Math 3A03 - Tutorial 9 Solutions - Winter 2019

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Problem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that*

$$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0. \tag{1}$$

Show that f has either an absolute maximum or an absolute minimum but not necessary both.

Solution. We assume that the function f is not identically zero on the real line, otherwise both absolute maximum and minimum of f is zero.

Since f is not identically zero on the real line, there exists a point $p \in \mathbb{R}$ satisfying $f(p) \neq 0$. Set $\epsilon := |f(p)| > 0$.

Consider the set $S := \{x : |f(x)| \geq \epsilon\}$. Note that the set S is bounded from below. Otherwise, there exists a sequence $\{x_n\} \subseteq S$ such that $x_n \to -\infty$ and $f(x_n) \to \epsilon$, which contradict to the condition (1) since $\epsilon > 0$. Similarly, we prove that S is also bounded from above. Therefore, both infimum and supremum of S exist.

Consider any accumulation point of the set S, i.e. consider a sequence $\{x_n\} \subseteq S$ such that $x_n \to x \in \mathbb{R}$. Since f is continuous, $f(x_n) \to f(x)$. Note that for each $n \in \mathbb{N}$, $f(x_n) \geq \epsilon$, and so $f(x) \geq \epsilon$ also. Then, $x \in S$, which means that the set S is closed. Therefore, $\inf(S) \in S$ and $\sup(S) \in S$. Set $c_1 := \inf(S)$ and $c_2 := \sup(S)$.

By the above construction,

$$|f(x)| < \epsilon = |f(c_1)| \quad \text{for all} \quad x < c_1, \quad \text{and}$$

$$|f(x)| < \epsilon = |f(c_2)| \quad \text{for all} \quad x > c_2.$$

Therefore, $f(c_1)$ is either an absolute maximum or an absolute minimum of f on the interval $(-\infty, c_1]$, and $f(c_2)$ is either an absolute maximum or an absolute minimum of f on the interval $[c_2, +\infty)$.

If $c_1 = c_2$, then $f(c_1)$ is an absolute extremum of f on $\mathbb{R}$. If $c_1 \neq c_2$, then the interval $[c_1, c_2]$ is nonempty, and by Extreme Value Theorem, there exist the absolute maximum M and the absolute minimum m of f on the interval $[c_1, c_2]$. If $\max\bigl(f(c_1), f(c_2)\bigr) > 0$, then $\max\bigl(f(c_1), f(c_2), M\bigr)$ is an absolute maximum of f on $\mathbb{R}$. If $\min\bigl(f(c_1), f(c_2)\bigr) < 0$, then $\min\bigl(f(c_1), f(c_2), m\bigr)$ is an absolute minimum of f on $\mathbb{R}$. If

$$\max\bigl(f(c_1), f(c_2)\bigr) = \min\bigl(f(c_1), f(c_2)\bigr) = 0,$$

then $\max\bigl(f(c_1), f(c_2), M\bigr)$ is an absolute max of f on $\mathbb{R}$ and $\min\bigl(f(c_1), f(c_2), m\bigr)$ is an absolute min of f on $\mathbb{R}$.

There exist cases when there is no absolute max or absolute min. For example, the function $f(x) = \exp(-x^2)$ does not have absolute min, whereas the function $f(x) = -\exp(-x^2)$ does not have absolute max.

Problem 2. *Let f be a continuous, one-to-one function defined on the interval $[a, b]$ with $f(a) < f(b)$. Show that, for all $x, y \in [a, b]$, if $x < y$ then $f(x) < f(y)$.*

Solution. Assume, by contradiction, that there exist $x, y \in [a, b]$ such that $x < y$ and $f(x) > f(y)$. Note that $a \leq x < y \leq b$, and so $a \neq y$. We divide the proof into three cases.

Case 1: $f(y) = f(a)$. Then, since $a \neq y$, we get a contradiction to the injectivity of f.

Case 2: $f(y) < f(a)$. We consider two intervals (a, y) and (y, b). The second interval is nonempty since $f(y) < f(a) < f(b)$, and so $y < b$. Then, for the interval (a, y), choose any $d \in (f(y), f(a))$. By Intermediate Value Theorem, there exist some $c_1 \in (a, y)$ such that $f(c_1) = d$. For the interval (y, b), since $f(y) < f(a) < f(b)$ holds, our previously chosen $d \in (f(y), f(b))$. Then, by Intermediate Value Theorem, there exist some $c_2 \in (y, b)$ such that $f(c_2) = d$. Since $(a, y) \cap (y, b) = \emptyset$, $c_1 \neq c_2$. Therefore, $f(c_1) = d = f(c_2)$ contradicts to the injectivity of f.

Case 3: $f(y) > f(a)$. We consider two intervals (a, x) and (x, y). The first interval is nonempty since $f(a) < f(y) < f(x)$, and so $a < x$. Then, for the interval (a, x), choose any $d \in (f(y), f(x)) \subset (f(a), f(x))$. By Intermediate Value Theorem, there exist some $c_1 \in (a, x)$ such that $f(c_1) = d$. For the interval (x, y), our previously chosen $d \in (f(y), f(x))$. Then, by Intermediate Value Theorem, there exist some $c_2 \in (x, y)$ such that $f(c_2) = d$. Since $(a, x) \cap (x, y) = \emptyset$, $c_1 \neq c_2$. Therefore, $f(c_1) = d = f(c_2)$ contradicts to the injectivity of f.

In each case we have a contradiction. Therefore, for all $x, y \in [a, b]$, if $x < y$ then $f(x) < f(y)$.

Problem 3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0,1)$ and with $f(0) = 0$ and $f(1) = 1$. Show there must exist distinct numbers ξ_1 and ξ_2 in that interval such that*

$$f'(\xi_1)f'(\xi_2) = 1.$$

Solution. By Mean Value Theorem for the interval $[0,1]$, there exist some $a \in (0,1)$ such that $f'(a) = \frac{f(1)-f(0)}{1-0} = 1$. We don't know the value of $f(a)$, so we divide the proof into three cases.

Case 1: $f(a) = a > 0$. Then, by Mean Value Theorem for $[0,a]$, there exists some $b \in (0,a)$ such that $f'(b) = \frac{f(a)-f(0)}{a-0} = \frac{f(a)}{a} = 1$. Since $a \neq b$, we got the desired $f'(a)f'(b) = 1$.

Case 2: $f(a) > a > 0$. Then, by Mean Value Theorem for the intervals $[0,a]$ and $[a,1]$, there exist $b \in (0,a)$ and $c \in (a,1)$ such that

$$f'(b) = \frac{f(a)-f(0)}{a-0} = \frac{f(a)}{a} > 1 \quad \text{and} \quad f'(c) = \frac{1-f(a)}{1-a} < 1,$$

where the latter inequalities are obtained due to the assumption $f(a) > a$. Then, we can write $f'(b) = 1+p$ and $f'(c) = 1-q$ for some $p > 0$ and $q > 0$. Set $\delta := \min(p,q) > 0$. Hence, $(1, 1+\delta) \subseteq (1, 1+p) = (f'(a), f'(b))$ and $(1-\delta, 1) \subseteq (1-q, 1) = (f'(c), f'(a))$.

By Darboux's Theorem for (b,a), choose any $d \in (1, 1+\delta) \subseteq (f'(a), f'(b))$, then there exist some $\xi_1 \in (b,a)$ such that $f'(\xi_1) = d$.

Since $1 < d < 1+\delta$, we have $1 > \frac{1}{d} > \frac{1}{1+\delta} > 1-\delta$, where the last inequality comes from $1 > 1-\delta^2$. Therefore, we get that $\frac{1}{d} \in (1-\delta, 1) \subseteq (f'(c), f'(a))$. By Darboux's Theorem for (a,c), there exists some $\xi_2 \in (a,c)$ such that $f'(\xi_2) = \frac{1}{d}$.

Since $(b,a) \cap (a,c) = \emptyset$, we have $\xi_1 \neq \xi_2$ and $f'(\xi_1)f'(\xi_2) = d \times \frac{1}{d} = 1$.

Case 3: $f(a) < a$. Proof is similar to the Case 2.