



Research Internship

Probabilistic Constellation Shaping with Autoencoders

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Abstract

The abstract comes here.

1 Introduction

The constant demand for higher capacity digital links has motivated the development of communication schemes which approach closer and closer the analytical limit of the channel capacity. According to the definition of channel capacity, sending a single bit per time-frequency slot is inefficient. For this reason, higher-order modulations like amplitude shift keying (ASK) or quadrature amplitude modulation (QAM) are used for better efficiency. Under these modulation schemes, the receiver handles more than two signal points per real dimension — the set of spacial signal points is known as constellation. However, these schemes present a constant-width gap to the capacity limit. This is due to the usage of both uniform and discrete probability distributions for the occurrence of the constellation points. While it is not possible to move away from the discrete distributions because of the digital nature of the communication, constellations with specific non-uniform distributions and optimized spatial locations can lead to further capacity improvements.

Constellation shaping thus, is a technique which seeks to optimize the distribution of the transmit symbols. Furthermore, this optimization unfolds into the improvement of the constellation points' location, their occurrence probability or both simultaneously. The first is known as geometric shaping while the latter is known as probabilistic shaping. In both cases, the goal is to maximize the mutual information (MI) of the channel input X and output Y , which we denote with $I(X;Y)$, by optimizing the constellation. This optimization problem arises from the definition of channel capacity C :

$$C = \max_{p(X)} I(X;Y). \quad (1.1)$$

Currently, the optimal $p(x)$ has only be found for specific channels, such as the additive white gaussian noise (AWGN), since knowledge of the channel distribution $p(y|x)$ is required. Still, solving 1.1 can become mathematically intractable despite knowing $p(y|x)$.

Here is where the use of deep-learning is key for finding constellations which maximize $I(X;Y)$, without analytical knowledge of the channel or with very complex $p(y|x)$. As shown in [OH17], the complete communication system can be interpreted as an autoen-

coder. This approach tackles the physical layer by optimizing the end-to-end performance instead of the performance of the individual components. This is possible thanks to the careful election of the loss function against which the trainable parameters will be optimized. What is more, geometric shaping under the autoencoder framework has been already performed in [OH17] and [JEY+18]. Still, learning the optimal probability distribution presents an additional challenge — the statistics of the training set are altered during the learning of the probability distribution making numerical approximations unprecise. [SAAH19] and [AC21] address this problem with different proposals. Their results show that the joint application of probabilistic and geometric shaping outperform the PS-QAM scheme from [BSS15] and approach the limit to within 0.1 dB in the AWGN channel. Nonetheless, the existence of multiple and apparently different architectures invites for a comparative study. Consequently, in this work we implement and discuss the architectures proposed by [SAAH19] and [AC21].

The rest of this document is organized as follows. Chapter 2, Preliminaries, presents the fundamentals of classical Probabilistic constellation systems as well as of autoencoder systems. Chapter 3 presents a comparative study of two autoencoder architectures which can perform joint probabilistic and geometric constellation shaping and highlights their limitations. Chapter 4 wraps-up this work.

2 Preliminaries

2.1 Probabilistic Constellation Shaping

In this section our goal is to present the capacity limitations of the commonly used ASK and QAM modulation schemes. These schemes are penalized for two reasons:

1. They use uniform probability densities.
2. The constellation points are equidistant.

In the following we explain the nature of these penalties.

2.1.1 Introduction

We begin with an important result from Information theory. Under a second-moment constraint, also known as power constraint, the probability distribution which maximizes the differential entropy is the Gaussian distribution, denoted with p_G . We thus have

$$h(X) \leq \frac{1}{2} \log(2\pi e \sigma^2) \quad (2.1)$$

where $\sigma^2 = \mathbb{E}[X^2]$, and with equality if and only if X is Gaussian-distributed. More generally in the multi-dimensional case we have

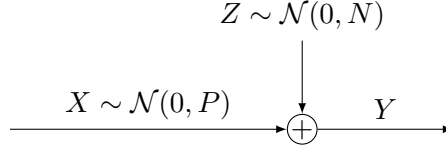
$$h(\underline{X}) \leq \frac{1}{2} \log((2\pi e)^n |\mathbf{Q}_{\underline{X}}|) \quad (2.2)$$

where we have considered a random column vector \underline{X} of dimension n , mean $\mathbb{E}[\underline{X}] = \underline{m}$ and covariance matrix

$$\mathbf{Q}_{\underline{X}} = \mathbb{E}[(\underline{X} - \underline{m})(\underline{X} - \underline{m})^\top] \quad (2.3)$$

and equality in (2.2) if only if the elements of \underline{X} are jointly Gaussian.

Lets now consider an AWGN channel with Gaussian input X , of zero mean and variance P ; Gaussian noise Z , of zero mean and variance N ; and output Y ; i.e. $Y = X + Z$.



Furthermore, the capacity of the AWGN is

$$C(P) = \max_{P_X: \mathbb{E}[X^2] \leq P} \mathbb{I}(X; Y) \quad (2.4)$$

$$= \max_{P_X: \mathbb{E}[X^2] \leq P} [h(Y) - h(Y|X)] \quad (2.5)$$

$$= \frac{1}{2} \log(2\pi e(P + N)) - \frac{1}{2} \log(2\pi eN) \quad (2.6)$$

$$= \frac{1}{2} \log \left(1 + \frac{P}{N} \right). \quad (2.7)$$

We can express the mutual information

$$\mathbb{I}(X; Y) = h(Y) - h(Y|X) \quad (2.8)$$

in two parts, the differential entropy of the output and the conditional differential entropy of the output given the input. We expand the second term as

$$h(Y|X) = h(Y - X|X) \quad (2.9)$$

$$= h(Z|X) \quad (2.10)$$

$$= h(Z) = \frac{1}{2} \log(2\pi e\sigma^2) \quad (2.11)$$

and observe that the term $h(Y|X)$ does not depend on how X is distributed. In contrast, $h(Y)$ does depend on how X is distributed by

$$p_Y(y) = \int_{-\infty}^{\infty} p_X(x) p_Z(y - x) dx = (p_X \star p_Z)(y). \quad (2.12)$$

To circumvent the fact that it is difficult to find a closed-form expression of $h(Y)$, we make use of the information divergence as

$$h(Y) \stackrel{(a)}{=} h(Y_G) - \mathbb{D}(p_Y \| p_G) \quad (2.13)$$

where (a) arises from the fact that $\mathbb{X}(p_X \| p_G) = h(Y_G)$ if and only if p_X has zero mean and variance P as p_G . (2.13) is very useful as it allows us to express the differential entropy of the output in terms of the information divergence between p_G and any other

distribution by means of the cross entropy.

Now we can rewrite 2.8 as

$$\mathbb{I}(X; Y) = h(Y) - h(Y|X) \quad (2.14)$$

$$= h(Y) - h(Z) \quad (2.15)$$

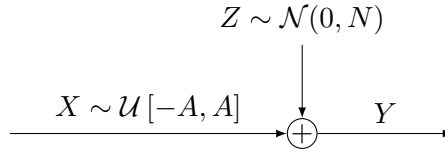
$$= h(Y_G) - \mathbb{D}(p_Y \| p_G) - h(Z) \quad (2.16)$$

$$= [h(Y_G) - h(Z)] - \mathbb{D}(p_Y \| p_G) \quad (2.17)$$

$$= C(P/\sigma^2) - \mathbb{D}(p_Y \| p_G). \quad (2.18)$$

This last result indicates that the loss of MI when using a distribution P_X different than P_G is the informational divergence $\mathbb{D}(p_Y \| p_G)$. In other words, if the gaussian distribution is not used, the capacity penalty is characterized by $\mathbb{D}(p_Y \| p_G)$.

2.1.2 Capacity Gap for Uniform Continuous Input



We would like now to understand how far a uniform distribution is from (2.1). To do this, we will follow the approach presented in [Bö18] to lower bound the MI. Start by defining X_u as a uniformly distributed input on the interval $[-A, A]$ where A is carefully chosen so that $\mathbb{E}[X_u^2] = P$. The corresponding output is Y_u and we proceed

$$\mathbb{I}(X_u; Y_u) = C(\text{snr}) - \mathbb{D}(p_{Y_u} \| p_{Y_G}) \quad (2.19)$$

$$\geq C(\text{snr}) - \mathbb{D}(p_{X_u} \| p_{X_G}) \quad (2.20)$$

$$= C(\text{snr}) - [h(X_G) - h(X_u)] \quad (2.21)$$

$$= C(\text{snr}) - \frac{1}{2} \log_2 \left(\frac{\pi e}{6} \right). \quad (2.22)$$

In Figure 2.1 we display the derived lower bound and observe that the capacity loss, originated from the use of a uniform input density, is never more than $\frac{1}{2} \log_2 \frac{\pi e}{6}$ independent of the signal-to-noise ratio (SNR). To show that the shaping gap is tight, it is necessary to proof an upper bound for $\mathbb{I}(X_u; Y_u)$ that approaches 2.22 with increasing SNR. We refer the reader to [Bö18], Section 4.3, for this proof.

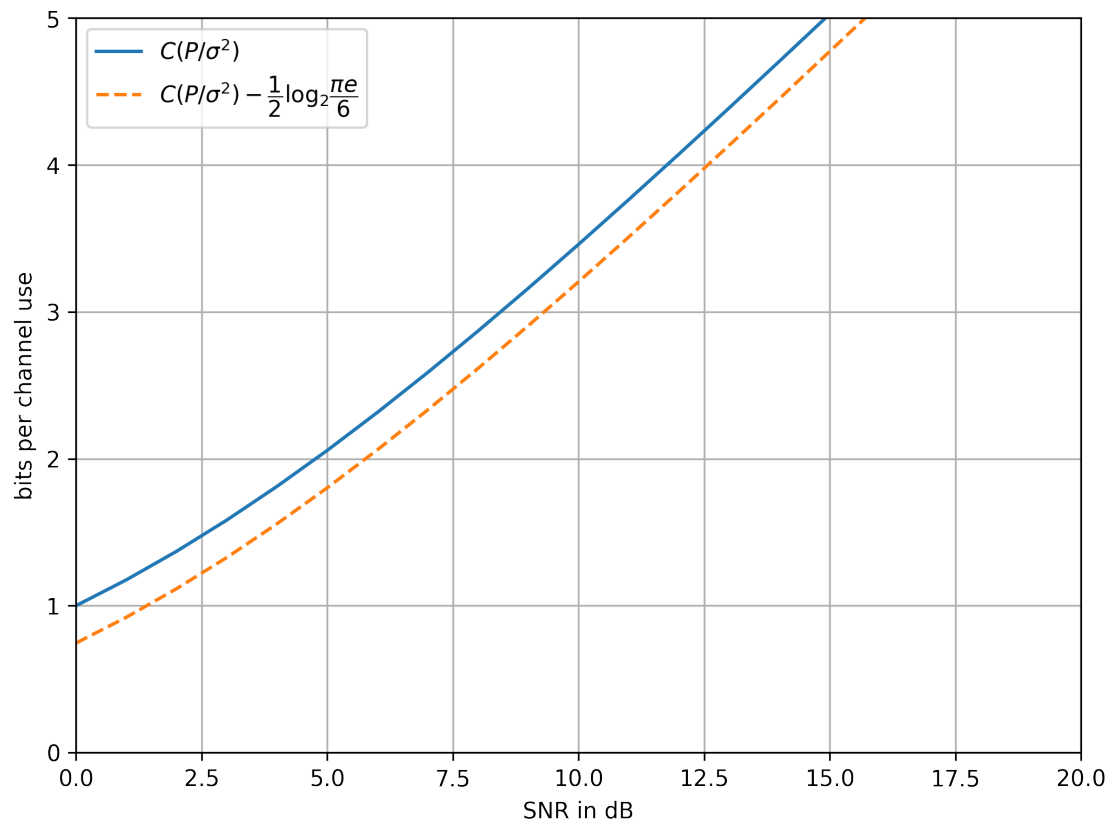


Figure 2.1: AWGN channel capacity gap. The orange line indicates the upper capacity bound for any uniformly distributed constellation.

2.1.3 Uniform Discrete Input Bound

We now show the penalty received for the use of an equidistant M-ASK constellation. We define the constellation points as

$$\mathcal{X} = \{\pm\Delta 1, \pm\Delta 3, \dots, \Delta(M-1)\}, \quad (2.23)$$

where Δ is the scaling factor of the constellation, so that the channel input is X . If X_M is uniformly distributed, the resulting power is

$$P = \mathbb{E}[X_M^2] = \Delta^2 \frac{M^2 - 1}{3}. \quad (2.24)$$

Theorem 1 (Uniform Discrete Input Bound). *The mutual information achieved by X_M is lower bounded by*

$$\mathbb{I}(X_M; Y_M) \geq \frac{1}{2} \log_2 \left(\frac{M^2}{M^2 - 1} \right) - \frac{1}{2} \log_2 \left[2\pi e \left(\frac{P}{M^2 - 1} + \frac{P}{1 + P/\sigma^2} \right) \right] \quad (2.25)$$

$$< C(\text{snr}) - \frac{1}{2} \log_2 \left(\frac{\pi e}{6} \right) - \frac{1}{2} \log_2 \left[1 + \left(\frac{2^{C(\text{snr})}}{M} \right)^2 \right] \quad (2.26)$$

$$< C(\text{snr}) - \text{Penalty}(\text{Uniform dist.}) - \text{Penalty}(\text{Equidistant dist.}) \quad (2.27)$$

where $\text{snr} = P/\sigma^2$.

We refer the reader to [Bö18], section 4.5, for the proof. Theorem 1 shows our goal, namely that both the usage of a uniform distribution and an equidistant constellation penalizes the capacity. We can additionally compute the relation between the constellation size, M , and $C(\text{snr})$ so that the resulting MI is within a constant gap of capacity. To make this result even more attractive, we increase the constraint for the gap to match the order of the distribution loss (0.255 bits). We obtain

$$-\frac{1}{2} \log_2 \left[1 + \left(\frac{2^{C(\text{snr})}}{M} \right)^2 \right] \leq -\frac{\log_2 e}{2} \left(\frac{2^{C(\text{snr})}}{M} \right)^2 = \frac{1}{4} \quad (2.28)$$

$$\Leftrightarrow M = 2^{C(\text{snr}) + \frac{1}{2} + \frac{1}{2} \log_2 \log_2 e} \quad (2.29)$$

by using $\log_e(x) \leq (1-x)$. So if

$$\log_2 M \approx C(\text{snr}) + 0.77, \quad (2.30)$$

then the mutual information is within 0.5 bit of capacity.

2.1.4 Capacity-achieving distributions

We now address the question of finding the discrete probability distribution which maximizes the capacity. Such distribution should be free of the $\frac{1}{2} \log_2 \left(\frac{\pi e}{6} \right)$ penalty. We use again an ASK constellation with M signal points (in practice, M is a power of 2) given by

$$\mathcal{X} = \{\pm 1, \pm 3, \dots, (M-1)\}. \quad (2.31)$$

Let X be a random variable with distribution P_X over \mathcal{X} . As before, we scale X by a $\Delta > 0$ and the resulting input/output relation for an AWGN channel becomes

$$Y = \Delta X + Z \quad (2.32)$$

In consequence, the MI of the channel input and output is

$$\mathbb{I}(\Delta X; Y) = \mathbb{I}(\Delta X; \Delta X + Z) \quad (2.33)$$

$$= \mathbb{I}(\Delta X; \Delta X + Z) \quad (2.34)$$

where the second equality follows because (ΔX) is a deterministic function of X and vice-versa. Under an input average power constraint P , the scaling Δ and the distribution P_X must be chosen to satisfy

$$\mathbb{E}[(\Delta X)^2] \leq P. \quad (2.35)$$

Formally, our optimization problem is the following

$$C(P/\sigma^2) = \max_{\Delta, P_X: \mathbb{E}[(\Delta X)^2] \leq P} \mathbb{I}(X; \Delta X + Z). \quad (2.36)$$

Maximizing the mutual information $\mathbb{I}(X; \Delta X + Z)$ both over the scaling of the constellation points and the input distribution requires a relatively high amount of power. Instead, as shown in [Bö18], section 5.3, we will use a suboptimal input distribution which follows from maximizing the input entropy.

We expand the mutual information as

$$\mathbb{I}(X, \Delta X + Z) = \mathbb{H}(X) - \mathbb{H}(X | \Delta X + Z) \quad (2.37)$$

and fixing Δ , we select the input distribution P_{X_Δ} that maximizes the input entropy

under our power constraint, i.e., we choose

$$P_{X_\Delta} = \arg \max_{P_X: \mathbb{E}[(\Delta X)^2] \leq P} \mathbb{H}(X). \quad (2.38)$$

Without the discrete constraint, the solution would be a Gaussian distribution. For this reason we explore sampled Gaussian distributions, also known as Maxwell-Boltzmann (MB) distributions. For each $\mathcal{X} = \{\pm 1, \pm 3, \dots, (M-1)\}$, define

$$P_{X_\nu}(x_i) = A_\nu e^{-\nu x_i^2}, \quad A_\nu = \frac{1}{\sum_{i=1}^M e^{-\nu x_i^2}} \quad (2.39)$$

We now show that P_{X_Δ} is given by

$$P_{X_\Delta}(x_i) = P_{X_\nu}(x_i) \text{ with } \nu : \mathbb{E}[(\Delta X_\nu)^2] = P \quad (2.40)$$

Proof. Consider the finite set $\mathcal{X} = x_1, x_2, \dots, x_n$. Let f be a function that assigns to each $x_i \in \mathcal{X}$ a positive cost $f(x_i) > 0$. Define the MB distribution

$$P_{X_\nu}(x_i) = A_\nu e^{-\nu f(x_i)}, \quad A_\nu = \frac{1}{\sum_{i=1}^M e^{-\nu f(x_i)}} \quad (2.41)$$

Let P_X be some distribution on \mathcal{X} with $\mathbb{E}[f(X)] = P$. Choose $\nu : \mathbb{E}[f(X_\nu)] = P$

$$0 \leq \mathbb{D}(P_X \| P_{X_\nu}) \quad (2.42)$$

$$= \sum_{x \in \text{Support}(P_{X_\nu})} P_X \log \left(\frac{P_X(x)}{P_{X_\nu}(x)} \right) \quad (2.43)$$

$$= -\mathbb{H}(X) - \sum_{x \in \text{Support}(P_{X_\nu})} P_X(x) \log(P_{X_\nu}(x)) \quad (2.44)$$

$$\stackrel{(*)}{=} -\mathbb{H}(X) - \sum_{x \in \text{Support}(P_{X_\nu})} P_{X_\nu}(x) \log(P_{X_\nu}(x)) \quad (2.45)$$

$$= -\mathbb{H}(X) + \mathbb{H}(X_\nu) \quad (2.46)$$

$$\mathbb{H}(X) \leq \mathbb{H}(X_\nu) \quad (2.47)$$

where the (*) marked step follows since both distributions produce the same moments for $\log(P_{X_\nu}(x))$. \square

2.2 Autoencoders

In this section we present the basics of autoencoders in the context of deep learning and communication systems, which was pioneered in [OH17]. The idea behind an autoencoder is to transmit a particular representation of the input data so that at the output, it can be reconstructed with minimal error. This means that the desired representations must be robust with respect to the channel impairments (i.e. noise, fading, distortion, etc.). To find such representations and the corresponding mappings \mathbf{x} to \mathbf{y} we train deep neural networks (NNs). Because NNs are universal function approximators [HSW89], this technique is particularly interesting for training over channels without a mathematically tractable model. A representation of an autoencoder system is shown in Figure ??.

To find fitting sets of parameters θ , the most used algorithm is stochastic gradient descent (SGD) which starts with a random set of initial values and then updates θ with each iteration as

$$\theta_{new} = \theta_{old} + \epsilon \frac{\partial}{\partial \theta_{old}} L(\theta_{old}) \quad (2.48)$$

One particular requirement for using SGD to train the autoencoders is that the loss gradient needs to be backpropagated all the way through receiver and channel to the transmitter. Otherwise, the transmitter parameters cannot be updated. This in turn means, that channel and receiver must be available as differentiable functions during training.

We can now set up the problem which we would like to address in this work. Namely, to train a deep NN-based autoencoder system to find a parametric distribution $p_{\theta}(s)$. By maximizing the MI during training, the output distribution must satisfy 2.18, and thus, approach the channel capacity.

3 Contribution

3.1 Notation

In the following we will use the notation:

$$\mathbb{I}(X, Y; D, P_M, C_M)$$

which expresses the mutual information between X and Y and D, P_M, C_M , separated by a semi-colon, are the trainable parameters of the system. D stands for the posterior probability distribution learnt by the demapper, P_M stands for the source's probability distribution learnt by the encoder, and C_M stands for the spatial distribution of the constellation points learnt by the mapper. These parameters can be seen as additional input to a function.

3.2 First implementation

In this section we break-down the autoencoder system presented by Stark *et al.* [SAAH19].

3.2.1 Optimization of trainable parameters

As we have seen in Chapter 2, the goal of probabilistic constellation shaping is to maximize the MI. To this end, defining an appropriate loss function is critical. Starting from the demodulator, the categorical cross entropy loss

$$L(D, P_M, C_M) \triangleq \mathbb{X}(P_{X|Y} || Q_{X|Y}; D) = \mathbb{E}[-\log_2(Q(X|Y; D))] \quad (3.1)$$

is appropriate for training D , but not for P_M and C_M . A modification of this loss function is necessary to ensure that the end-to-end MI is maximized. The following expansions will come handy

$$\mathbb{H}(X) = \mathbb{X}(P_{X|Y} || Q_{X|Y}) - \mathbb{D}(P_{X|Y} || Q_{X|Y}) \quad (3.2)$$

$$\mathbb{H}(X|Y = y) = \mathbb{X}(P_{X|y}||Q_{X|y}|Y = y) - \mathbb{D}(P_{X|y}||Q_{X|y}|Y = y) \quad (3.3)$$

$$\mathbb{H}(X|Y) = \mathbb{E}_y [\mathbb{X}(P_{X|y}||Q_{X|y}|Y = y)] - \mathbb{E}_y [\mathbb{D}(P_{X|y}||Q_{X|y}|Y = y)]. \quad (3.4)$$

Using the last expansion we can rewrite the mutual information in terms of the categorical cross entropy

$$\mathbb{I}(X, Y) = \mathbb{H}(X) - \mathbb{X}(P_{X|Y}||Q_{X|Y}) + \mathbb{D}(P_{X|Y}||Q_{X|Y}). \quad (3.5)$$

And the categorical cross entropy loss function becomes

$$L(D, P_M, C_M) \triangleq \mathbb{H}(X) - \mathbb{I}(X, Y) + \mathbb{D}(P_{X|Y}||Q_{X|Y}). \quad (3.6)$$

So, if L is minimized during training, the source entropy is unwantedly minimized. To avoid this effect, Stark *et al.* modify the loss function as

$$\hat{L}(D, P_M, C_M) \triangleq L(D, P_M, C_M) - \mathbb{H}(X). \quad (3.7)$$

With this correction the optimization problem

$$\min_{D, P_M, C_M} \hat{L}(D, P_M, C_M) = \max_{D, P_M, C_M} \{\mathbb{I}(X, Y) - \mathbb{D}(P_{X|Y}||Q_{X|Y})\} \quad (3.8)$$

maximizes the MI.

3.2.2 Autoencoder Architecture

Stark *et al.*'s autoencoder is made up from three major blocks: sampler, modulator, and demodulator. Fig 3.1 shows the complete architecture of the end-to-end system. While the modulator and demodulator blocks are similar to the proposal from [OH17], the simultaneous probabilistic shaping is possible thanks to the careful design of the sampler. By ensuring that the sampler mechanism is differentiable, the gradients with respect to each $p_i \in P_M$ are precise when calculated via SGD. In fact, the differentiability is gained by leveraging the so-called *Gumbel-Softmax trick* [JGP16], which circumvents the need for using the arg-max function to sample the discrete distribution P_M .

3.2.3 Autoencoder Performance

We have implemented the end-to-end system using [PyTorch]. First, for the

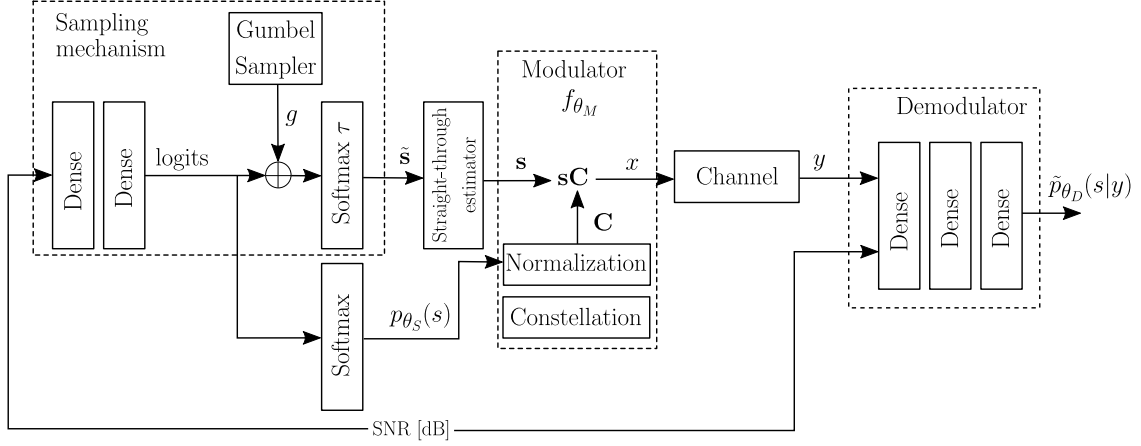


Figure 3.1: Proposed Autoencoder Architecture from [SAAH19]

3.3 Second implementation

In this section we break-down the autoencoder system presented by Aref et. al. in [AC21]. This approach is motivated by the fact that the implementation presented by Stark et.al. presents numerical instabilities.

3.3.1 Optimization of trainable parameters

As we have seen in chapter 2, the goal of probabilistic constellation shaping is to maximize the mutual information

$$\max_{D, P_M, C_M} \mathbb{I}(X, Y; D, P_M, C_M) = \mathbb{H}(X) - \mathbb{X}(P_{X|Y} \| Q_{X|Y}; D, P_M, C_M) \quad (3.9)$$

where the entropy is maximized when the symbols probabilities follow a MB distribution; and the cross-equivocation is minimum when $Q_{X|Y} = P_{X|Y}$.

Typically, the gradient descent (or ascent, as we intent to maximize) allows us to solve the optimization problem by adjusting the trainable parameters as:

$$\theta_{new} = \theta_{old} + \epsilon \frac{\partial}{\partial \theta_{old}} \mathbb{I}(X, Y; \theta_{old}) \quad (3.10)$$

for all trainable parameters $\theta \in P_M, C_M, D$. And the MI can be numerically approximated

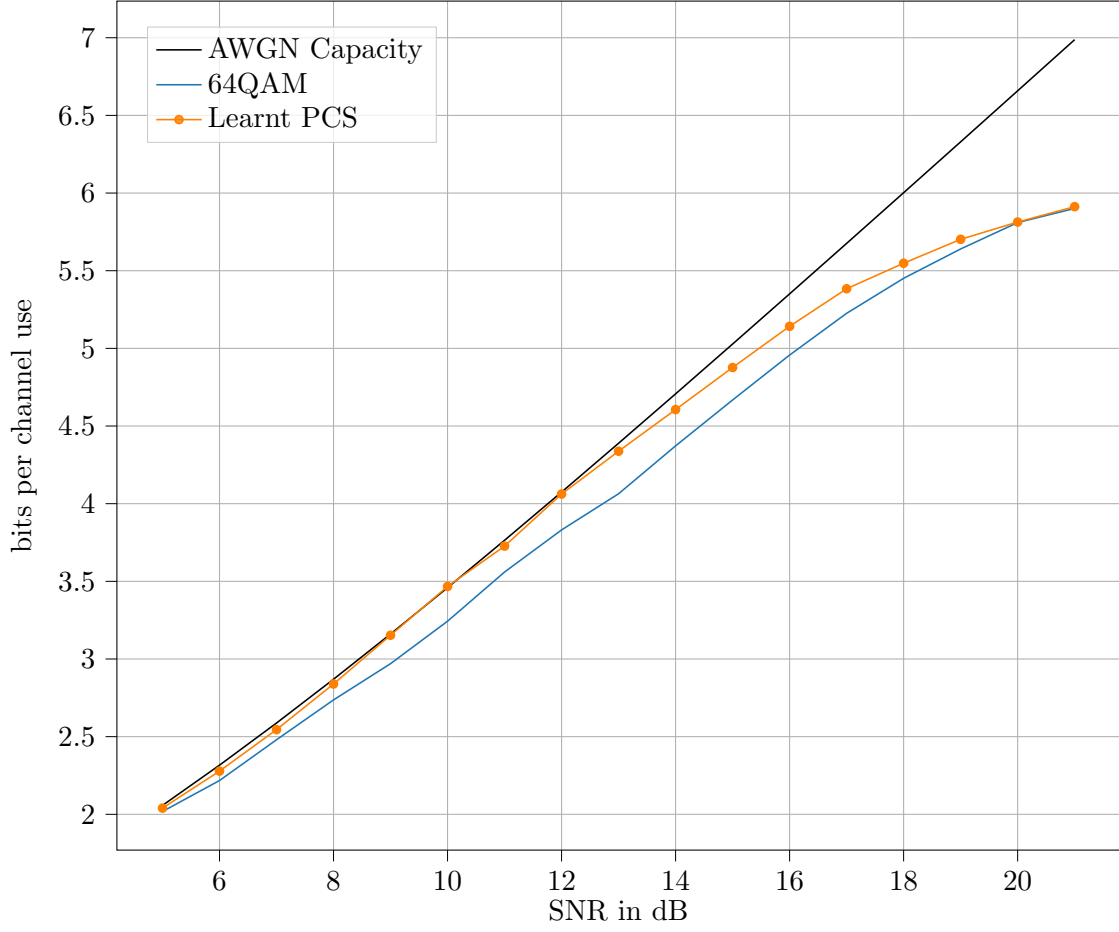


Figure 3.2: Mutual information learned by the probabilistic constellation shaping on the AWGN channel.

by

$$\mathbb{I}(X, Y) \approx \mathbb{I}(X, Y)_{\text{num}} = \frac{1}{B} \sum_{i=1}^B -\log_2(P(x_i)) + \log_2(Q_{X|Y}(x_i|y_i)) \quad (3.11)$$

$$= \frac{1}{B} \sum_{i=1}^B L(x_i, y_i). \quad (3.12)$$

Next, the following approximation usually allows to adjust the trainable parameters:

$$\frac{\partial}{\partial \theta} \mathbb{I}(X, Y; \theta) \approx \frac{\partial}{\partial \theta} \mathbb{I}(X, Y)_{\text{num}} = \frac{1}{B} \sum_{i=1}^B L(x_i, y_i). \quad (3.13)$$

However, Aref claims that although this is true for the constellation locations ($\theta \in C_M$)

and the demapper parameters ($\theta \in D$), it does not hold for the constellation probabilities $\{p_1, p_2, \dots, p_M\} = P_M$

$$\frac{\partial}{\partial p_j} \mathbb{I}(X, Y; P_M) \approx \frac{1}{B} \sum_{i=1}^B \frac{\partial}{\partial p_j} L(x_i, y_i) \quad (3.14)$$

as $\{p_1, p_2, \dots, p_M\}$ changes the statistics of the training set.

For this reason, (3.14) must be computed differently. On the one hand, to compute the derivative of the cross-equivocation, the following expansions are useful

$$\mathbb{X}(P_{X|Y} \| Q_{X|Y} | Y = b) = \sum_{a \in \text{Supp}(P_{X|Y}(\cdot|b))} P_{X|Y}(a|b) \log_2(Q_{X|Y}(a|b)) \quad (3.15)$$

$$\mathbb{X}(P_{X|Y} \| Q_{X|Y}) = \sum_{b \in \text{Supp}(P_Y)} P_Y(b) \mathbb{X}(P_{X|Y} \| Q_{X|Y} | Y = b) \quad (3.16)$$

as combined together and applying Bayes' theorem they yield

$$\mathbb{X}(P_{X|Y} \| Q_{X|Y}) = \sum_{(a,b) \in \text{Supp}(P_{XY})} P_X(a) P_{Y|X}(b|a) \log_2(Q_{X|Y}(a|b)). \quad (3.17)$$

And so, the derivative results

$$\frac{\partial}{\partial p_j} \mathbb{X}(P_{X|Y} \| Q_{X|Y}) = \sum_{b \text{ if } x=j} P_{Y|X}(b|j) \log_2 Q_{X|Y}(j|b) \quad (3.18)$$

$$+ \sum_{(a,b) \in \text{Supp}(P_{XY})} P_{XY}(a,b) \frac{\partial}{\partial p_j} \log_2 Q_{X|Y}(a|b) \quad (3.19)$$

which can be rewritten using the expectation operator as

$$\frac{\partial}{\partial p_j} \mathbb{X}(P_{X|Y} \| Q_{X|Y}) = \mathbb{E}_{Y|X}[\log_2 Q_{X|Y}(j|b) | X = j] \quad (3.20)$$

$$+ \mathbb{E}_{XY}[\frac{\partial}{\partial p_j} \log_2 Q_{X|Y}(a|b)]. \quad (3.21)$$

The terms can now be numerically computed as

$$\mathbb{E}_{Y|X}[\log_2 Q_{X|Y}(j|b) | X = j] \approx \frac{1}{B p_j} \sum_{b \text{ if } x=j} \log_2 Q_{X|Y}(j|b) \quad (3.22)$$

$$\mathbb{E}_{XY} \left[\frac{\partial}{\partial p_j} \log_2 Q_{X|Y}(a|b) \right] \approx \frac{1}{B} \sum_{(a,b) \in \text{Supp}(P_{XY})} \log_2 Q_{X|Y}(a|b). \quad (3.23)$$

On the other hand, the derivative of the entropy w.r.t. p_j is

$$\frac{\partial}{\partial p_j} \mathbb{H}(X) = \frac{\partial}{\partial p_j} \sum_{i=1}^B -p_i \log_2(p_i) = -\log_2(p_j) - \log_2(e). \quad (3.24)$$

Now, combining (3.22), (3.23), and (3.24) the derivative of the mutual information w.r.t. p_j , (3.14), can be computed as

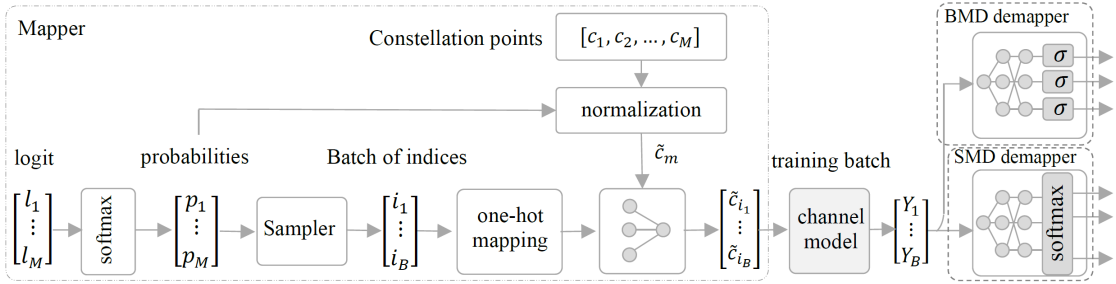
$$\frac{\partial}{\partial p_j} \mathbb{I}(X, Y; P_M) \approx -\log_2(p_j) - \log_2(e) + \frac{1}{B p_j} \sum_{b \text{ if } x=j} \log_2 Q_{X|Y}(j|b) + \frac{1}{B} \sum_{(a,b)} \log_2 Q_{X|Y}(a|b) \quad (3.25)$$

Aref now indicates that the following terms can be computed via backpropagation

$$-\log_2(p_j) + \frac{1}{B p_j} \sum_{b \text{ if } x=j} \log_2 Q_{X|Y}(j|b) = \frac{1}{B} \sum_{i=1}^B \frac{\partial}{\partial p_j} L(x_i, y_i) \quad (3.26)$$

while the remaining ones must be explicitly computed and added to the gradient after backpropagating. We call this step *gradient correction* and it is due to the change of statistics in the sampled batch.

3.3.2 Autoencoder Architecture



3.3.3 Autoencoder Performance

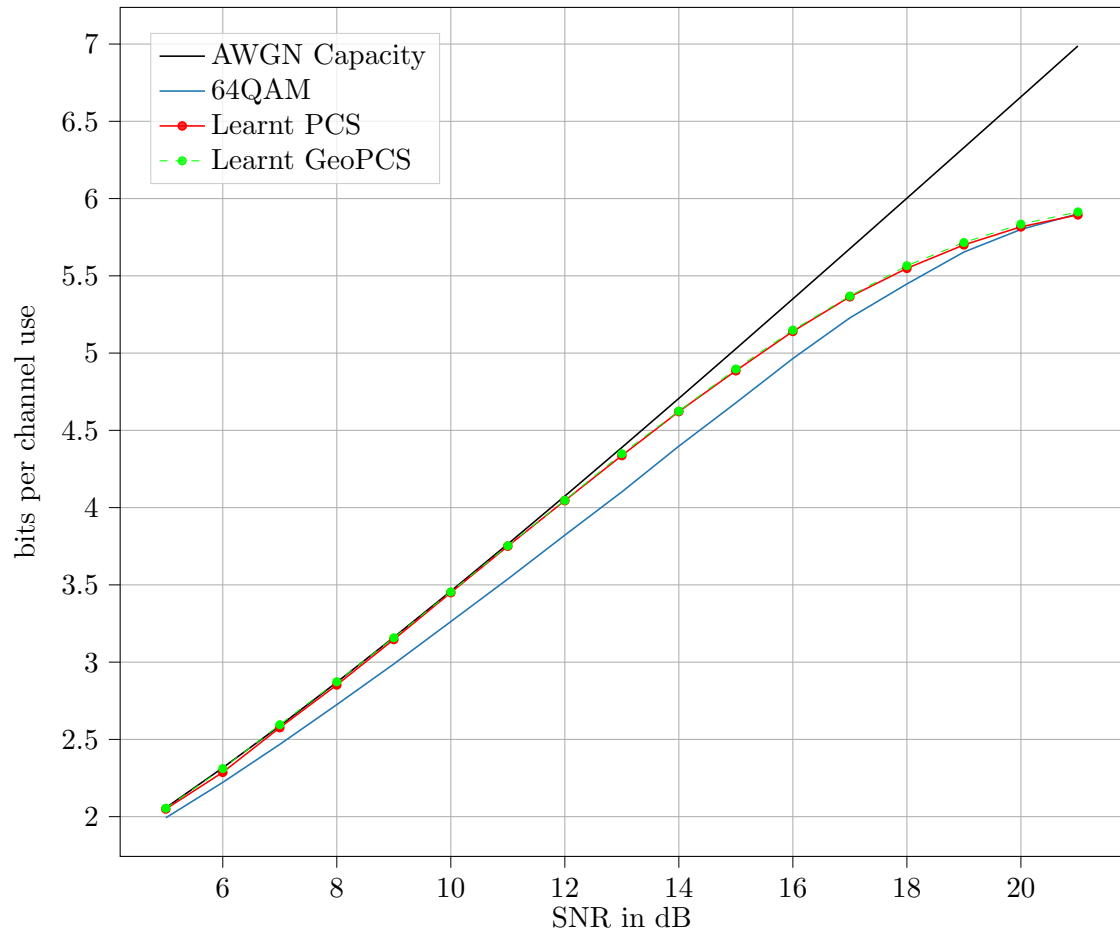


Figure 3.3: Mutual information learned by the probabilistic constellation shaping on the AWGN channel.

Appendix A

First appendix goes here.

Appendix B

Second appendix goes here.

List of Abbreviations

ASK	amplitude shift keying.
AWGN	additive white gaussian noise.
MB	Maxwell-Boltzmann.
MI	mutual information.
NN	neural network.
QAM	quadrature amplitude modulation.
SGD	stochastic gradient descent.
SNR	signal-to-noise ratio.

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