Laboratorator 11

luni, 9 decembrie 2024 10:18

Variabile aleatoare continue

$X = \begin{pmatrix} 1 \dots & 2 \dots & \dots & m & \dots \\ p_1 & p_2 & \dots & p_m & \dots \end{pmatrix}$

$p_m = \frac{1}{2^m}$, $\sum_{k=1}^{\infty} \frac{1}{2^k}$ serie convergentă cu $q \in (-1, 1)$

$S = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1$

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} k \cdot (x^3 - x) & x \in [1, 2] \\ 0 & \text{în rest} \end{cases}$

determinați k astfel încât f densitate de probabilitate

Densitate de probabilitate : $f(x) \geq 0$ și $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$f \geq 0 \iff k\, x(x-1)(x+1) \geq 0 \iff k \geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} k(x^3 - x)\,dx = k\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-\infty}^{\infty} = k x^2 \cdot \frac{x^2-2}{4}\Big|_{-\infty}^{\infty} =$

$= k\, \frac{x^4 - 2x^2}{4}\Big|_1^2 = k \cdot \left(\frac{8}{4} + \frac{1}{4}\right) = \frac{9k}{4} \Rightarrow k = \frac{4}{9}$

$1\!\!1_A\, f(x) = \{x \in X \mid f(x) \neq 0\}$

↳ funcția evaluatoare (suportul funcției)

$f(x) = k(x^3 - x) = 1\!\!1_{[1,2]}(x)$

orice integrală se restrânge la integrală pe suportul funcției

determinați media și varianța

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

$Var(X) = E(X^2) - (E(X))^2$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_1^2 x \cdot \frac{4}{9}(x^3 - x) = \frac{4}{9}\left(\frac{x^5}{5} - \frac{x^2}{2}\right)\Big|_1^2 = \frac{4}{9}\left(\frac{32}{5} - \frac{4}{2} - \frac{1}{5} + \frac{1}{2}\right) =$

$= \frac{4}{9}\left(\frac{31}{5} - \frac{3}{2}\right) = \frac{4}{9} \cdot \frac{62-15}{10} = \frac{232}{135}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{4}{9}\left(\int_{-\infty}^{\infty} x^5 - x^3\right) = \frac{4}{9}\left(\frac{x^6}{6} - \frac{x^4}{4}\right)\Big|_1^2 =$

$= \frac{4}{9}\left(\frac{64-1}{6} - \frac{15}{4}\right) = \frac{4}{9} \cdot \left(\frac{63}{6} - \frac{15}{4}\right) = \frac{4}{9}\left(\frac{21}{3} - \frac{15}{4}\right) = \frac{4}{9} \cdot \frac{84-60}{12} =$

$= \frac{4}{9} \cdot 2 = \frac{8}{9}$

$Var(X) = \frac{8}{9} - \left(\frac{232}{135}\right)^2$

$E(2-3X) = 2 - 3E(X) = 2 - 3 \cdot \frac{232}{135} \dots$

$Var(2-3X) = 3^2\, Var(X) = 9\, Var(X) = 9\left(\frac{8}{9} - \left(\frac{232}{135}\right)^2\right) \dots$

$\mathbb{P}(0.5 \leq x \leq 1.5) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)\,dx = \int_1^{\frac{3}{2}} \frac{4}{9}(x^3 - x)\,dx - \int_{\frac{1}{2}}^{1} 0\,dx = \frac{4}{9}\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_1^{\frac{3}{2}} - 0 =$

$= \frac{4}{9}\left(\frac{81}{16} \cdot \frac{1}{4} - \frac{9}{8} - \frac{1}{4} + \frac{1}{2}\right) = \frac{4}{9}\left(\frac{81}{64} - \frac{56}{64} - \frac{16}{64} + \frac{32}{64}\right) = \frac{4}{9} \cdot \frac{113-72}{\cancel{64}_{16}} = \frac{41}{9}$

$\mathbb{P}(X > \frac{3}{4} \mid X \leq 1.5) = \frac{\mathbb{P}(\frac{3}{4} < x \leq \frac{3}{2})}{\mathbb{P}(x \leq 1.5)} = \frac{\int_{\frac{3}{4}}^{3/2} \frac{4}{9}(x^3 - x)\,dx}{\int_1^{\frac{3}{2}} \frac{4}{9}(x^3 - x)\,dx} =$

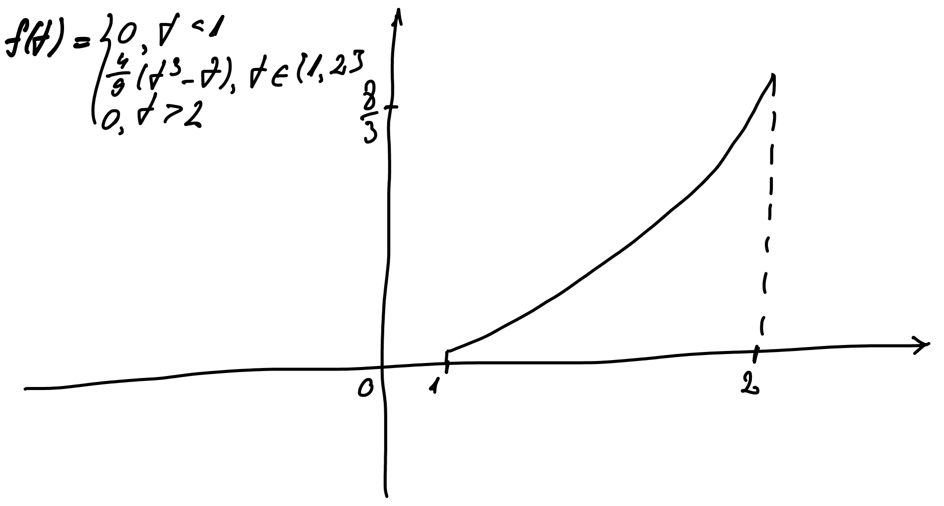
$4^5 = 2^{10}$

$= \frac{\frac{x^4}{4} - \frac{x^2}{2}\Big|_{3/4}^{3/2}}{\frac{x^4}{4} - \frac{x^2}{2}\Big|_1^{3/2}} = \frac{\frac{81}{64} - \frac{9}{8} - \frac{81}{1024} + \frac{9}{32}}{\frac{81}{64} - \frac{9}{8} - \frac{1}{4} + \frac{1}{2}} = \frac{\frac{2^6 \cdot 81 - 9 \cdot 2^7 - 81 + 9 \cdot 2^5}{1024}}{\frac{81 - 72 - 16 + 32}{64}} =$

$= \frac{64 \cdot 81 - 128 \cdot 9 - 81 + 288}{1024} \cdot \frac{64}{25} = \frac{4239}{\cancel{1024}_{16}} \cdot \frac{\cancel{64}^{1}}{25} = \frac{319}{400}$

$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{\infty} f(t)\,dt$

$f(t) = \begin{cases} 0, & t < 1 \\ \frac{4}{9}(t^3 - t), & t \in [1, 2] \\ 0, & t > 2 \end{cases}$



1. $x < 1$, $F(x) = 0$

2. $x \in [1, 2]$, $F(x) = \int_{-\infty}^{1} 0\,dt + \int_1^2 \frac{4}{9}(t^3 - t)\,dt = \frac{4}{9}\left(\frac{t^4}{4} - \frac{x^2}{2}\right)\Big|_1^2 = 1$

3. $x > 2$, $F(x) = \int_{-\infty}^{1} 0\,dt + \int_1^2 \frac{4}{9}(t^3 - t)\,dt + \int_2^{\infty} 0\,dt = 1$

funcția de repartiție $F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{9}(x^2 - 1)^2, & x \in [1, 2] \\ 1, & x > 2 \end{cases}$

$\mathbb{P}(1 \leq x \leq \frac{3}{2}) = F(\frac{3}{2}) - F(1) = \frac{1}{9}\left(\frac{9}{4} - 1\right)^2 - \frac{1}{9}(1-1)^2 = \frac{1}{9} \cdot \frac{25}{16} = \frac{25}{144}$

$\mathbb{P}(X > \frac{5}{4} \mid X \leq \frac{3}{2}) = \frac{\mathbb{P}(\frac{5}{4} \leq x \leq \frac{3}{2})}{\mathbb{P}(x \leq \frac{3}{2})} = \frac{F(\frac{3}{2}) - F(\frac{5}{4})}{F(\frac{3}{2})} = \frac{\frac{25}{144} - \frac{81}{256}}{\frac{25}{144}} \dots$