

1 Introduction



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

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Lecture 1
Introduction
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Where to find course information

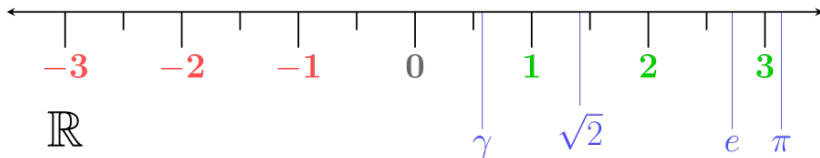
- The course web site:
<http://www.math.mcmaster.ca/earn/3A03>
- Click on [Course information](#) to download course information as pdf file. *You are expected to read and pay attention to every word of this file.*
- Let's have a look now. . .

What is a “real” number?



What is a “real” number?

- The “Reals” (\mathbb{R}) are all the numbers that are needed to fill in the “number line” (so it has no “gaps” or “holes”).
- Why aren’t the rational numbers (\mathbb{Q}) sufficient?



- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
Approach: “Proof by contradiction.”

$\sqrt{2}$ is irrational

Theorem

$$\sqrt{2} \notin \mathbb{Q}.$$

Proof.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m, n) = 1$ such that $m/n = \sqrt{2}$.

$$\therefore \left(\frac{m}{n}\right)^2 = (\sqrt{2})^2 \implies \frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.$$

$\therefore m^2$ is even $\implies m$ is even (\because odd numbers have odd squares).

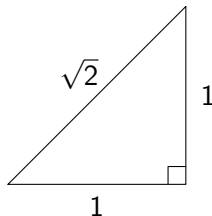
$\therefore m = 2k$ for some $k \in \mathbb{N}$.

$$\therefore 4k^2 = m^2 = 2n^2 \implies 2k^2 = n^2 \implies n \text{ is even.}$$

$\therefore 2$ is a factor of both m and n . **Contradiction!** $\therefore \sqrt{2} \notin \mathbb{Q}$. \square

Does $\sqrt{2}$ exist?

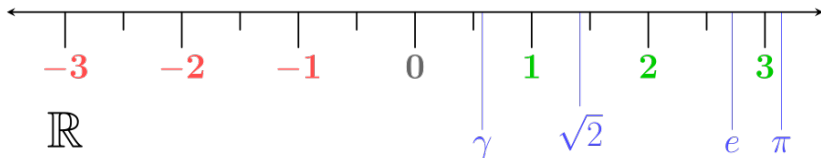
- We have established that $\sqrt{2}$ is not rational.
- But do we really know it exists?
- Can we do without it?
- No. Objects with side length $\sqrt{2}$ exist!



- So irrational numbers are “real”.

What exactly *are* non-rational real numbers?

- We have solid intuition for what rational numbers are. (Ratios of integers.)
- The number line contains numbers that are not rational.



- Can we *construct* irrational numbers? (Just as we construct rationals as ratios of integers?)
- Do we need to *construct* integers first?
- Maybe we should start with 0, 1, 2, ...
- But what exactly are we supposed to *construct* numbers from?

Informal introduction to construction of numbers (\mathbb{N})

- Assume we know what a **set** is.
- Define $0 \equiv \emptyset = \{\}$ (the empty set)
- Define $1 \equiv \{0\} = \{\emptyset\} = \{\{\}\}$
- Define $2 \equiv \{0, 1\} = \{\{\}, \{\{\}\}\}$
- Define $n + 1 \equiv n \cup \{n\}$ (successor function)
- Define **natural numbers** $\mathbb{N} = \{1, 2, 3, \dots\}$
 - Some books define $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \dots\}$.
 - It is more common to define \mathbb{N} to start with 1.
- Thus, n is defined to be a set containing n elements.

Informal introduction to construction of numbers (\mathbb{N})

Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as “the set of all sets containing n elements”.
- The earlier definition possibly better captures our intuitive notion of what n “really is”, but such “sets” are unweildy and create serious challenges for development of mathematical foundations.

Informal introduction to construction of numbers (\mathbb{N})

Order of natural numbers:

- Natural numbers defined as above have the right order:

$$m \leq n \iff m \subseteq n$$

Note: we define " \leq " on natural numbers via " \subseteq " on sets.

Addition and multiplication of natural numbers:

- Still possible to define in terms of sets, but trickier.
- We'll defer this for later, after gaining more experience with rigorous mathematical concepts.
- If you can't wait, see this free e-book:

"Transition to Higher Mathematics"
<http://openscholarship.wustl.edu/books/10/>.