McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 1
Introduction
Tuesday 3 September 2019

Where to find course information

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- The course web site: http://ms.mcmaster.ca/earn/3A03

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- Let's have a look now. . .

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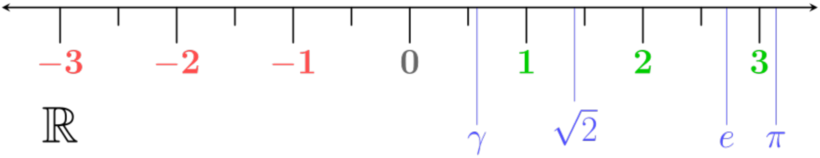

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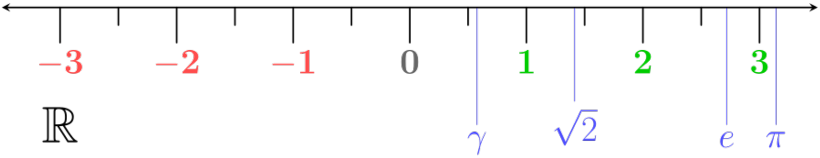

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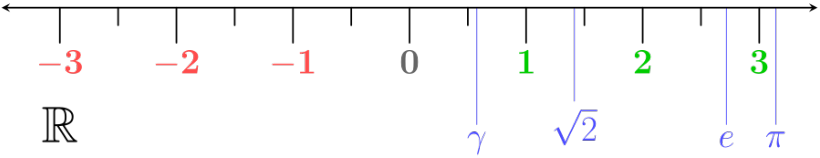

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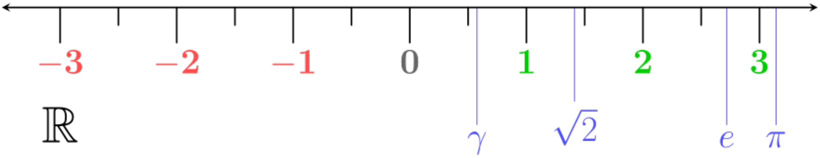

- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
  Approach: "Proof by contradiction."

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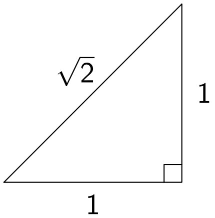

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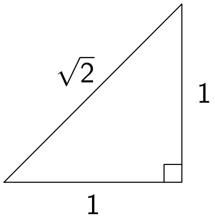

- So irrational numbers are “real”.

Poll on rationality

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- Let's Deactivate the poll and View Results

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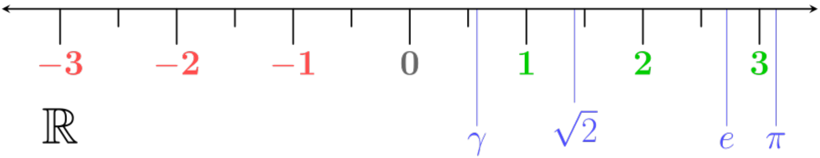

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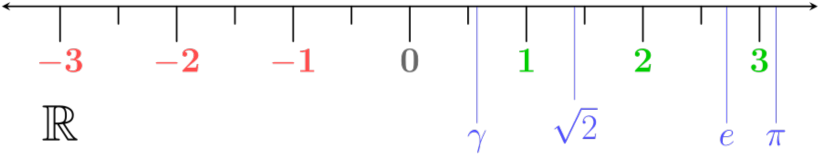

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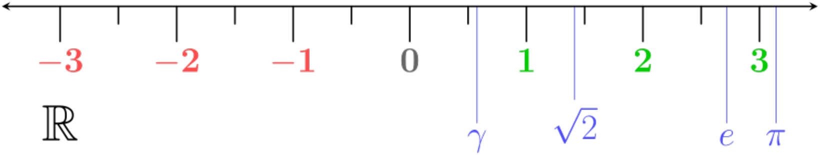

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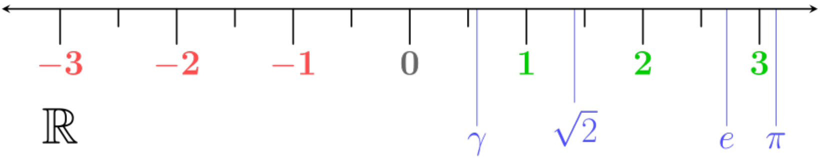

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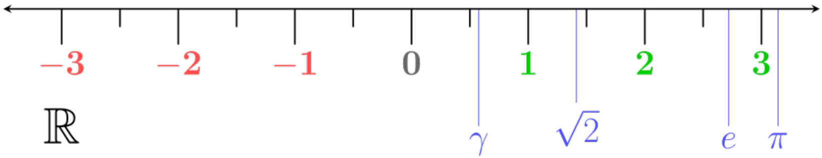

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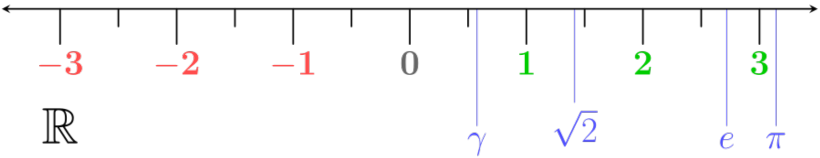

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- But what exactly are we supposed to *construct* numbers from?

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- Thus, n is defined to be a set containing n elements.

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- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Properties of $\mathbb{R}$
Thursday 5 September 2019

Where to find course information

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- The course web site: http://ms.mcmaster.ca/earn/3A03

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- **Assignment 1:** You should have received an e-mail from crowdmark. If not, please e-mail earn@math.mcmaster.ca ASAP stating your full name, student number, and when you registered in the course.

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Poll

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The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

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The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

Note: M3 ensures $0 \neq 1$ to exclude the uninteresting case of a field with only one element.

Standard Mathematical Shorthand

Standard Mathematical Shorthand

Quantifiers

Standard Mathematical Shorthand

Quantifiers

$\forall$

Standard Mathematical Shorthand

Quantifiers

$\forall$ for all

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
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Logical operands

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
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Logical operands

$\wedge$

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and
$\vee$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
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$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and
$\vee$	logical or

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	

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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not

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$\forall$	for all
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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not
$\veebar$	

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$\forall$	for all
$\exists$	there exists
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Logical operands

$\wedge$	logical and
$\vee$	logical or
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Standard Mathematical Shorthand

Quantifiers	
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Logical operands	
$\wedge$	logical and
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Note: $A \veebar B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

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Other shorthand

Standard Mathematical Shorthand

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Other shorthand

$\therefore$

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Other shorthand

$\therefore$	therefore
$\ni$	

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Logical operands

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Other shorthand

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$\ni$	such that
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Other shorthand

$\therefore$	therefore
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$\equiv$	equivalent

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$\ni$	such that	$\iff$	
$\equiv$	equivalent		

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$\therefore$	therefore
$\ni$	such that
$\equiv$	equivalent
$\because$	because
$\iff$	if and only if

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	contradiction

The field axioms (in mathematical shorthand) for field $\mathbb{F}$

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Addition axioms

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A1 *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(x+y) \in \mathbb{F} \,\wedge\, (x+y) = (y+x)$.

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Distribution axiom

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Any collection $\mathbb{F}$ of mathematical objects is called a *field* if it satisfies these 9 algebraic properties.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are Fields?**
- $\boxed{\texttt{Submit}}$.

The integers modulo 3 ($\mathbb{Z}_3$)

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Imagine a clock that repeats after 3 hours rather than 12 hours.

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Examples of fields

Examples of fields

Set	Field?	Why?

Examples of fields

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rationals ($\mathbb{Q}$)		

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)		

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
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reals ($\mathbb{R}$)		

Examples of fields

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Examples of fields

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rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses
reals ($\mathbb{R}$)	**YES**	
complexes ($\mathbb{C}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**YES**	$2^{-1} = 2$

Ordered fields

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$\forall x, y \in \mathbb{F},\ \big((x = y) \wedge \neg(x < y) \wedge \neg(y < x)\big) \veebar \big((x \neq y) \wedge [(x < y) \veebar (y < x)]\big)$

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O2 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $y < z$ is true, then $x < z$ is true

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are ORDERED Fields?**
- $\boxed{\texttt{Submit}}$.

Examples of ordered fields

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Field	Ordered?	Why?

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rationals ($\mathbb{Q}$)		

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Food for thought: Is it possible for any finite field be ordered?

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  It is related to *upper and lower bounds*...

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 3
Properties of $\mathbb{R}$ II
Friday 6 September 2019

Putnam Competition

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- The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. It is organized by the Mathematical Association of America and is taken by over 4,000 participants at more than 500 colleges and universities. More information can be found at

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  Usually a slightly redundant set of axioms is stated to emphasize all the key properties.

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- To prove that $\mathbb{C}$ is not an ordered field, it is not sufficient to prove that the standard order on $\mathbb{R}$ cannot be extended to $\mathbb{C}$. You must show that it is not possible to define *any* order on $\mathbb{C}$ that makes it an ordered field.

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  - These videos go at a slower pace than we do, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

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A set that is bounded above and below is said to be ***bounded***.

Maxima and Minima

Maxima and Minima

Definition (Maximum)

Let $E \subseteq \mathbb{R}$.

Maxima and Minima

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Let $E \subseteq \mathbb{R}$. A number M is said to be ***the maximum*** of E if

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 3: bounded sets**
- Submit.

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$					

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$	**YES**				

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1,1]$	**YES**	**YES**			

Bounds, maxima and minima

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Set	bounded below	bounded above	bounded	min	max
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Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$	**YES**	**YES**	**YES**	-1	

Bounds, maxima and minima

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Bounds, maxima and minima

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$[-1,1)$					

Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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$\varnothing$					

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$\varnothing$	**YES**				

Bounds, maxima and minima

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Set	bounded below	bounded above	bounded	min	max
$[-1,1]$	YES	YES	YES	-1	1
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$\mathbb{N}$	YES	NO	NO	1	$\nexists$
$\mathbb{R}$	NO	NO	NO	$\nexists$	$\nexists$
$\varnothing$	YES	YES			

Bounds, maxima and minima

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$\varnothing$	**YES**	**YES**	**YES**	$\nexists$	$\nexists$

Least upper bounds

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Definition (Least Upper Bound/Supremum)

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What sets have least upper bounds?

Least upper bounds

Least upper bounds

Example

Least upper bounds

Example

Set	bounded above	sup

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$		

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$		

Least upper bounds

Example

Set	bounded above	sup
$[-1,1]$	**YES**	1
$[-1,1)$	**YES**	

Least upper bounds

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Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	1

Least upper bounds

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Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	1
$\varnothing$		

Least upper bounds

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Set	bounded above	sup
$[-1,1]$	**YES**	1
$[-1,1)$	**YES**	1
$\varnothing$	**YES**	

Least upper bounds

Example

Set	bounded above	sup
$[-1,1]$	**YES**	1
$[-1,1)$	**YES**	1
$\varnothing$	**YES**	$\nexists$

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- Extra Challenge Problem:
  Prove that $\mathbb{R}$ is the only complete ordered field.

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If m is the greatest lower bound of E then we write $m = \inf E$.

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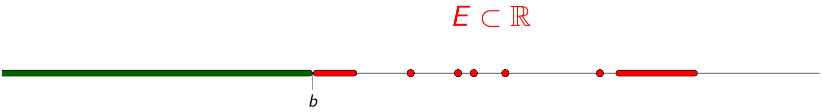

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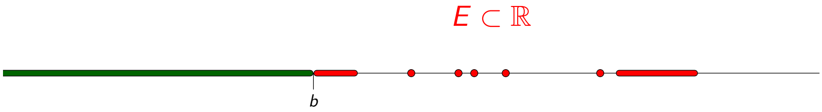

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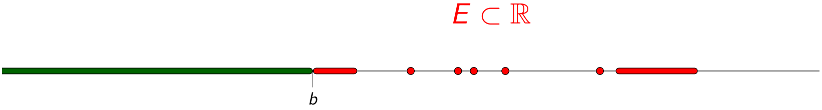

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Comment on least upper bounds and greatest lower bounds

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If you are asked "*What is the least upper bound of* $\mathbb{R}$*?*" how should you answer?
Correct answer: "$\mathbb{R}$ is not bounded above so it does not have a least upper bound."

Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Archimedean property)

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Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$.

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Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$,

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