McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 1
Introduction
Tuesday 3 September 2019

Where to find course information

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- The course web site: http://ms.mcmaster.ca/earn/3A03

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- Let's have a look now. . .

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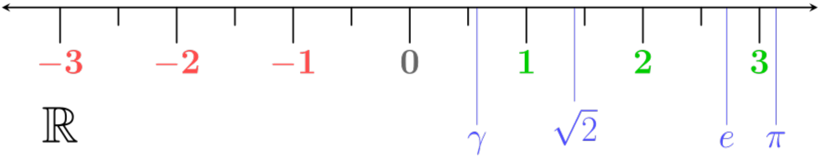

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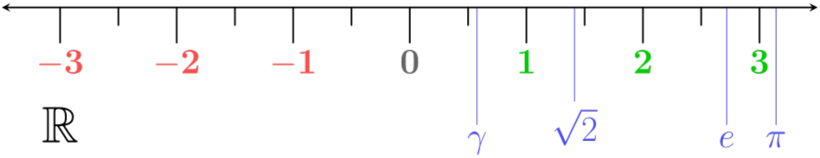

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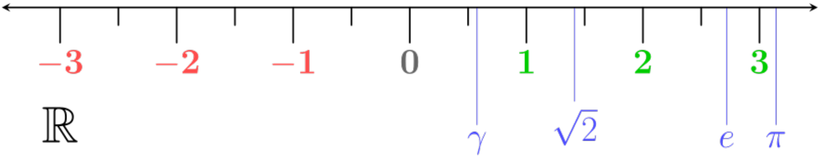

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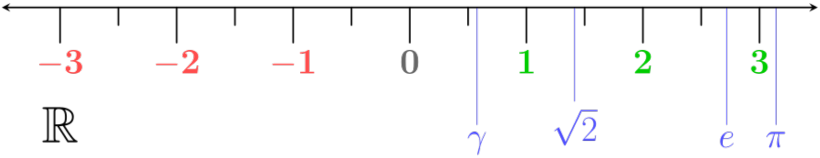

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  <u>*Approach*</u>: “Proof by contradiction.”

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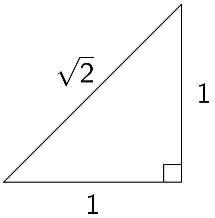

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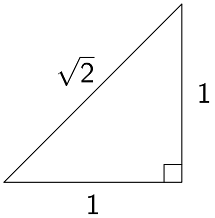

- So irrational numbers are “real”.

Poll on rationality

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- Let's Deactivate the poll and View Results

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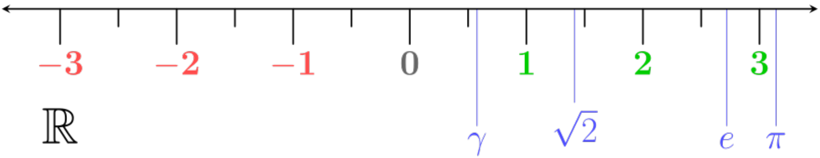

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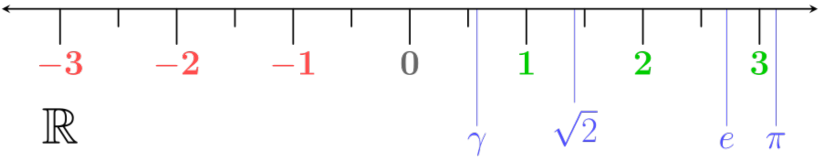

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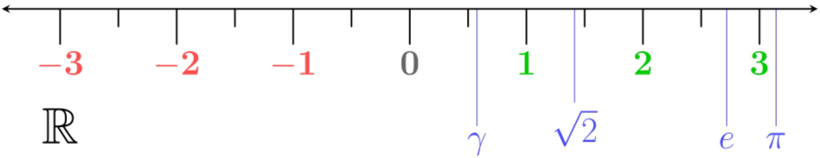

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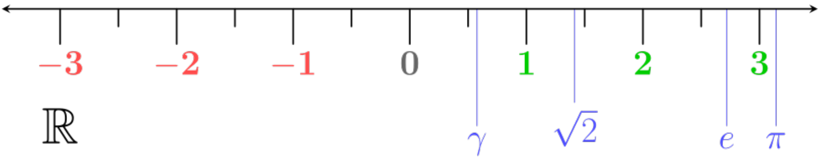

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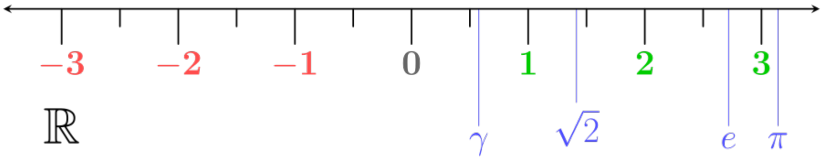

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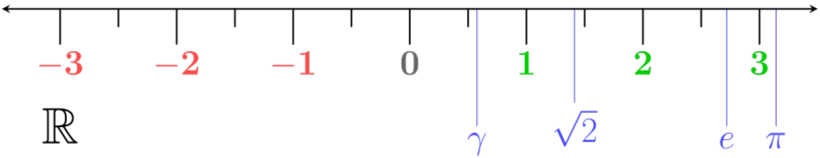

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- Do we need to *construct* integers first?
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- But <u>what</u> exactly are we supposed to *construct* numbers <u>from</u>?

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- Assume we know what a ***set*** is.

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Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

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Addition and multiplication of natural numbers:

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- We can then *construct* the rationals $\mathbb{Q}$. . .

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Properties of $\mathbb{R}$
Thursday 5 September 2019

Where to find course information

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- The course web site: http://ms.mcmaster.ca/earn/3A03

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- **Assignment 1:** You should have received an e-mail from crowdmark. If not, please e-mail earn@math.mcmaster.ca ASAP stating your full name, student number, and when you registered in the course.

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Poll

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The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

Properties of the rational numbers ($\mathbb{Q}$)

Addition and multiplication together:

AM1 *Distributive law.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y)z = xz + yz$$

is true.

The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

Note: M3 ensures $0 \neq 1$ to exclude the uninteresting case of a field with only one element.

Standard Mathematical Shorthand

Standard Mathematical Shorthand

Quantifiers

Standard Mathematical Shorthand

Quantifiers

$\forall$

Standard Mathematical Shorthand

Quantifiers

$\forall$ for all

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
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Logical operands

Standard Mathematical Shorthand

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$\forall$	for all
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Logical operands

$\wedge$

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
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Logical operands

$\wedge$	logical and
$\vee$	

Standard Mathematical Shorthand

Quantifiers

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Logical operands

$\wedge$	logical and
$\vee$	logical or

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$\wedge$	logical and
$\vee$	logical or
$\neg$	

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Logical operands

$\wedge$	logical and
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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not
$\veebar$	

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Note: $A \veebar B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

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Other shorthand

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$\therefore$

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<u>*Note:*</u> $A \veebar B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

Other shorthand

$\therefore$	therefore
$\ni$	

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Other shorthand

$\therefore$	therefore
$\ni$	such that

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$\equiv$	

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$\therefore$	therefore	$\because$	because
$\ni$	such that	$\Longleftrightarrow$	
$\equiv$	equivalent		

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Other shorthand

$\therefore$	therefore
$\ni$	such that
$\equiv$	equivalent
$\because$	because
$\iff$	if and only if

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	contradiction

The field axioms (in mathematical shorthand) for field $\mathbb{F}$

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Addition axioms

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- **A1** *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(x+y) \in \mathbb{F} \;\wedge\; (x+y) = (y+x)$.

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Any collection $\mathbb{F}$ of mathematical objects is called a *field* if it satisfies these 9 algebraic properties.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are Fields?**
- $\boxed{\texttt{Submit}}$.

The integers modulo 3 ($\mathbb{Z}_3$)

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Imagine a clock that repeats after 3 hours rather than 12 hours.

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$+$	0	1	2
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Examples of fields

Examples of fields

Set	Field?	Why?

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)		

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)		

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses

Examples of fields

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rationals ($\mathbb{Q}$)	**YES**	
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reals ($\mathbb{R}$)		

Examples of fields

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Examples of fields

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Examples of fields

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rationals ($\mathbb{Q}$)	**YES**	
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reals ($\mathbb{R}$)	**YES**	
complexes ($\mathbb{C}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**YES**	$2^{-1} = 2$

Ordered fields

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$\forall x, y \in \mathbb{F},\ \big((x = y) \wedge \neg(x < y) \wedge \neg(y < x)\big) \veebar \big((x \neq y) \wedge [(x < y) \veebar (y < x)]\big)$

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O2 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $y < z$ is true, then $x < z$ is true

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are ORDERED Fields?**
- $\boxed{\texttt{Submit}}$.

Examples of ordered fields

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Field	Ordered?	Why?

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rationals ($\mathbb{Q}$)		

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		Extra Challenge Problem: *Prove the field $\mathbb{C}$ cannot be ordered.*

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Food for thought: Is it possible for any finite field be ordered?

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  It is related to *upper and lower bounds*...

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 3
Properties of $\mathbb{R}$ II
Friday 6 September 2019

Putnam Competition

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  Usually a slightly redundant set of axioms is stated to emphasize all the key properties.

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- To prove that $\mathbb{C}$ is not an ordered field, it is not sufficient to prove that the standard order on $\mathbb{R}$ cannot be extended to $\mathbb{C}$. You must show that it is not possible to define *any* order on $\mathbb{C}$ that makes it an ordered field.

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  - These videos go at a slower pace than we do, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

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A set that is bounded above and below is said to be ***bounded***.

Maxima and Minima

Maxima and Minima

Definition (Maximum)

Let $E \subseteq \mathbb{R}$.

Maxima and Minima

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 3: bounded sets**
- $\boxed{\texttt{Submit}}$.

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$					

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$	**YES**				

Bounds, maxima and minima

Example

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$[-1,1]$	**YES**	**YES**			

Bounds, maxima and minima

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Set	bounded below	bounded above	bounded	min	max
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Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1,1]$	**YES**	**YES**	**YES**	-1	

Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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$\varnothing$					

Bounds, maxima and minima

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$\mathbb{N}$	**YES**	**NO**	**NO**	1	$\nexists$
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$\varnothing$	**YES**				

Bounds, maxima and minima

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$\mathbb{N}$	YES	NO	NO	1	$\nexists$
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$\varnothing$	YES	YES	YES	$\nexists$	

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$\varnothing$	**YES**	**YES**	**YES**	$\nexists$	$\nexists$

Least upper bounds

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Definition (Least Upper Bound/Supremum)

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What sets have least upper bounds?

Least upper bounds

Least upper bounds

Example

Least upper bounds

Example

Set	bounded above	sup

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$		

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$		

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	1

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	1
$\varnothing$		

Least upper bounds

Example

Set	bounded above	sup
$[-1,1]$	**YES**	1
$[-1,1)$	**YES**	1
$\varnothing$	**YES**	

Least upper bounds

Example

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$[-1, 1]$	**YES**	1
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- Extra Challenge Problem:
  Prove that $\mathbb{R}$ is the only complete ordered field.

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If m is the greatest lower bound of E then we write $m = \inf E$.

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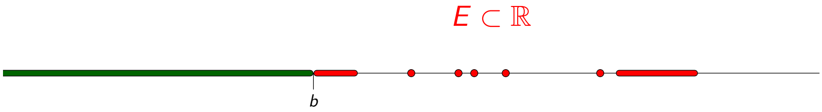

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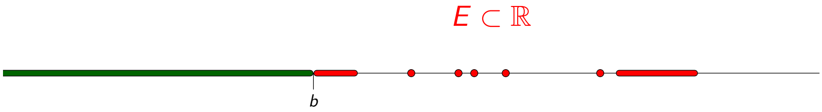

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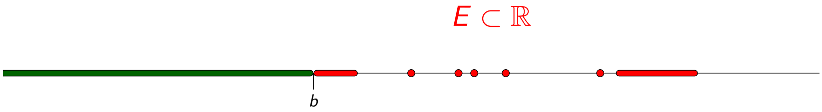

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Comment on least upper bounds and greatest lower bounds

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Some notational abuse concerning sup and inf

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If you are asked "*What is the least upper bound of* $\mathbb{R}$*?*" how should you answer?

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Correct answer:

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This is an ***abuse of notation***, since $\varnothing$ and $\mathbb{R}$ do not have least upper or greatest lower bounds in $\mathbb{R}$. ∞ *is not a real number.*

If you are asked "*What is the least upper bound of* $\mathbb{R}$*?*" how should you answer?
Correct answer: "$\mathbb{R}$ is not bounded above so it does not have a least upper bound."

Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Archimedean property)

The set of natural numbers $\mathbb{N}$ has no upper bound.

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Proof.

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Suppose $\mathbb{N}$ is bounded above.

Consequences of the real number axioms (§§1.7–1.9)

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Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$.

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Archimedean property)

The set of natural numbers $\mathbb{N}$ has no upper bound.

Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$.

Consequences of the real number axioms (§§1.7–1.9)

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Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$,

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Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$, hence $n + 1 \leq B$ for all $n \in \mathbb{N}$, *i.e.,* $n \leq B - 1$ for all $n \in \mathbb{N}$. Thus, $B - 1$ is an upper bound for $\mathbb{N}$,

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Proof.

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Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Equivalences of the Archimedean property)

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Consequences of the real number axioms (§§1.7–1.9)

Theorem (Equivalences of the Archimedean property)

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1. *The set of natural numbers $\mathbb{N}$ has no upper bound.*
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i.e., No matter how large a real number x is, there is always a natural number n that is larger.

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   i.e., No matter how large a real number x is, there is always a natural number n that is larger.
3. *Given any* $x > 0$ *and* $y > 0$*, there exists* $n \in \mathbb{N}$ *such that* $nx > y$.

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1. *The set of natural numbers $\mathbb{N}$ has no upper bound.*
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3. *Given any $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

   i.e., Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in \mathbb{N}$).

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Theorem (Equivalences of the Archimedean property)

1. *The set of natural numbers* $\mathbb{N}$ *has no upper bound.*
2. *Given any* $x \in \mathbb{R}$*, there exists* $n \in \mathbb{N}$ *such that* $n > x$*.*

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   i.e., Given any positive number y*, no matter how large, and any positive number* x*, no matter how small, one can add* x *to itself sufficiently many times so that the result exceeds* y *(i.e.,* $nx > y$ *for some* $n \in \mathbb{N}$*).*
4. *Given any* $x > 0$*, there exists* $n \in \mathbb{N}$ *such that* $\frac{1}{n} < x$*.*

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1. *The set of natural numbers $\mathbb{N}$ has no upper bound.*

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3. *Given any $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

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4. *Given any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

   i.e., Given any positive number x, no matter how small, one can always find a fraction $1/n$ that is smaller than x.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 4
Properties of $\mathbb{R}$ III
Tuesday 10 September 2019

Comments arising. . .

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- TA Math Help Centre hours are now listed on course information sheet.

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- Remember Assignment 1 is due Tuesday 17 Sep 2019 @ 2:25pm via crowdmark.
- Last time we ended with some equivalent conditions relating $\mathbb{R}$ and $\mathbb{N}$.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 4: Theorem or Axiom?**
- $\boxed{\texttt{Submit}}$.

Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Well-Ordering Property)

Every nonempty subset of $\mathbb{N}$ has a smallest element.

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Proof.

Consequences of the real number axioms (§§1.7–1.9)

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Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$.

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Proof.

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Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound

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Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound (in $\mathbb{R}$).

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Suppose $b \notin S$.

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Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b)

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Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$

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Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$ (since $b \notin S$). $\therefore$ $n \in S \cap (b, b+1)$.

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Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$ (since $b \notin S$). $\therefore$ $n \in S \cap (b, b + 1)$. But just as $b + 1$ cannot be a lower bound for S, n cannot be a lower bound for S (since it too would be a lower bound greater than $b = \inf S$). $\therefore \exists m \in S \cap (b, n)$.

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Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound (in $\mathbb{R}$). Let $b = \inf S$. If $b \in S$ then $b = \min S$ and we are done.

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Every nonempty subset of $\mathbb{N}$ has a smallest element.

Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound (in $\mathbb{R}$). Let $b = \inf S$. If $b \in S$ then $b = \min S$ and we are done.

Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$ (since $b \notin S$). $\therefore$ $n \in S \cap (b, b+1)$. But just as $b + 1$ cannot be a lower bound for S, n cannot be a lower bound for S (since it too would be a lower bound greater than $b = \inf S$). $\therefore \exists m \in S \cap (b, n)$. But we now have $b < m < n < b + 1$, which is impossible because m and n are both integers. $\Rightarrow\Leftarrow$ Therefore $b \in S$,

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Well-Ordering Property)

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound (in $\mathbb{R}$). Let $b = \inf S$. If $b \in S$ then $b = \min S$ and we are done.

Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$ (since $b \notin S$). $\therefore$ $n \in S \cap (b, b+1)$. But just as $b + 1$ cannot be a lower bound for S, n cannot be a lower bound for S (since it too would be a lower bound greater than $b = \inf S$). $\therefore \exists m \in S \cap (b, n)$. But we now have $b < m < n < b + 1$, which is impossible because m and n are both integers. $\Rightarrow\Leftarrow$ Therefore $b \in S$, so $b = \min S$. $\square$

Consequences of the real number axioms (§§1.7–1.9)

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Corollary

Every nonempty subset of $\mathbb{Z}$ that is bounded below (in $\mathbb{R}$) has a smallest element.

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Corollary

Every nonempty subset of $\mathbb{Z}$ that is bounded below (in $\mathbb{R}$) has a smallest element.

Proof.

The proof is identical to the proof of the well-ordering property for $\mathbb{N}$ except that we start with a set of integers that is bounded below, rather than having to first identify a lower bound for the set. $\square$

Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Principle of Mathematical Induction)

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Consequences of the real number axioms (§§1.7–1.9)

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Let $S \subseteq \mathbb{N}$. Suppose that $1 \in S$ and, for every $n \in \mathbb{N}$,

Consequences of the real number axioms (§§1.7–1.9)

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Let $S \subseteq \mathbb{N}$. Suppose that $1 \in S$ and, for every $n \in \mathbb{N}$, if $n \in S$ then $n+1 \in S$.

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Principle of Mathematical Induction)

Let $S \subseteq \mathbb{N}$. Suppose that $1 \in S$ and, for every $n \in \mathbb{N}$, if $n \in S$ then $n + 1 \in S$. Then $S = \mathbb{N}$.

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Proof.

Let $E = \mathbb{N} \setminus S$ and suppose $E \neq \varnothing$.

Consequences of the real number axioms (§§1.7–1.9)

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Let $S \subseteq \mathbb{N}$. Suppose that $1 \in S$ and, for every $n \in \mathbb{N}$, if $n \in S$ then $n + 1 \in S$. Then $S = \mathbb{N}$.

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Let $E = \mathbb{N} \setminus S$ and suppose $E \neq \varnothing$. Since $E \subset \mathbb{N}$ and $E \neq \varnothing$,

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Proof.

Let $E = \mathbb{N} \setminus S$ and suppose $E \neq \varnothing$. Since $E \subset \mathbb{N}$ and $E \neq \varnothing$, the well-ordering property implies

Consequences of the real number axioms (§§1.7–1.9)

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Proof.

Let $E = \mathbb{N} \setminus S$ and suppose $E \neq \varnothing$. Since $E \subset \mathbb{N}$ and $E \neq \varnothing$, the well-ordering property implies E has a smallest element, say m.

Consequences of the real number axioms (§§1.7–1.9)

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Now $1 \in S$,

Consequences of the real number axioms (§§1.7–1.9)

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Consequences of the real number axioms (§§1.7–1.9)

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Proof.

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Consequences of the real number axioms (§§1.7–1.9)

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Consequences of the real number axioms (§§1.7–1.9)

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Definition (Dense Sets)

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A set E of real numbers is said to be ***dense*** (or ***dense in*** $\mathbb{R}$) if

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If $a, b \in \mathbb{R}$ *and* $a < b$ *then there is a rational number in the interval* (a, b).

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Every real number can be approximated arbitrarily well by a rational number.

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Corollary

Every real number can be approximated arbitrarily well by a rational number.

Given $x \in \mathbb{R}$, consider the interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ for $n \in \mathbb{N}$.

The metric structure of $\mathbb{R}$ (§1.10)

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Definition (Absolute Value function)

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For any $x \in \mathbb{R}$,

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The metric structure of $\mathbb{R}$ (§1.10)

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The metric structure of $\mathbb{R}$ (§1.10)

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4. $|x| - |y| \leq |x - y|$.

The metric structure of $\mathbb{R}$ (§1.10)

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Definition (Distance function or metric)

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The distance between two real numbers x and y is

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Theorem (Properties of distance function or metric)

1. $d(x,y) \geq 0$ — *distances are positive or zero*
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3. $d(x,y) = d(y,x)$ — *distance is symmetric*
4. $d(x,y) \leq d(x,z) + d(z,y)$ — *the triangle inequality*

Note: Any function satisfying these properties can be considered a "distance" or "metric".

The metric structure of $\mathbb{R}$ (§1.10)

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Given $d(x, y) = |x - y|$, the properties of the distance function are equivalent to:

The metric structure of $\mathbb{R}$ (§1.10)

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The metric structure of $\mathbb{R}$ (§1.10)

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The metric structure of $\mathbb{R}$ (§1.10)

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The metric structure of $\mathbb{R}$ (§1.10)

Given $d(x,y) = |x-y|$, the properties of the distance function are equivalent to:

Theorem (Metric properties of the absolute value function)

For all $x, y \in \mathbb{R}$:

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Slick proof of the triangle inequality

Theorem (The Triangle Inequality)

$|x+y| \leq |x|+|y|$ *for all* $x, y \in \mathbb{R}$.

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Proof.

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$\square$

A non-standard metric on $\mathbb{R}$

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Example (finite distance between every pair of real numbers)

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and define

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Example (finite distance between every pair of real numbers)

Let

$$f(x) = \frac{|x|}{1 + |x|}\,,$$

and define

$$d(x, y) = f(x - y)\,.$$

Prove that $d(x, y)$ can be interpreted as a distance between x and y because it satisfies all the properties of a metric.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 5
Properties of $\mathbb{R}$ IV
Thursday 12 September 2019

Announcements

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- A typo has been corrected in Question 3(c) of Assignment 1 on crowdmark.

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- Distance/metric definitions.

Plan for today's class

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- Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.
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- Begin discussing sequences.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 5: Dense sets**
- $\boxed{\texttt{Submit}}$.

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Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

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- We will first develop the ideas for the proof in the way that you might proceed if you were trying to discover a proof from scratch.
- We will then look at a "clean proof" that you might construct after discovering an argument that works.

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We want to find a number $r \in \mathbb{Q}$ such that

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i.e., we want to find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

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Given such m and n, it will follow that

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If we can find integers m and n that satisfy this inequality, then we can work backwards to get what we want.

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We need to find $n \in \mathbb{N}$ such that $nb - na > 1$

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So, to construct a complete proof, the only missing piece is to prove that *if $y - x > 1$ then there is an integer in the interval (x, y).*

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If $a, b \in \mathbb{R}$ and $a < b$ then there is a rational number in the interval (a, b).

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If $a, b \in \mathbb{R}$ and $a < b$ then there is a rational number in the interval (a, b).

If $y - x > 1$, *i.e.*, $x < y - 1$, then we have $[y-1, y) \subset (x, y)$.

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If $a, b \in \mathbb{R}$ and $a < b$ then there is a rational number in the interval (a, b).

If $y - x > 1$, *i.e.*, $x < y - 1$, then we have $[y-1, y) \subset (x, y)$. So it is enough to *show there is an integer in the interval $[y-1, y)$ for any $y \in \mathbb{R}$.*

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If $y \in \mathbb{Z}$ then

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If $y \in \mathbb{Z}$ then $y - 1 \in \mathbb{Z}$, so we are done.

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