{x}}_n^k = [\hat{x}_n^k \quad \hat{y}_n^k \quad \hat{z}_n^k \quad \hat{b}_n^k]$ is the **state estimate** (Position and Time) at the k-th iteration
- The **number of visible satellites**, J, is **not constant** over the observation timespan
- $\Delta\boldsymbol{\rho}_n$ is the vector of the J_n **pseudoranges differences** (between observed/measured and calculated w.r.t. the linearization point $\hat{\mathbf{x}}_n^0$) at the k-th iteration
- At each iteration k, $\mathbf{H}_n^k$ is the geometrical matrix obtained from the previous estimated position:

Satellite coordinates are kept fixed for any k

$$\mathbf{H}_n^k = \begin{bmatrix} a_{x,1_n}^k & a_{y,1}^k & a_{z,1_n}^k & 1 \\ a_{x,2_n}^k & a_{y,2_n}^k & a_{z,2_n}^k & 1 \\ a_{x,3_n}^k & a_{y,3_n}^k & a_{z,3_n}^k & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{x,J_n}^k & a_{y,J_n}^k & a_{z,J_n}^k & 1 \end{bmatrix}$$

$$a_{x,j_n}^k = \left[\frac{x_{j,n} - \hat{x}_n^k}{\hat{\rho}_{j,n}}\right],\ a_{y,j_n} = \cdots,\ a_{z,j_n} = \cdots$$

$$\hat{\rho}_{j,n} = \sqrt{\left(x_{j,n} - \hat{x}_n^k\right)^2 + \left(y_{j,n} - \hat{y}_n^k\right)^2 + \left(z_{j,n} - \hat{z}_n^k\right)^2}$$

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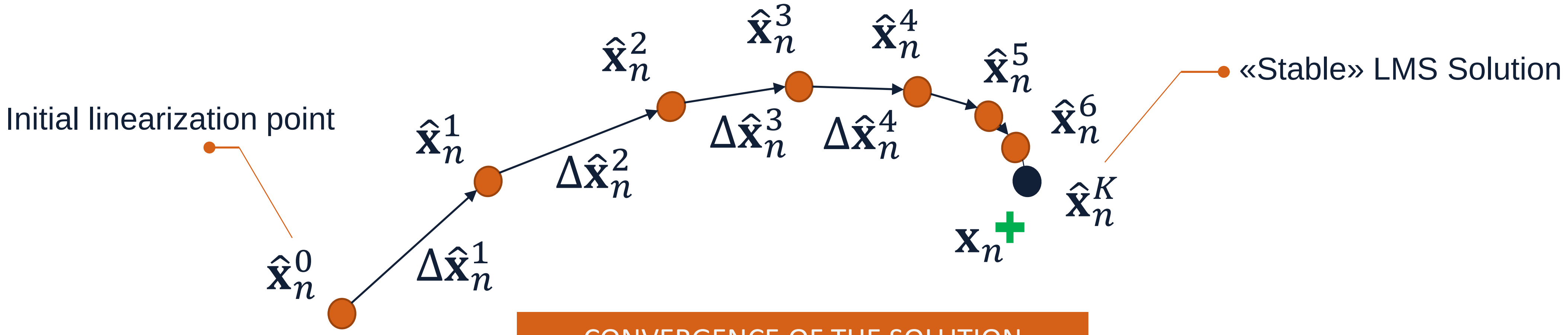
Recursive Least Mean Square

- By adding $\Delta\hat{\mathbf{x}}_n^k$ to the estimated position at the previous iteration, the position estimate is updated, and a new solution is found.

$$\boxed{\hat{\mathbf{x}}_n^k = \hat{\mathbf{x}}_n^{k-1} + \Delta\hat{\mathbf{x}}_n^k}$$



CONVERGENCE OF THE SOLUTION

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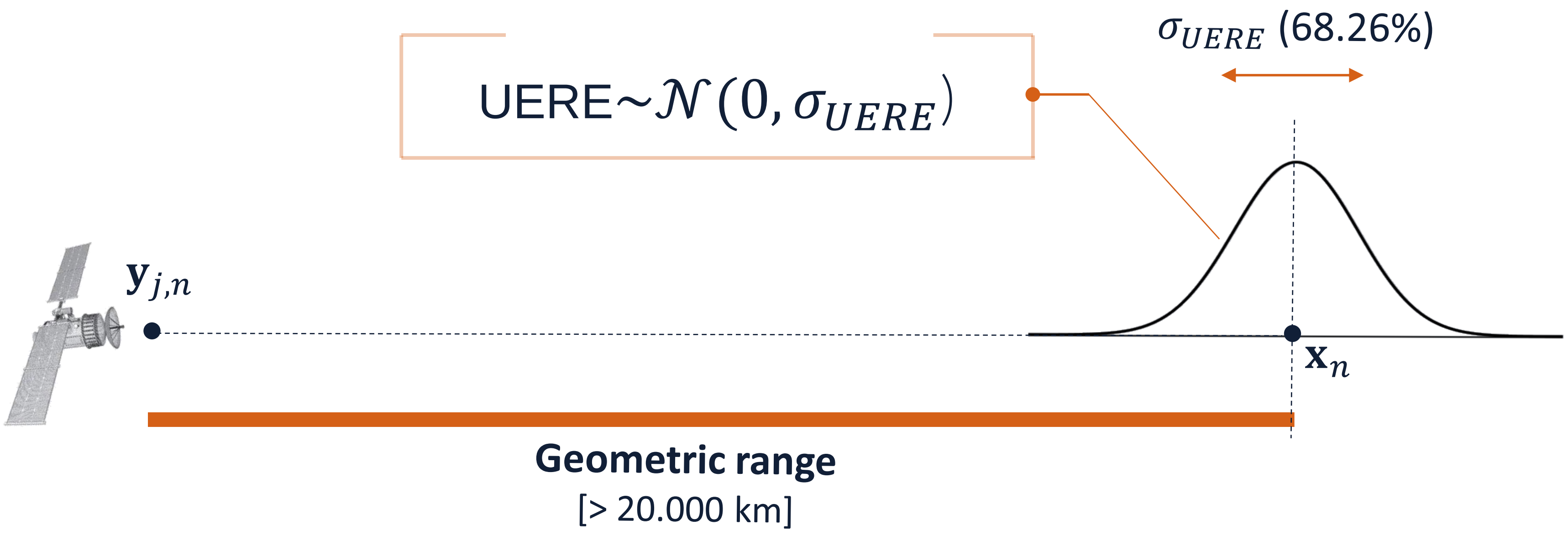
From raw to corrected pseudorange measurements

- In real applications, the measurement of the pseudoranges must be corrected of all the predictable error contributions (bias terms).
- After the correction, a **random contribution** is left that is modeled by the User Equivalent Range Error (UERE). This error along with the geometrical dilution of precision are responsible for the **state estimation error** and condition the precision of the solution.

$$\text{UERE} \sim \mathcal{N}(0, \sigma_{UERE})$$



Different UERE for different satellites

A naive LMS algorithm attributes the **same relevance to all the measurements** but actual pseudorange measurements are **not equally precise**.

- In real cases, each pseudorange may be characterized by a different value of standard deviation, $\sigma_{j,\mathrm{UERE}}$
- Pseudorange errors are assumed uncorrelated such that their covariance matrix is written as:

Measurements Error Covariance

$$\mathbf{R} = \mathrm{Cov}(\boldsymbol{\rho}) = \begin{bmatrix} \sigma^2_{1,UERE} & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_{2,UERE} & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_{3,UERE} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma^2_{J,UERE} \end{bmatrix}$$

Different errors for different satellites

- The direct estimation of the measurements error covariance matrix is not possible since each $\rho_{j,n}$ is time-dependent (multiple realizations for the same instant n are not available)
- We can estimate the $\sigma_{j,\mathrm{UERE}}$ for each pseudorange, ρ_j, analyzing the error on the pseudorange itself along the time (assuming it is an ergodic process)
- Dependency on time must be removed (measurement de-trending)
- Since pseudorange measurement follows a nearly quadratic trend, the second order derivative is sufficient to remove their trend along the time

Use MATLAB function `diff(X,`*`order`*`)`

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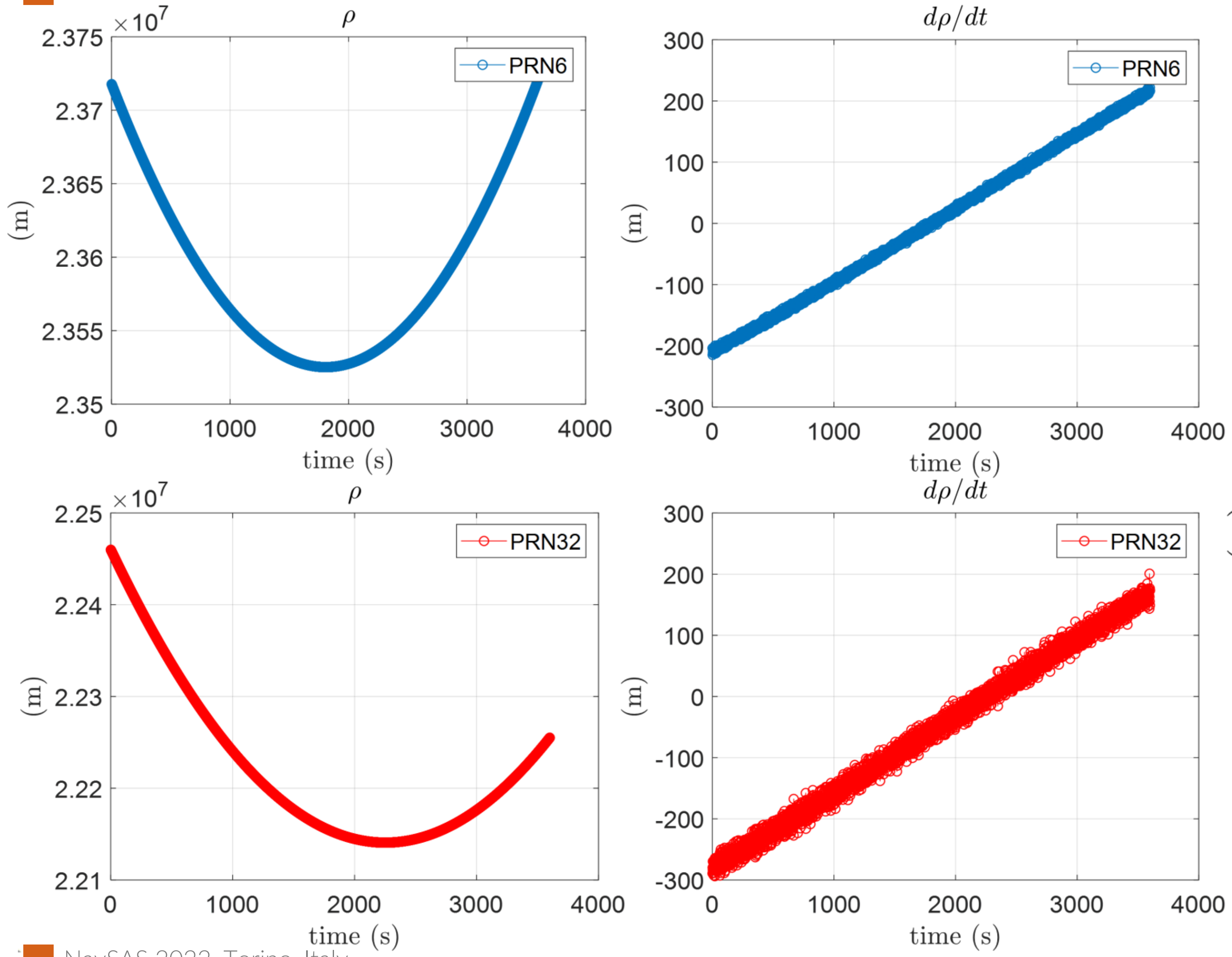
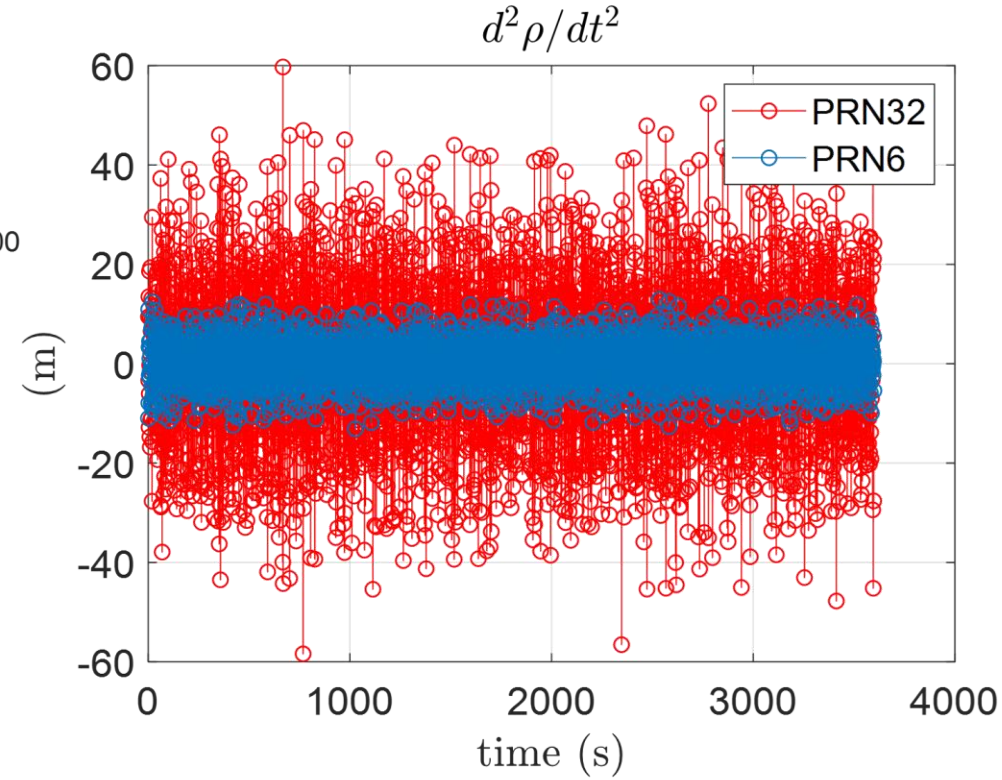
Pseudorange de-trending by differentiation



Dentrended time series allows to observe the different UEREs affecting the pseudoranges measured via different signals

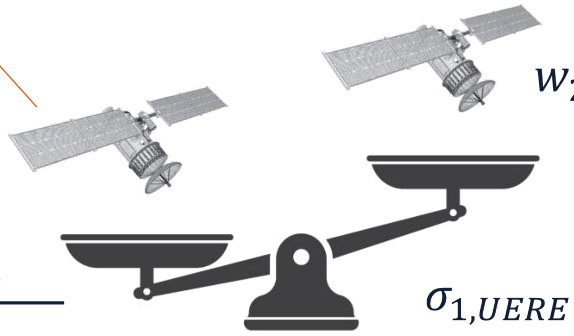


Weighted Least Mean Square (WLMS/WLS)

- A second approach foresees to solve the system by means of a **Weighted Least Mean Squares, or simply Weighted Least Square** (WLS) algorithm, characterized by the introduction of a weighting matrix $\mathbf{W}$, which is a positive definite matrix
- Since some measurements may be known to be more accurate than others, the measurement accuracy is known to be characterized by the inverse of the measurement errors covariance matrix $\mathbf{R}$
- It is natural to select $\mathbf{W} = \mathbf{R}^{-1}$ to give the least weighting to the most uncertain measurements.

Low-noise measurements
(e.g., PRN 6)

Higher noise measurements
(e.g., PRN 32)



$$w_2 = \frac{1}{\sigma_{2,UERE}}$$

$$w_1 = \frac{1}{\sigma_{1,UERE}}$$

$$\sigma_{1,UERE} < \sigma_{2,UERE} \Leftrightarrow w_1 > w_2$$

Uncorrelated measurements

- The weight matrix can be estimated from the measurements, thus designing a new **weighted geometrical matrix** $\overline{\mathbf{H}}_n^k$
- For each time instant n, a number of iterations $k = 1, \ldots, K$ is performed for the same measurements vector $\boldsymbol{\rho}_n$

Initialization	$\hat{\mathbf{x}}_n^0$
Procedure for k-th iteration	$\Delta\hat{\boldsymbol{\rho}}_n^k = \hat{\boldsymbol{\rho}}_n^k - \boldsymbol{\rho}_n$
	$\overline{\mathbf{H}}_n^k = \left((\mathbf{H}_n^k)^{\mathrm{T}}\mathbf{W}\mathbf{H}_n^k\right)^{-1}(\mathbf{H}_n^k)^{\mathrm{T}}\mathbf{W}$
	$\Delta\hat{\mathbf{x}}_n^k = \overline{\mathbf{H}}_n^k\,\Delta\hat{\boldsymbol{\rho}}_n^k$
	$\hat{\mathbf{x}}_n^k = \hat{\mathbf{x}}_n^{k-1} + \Delta\hat{\mathbf{x}}_n^k$

Weigthing Matrix, $\mathbf{W} = \mathbf{R}^{-1}$

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[03LPXBG, 02LPXQW] – Satellite Navigation Systems

Lab Session 1: State estimation

Laboratory on Least Mean Square position and bias estimation | **DATASETS AND TASKS**

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Variables for GPS constellation

USER and SATELLITE STATE VECTORS (position and clock bias)

Unknown User Position

$$\mathbf{x}_n = [x_n \quad y_n \quad z_n \quad b_{n,GPS}]$$

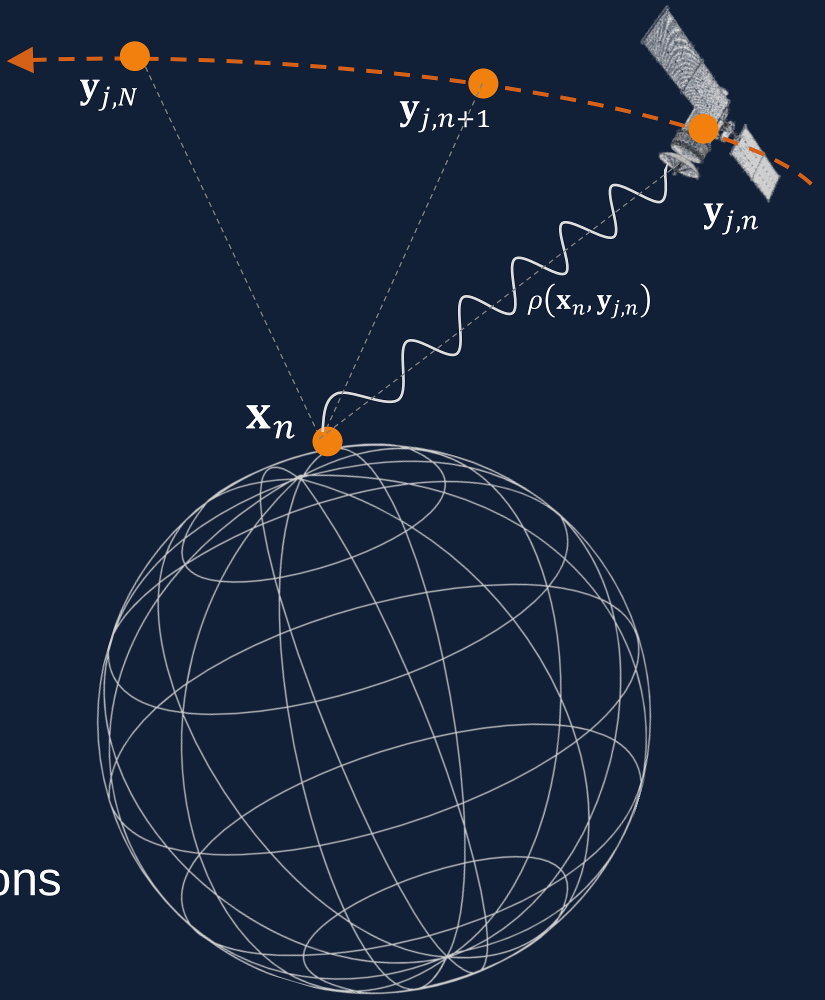
Known Satellite Position

$$\mathbf{y}_{j,n} = [x_{j,n} \quad y_{j,n} \quad z_{j,n}\ ]$$

PSEUDORANGE EQUATION

Measured range distance from the satellite

$$\rho_{j_n} = \rho(\mathbf{x}_n, \mathbf{y}_{j,n}) = \sqrt{(x_n - x_{j,n})^2 + (y_n - y_{j,n})^2 + (z_n - z_{j,n})^2} + b_{n,GPS}$$



- $n \in (1,2,\dots,N)$ is the **time index**
- $j \in (1,2,\dots,J)$ is the **satellite identifier**
- Data collections include **3600 seconds** (1 hour) of satellites observations from a static position $\mathbf{x}_n$: $\mathbf{x}_n = \mathbf{x}_{n+1} = \cdots = \mathbf{x}_N$, where $N = 3600$

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Observables Data Structures

PSEUDORANGE MEASUREMENTS (GPS)

RHO.GPS				
	$n = 1$	$n = 2$	..	$n = N$
GPS.PRN_1	$\rho(\mathbf{x}_1, \mathbf{y}_{1,1})$	$\rho(\mathbf{x}_2, \mathbf{y}_{1,2})$	…	$\rho(\mathbf{x}_N, \mathbf{y}_{1,N})$
GPS.PRN_2	$\rho(\mathbf{x}_1, \mathbf{y}_{2,1})$	$\rho(\mathbf{x}_2, \mathbf{y}_{2,2})$	…	$\rho(\mathbf{x}_N, \mathbf{y}_{2,N})$
GPS.PRN_3	$\rho(\mathbf{x}_1, \mathbf{y}_{3,1})$	$\rho(\mathbf{x}_2, \mathbf{y}_{3,2})$	…	$\rho(\mathbf{x}_N, \mathbf{y}_{3,N})$
…	…	…	…	…
GPS.PRN_J	$\rho(\mathbf{x}_1, \mathbf{y}_{J,1})$	$\rho(\mathbf{x}_2, \mathbf{y}_{J,2})$	…	$\rho(\mathbf{x}_N, \mathbf{y}_{J,N})$

SATELLITE POSITIONS FROM EPHEMERIS (GPS)

SV_ECEF.GPS	
SV ID	$\mathbf{Y}_j$
GPS.PRN_1	$\mathbf{Y}_1$
GPS.PRN_2	$\mathbf{Y}_2$
GPS.PRN_3	$\mathbf{Y}_3$
…	…
GPS.PRN_J	$\mathbf{Y}_J$

Ex: SV Position State Vector History			
$\mathbf{Y}_1$	x	y	z
$n = 1$	$x_{1,1}$	$y_{1,1}$	$z_{1,1}$
$n = 2$	$x_{1,2}$	$y_{1,2}$	$z_{1,2}$
$n = 3$	$x_{1,3}$	$y_{1,3}$	$z_{1,3}$
…	…	…	…
$n = N$	$x_{1,N}$	$y_{1,N}$	$z_{1,N}$

- The number of visible satellites is not constant over observation time n
- The folder named «NominalUERE» contains pseudorange measurements with the same σ_{UERE}
- The folder named «RealisticUERE» contains pseudorange measurements with satellite-dependent σ_{UERE}
- Pseudorange measurements can be considered an ergodic random process over short time periods
- It is possible to select different constellations using strings GPS, GLO, BEI, GAL in the data structure

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Lab session | Evaluation of PVT

TASK 1

Load data from the *NominalUERE* folder. Check the satellites visibility at each time instant n, for all the constellations. Plot the number of the visible satellites as well as the measured pseudoranges along the time.

TASK 2

Choose a dataset and a constellation (e.g., GPS, Galileo) and implement a Least Mean Square (LMS) positioning algorithm to estimate the user state, $\hat{\mathbf{x}}_n^K$, at each time instant n. Convert the estimated solution from ECEF to LLA coordinates and verify the position on Google Earth by means of *writeKML_GoogleEarth.m (provided in Utilities)* or MATLAB's geoplot() and refmap().

TASK 3

a) Compute the average position; *b)* compute the *position error* for each time instant with respect to the average position; *c)* compute the *standard deviation of the position error* and *d)* compare the quality of the obtained solution for different datasets and constellations. Motivate the results.

TASK 4

Estimate the $\sigma_{j,\mathrm{UERE}}$ for all satellites using the dataset from the *realisticUERE* folder. Implement the **Weigthed Least Mean Square** (WLMS) and repeat Tasks 2 and 3. Compare the performance of LMS and WLMS on *realisticUERE* dataset.

TASK 5

Write a MATLAB function to draw the error ellipsoid on the set of positioning solutions and graphically compare the error ellipsoids for LMS and WLMS solutions.

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Simulation hint (LMS/WLMS implementation)

For each epoch $n = 1:N$

- Find the available pseudorange measurements
- Build the measurement vector $\boldsymbol{\rho}_n = [\rho_{1,n} \quad \cdots \quad \rho_{J,n}]$
- Retrieve the corresponding satellites coordinates $\mathbf{y}_{j,n}$
- Compute PVT solution (K iterations)

end

Number of LMS/WLMS iterations

Use a reasonable number of iterations, (i.e. $K < 10$)

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