Temă săpt. 14 - Alexandrescu Andra

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1.

a) $f_\theta(x) = e^{-2\theta} \cdot \frac{(2\theta)^x}{x!}$, $x \in \mathbb{N}$, $\theta \in \mathbb{R}$

I. Metoda momentelor (MM)

$E(X) = \overline{X}$ (media empirică) (media teoretică) cu X variabilă aleatoare discretă $\Rightarrow E(X) = \overline{X} = \sum_{x=0}^{\infty} x \cdot P(X=x)$

Rescriem funcția de masă a probabilității:

$P(X=x) = f(x) = e^{-2\theta} \cdot \frac{2\theta^x}{x!}$, $x \in \mathbb{N}$

De fapt, se observă că e o distribuție Poisson (probabilitatea a k evenimente într-un interval $\frac{\lambda^k \cdot e^{-\lambda}}{k!}$)

În cazul acesta, e vorba despre x evenimente (substituim k cu x) și unde $\lambda = 2\theta$: $\frac{2\theta^x \cdot e^{-2\theta}}{x!}$

Distribuția Poisson cu $\lambda = 2\theta$

are media variabilei aleatoare Poisson $E(X) = \lambda = 2\theta$

(Dem. caz general:

$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} = \lambda \cdot e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$)

Revenind:

$E(X) = \sum_{x=0}^{\infty} x \cdot e^{-2\theta} \cdot \frac{2\theta^x}{x!} = e^{-2\theta} \sum_{x=1}^{\infty} x \cdot \frac{2\theta^x}{x!} = 2\theta \cdot e^{-2\theta} \cdot \sum_{y=0}^{\infty} \frac{2\theta^y}{y!}$

nu depinde de x

$x \cdot \frac{2\theta^x}{x!} = \frac{2\theta^x}{(x-1)!} = 2\theta \cdot \frac{2\theta^{x-1}}{(x-1)!}$

Substituim x cu y unde $y = x-1$

$x = 1 \Rightarrow y = 0$

$x \to \infty \Rightarrow y \to \infty$

$= 2\theta \cdot e^{-2\theta} \cdot e^{2\theta} = 2\theta \cdot e^{\cancel{2\theta}\cancel{-2\theta}} = 2\theta \cdot 1 = 2\theta$

Obs. Caz general funcția exponențială: $e^x = \sum_{y=0}^{\infty} \frac{x^y}{y!}$

$\Rightarrow e^{2\theta} = \sum_{y=0}^{\infty} \frac{2\theta^y}{y!}$

Așadar, $E(X) = 2\theta$, iar estimarea lui λ presupune să egalez media teoretică $E(X)$ cu cea empirică $\overline{X}$

unde $\overline{X} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$; $2\theta = \overline{X} \Leftrightarrow \theta = \frac{\overline{X}}{2} = \frac{1}{2n} \cdot \sum_{i=1}^{n} X_i$

acest set de date $x_1 \dots x_n$ va da la media empirică să fie o val. apropiată de 2θ

II. Metoda Verosimilității Maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \dots, x_n$

[Caz general:

Funcția de verosimilitate pt distribuția Poisson:

$L(\lambda) \overset{not}{=} L(\lambda | x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} P(\lambda | X = x_i) = \prod_{i=1}^{n} e^{-\lambda} \cdot \frac{\lambda^{x_i}}{x_i!}$]

Așadar, $L(2\theta | x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} e^{-2\theta} \cdot \frac{2\theta^{x_i}}{x_i!} = e^{-2n\theta} \cdot \prod_{i=1}^{n} \frac{2\theta^{x_i}}{x_i!}$

$= e^{-2n\theta} \cdot (2\theta)^{\sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} \frac{1}{x_i!}$

Logaritmarea verosimilității:

$\ln L(\lambda) \xrightarrow[\log_n(a \cdot b) = \log_n a + \log_n b]{\text{devine sumă}} \ln(e^{-2n\theta}) + \ln\left((2\theta)^{\sum_{i=1}^{n} x_i}\right) + \ln\left(\prod_{i=1}^{n} \frac{1}{x_i!}\right) = -2n\theta + \sum_{i=1}^{n} x_i \cdot \ln(2\theta) + \sum_{i=1}^{n} \ln\left(\frac{1}{x_i!}\right)$

$= -2n\theta + \sum_{i=1}^{n} x_i \cdot \ln(2\theta) - \sum_{i=1}^{n} \ln(x_i!) = -2n\theta + \sum_{i=1}^{n} x_i \ln 2 + \sum_{i=1}^{n} x_i \ln\theta - \sum_{i=1}^{n} \ln(x_i!)$

Derivarea log-verosimilității în raport cu λ:

$\frac{\partial \ln L}{\partial \theta} = -2n + \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta}$

obs. $\frac{\partial}{\partial \theta}(-2n\theta) = -2n$

$\frac{\partial}{\partial \theta}\left(\sum_{i=1}^{n} x_i \cdot \ln 2\theta\right) = \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta}$

Egalarea derivatei cu 0 pt a afla maximul:

$-2n + \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta} = 0 \Leftrightarrow \frac{\sum_{i=1}^{n} x_i}{\theta} = 2n \Leftrightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{2n} = \frac{1}{2} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} x_i = \frac{\overline{X}}{2}$

Așadar, estimarea lui θ este $\hat{\theta} = \frac{\overline{X}}{2}$

Proprietăți estimatori (nedeplasare, consistență, eficiență):

1) Nedeplasare

Def. Un estimator e nedeplasat dacă media estimatorilor e egală cu valoarea reală a param. pe care îl estimează

Am aflat $\hat{\theta} = \frac{\overline{X}}{2}$

$E(\hat{\theta}) = E\left(\frac{1}{2n} \sum_{i=1}^{n} X_i\right) = \frac{1}{2n} \cdot \sum_{i=1}^{n} E(x_i) = \frac{1}{\cancel{2n}} \cdot \cancel{n} \cdot \cancel{2}\theta = \theta$

Deci $\hat{\lambda}$ e nedeplasat.

2) Consistență

Def. Un estimator e consistent dacă converge către val. reală a parametrului (dimensiune eșantion crește)

[Caz general $Var(X)$ pentru distribuția Poisson:

$E(X^2) = \sum_{i=1}^{n} x_i^2 \cdot P(X = x_i) = \sum_{k=0}^{\infty} k^2 \cdot \frac{1}{k!} \cdot \lambda^k \cdot e^{-\lambda} = \dots = \lambda^2 + \lambda$

$Var(X) = E(X^2) - (E(X))^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$]

https://proofwiki.org/wiki/Variance_of_Poisson_Distribution

$Var(\hat{\theta}) = Var\left(\frac{1}{2n} \cdot \sum_{i=1}^{n} X_i\right) = \frac{1}{4n^2} \cdot \sum_{i=1}^{n} Var(X_i) = \frac{1}{4n^{\cancel{2}}} \cdot \cancel{n} \cdot Var(X) = \frac{1}{2\cancel{4}n} \cdot \cancel{2}\theta = \frac{\theta}{2n}$

$n \to \infty \Rightarrow \frac{\theta}{2n} \to 0$

Așadar, varianța estimatorului scade, deci valorile devin mai concentrate către 2θ, deci $\hat{\theta}$ e consistent.

b) $f_p(x) = C_n^x p^x (1-p)^{n-x}$, $n \in \mathbb{N}^*$, $x \in \{0, 1, \dots, n\}$, $p \in (0,1)$

coef binomial; prob. de succes; nr. total încercări; nr. total succese

X val. aleatoare discretă

I. Metoda momentelor

Caz general media distribuției binomiale:

$E(X) = \overline{X} = \sum_{x=0}^{n} x \cdot C_n^x \cdot p^x \cdot (1-p)^{n-x} \xrightarrow[\Rightarrow \text{primul termen al sumei e tot } 0]{\text{Pt. } x=0} \sum_{x=1}^{n} x \cdot C_n^x \cdot p^x \cdot (1-p)^x$

$= n \sum_{x=1}^{n} C_{n-1}^{x-1} \cdot p^x \cdot (1-p)^{n-x} = n \cdot \sum_{x=1}^{n} C_{n-1}^{x-1} \, p \cdot p^{x-1} \cdot (1-p)^{(n-1)-(x-1)} =$

$x \cdot C_n^x = x \cdot \frac{n!}{x!(n-x)!} = \cancel{x} \cdot \frac{n}{\cancel{x}} \cdot \frac{(n-1)!}{(x-1)!(n-x)!} = n \cdot C_{n-1}^{x-1}$

$= np \cdot \sum_{x=1}^{n} C_{n-1}^{x-1} \cdot p^{x-1} \cdot (1-p)^{(n-1)-(x-1)} = np \cdot \sum_{y=0}^{n-1} C_{n-1}^{y} \cdot p^y \cdot (1-p)^{(n-1)-y} =$

substituim x cu $y = x-1$

$x = 1 \Rightarrow y = 0$

$x = n \Rightarrow y = n-1$

$= n \cdot p \cdot 1 = np$

Știu

$\sum_{j=0}^{n-1} C_{n-1}^{j} p^j (1-p)^{(n-1)-j} = (p + (1-p))^{n-1} = 1$

$E(X) = np = \overline{X} \Rightarrow \hat{p} = \frac{\overline{X}}{n} = \frac{1}{n^2} \cdot \sum_{i=1}^{n} x_i$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \dots, x_n$

Funcția de verosimilitate:

$L(p) = L(p | x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} P(p | X = x_i) = \prod_{i=1}^{n} C_n^{x_i} \cdot p^{x_i} \cdot (1-p)^{n-x_i}$

Log-verosimilitate:

$\ln L(p) \xrightarrow[\log_n(a \cdot b) = \log_n a + \log_n b]{} \sum_{i=1}^{n} \ln C_n^{x_i} + \sum_{i=1}^{n} x_i \ln(p) + \sum_{i=1}^{n} (n - x_i) \cdot \ln(1-p) =$

Derivarea log-verosimilității în raport cu p.

$\frac{\partial \ln L(p)}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n} (n - x_i)}{1-p}$

$\frac{\partial}{\partial p}\left(\ln C_n^{x_i}\right) = 0$

$\frac{\partial}{\partial p}\left(\ln(p) \cdot \sum_{i=1}^{n} x_i\right) = \frac{\sum_{i=1}^{n} x_i}{p}$

$\frac{\partial}{\partial p}\left(\ln(1-p) \cdot \sum_{i=1}^{n} (n - x_i)\right) = \frac{-1}{1-p} \cdot \sum_{i=1}^{n} (n - x_i)$

Egalarea derivatei cu 0 pt a afla maximul:

$\sum x_i \quad \sum (n - x_i) \quad \dots \quad \sum (n - x_i) \Rightarrow \sum_{}^{n} x_i \dots \sum \dots \sum_{}^{n} (n - x_i)$

$$\frac{\sum_{i=1}}{p} = \frac{\sum_{i=1} \dots}{1-p} \Leftrightarrow (1-p)\sum_{i=1} x_i = p \sum_{i=1} (\dots) \Leftrightarrow \sum_{i=1} x_i - p \sum_{i=1} x_i = p \sum_{i=1} (\dots)$$

$$\Leftrightarrow \quad p = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(n-x_i) + \sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} x_i}{n \cdot n} = \frac{\bar{x}}{n} = \frac{1}{n^2} \cdot \sum_{i=1}^{n} x_i$$

1) Nedeplasarea

$$E(\hat{p}) = E\left(\frac{1}{n^2} \cdot \sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} \cdot \sum_{i=1}^{n} E(X_i) = \frac{1}{\cancel{n^2}} \cdot \cancel{n \cdot n} \cdot p = p \Rightarrow \hat{p} \text{ nedeplasat}$$

2) Consistența

În general distribuția binomială:

$$E(X^2) = \sum_{x=0}^{n} x^2 \cdot C_n^x \cdot p^x \cdot (1-p)^{n-x} \underset{\text{primul termen e 0}}{=} \sum_{x=1}^{n} x^2 \cdot C_n^x \cdot p^x \cdot (1-p)^{n-x} = \dots = (np)^2 - np^2 + np = n^2p^2 + np(1-p)$$

$$Var(X) = E(X^2) - (E(X))^2 = E(X^2) - n^2p^2 = \cancel{n^2p^2} + np(1-p) - \cancel{n^2p^2} = np(1-p)$$

$$Var(\hat{p}) = Var\left(\frac{1}{n^2} \cdot \sum_{i=1}^{n} x_i\right) = \frac{1}{n^4} \cdot \sum_{i=1}^{n} Var(x_i) = \frac{1}{n^{\cancel{4}3}} \cdot \cancel{n} \cdot Var(X) = \frac{1}{n^{\cancel{3}2}} \cdot \cancel{n}p(1-p)$$

$$= \frac{1}{n^2} \cdot p(1-p) = \frac{p(1-p)}{n^2}$$

$$n \to \infty \Rightarrow Var(\hat{p}) \to 0 \Rightarrow p \text{ consistent}$$

c) $f_\lambda(x) = \frac{x^2}{2\lambda^3} \cdot e^{-\frac{x}{\lambda}} \cdot \mathbb{1}_{(0,\infty)}$, $\lambda > 0$ — fct. indicator

X variabilă aleatoare continuă

I. Metoda momentelor

$$E(X) = \bar{X} = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} \frac{x^3}{2\lambda^3} \cdot e^{-\frac{x}{\lambda}}\,dx = \frac{1}{2\lambda^3} \int_0^{\infty} t^3 \lambda^3 \cdot e^{-t} \lambda\,dt = \frac{\lambda^{\cancel{4}}}{2\cancel{\lambda^3}} \int_0^{\infty} t^3 \cdot e^{-t}\,dt$$

$t \overset{\text{not}}{=} \frac{x}{\lambda} \Rightarrow x = \lambda t$
$dx = \lambda\,dt$
$x = 0 \Rightarrow t \to 0$
$x \to \infty \Rightarrow t \to \infty$

$$= \frac{\lambda}{2} \cdot \Gamma(4) = \frac{\lambda}{2} \cdot 6 = 3\lambda$$

$\Gamma(a) = \int_0^{\infty} x^{a-1} \cdot e^{-x}\,dx = (a-1)!$

$$E(X) = \bar{X} = 3\lambda \Rightarrow \hat{\lambda} = \frac{\bar{X}}{3}$$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \dots, x_n$

Funcția de verosimilitate:

$$L(\lambda) = L(\lambda \mid x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \left(\frac{x_i^2}{2\lambda^3} \cdot e^{-\frac{x_i}{\lambda}}\right) = \left(\frac{1}{2\lambda^3}\right)^{n} \cdot e^{-\frac{1}{\lambda} \cdot \sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} x_i^2$$

Log-verosimilitate:

$$\ln L(\lambda) \overset{\dots}{=} -n\ln 2 - 3n\ln\lambda - \frac{1}{\lambda}\sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n} \ln x_i$$

Derivarea log-verosimilității în raport cu p.

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = -\frac{3n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} x_i$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{3n}{\lambda} = \frac{1}{\lambda^2} \cdot \sum_{i=1}^{n} x_i \;\Big|\cdot \lambda^2 \Leftrightarrow 3n\lambda = \sum_{i=1}^{n} x_i$$

Deci $\hat{\lambda} = \frac{1}{3n} \cdot \sum_{i=1}^{n} x_i = \frac{\bar{x}}{3}$

1) Nedeplasare

$$E(\hat{\lambda}) = E\left(\frac{\bar{X}}{3}\right) = \frac{1}{3}E(\bar{X}) = \frac{1}{3}E\left(\frac{1}{n} \cdot \sum_{i=1}^{n} x_i\right) = \frac{1}{3n} \cdot \sum_{i=1}^{n} E(X_i) =$$

$$= \frac{1}{3\cancel{n}} \cdot \cancel{n} \cdot E(X_i) = \frac{1}{3} \cdot E(X) = \frac{\cancel{3}\lambda}{\cancel{3}} = \lambda \Rightarrow \hat{\lambda} \text{ e nedeplasat}$$

2) Consistența

$$E(X^2) = \int_0^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \cdot \frac{x^2}{2\lambda^3} \cdot e^{-\frac{x}{\lambda}}\,dx = \frac{1}{2\lambda^3} \int_0^{\infty} x^4 \cdot e^{-\frac{x}{\lambda}}\,dx \overset{\dots}{=} 12\lambda^2$$

$$Var(X) = E(X^2) - (E(X))^2 = 12\lambda^2 - (3\lambda)^2 = 12\lambda^2 - 9\lambda^2 = 3\lambda^2$$

$$Var(\hat{\lambda}) = \frac{1}{9}Var(\bar{X}) = \frac{1}{9} \cdot Var\left(\frac{1}{n} \cdot \sum_{i=1}^{n} x_i\right) = \frac{1}{9n^{\cancel{2}}} \cdot \cancel{n} \cdot Var(X) = \frac{1}{{}_3\cancel{9}n} \cdot \cancel{3}\lambda^2 = \frac{\lambda^2}{3n}$$

$$n \to \infty \Rightarrow Var(\hat{\lambda}) \to 0 \Rightarrow \hat{\lambda} \text{ e consistent}$$

d) $f_\theta(x) = \theta(1-\theta)^x$, $x \in \mathbb{N}$, $\theta \in (0,1)$

X variabilă aleatoare discretă

I. Metoda momentelor

$$E(X) = \bar{X} = \sum_{x=0}^{\infty} x \cdot f(x) = \sum_{x=0}^{\infty} x \cdot \theta(1-\theta)^x \underset{\text{distribuție geometrică}}{=} \theta\sum_{x=0}^{\infty} x \cdot (1-\theta)^x = \theta(1-\theta)\sum_{x=0}^{\infty} x \cdot (1-\theta)^{x-1} = -\theta(1-\theta)\sum_{x=0}^{\infty} \left((1-\theta)^x\right)' =$$

$\frac{\partial}{\partial x}\left((1-x)^{a}\right) = -a \cdot (1-x)^{a-1}$ Atenție la minus!

derivare în funcție de θ $\left(\frac{\partial}{\partial \theta}\right)$

$$= -\theta(1-\theta) \cdot \frac{\partial}{\partial \theta}\left(\frac{1}{\cancel{1}-\cancel{(1}-\theta)}\right) = -\theta(1-\theta) \cdot \frac{\partial}{\partial \theta}\left(\frac{1}{\theta}\right) = -\cancel{\theta} \cdot (1-\theta) \cdot \frac{-1}{\theta^{\cancel{2}}} = \frac{1-\theta}{\theta}$$

Știm seria geometrică:

$$\sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q} \quad (q \in (-1,1) \setminus \{0\})$$

Deci, $\frac{\partial}{\partial \theta}\left(\sum_{x=0}^{\infty} (1-\theta)^x\right) = \frac{\partial}{\partial \theta}\left(\frac{1}{1-(1-\theta)}\right)$

$$\bar{X} = \frac{1-\theta}{\theta} \;\Big|\cdot \theta \Leftrightarrow \bar{X} \cdot \theta = 1-\theta \Leftrightarrow \bar{X}\cdot\theta + \theta = 1 \Leftrightarrow \theta(\bar{X}+1) = 1 \Leftrightarrow \hat{\theta} = \frac{1}{\bar{X}+1}$$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \dots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i} = \theta^n \cdot (1-\theta)^{\sum_{i=1}^{n} x_i}$$

Log-verosimilitate:

$$\ln L(\theta) = n\ln\theta + \left(\sum_{i=1}^{n} x_i\right) \cdot \ln(1-\theta)$$

Derivarea log-verosimilității în raport cu p:

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta}$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i}{1-\theta} \Leftrightarrow n(1-\theta) = \theta \cdot \sum_{i=1}^{n} x_i \Leftrightarrow \theta\left(\left(\sum_{i=1}^{n} x_i\right) + n\right) = n \Leftrightarrow \hat{\theta} = \frac{n}{n + \underbrace{\sum_{i=1}^{n} x_i}_{= n\bar{x}}}$$

$$\Leftrightarrow \hat{\theta} = \frac{n}{n + n\bar{x}} \overset{(n}{=} \frac{1}{1+\bar{x}}$$

e) $f_\theta(x) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\theta}}}{\Gamma(\alpha) \cdot \theta^\alpha} \cdot \mathbb{1}_{(0,\infty)}$, $\theta > 0$, $\alpha > 0$ cunoscut, Distribuția Gamma

X variabilă aleatoare continuă

I. Metoda momentelor

$$E(X) = \bar{X} = \int_{-\infty}^{\infty} x \cdot f_\theta(x)\,dx = \int_0^{\infty} x \cdot \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\theta}}}{\Gamma(\alpha) \cdot \theta^\alpha} = \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \cdot \int_0^{\infty} x^\alpha \cdot e^{-\frac{x}{\theta}}\,dx = \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \cdot \int_0^{\infty} \theta^\alpha t^\alpha \cdot e^{-t}\,\theta\,dt$$

$t = \frac{x}{\theta} \Rightarrow x = \theta t$
$dx = \theta\,dt$

$x = 0 \Rightarrow t = 0$
$x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \cdot \theta^{\alpha+1} \cdot \underbrace{\int_0^{\infty} t^\alpha \cdot e^{-t}\,dt}_{\text{fct. gamma } \Gamma(\alpha+1)} = \frac{\theta}{\Gamma(\alpha)} \cdot \Gamma(\alpha+1) = \frac{\theta \cdot \alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = \alpha\theta \Rightarrow \hat{\theta} = \frac{\bar{X}}{\alpha}$$

Prop. (recursivitate) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de ...

Fie $X_1, X_2, \ldots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \ldots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1} \cdot e^{-\frac{x_i}{\theta}}}{\Gamma(\alpha)\cdot\theta^{\alpha}} = \left(\frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}}\right)^n \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)\cdot e^{-\frac{1}{\theta}\cdot\sum_{i=1}^{n} x_i}$$

Log-verosimilitate:

$$\ln L(\theta) = -n\ln(\Gamma(\alpha)) - n\alpha\ln(\theta) - \frac{\sum_{i=1}^{n} x_i}{\theta}$$

Derivarea log-verosimilității în raport cu p:

$$\frac{\partial \ln L(\theta)}{\partial\theta} = -\frac{n\alpha}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2}$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{\overbrace{\sum_{i=1}^{n} x_i}^{= n\bar{x}}}{\theta^2} = \frac{n\alpha}{\theta} \Big| \cdot\theta \Leftrightarrow \frac{n\bar{x}}{\theta} = n\alpha \Big| \cdot \frac{1}{n} \Leftrightarrow \frac{\bar{X}}{\theta} = \alpha \Leftrightarrow \hat{\theta} = \frac{\bar{X}}{\alpha}$$

f) $f_\theta(x) = \frac{\theta}{1-\theta}\cdot x^{\frac{2\theta-1}{1-\theta}}\cdot 1_{(0,1)}$, $\theta \in \left(\frac{1}{2}, 1\right)$

X variabilă aleatoare continuă

I Metoda momentelor

$$E(X) = \bar{X} = \int_{-\infty}^{1} x\cdot f(x)\,dx = \int_0^1 x\cdot\frac{\theta}{1-\theta}\cdot x^{\frac{2\theta-1}{1-\theta}}\,dx = \frac{\theta}{1-\theta}\cdot\int_0^1 x^{\frac{2\theta-1+1-\theta}{1-\theta}}\,dx = \frac{\theta}{1-\theta}\cdot\int_0^1 x^{\frac{\theta}{1-\theta}}\,dx = \frac{\theta}{\cancel{1-\theta}}\cdot(\cancel{1-\theta}) = \theta$$

$\Rightarrow \hat{\theta} = \bar{x}$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \ldots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \ldots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{\theta}{1-\theta}\cdot x_i^{\frac{2\theta-1}{1-\theta}} = \left(\frac{\theta}{1-\theta}\right)^n\cdot\prod_{i=1}^{n} x_i^{\frac{2\theta-1}{1-\theta}}$$

Log-verosimilitate:

$$\ln L(\theta) = n\ln\left(\frac{\theta}{1-\theta}\right) + \sum_{i=1}^{n}\frac{2\theta-1}{1-\theta}\ln x_i$$

Derivarea log-verosimilității în raport cu p:

$$\frac{\partial \ln L(\theta)}{\partial\theta} = n\left(\frac{\partial}{\partial\theta}\ln\theta - \frac{\partial}{\partial\theta}\ln(1-\theta)\right) + \sum_{i=1}^{n}\frac{\partial}{\partial\theta}\left(\frac{2\theta-1}{1-\theta}\ln(x_i)\right) = \frac{n}{\theta} - \frac{n}{1-\theta} + n\cdot\frac{1}{(1-\theta)^2}\left(\sum_{i=1}^{n}\ln x_i\right)' = ?$$

x_i comportat ca constantă; $g(\theta) = \frac{2\theta-1}{1-\theta}$

$$g'(\theta) = \frac{(2\theta-1)'(1-\theta) - (2\theta-1)(1-\theta)'}{(1-\theta)^2} = \frac{2(1-\theta)+2\theta-1}{(1-\theta)^2} = \frac{2-\cancel{2\theta}+\cancel{2\theta}-1}{(1-\theta)^2} = \frac{1}{(1-\theta)^2}$$

Egalarea derivatei cu 0 pt a afla maximul:

g) $f_\theta(x) = \frac{\alpha\cdot\theta^{\alpha}}{x^{\alpha+1}}\cdot 1_{(\theta,\infty)}$, $\theta>0$, $\alpha>0$, cunoscut (pt. convergență)

X variabilă aleatoare continuă

I. Metoda momentelor

(pt $\alpha\in(0,1)$, diverge, nu ăsta converge)

$$E(X) = \bar{X} = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \int_\theta^\infty x\cdot\frac{\alpha\cdot\theta^\alpha}{x^{\alpha+1}}\,dx = \alpha\cdot\theta^\alpha\cdot\int_\theta^\infty\frac{1}{x^\alpha}\,dx = \alpha\cdot\theta^\alpha\cdot\int_\theta^\infty x^{-\alpha}\,dx = \alpha\cdot\theta^\alpha\cdot\frac{x^{-\alpha+1}}{-\alpha+1}\Big|_\theta^\infty = \alpha\cdot\theta^\alpha\left(\lim_{x\to\infty}\frac{x^{-\alpha+1}}{-\alpha+1} - \frac{\theta^{-\alpha+1}}{-\alpha+1}\right) = \alpha\cdot\theta^\alpha\cdot\frac{\theta^{-\alpha+1}}{\alpha-1}$$

($-\alpha+1 \neq 0 \Leftrightarrow \alpha\neq 1$; $\alpha>1 \Rightarrow x^{-\alpha+1} \xrightarrow[x\to\infty]{} 0$)

$$= \alpha\cdot\frac{\theta^{\cancel{\alpha}-\cancel{\alpha}+1}}{\alpha-1} = \frac{\alpha}{\alpha-1}\cdot\theta \Rightarrow \hat{\theta} = \frac{\alpha-1}{\alpha}\cdot\bar{X}$$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \ldots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \ldots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n}\frac{\alpha\,\theta^\alpha}{x_i^{\alpha+1}} = \alpha^n\cdot\theta^{\alpha n}\cdot\prod_{i=1}^{n} x_i^{-(\alpha+1)}$$

Log-verosimilitate:

$$\ln L(\theta) = n\ln(\alpha) + n\alpha\cdot\ln(\theta) - (\alpha+1)\sum_{i=1}^{n}\ln(x_i)$$

Derivarea log-verosimilității în raport cu p:

$$\frac{\partial\ln L(\theta)}{\partial\theta} = \underbrace{0}_{\text{constantă}} + \frac{n\alpha}{\theta} - \underbrace{0}_{\text{constantă}} = \frac{n\alpha}{\theta}$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{\overbrace{n\alpha}^{>0}}{\theta} = 0 \Rightarrow \nexists \text{ sol. pt } \hat{\theta} \text{ (prin MVM)}$$

h) $f_\theta(x) = \frac{5\theta^5}{x^6}\cdot 1_{(\theta,\infty)}$, $\theta>0$

X variabilă aleatoare continuă

I. Metoda momentelor

$$E(X) = \bar{X} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_\theta^\infty x\cdot\frac{5\theta^5}{x^6}\,dx = 5\theta^5\int_\theta^\infty x^{-5}\,dx = 5\theta^5\left(\lim_{x\to\infty}\left(-\frac{1}{4x^4}\right) - \left(-\frac{1}{4\theta^4}\right)\right) = 5\theta^5\cdot\left(0+\frac{1}{4\theta^4}\right) = \frac{5}{4}\theta$$

$\Rightarrow \bar{X} = \frac{5}{4}\hat{\theta} \Leftrightarrow \hat{\theta} = \frac{4}{5}\bar{X}$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \ldots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \ldots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n}\frac{5\theta^5}{x_i^6} = 5^n\theta^{5n}\cdot\prod_{i=1}^{n} x_i^{-6}$$

Log-verosimilitate:

$$\ln L(\theta) = n\ln(5) + 5n\ln(\theta) - 6\sum_{i=1}^{n}\ln(x_i)$$

Derivarea log-verosimilității:

$$\frac{\partial\ln L(\theta)}{\partial\theta} = \frac{5n}{\theta}$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{\overbrace{5n}^{>0}}{\theta} = 0 \Rightarrow \nexists \text{ sol. pt } \hat{\theta} \text{ (prin MVM)}$$

i) $f_\theta(x) = \frac{\theta^x}{(\theta+1)^{x+1}}$, $x\in\mathbb{N}$, $\theta>0$

X variabilă aleatoare discretă

I. Metoda momentelor

$$E(X) = \bar{X} = \sum_{x=0}^{\infty} x\cdot f_\theta(x) = \sum_{x=0}^{\infty} x\cdot\frac{\theta^x}{(\theta+1)^{x+1}} = \frac{1}{\theta+1}\cdot\sum_{x=0}^{\infty} x\cdot\left(\frac{\theta}{\theta+1}\right)^x = \frac{1}{\cancel{\theta+1}}\cdot\theta(\cancel{\theta+1}) = \theta \Rightarrow \hat{\theta} = \bar{X}$$

Știm seria geometrică: $\sum_{x=1}^{\infty} p^{x-1} = \frac{1}{1-p}$ (pt $p\in(-1,1)\setminus\{0\}$), p = prob. de succes

$p = \frac{\theta}{\theta+1}$; $\sum_{x=0}^{\infty} x\cdot p^x = ?$

Știu $\sum_{x=0}^{\infty} p^x = \frac{1}{1-p}$ $\Big|\frac{\partial}{\partial p}$ derivăm față de p $\Leftrightarrow \frac{\partial}{\partial p}\left(\sum_{x=0}^{\infty} p^x\right) = \frac{1}{(1-p)^2}$

$\Leftrightarrow \sum_{x=0}^{\infty} x\,p^{x-1} = \frac{1}{(1-p)^2}$; Așadar $\sum_{x=0}^{\infty} x\cdot p^x = p\cdot\sum_{x=0}^{\infty} x\cdot p^{x-1} = \frac{p}{(1-p)^2}$

Deci: $\sum_{x=0}^{\infty} x\cdot\left(\frac{\theta}{\theta+1}\right)^x = \frac{\frac{\theta}{\theta+1}}{\left(1-\frac{\theta}{\theta+1}\right)^2} = \frac{\theta}{(\theta+1)\cdot\left(\frac{\cancel{\theta}+1-\cancel{\theta}}{\theta+1}\right)^2} = \frac{\theta}{\cancel{(\theta+1)}\cdot\frac{1}{(\theta+1)^{\cancel{2}}}} = \theta(\theta+1) = \theta^2+\theta$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \ldots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \ldots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n}\frac{\theta^{x_i}}{(\theta+1)^{x_i+1}} = \frac{\theta^{\sum_{i=1}^{n} x_i}}{(1+\theta)^{\sum_{i=1}^{n} x_i + n}}$$

Log-verosimilitate:

$$\ln L(\theta) = \underbrace{\left(\sum_{i=1}^{n} x_i\right)}_{= n\bar{x}}\cdot\ln\theta - \left(\left(\sum_{i=1}^{n} x_i\right)+n\right)\cdot\ln(\theta+1)$$

Derivarea log-verosimilității:

$$\frac{\partial\ln L(\theta)}{\partial\theta} = \frac{n\bar{x}}{\theta} - \frac{n\bar{x}+n}{\theta+1}$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{n\bar{x}}{\theta} = \frac{n(\bar{x}+1)}{\theta+1}\Big|\cdot\frac{1}{n} \Leftrightarrow \frac{\bar{x}}{\theta} = \frac{\bar{x}+1}{\theta+1} \Leftrightarrow \cancel{\bar{x}\theta}+\bar{x} = \cancel{\theta\bar{x}}+\theta$$

$\Leftrightarrow \hat{\theta} = \bar{x}$

j) făcut la clasă

k) $f_\theta(x) = \frac{\lambda}{\theta}\cdot x^{\lambda-1}\cdot e^{-\frac{x^\lambda}{\theta}}\cdot 1_{(0,\infty)}$, $\theta>0$, $\lambda>0$ cunoscut

X variabilă aleatoare continuă

I. Metoda momentelor

$$E(X) = \bar{X} = \int_0^\infty x\cdot\frac{\lambda}{\theta}\cdot x^{\lambda-1}\cdot e^{-\frac{x^\lambda}{\theta}}\,dx = \frac{\lambda}{\theta}\int_0^\infty x^\lambda\cdot e^{-\frac{x^\lambda}{\theta}}\,dx = \frac{\cancel{\lambda}}{\cancel{\theta}}\cdot\int_0^\infty \cancel{\theta} t\cdot e^{-t}\cdot\frac{1}{\cancel{\lambda}}\cdot\theta^{\frac{1}{\lambda}}\cdot t^{\frac{1-\lambda}{\lambda}}\,dt = \theta^{\frac{1}{\lambda}}\int_0^\infty t^{\frac{1-\lambda+\lambda}{\lambda}}\cdot e^{-t}\,dt$$

$t = \frac{x^\lambda}{\theta} \Rightarrow x = (t\theta)^{\frac{1}{\lambda}}$; $dx = \frac{1}{\lambda}\theta^{\frac{1}{\lambda}}\cdot t^{\frac{1-\lambda}{\lambda}}\,dt$

$$= \theta^{\frac{1}{\lambda}}\int_0^\infty t^{\frac{1}{\lambda}}\cdot e^{-t}\,dt = \theta^{\frac{1}{\lambda}}\cdot\Gamma\left(\frac{1}{\lambda}+1\right) = \theta^{\frac{1}{\lambda}}\cdot\frac{1}{\lambda}\Gamma\left(\frac{1}{\lambda}\right)$$

$\int_0^\infty x^k\cdot e^{-x}\,dx = k! = \Gamma(k+1)$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \ldots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \ldots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n}\left(\frac{\lambda}{\theta}x_i^{\lambda-1}\cdot e^{-\frac{x_i^\lambda}{\theta}}\right) = \left(\frac{\lambda}{\theta}\right)^n\left(\prod_{i=1}^{n} x_i^{\lambda-1}\right)\cdot e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i^\lambda\right)}$$

Log-verosimilitate:

$$\ln L(\theta) = \underbrace{n\ln\lambda}_{\text{constantă}} - n\ln\theta + \underbrace{(\lambda-1)\left(\sum_{i=1}^{n}\ln x_i\right)}_{\text{constantă}} - \frac{1}{\theta}\cdot\sum_{i=1}^{n} x_i^{\lambda}$$

Derivarea log-verosimilității:

$$\frac{\partial \ln L(\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^{n} x_i^{\lambda}\right)$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{1}{\theta^2}\left(\sum_{i=1}^{n} x_i^{\lambda}\right) = \frac{n}{\theta} \Leftrightarrow \dots\ \hat{\theta} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{\lambda}\right)$$

l) $f_{\theta,m}(x) = \frac{\theta}{\sqrt{\pi}}\cdot e^{-\theta^2(x-m)^2}$, $m\in\mathbb{R}$, $\theta>0$

X variabilă aleatoare continuă

I. Metoda momentelor

$$E(X) = \overline{X} = \int_{-\infty}^{\infty} x\cdot\frac{\theta}{\sqrt{\pi}}\cdot e^{-\theta^2(x-m)^2}dx = \frac{\theta}{\sqrt{\pi}}\int_{-\infty}^{\infty} x\cdot e^{-\theta^2(x-m)^2}dx = \frac{\theta}{\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}(m+t)\cdot e^{-\theta^2\cdot t^2}dt = \frac{\theta}{\sqrt{\pi}}\left(m\underbrace{\int_{-\infty}^{\infty} e^{-\theta^2\cdot t^2}dt}_{} + \int_{-\infty}^{\infty} t\cdot e^{-\theta^2\cdot t^2}dt\right)$$

$t \overset{\text{not}}{=} x-m \Rightarrow x = m+t$, $dx = dt$

$x\to-\infty \Rightarrow t\to-\infty$

$x\to+\infty \Rightarrow t\to+\infty$

$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

$\Rightarrow \int_{-\infty}^{\infty} e^{-\theta^2\cdot t^2}dt = \sqrt{\frac{\pi}{\theta^2}} = \frac{\sqrt{\pi}}{\theta}$

fct. „pară" $=0$" $(g(x)=g(-x))$

$$= \frac{\cancel{\theta}}{\cancel{\sqrt{\pi}}}\cdot\left(m\cdot\frac{\sqrt{\pi}}{\cancel{\theta}} + 0\right) = m \Rightarrow \hat{m} = \overline{x}$$

$\hat{\theta}$ nu poate fi scos de aici

II. Metoda verosimilității maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \dots, x_n$

Funcția de verosimilitate:

$$L(\theta) = L(\theta_m | x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f_{\theta_m}(x_i) = \prod_{i=1}^{n}\frac{\theta}{\sqrt{\pi}}\cdot e^{-\theta^2(x_i-m)^2} = \left(\frac{\theta}{\sqrt{\pi}}\right)^n\cdot e^{-\theta^2\cdot\sum_{i=1}^{n}(x_i-m)^2}$$

Log-verosimilitate:

$$\ln L(\theta)_m = n\ln\theta - \frac{n}{2}\ln\pi - \theta^2\cdot\sum_{i=1}^{n}(x_i-m)^2$$

Derivarea log-verosimilității: $\left[\frac{\partial}{\partial y}(a-y)^2 = -2a+2y\right]$

$$\frac{\partial \ln L(\theta)_m}{\partial m} = -\theta^2(-2)\cdot\sum_{i=1}^{n}(x_i-m) = 2\theta^2\sum_{i=1}^{n}(x_i-m)$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\overbrace{2\theta^2}^{>0}\sum_{i=1}^{n}(x_i-m) = 0 \Leftrightarrow \sum_{i=1}^{n}(x_i-m) = 0 \Leftrightarrow \left(\sum_{i=1}^{n} x_i\right) - n\cdot m = 0 \Leftrightarrow m = \overline{X}$$

$$\frac{\partial \ln L(\theta)_m}{\partial\theta} = \frac{n}{\theta} - 2\theta\cdot\sum_{i=1}^{n}(x_i-m)^2$$

Egalarea derivatei cu 0 pt a afla maximul:

$$\frac{n}{\theta} = 2\theta\sum_{i=1}^{n}(x_i-m)^2$$

$$\frac{n}{2\sum_{i=1}^{n}(x_i-m)^2} = \theta^2 \Leftrightarrow \hat{\theta} = \sqrt{\frac{n}{2\sum_{i=1}^{n}(x_i-m)^2}}$$

m) $f_{\theta_1,\theta_2}(x) = \frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\cdot x^{\theta_1-1}\cdot e^{-\frac{x}{\theta_2}}\cdot 1_{(0,\infty)}$, $\theta_1,\theta_2>0$, Distribuția Gamma

X variabilă aleatoare continuă

I. Metoda momentelor

Primul moment:

$$E(X) = \overline{X} = \int_0^{\infty} x\cdot f_{\theta_1,\theta_2}(x)dx = \int_0^{\infty} x\cdot\frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\cdot x^{\theta_1-1}\cdot e^{-\frac{x}{\theta_2}}dx = \frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\cdot\int_0^{\infty} x^{\theta_1}\cdot e^{-\frac{x}{\theta_2}}dx = \frac{1}{\cancel{\theta_2^{\theta_1}}\cdot\Gamma(\theta_1)}\cdot\int_0^{\infty}(t\cdot\cancel{\theta_2})^{\cancel{\theta_1}}\cdot e^{-t}\cdot\theta_2\,dt$$

$t \overset{\text{not}}{=} \frac{x}{\theta_2} \Rightarrow x = t\cdot\theta_2$, $dx = \theta_2\cdot dt$

$x\to0 \Rightarrow t\to0$

$x\to\infty \Rightarrow t\to\infty$

$$= \frac{\theta_2}{\Gamma(\theta_1)}\cdot\int_0^{\infty} t^{\theta_1}\cdot e^{-t}dt \underset{\int_0^{\infty} x^k\cdot e^{-x}dx = k! = \Gamma(k+1)}{=} \frac{\theta_2}{\Gamma(\theta_1)}\cdot\Gamma(\theta_1+1) = \frac{\theta_2}{\cancel{\Gamma(\theta_1)}}\cdot\theta_1\cdot\cancel{\Gamma(\theta_1)} = \theta_1\cdot\theta_2 \Rightarrow \overline{X} = \theta_1\cdot\theta_2$$

Al doilea moment:

$$E(X^2) = \int_0^{\infty} x^2\cdot f_{\theta_1,\theta_2}(x)dx = \frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\int_0^{\infty} x^{\theta_1+1}\cdot e^{-\frac{x}{\theta_2}}dx = \frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\cdot\int_0^{\infty}(t\theta_2)^{\theta_1+1}\cdot e^{-t}\cdot\theta_2\cdot dt = \frac{\theta_2^2}{\Gamma(\theta_1)}\cdot\int_0^{\infty} t^{\theta_1+1}\cdot e^{-t}dt$$

$t \overset{\text{not}}{=} \frac{x}{\theta_2} \Rightarrow x = t\cdot\theta_2$, $dx = \theta_2 dt$

$$\frac{\theta_2^2}{\Gamma(\theta_1)}\cdot\Gamma(\theta_1+2) = \frac{\theta_2^2}{\Gamma(\theta_1)}\cdot(\theta_1+1)\cdot\Gamma(\theta_1+1) = \frac{\theta_2^2}{\cancel{\Gamma(\theta_1)}}(\theta_1+1)\theta_1\cdot\cancel{\Gamma(\theta_1)} = \theta_2^2(\theta_1+1)\theta_1$$

$\int_0^{\infty} x^k\cdot e^{-x}dx = k! = \Gamma(k+1)$

II. Metoda verosimilității maxime

Fie $X_1, X_2, \dots, X_n$ un eșantion cu valorile de selecție $x_1, x_2, \dots, x_n$

Funcția de verosimilitate:

$$L(\theta_1,\theta_2) = L(\theta_1,\theta_2 | x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f_{\theta_1,\theta_2}(x_i) = \prod_{i=1}^{n}\frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\cdot x_i^{\theta_1-1}\cdot e^{-\frac{x_i}{\theta_2}} = \left(\frac{1}{\theta_2^{\theta_1}\cdot\Gamma(\theta_1)}\right)^n\cdot\prod_{i=1}^{n} x_i^{\theta_1-1}\cdot e^{-\frac{x_i}{\theta_2}} = \frac{1}{(\theta_2^{\theta_1})^n\cdot(\Gamma(\theta_1))^n}\left(\prod_{i=1}^{n} x_i^{\theta_1-1}\right)\cdot e^{-\frac{1}{\theta_2}\cdot\sum_{i=1}^{n} x_i}$$

Log-verosimilitate:

$$\ln L(\theta_1,\theta_2) = -n\theta_1\ln(\theta_2) - n\ln(\Gamma(\theta_1)) + (\theta_1-1)\cdot\sum_{i=1}^{n}\ln(x_i) - \frac{1}{\theta_2}\sum_{i=1}^{n} x_i$$

$\frac{\partial}{\partial t}\ln\Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}$

Derivarea log-verosimilității:

$$\frac{\partial \ln L(\theta_1,\theta_2)}{\partial\theta_1} = -n\ln(\theta_2) - n\cdot\frac{\Gamma'(\theta_1)}{\Gamma(\theta_1)} + \sum_{i=1}^{n}\ln(x_i) - 0$$

$$\frac{\partial \ln L(\theta_1,\theta_2)}{\partial\theta_2} = -\frac{n\theta_1}{\theta_2} + \frac{n}{\theta_2^2}\cdot\sum_{i=1}^{n} x_i$$

Egalarea derivatei cu 0 pt a afla maximul: $-n\ln(\theta_2) - n\cdot\frac{\Gamma'(\theta_1)}{\Gamma(\theta_1)} + \sum_{i=1}^{n}\ln(x_i) = 0$

Egalarea derivatei cu 0 pt a afla maximul: $-\frac{n\theta_1}{\theta_2} + \underbrace{\frac{n}{\theta_2^2}\cdot\sum_{i=1}^{n} x_i}_{n^2\cdot\overline{x}} = 0$

$$-\frac{n\theta_1}{\theta_2} + \frac{n^2\overline{x}}{\theta_2^2} = 0$$

$\Rightarrow$ de aici se scot estimatorii $\hat{\theta_1}$ și $\hat{\theta_2}$