Wednesday, January 23, 2019 11:24 AM

Theorem: every convergent sequence is bounded.

If s_n converges to i.e. $\lim_{n\to\infty} s_n = L$, $L \in \mathbb{R}$ then

$\exists\, M \geq 0$ such that $|s_n| \leq M$

Proof: Suppose that s_n is a convergent sequence with $\lim_{n\to\infty} s_n = L$

more formally for any $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $|s_n - L| < \varepsilon$

$\forall n \geq N$, in particular such an N exists when

$\varepsilon = 1$.

$$|s_n - L| < 1 \quad \forall n \geq N$$

$$|s_n| - |L| \leq \big||s_n| - |L|\big| \leq |s_n - L| < 1$$

reverse Δ inequality

hence: $|s_n| < 1 + |L| \quad \forall n \geq N.$

This acts as a bound when $n \geq N$ but what about

$n \in \{1, 2, \ldots, N-1\}$?

Notice that the set $\{|s_1|, |s_2|, \ldots, |s_{N-1}|\}$ is a finite set of real numbers, in particular finite sets of real numbers have a maximum element.

look at $\max(\{|s_1|, |s_2|, \ldots, |s_{N-1}|, 1+|L|\}) = M.$

then $|s_n| \leq M$ for $n \in \mathbb{N}$, hence s_n is bounded then $|s_n| \le M$ for $n \in \mathbb{N}$, hence s_n is bounded

---

proof of algebra of limits properties.

$$\lim_{n\to\infty} s_n = L$$

$$\lim_{n\to\infty} t_n = K.$$

(1) $\lim_{n\to\infty} cs_n = c\lim_{n\to\infty} s_n = cL.$

Since s_n converges to L we know that given $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that

$$|s_n - L| < \frac{\varepsilon}{|c|} \quad \forall n \ge N.$$

then $|cs_n - cL| = |c|\,|s_n - L| < \cancel{|c|} \cdot \frac{\varepsilon}{\cancel{|c|}} = \varepsilon \quad \forall n \ge N.$

so $\lim_{n\to\infty} cs_n = cL.$

(2) Given $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ s.t.

since s_n and t_n are convergent sequences.

$$|s_n - L| < \frac{\varepsilon}{2} \quad \forall n \ge N_1 \ (1)$$

$$|t_n - K| < \frac{\varepsilon}{2} \quad \forall n \ge N_2 \ (2)$$

let $N = \max(N_1, N_2)$

$|(s_n + t_n) - (L + K)| = |(s_n - L) + (t_n - K)|$

$$\left|(s_n+t_n)-(L+k)\right| = \left|(s_n-L)+(t_n-k)\right|$$

$$\Delta \text{ inequality} \leq |s_n-L|+|t_n-k|$$

$$\forall n \geq N \quad < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \quad \text{by } (1),(2)$$

$$\lim_{n\to\infty}(s_n+t_n) = L+k$$

(3) straightforward proof: $a_n = -t_n$, $J = -K$ and apply (2) to $\lim_{n\to\infty}(s_n+a_n)$.

Exersize: copy the idea above, making changes as necessary.

(4) Given $\varepsilon > 0$, since $s_n \xrightarrow[n\to\infty]{} L$, $t_n \xrightarrow[n\to\infty]{} k$ $\exists N_1, N_2 \in \mathbb{N}$ such that

$$|s_n-L| < \frac{\varepsilon}{2(|M|+1)} \quad \forall n \geq N_1 \quad (3)$$

$$|t_n-k| < \frac{\varepsilon}{2(|L|+1)} \quad \forall n \geq N_2. \quad (4)$$

and choose $N = \max(N_1, N_2)$

$$|s_n t_n - Lk| = |s_n t_n - Lt_n + Lt_n - Lk|$$

$$= |(s_n-L)t_n + L(t_n-k)|$$

$$\text{by } \Delta \text{ ineq.} \leq |t_n||s_n-L| + |L||t_n-k|$$

$$\leq |t_n|\,|s_n-L| + (|L|+1)\,|t_n-k|$$

$\le |t_n|\,|s_n - L| + |L| \cdot |t_n - \ldots$

since t_n converges $\exists M \in \mathbb{R}^{\ge 0}$ such that

$$|t_n| \le M \quad \forall n \in \mathbb{N}.$$

then $$|s_n t_n - Lk| \le M\,|s_n - L| + (|L|+1)\,|t_n - k|$$

by (3) and (4) $$< M \cdot \frac{\varepsilon}{2(M+1)} + \cancel{(|L|+1)} \cdot \frac{\varepsilon}{2\cancel{(|L|+1)}}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \forall n \ge N.$$

$\square$