McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 1
Introduction
Monday 7 January 2019

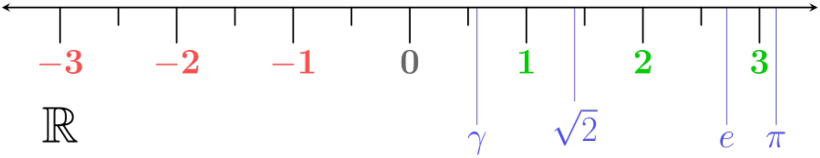
Where to find course information

- The course web site:
  `http://www.math.mcmaster.ca/earn/3A03`
- Click on Course information to download course information as pdf file. *You are expected to read and pay attention to every word of this file.*
- Let's have a look now. . .

What is a "real" number?

What is a "real" number?

- The "Reals" ($\mathbb{R}$) are all the numbers that are needed to fill in the "number line" (so it has no "gaps" or "holes").
- Why aren't the rational numbers ($\mathbb{Q}$) sufficient?



- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
  Approach: "Proof by contradiction."

$\sqrt{2}$ is irrational

Theorem

$\sqrt{2} \notin \mathbb{Q}$.

Proof.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m,n) = 1$ such that $m/n = \sqrt{2}$.

$$\therefore \left(\frac{m}{n}\right)^2 = (\sqrt{2})^2 \implies \frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.$$

$\therefore m^2$ is even $\implies$ m is even ($\because$ odd numbers have odd squares).

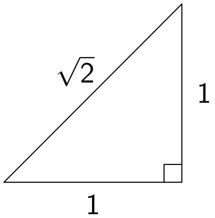
$\therefore m = 2k$ for some $k \in \mathbb{N}$.

$$\therefore 4k^2 = m^2 = 2n^2 \implies 2k^2 = n^2 \implies n \text{ is even}.$$

$\therefore$ 2 is a factor of both m and n. **Contradiction!** $\therefore \sqrt{2} \notin \mathbb{Q}$. $\square$

Does $\sqrt{2}$ exist?

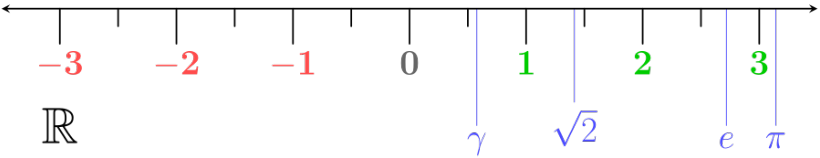
- We have established that $\sqrt{2}$ is not rational.
- But do we really know it exists?
- Can we do without it?
- No. Objects with side length $\sqrt{2}$ exist!



- So irrational numbers are "real".

What exactly *are* non-rational real numbers?

- We have solid intuition for what rational numbers are. (Ratios of integers.)
- The number line contains numbers that are not rational.



- Can we *construct* irrational numbers? (Just as we construct rationals as ratios of integers?)
- Do we need to *construct* integers first?
- Maybe we should start with 0, 1, 2, . . .
- But <u>what</u> exactly are we supposed to *construct* numbers <u>from</u>?

Informal introduction to construction of numbers ($\mathbb{N}$)

- Assume we know what a **set** is.
- Define $0 \equiv \varnothing = \{\}$ (the empty set)
- Define $1 \equiv \{0\} = \{\varnothing\} = \{\{\}\}$
- Define $2 \equiv \{0, 1\} = \{\{\}, \{\{\}\}\}$
- Define $n + 1 \equiv n \cup \{n\}$ (successor function)
- Define **natural numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$
  - Some books define $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$.
  - It is more common to define $\mathbb{N}$ to start with 1.
- Thus, n is defined to be a set containing n elements.

Informal introduction to construction of numbers ($\mathbb{N}$)

Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

Informal introduction to construction of numbers ($\mathbb{N}$)

Order of natural numbers:

- Natural numbers defined as above have the right order:

$$m \leq n \iff m \subseteq n$$

Note: we *define* "$\leq$" on natural numbers via "$\subseteq$" on sets.

Addition and multiplication of natural numbers:

- Still possible to define in terms of sets, but trickier.
- We'll defer this for later, after gaining more experience with rigorous mathematical concepts.
- If you can't wait, see this free e-book:

  "Transition to Higher Mathematics"
  http://openscholarship.wustl.edu/books/10/.

Informal introduction to construction of numbers ($\mathbb{Z}$)

Integers:

- Need additive inverses for all natural numbers.
- Need to define $\cdot, +, -,$ for all pairs of integers.
- Again, possible to define everything via set theory.
- Again, we'll defer this for later.

- For now, we'll assume we "know" what the naturals $\mathbb{N}$ and the integers $\mathbb{Z}$ "are".
- We can then *construct* the rationals $\mathbb{Q}\ldots$

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Properties of $\mathbb{R}$
Wednesday 9 January 2019

Where to find course information

- The course web site:
  `http://www.math.mcmaster.ca/earn/3A03`
- Click on Course information to download pdf file.
  - **Read it!!**
- Check the course web site regularly!

What we did last class

- The "Reals" ($\mathbb{R}$) are all the numbers that are needed to fill in the "number line" (so it has no "gaps" or "holes").
- The rationals ($\mathbb{Q}$) have "holes", *e.g.*, $\sqrt{2}$.
- Numbers can be constructed using sets. We will discuss this *informally*. A more formal approach is taken in Math 4L03 (Mathematical Logic) or in this online e-book.
  - The naturals ($\mathbb{N} = \{1, 2, 3, \ldots\}$) can be constructed from $\varnothing$: $0 = \varnothing,\ 1 = \{0\},\ 2 = \{0, 1\}, \ldots,\ n + 1 = n \cup \{n\}$.
  - The integers ($\mathbb{Z}$), and operations on them $(+, -, \cdot)$, can also be constructed from sets and set operations (but we deferred that for later).
  - Given $\mathbb{N}$ and $\mathbb{Z}$, we can construct $\mathbb{Q}$...

Informal introduction to construction of numbers ($\mathbb{Q}$)

Rationals:

- *Idea:* Associate $\mathbb{Q}$ with $\mathbb{Z} \times \mathbb{N}$
- Use notation $\dfrac{a}{b} \in \mathbb{Q}$ if $(a, b) \in \mathbb{Z} \times \mathbb{N}$.
- Define equivalence of rational numbers:

$$\frac{a}{b} = \frac{c}{d} \quad \overset{\textbf{def}}{=} \quad a \cdot d = b \cdot c$$

- Define order for rational numbers:

$$\frac{a}{b} \leq \frac{c}{d} \quad \overset{\textbf{def}}{=} \quad a \cdot d \leq b \cdot c$$

Informal introduction to construction of numbers ($\mathbb{Q}$)

Rationals, continued:

- Define operations on rational numbers:

$$\frac{a}{b} + \frac{c}{d} \quad \stackrel{\textbf{def}}{=} \quad \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} \quad \stackrel{\textbf{def}}{=} \quad \frac{a \cdot c}{b \cdot d}$$

- Constructed in this way (ultimately from the empty set), $\mathbb{Q}$ satisfies all the standard properties we associate with the rational numbers.

- Formally, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$. Extra Challenge Problem: Are "$+$" and "$\cdot$" well-defined on $\mathbb{Q}$?

Properties of the rational numbers ($\mathbb{Q}$)

Addition:

A1 *Closed and commutative under addition.* For any $x, y \in \mathbb{Q}$ there is a number $x + y \in \mathbb{Q}$ and $x + y = y + x$.

A2 *Associative under addition.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y) + z = x + (y + z)$$

is true.

A3 *Existence and uniqueness of additive identity.* There is a unique number $0 \in \mathbb{Q}$ such that, for all $x \in \mathbb{Q}$,

$$x + 0 = 0 + x = x.$$

A4 *Existence of additive inverses.* For any number $x \in \mathbb{Q}$ there is a corresponding number denoted by $-x$ with the property that

$$x + (-x) = 0.$$

Properties of the rational numbers ($\mathbb{Q}$)

Multiplication:

M1 *Closed and commutative under multiplication.* For any $x, y \in \mathbb{Q}$ there is a number $xy \in \mathbb{Q}$ and $xy = yx$.

M2 *Associative under multiplication.* For any $x, y, z \in \mathbb{Q}$ the identity $(xy)z = x(yz)$ is true.

M3 *Existence and uniqueness of multiplicative identity.* There is a unique number $1 \in \mathbb{Q} \setminus \{0\}$ such that, for all $x \in \mathbb{Q}$, $x1 = 1x = x$.

M4 *Existence of multiplicative inverses.* For any non-zero number $x \in \mathbb{Q}$ there is a corresponding number denoted by x^{-1} with the property that $xx^{-1} = 1$.

Properties of the rational numbers ($\mathbb{Q}$)

Addition and multiplication together:

AM1 *Distributive law.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y)z = xz + yz$$

is true.

The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a **field**.

Note: M3 ensures $0 \neq 1$ to exclude the uninteresting case of a field with only one element.

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not
$\veebar$	logical exclusive or

Note: $A \veebar B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	contradiction

The field axioms (in mathematical shorthand) for field $\mathbb{F}$

Addition axioms

A1 *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(x+y) \in \mathbb{F} \wedge (x+y) = (y+x)$.

A2 *Associative.* $\forall x, y, z \in \mathbb{F}$, $(x+y)+z = x+(y+z)$.

A3 *Identity.* $\exists!\ 0 \in \mathbb{F} \ni \forall x \in \mathbb{F}$, $x+0 = 0+x = x$.

A4 *Inverses.* $\forall x \in \mathbb{F},\ \exists\,(-x) \in \mathbb{F} \ni$ $x+(-x) = 0$.

Multiplication axioms

M1 *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(xy) \in \mathbb{F} \ \wedge\ (xy) = (yx)$.

M2 *Associative.* $\forall x, y, z \in \mathbb{F}$, $(xy)z = x(yz)$.

M3 *Identity.* $\exists!\ 1 \in \mathbb{F}\setminus\{0\} \ni$ $\forall x \in \mathbb{F},\ x1 = 1x = x$.

M4 *Inverses.* $\forall x \in \mathbb{F}\setminus\{0\}$, $\exists\, x^{-1} \in \mathbb{F} \ni xx^{-1} = 1$.

Distribution axiom

AM1 *Distribution.* $\forall x, y, z \in \mathbb{F},\ (x+y)z = xz + yz$.

Any collection $\mathbb{F}$ of mathematical objects is called a *field* if it satisfies these 9 algebraic properties.

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses
reals ($\mathbb{R}$)	**YES**	
complexes ($\mathbb{C}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**YES**	$2^{-1} = 2$

The integers modulo 3 ($\mathbb{Z}_3$)

Imagine a clock that repeats after 3 hours rather than 12 hours.

$\mathbb{Z}_3$ contains the three elements $\{0, 1, 2\}$, with addition and multiplication defined as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Ordered fields

A field $\mathbb{F}$ is said to be **ordered** if the following properties hold:

Order axioms

O1 For any $x, y \in \mathbb{F}$, exactly one of the statements $x = y$, $x < y$ or $y < x$ is true ("**trichotomy**"), *i.e.,*
$\forall x, y \in \mathbb{F},\ ((x = y) \wedge \neg(x < y) \wedge \neg(y < x)) \veebar ((x \neq y) \wedge [(x < y) \veebar (y < x)])$

O2 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $y < z$ is true, then $x < z$ is true, *i.e.,* $\forall x, y, z \in \mathbb{F},\ (x < y) \wedge (y < z) \implies (x < z)$

O3 For any $x, y \in \mathbb{F}$, if $x < y$ is true, then $x + z < y + z$ is also true for any $z \in \mathbb{F}$, *i.e.,* $\forall x, y \in \mathbb{F},\ (x < y) \implies x + z < y + z,\ \forall z \in \mathbb{F}$

O4 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $z > 0$ is true, then $xz < yz$ is also true,
i.e., $\forall x, y, z \in \mathbb{F},\ (x < y) \wedge (0 < z) \implies (xz < yz)$

Examples of ordered fields

Field	Ordered?	Why?
rationals ($\mathbb{Q}$)	**YES**	
reals ($\mathbb{R}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**NO**	Next slide...
complexes ($\mathbb{C}$)	**NO**	Extra Challenge Problem: *Prove the field $\mathbb{C}$ cannot be ordered.*

The field of integers modulo 3 cannot be ordered

Proposition

$\mathbb{Z}_3$ is not an ordered field.

Proof.

Approach: proof by contradiction.

If $\mathbb{Z}_3$ is ordered, then O1 (trichotomy) implies that either $0 < 1$ or $1 < 0$ (and not both).

Suppose $0 < 1$ and $1 \not< 0$. Then O3 $\implies$ $0 + 1 < 1 + 1$,
i.e., $1 < 2$. $\therefore$ O2 (transitivity) $\implies$ $0 < 2$.
Using O3 again, we have $0 + 1 < 2 + 1$, *i.e.*, $1 < 0$. $\Rightarrow\Leftarrow$

Now suppose $1 < 0$. Similarly reach a contradiction (check!).
$\therefore$ $\mathbb{Z}_3$ cannot be ordered. $\square$

Food for thought: Is it possible for any finite field be ordered?

What other properties does $\mathbb{R}$ have?

- $\mathbb{R}$ is an ordered field.
- $\mathbb{R}$ includes numbers that are not in $\mathbb{Q}$, *e.g.*, $\sqrt{2}$.
- What additional properties does $\mathbb{R}$ have?
- Only one more property is required to fully characterize $\mathbb{R}$...
  It is related to *upper and lower bounds*...

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Mathematics 3A03
Real Analysis I

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Lecture 3
Properties of $\mathbb{R}$ II
Friday 11 January 2019

Announcements and comments arising from Lecture 2

- My office hours will be Mondays 1:30pm–2:20pm going forward. (Or by appointment.)
- Questions for next week's tutorials and some "Logic Notes" are posted on the Tutorials page of the course web site.
- Field Axiom M3 was corrected before posting slides for Lecture 2.
- No claim is being made that the field axioms as stated are absolutely minimal (*i.e.*, that there are no redundancies). In fact, we don't need to assume:
  - Identities are unique.
  - Inverses are unique.
  - Commutivity under addition (!).

  Usually a slightly redundant set of axioms is stated to emphasize all the key properties.

An additional online resource

A sequence of 15 short (3–7 minute) videos covering the very basics of mathematical logic and theorem proving has been posted associated with a course at the University of Toronto:

- Go to http://uoft.me/MAT137, click on the **Videos** tab and then on **Playlist 1**.

These videos go at a slower pace than we do, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

More comments arising from Lecture 2

- The property that completes the specification of $\mathbb{R}$ has to somehow fill in all the "holes" in $\mathbb{Q}$.
- It is true that if $x, y \in \mathbb{Q}$ then $\exists r \in \mathbb{R} \setminus \mathbb{Q}$ with $x < r < y$. But this property is not sufficient to characterize $\mathbb{R}$, because it is satisfied by subsets of $\mathbb{R}$.

Bounds

Definition (Upper Bound)

Let $E \subseteq \mathbb{R}$. A number M is said to be an **upper bound** for E if $x \leq M$ for all $x \in E$.

A set that has an upper bound is said to be **bounded above**.

Definition (Lower Bound)

Let $E \subseteq \mathbb{R}$. A number m is said to be a **lower bound** for E if $m \leq x$ for all $x \in E$.

A set that has a lower bound is said to be **bounded below**.

A set that is bounded above and below is said to be **bounded**.

Maxima and Minima

Definition (Maximum)

Let $E \subseteq \mathbb{R}$. A number M is said to be **the maximum** of E if M is an upper bound for E and $M \in E$. If such an M exists we write $M = \max E$.

Definition (Minimum)

Let $E \subseteq \mathbb{R}$. A number m is said to be **the minimum** of E if m is a lower bound for E and $m \in E$. If such an m exists we write $m = \min E$.

We refer to "the" maximum and "the" minimum of E because there cannot be more than one of each. *(Proof?)*

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1,1]$	YES	YES	YES	-1	1
$[-1,1)$	YES	YES	YES	-1	$\nexists$
$[-1,\infty)$	YES	NO	NO	-1	$\nexists$
$[-1,-\frac{1}{4})\cup(\frac{1}{2},1]$	YES	YES	YES	-1	1
$\mathbb{N}$	YES	NO	NO	1	$\nexists$
$\mathbb{R}$	NO	NO	NO	$\nexists$	$\nexists$
$\varnothing$	YES	YES	YES	$\nexists$	$\nexists$

Least upper bounds

Definition (Least Upper Bound/Supremum)

A number M is said to be the **least upper bound** or **supremum** of a set E if

(i) M is an upper bound of E, and

(ii) if $\widetilde{M}$ is an upper bound of E then $M \leq \widetilde{M}$.

If M is the least upper bound of E then we write $M = \sup E$.

Note: We can refer to "the" least upper bound of E because there cannot be more than one. (Proof?)

What sets have least upper bounds?

Least upper bounds

Example

Set	bounded above	sup
$[-1,1]$	**YES**	1
$[-1,1)$	**YES**	1
$\varnothing$	**YES**	$\nexists$
$\{x \in \mathbb{R} : x^2 < 2\}$	**YES**	$\sqrt{2}$
$\{x \in \mathbb{Q} : x^2 < 2\}$	**YES**	$\notin \mathbb{Q}$

Least upper bounds

The property that any set that is bounded above has a least upper bound is what distinguishes the real numbers $\mathbb{R}$ from the rational numbers $\mathbb{Q}$.

Does this realization allow us to finish constructing $\mathbb{R}$?

YES, but we will delay the construction until later in the course.

For now, we will simply annoint the least upper bound property as an axiom:

Completeness Axiom

If $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and E is bounded above, then E has a least upper bound (*i.e.*, $\sup E$ exists and $\sup E \in \mathbb{R}$).

$\mathbb{R}$ is a complete ordered field

- Any field $\mathbb{F}$ that satisfies the order axioms and the completeness axiom is said to be a **complete ordered field**.
- $\mathbb{R}$ is a complete ordered field.
- Are there any other complete ordered fields?
- Extra Challenge Problem:
  Prove that $\mathbb{R}$ is the only complete ordered field.

Greatest lower bounds

Definition (Greatest Lower Bound/Infimum)

A number m is said to be the **greatest lower bound** or **infimum** of a set E if

(i) m is a lower bound of E, and

(ii) if $\widetilde{m}$ is a lower bound of E then $\widetilde{m} \leq m$.

If m is the greatest lower bound of E then we write $m = \inf E$.

Greatest lower bounds

- The existence of least upper bounds was taken as an axiom.
- The existence of greatest lower bounds then follows.

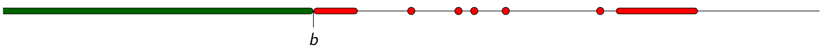
Theorem

If $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and E is bounded below, then E has a greatest lower bound (i.e., $\inf E$ exists and $\inf E \in \mathbb{R}$).

Proof?

Idea of proof:

$E \subset \mathbb{R}$



$L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound of } E\}$

Greatest lower bounds

Theorem

If $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and E is bounded below, then E has a greatest lower bound (i.e., inf E *exists and* inf $E \in \mathbb{R}$*).*

Proof. Recall graphical idea of proof.

Let $L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound of } E\}$. Then:

- $L \neq \varnothing$ ($\because$ E is bounded below).
- L is bounded above ($\because x \in E \implies x$ an upper bound for L).
- $\therefore$ L has a least upper bound, say $b = \sup L$.

Now show $b = \inf E$. First show $b \in L$ (*i.e.*, $x \in E \implies b \leq x$). Suppose $x \in E$ and $b \not\leq x$; then by O1 (trichotomy), we must have $b > x$. Now $b = \sup L$ and $x < b$, so x is not an upper bound of L, *i.e.*, there is some $\ell \in L$ such that $x < \ell$. But then ℓ is not a lower bound of E. $\Rightarrow\Leftarrow$ $\therefore$ $b \in L$ and b is also $\max L$, *i.e.*, $b = \inf E$. $\square$

Comment on least upper bounds and greatest lower bounds

- The proof above shows that:

$$\inf E = \sup\{x \in \mathbb{R} : x \text{ is a lower bound of } E\}$$

- Similarly:

$$\sup E = \inf\{x \in \mathbb{R} : x \text{ is a upper bound of } E\}$$

Some notational abuse concerning sup and inf

By convention, for convenience, we (and your textbook) sometimes write:

$$\begin{aligned} \inf \mathbb{R} &= -\infty \\ \sup \mathbb{R} &= \infty \\ \inf \varnothing &= \infty \\ \sup \varnothing &= -\infty \end{aligned}$$

This is an **abuse of notation**, since $\varnothing$ and $\mathbb{R}$ do not have least upper or greatest lower bounds in $\mathbb{R}$. ∞ *is not a real number.*

If you are asked "*What is the least upper bound of $\mathbb{R}$?*" how should you answer?
Correct answer: "$\mathbb{R}$ is not bounded above so it does not have a least upper bound."

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Archimedean property)

The set of natural numbers $\mathbb{N}$ has no upper bound.

Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$, hence $n + 1 \leq B$ for all $n \in \mathbb{N}$, *i.e.*, $n \leq B - 1$ for all $n \in \mathbb{N}$. Thus, $B - 1$ is an upper bound for $\mathbb{N}$, contradicting B being the least upper bound. $\square$

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Equivalences of the Archimedean property)

1. *The set of natural numbers $\mathbb{N}$ has no upper bound.*

2. *Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

   i.e., No matter how large a real number x is, there is always a natural number n that is larger.

3. *Given any $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

   i.e., Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in \mathbb{N}$).

4. *Given any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

   i.e., Given any positive number x, no matter how small, one can always find a fraction $1/n$ that is smaller than x.