Mathematics 4MB3/6MB3 Mathematical Biology
2018 ASSIGNMENT 3

This assignment is **due in class** on **Wednesday 28 February 2018 at 11:30am**.

Notes

(i) **Please re-read all the General Notes listed for previous assignments.**

(ii) **Please re-read all the Technical Notes listed for previous assignments.**

(iii) *A technical note:* In order use `knitr`, you must select `knitr` (rather than `Sweave`) as the `Sweave` interpretter in `RStudio`. To check this setting in `RStudio`, go to

`Preferences` → `Sweave` → `Weave Rnw files using`

and choose `knitr`.

Analysis of the standard SIR model with vital dynamics

Consider the standard SIR model with vital dynamics,

$$\frac{dS}{dt} = \mu N - \frac{\beta}{N} SI - \mu S \tag{1a}$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \gamma I - \mu I \tag{1b}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{1c}$$

where S, I and R denote the numbers of susceptible, infectious and removed individuals, respectively, and $N = S + I + R$ is the total population size. The *per capita* rates of birth and death are the same (both are equal to μ). As usual, β is the transmission rate and γ is the recovery rate.

(a) Prove that the population size N is constant and that equations (1) are biologically well-defined, *i.e.,* the set Δ of biologically meaningful states is forward-invariant. (Note that you will need to begin by defining precisely the set Δ and the term "forward-invariant".)

(b) Show that equations (1) are equivalent dynamically to equations for the proportions (rather than numbers) of individuals in each disease state. For the remainder of this problem, use the equations in proportional form.

(c) Re-express the equations for proportions in *dimensionless* form using the dimensionless time coordinate

$$\tau = (\gamma + \mu)t\,, \tag{2a}$$

and the dimensionless parameters

$$\mathcal{R}_0 = \frac{\beta}{\gamma + \mu}\,, \tag{2b}$$

$$\varepsilon = \frac{\mu}{\gamma + \mu}\,. \tag{2c}$$

What are the biological meanings of τ, $\mathcal{R}_0$ and ε? Why are they good choices for non-dimensionalizing the equations? For a few diseases that you are familiar with, what is the order of magnitude of ε?

(d) Show that there are exactly two equilibria: the disease free equilibrium (DFE) at $(S, I) = (1, 0)$ and an endemic equilibrium (EE) at $(S, I) = (\hat{S}, \hat{I})$, where $\hat{S}$ and $\hat{I}$ can be expressed compactly in terms of $\mathcal{R}_0$ and ε. Are both equilibria always biologically relevant?

(e) Show that the DFE is locally asymptotically stable (LAS) if $\mathcal{R}_0 < 1$ and the EE is LAS if $\mathcal{R}_0 > 1$.

(f) Prove that the DFE is, in fact, globally asymptotically stable (GAS) if $\mathcal{R}_0 \leq 1$. *Hint:* This requires some careful analysis. Begin by using the function $L(S, I) = I$, and Theorem 2 stated in Assignment 1 under "Notes on Lyapunov Functions", to prove that all initial states in Δ are attracted to the S axis.

(g) Prove that the EE is GAS if $\mathcal{R}_0 > 1$. *Hint:* Consider

$$L(S, I) = S - \hat{S}\log S + I - \hat{I}\log I\,, \tag{3}$$

and convince yourself that condition (a) in Theorem 1 stated in Assignment 1 under "Notes on Lyapunov Functions" can be replaced with

$$L(X) > L(X_*) \quad \text{for all} \quad X \in \mathcal{O} \setminus \{X_*\}\,. \tag{4}$$

Note: By GAS we mean here that *almost all* initial states are attracted to the EE. One way of making this precise is to say that the *basin of attraction* of the EE is an open, dense subset of Δ. You should completely describe the basins of attraction of both the EE and the DFE. Do your results make biological sense?

(h) Prove that the approach to the EE occurs via damped oscillations if and only if $\varepsilon < \varepsilon^*$, where

$$\varepsilon^* = \frac{4(\mathcal{R}_0 - 1)}{\mathcal{R}_0^2}\,. \tag{5}$$

For which diseases that you are familiar with would you expect damped oscillations versus monotonic convergence to the equilibrium?

(i) For $\varepsilon < \varepsilon^*$, derive expressions for the period of damped oscillations onto the EE and the e-folding time for decay of the amplitude of oscillation. Use R to make a plot that displays your results graphically for some biologically relevant and illustrative parameter values.

(j) Prove that as $\mathcal{R}_0$ is increased from 0 to ∞, three "bifurcations" occur. In addition, use R to make a four-panel plot that illustrates the different dynamics (phase portraits) in each of the four $\mathcal{R}_0$ intervals that have distinct dynamics. (*Hint:* I suggest you choose $\varepsilon = 8/9$ for this figure, but you should explain why this is a good choice.)

Theoretical note: The word "bifurcation" is in quotes above because many dynamicists would consider only one of the three transitions to be a genuine bifurcation (it happens to be a *transcritical bifurcation*). The other two dynamical transitions yield biologically relevant qualitative changes, but the phase portraits on either side of the "bifurcation point" are actually topologically conjugate.

R *note:* Computing phase portraits should be easy based on code you've written for solving ODEs in R "from scratch". However, if you wish, you can use the `phaseR` package (or another R package of your choice) to make the phase portraits.

(k) Are there real diseases that display recurrent epidemics for which the standard SIR model that you have studied in this problem might be adequate to explain the observed epidemic dynamics? If so, which diseases? If not, why not?

— END OF ASSIGNMENT —

Compile time for this document: February 6, 2018 @ 22:38