Math 3A03 - Tutorial 6 Questions - Winter 2019

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Problem 1. *(a) Find the closure, interior points, accumulation points, and boundary points of the set $S = [0, \sqrt{5}] \cap \mathbb{Q}$. Is this set open? Is it closed?*

(b) Show that $O = (0, 3)$ is open.

(c) Show that $C = [0, 3]$ is closed.

Solution. (a) The closure of the set is $[0, \sqrt{5}]$. The interior points of the set are $\emptyset$, since for any point q in the set and any $c > 0$ we have points in $[q - c, q + c]$ which are not in the set since between any two real numbers there is an irrational number. The accumulation points of the set are $[0, \sqrt{5}]$, since by density of the real numbers given a $c > 0$ and $x \in [0, \sqrt{5}]$ we can find a rational number in $[x - c, x + c] \cap (S \setminus \{x\})$. By similar reasoning $[0, \sqrt{5}]$ are the boundary points of S. The set is not open since there are points in the set which are not interior points. S is not closed since it does not contain all of its accumulation points.

(b) To show that the set is open we need to verify that each $x \in (0, 3)$ is an interior point. Suppose that $x \in (0, 3)$, then $0 < x < 3$. To verify this is an interior point we need to find a $c > 0$ such that $(x - c, x + c) \subseteq (0, 3)$. Note that if $0 < c \leq \frac{x}{2}$ then $x - c \geq x - \frac{x}{2} = \frac{x}{2} > 0$, further if $0 < c \leq \frac{3-x}{2}$ then $x + c \leq \frac{3+x}{2} < 3$. Then if we pick $c = \min(\frac{x}{2}, \frac{3-x}{2})$ the requirements that $0 < c \leq \frac{x}{2}$ and $0 < c \leq \frac{3-x}{2}$ are satisfied, hence $0 < x - c < x + c < 3$, and $(x - c, x + c) \subseteq (0, 3)$.

(c) We have 2 options, one is to show that the complement $[0, 3]^C = (-\infty, 0) \cup (3, \infty)$ is open, or we can show that the set contains all of its accumulation points. Let's do the latter. First note that if $x \in [0, 3]$, then every neighbourhood of x contains infinitely many points of the interval, so x is an accumulation point. We need to show that there are no accumulation points outside of the set. i.e. we need to show that if $x > 3$ or $x < 0$ then x is **not** an accumulation point of the set. What we need to prove is for such an x there exists a $c > 0$ such that $(x - c, x + c) \cap ([0, 3] \setminus \{x\}) = \emptyset$. Suppose that

$x < 0$ and let $c = \frac{|x|}{2}$, then $x+c < 0$ as well, hence $(x-c, x+c) \cap ([0,3] \setminus \{x\}) = \emptyset$ since the sets are disjoint (note that $\sup(x-c, x+c) = -\frac{|x|}{2} < 0 = \inf[0,3]$). A similar argument shows that each $x > 3$ is also not an accumulation point.

Problem 2. *Show that* $S = [0,1] \cap \mathbb{Q}$ *is not compact by verifying explicitly that the Heine-Borel property does not hold. Hint: How was this approached in class for* $(0,1]$*? How can that proof be modified?*

Solution. In class for the set $(0,1]$ we formed an open cover of the set (with no finite subcover) by approaching one of the limit points (namely 0) which is not inside the set with a family of sets. An example of a family of sets which form an open cover is $O_n = (\frac{1}{n}, 2)$. We can do take a similar approach by using a limit point of the rational numbers between 0 and 1 not in the set, namely irrational numbers between 0 and 1. So for example $\frac{\sqrt{2}}{2} \notin S$, but it is a limit point of S. Notice that

$$S = [0,1] \cap \mathbb{Q} = ([0, \frac{\sqrt{2}}{2}) \cap \mathbb{Q}) \cup ((\frac{\sqrt{2}}{2}, 1] \cap \mathbb{Q}).$$

Let $O_0 = (\frac{\sqrt{2}}{2}, 2), O_n = (-1, \frac{\sqrt{2}}{2} - \frac{1}{n})$, notice that $S \subseteq \cup_{n=0}^{\infty} O_n$, so this is an open cover of S. However if we fix any $N \in \mathbb{N}$ we have $\cup_{n=0}^{N} O_n = (-1, \frac{\sqrt{2}}{2} - \frac{1}{N}) \cup (\frac{\sqrt{2}}{2}, 2) \not\supseteq S$, so this cover has no finite subcover of S.

Problem 3. *Suppose that* K *is a compact set, and* $D \subseteq K$ *is a closed subset. Prove that* D *is also compact by verifying that it has the Heine-Borel property.*

Solution. Since K is compact we know that for any open cover of K there exists a finite subcove. We wish to begin with an open cover , $\{O_n\}_{n=1}^{\infty}$ of D and extract a finite subcover, $\{O_{n_k}\}_{k=1}^{N}$. i.e. if $D \subseteq \cup_{n=1}^{\infty} O_n$ then $D \subseteq \cup_{k=1}^{N} O_{n_k}$.

Since K is closed then D^C is open, note that $K \subseteq \mathbb{R} = D \cup D^C$, hence $(\cup_{n=1}^{\infty} O_n) \cup D^C$ is an open cover of K. Since K is compact there is a finite subcover, i.e. there is an $N \in \mathbb{N}$ so that $D \subseteq K \subseteq \left(\cup_{k=1}^{N} O_{n_k}\right) \cup D^C$. Is this a finite subcover of D? Yes, however note we added an extra set, so we haven't extracted an finite subcover yet. However the set we added, D^C, is the complement of D, so $D \cap D^C = \emptyset$. If we take the intersection of D we get $D = D \cap D \subseteq \left(\cup_{k=1}^{N} O_{n_k}\right) \cup D^C \cap D = \cup_{k=1}^{N} O_{n_k}$, so $\{O_{n_k}\}_{k=1}^{N}$ is the desired finite subcover and D has teh Heine-Borel property.