Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 26
Integration
Friday 15 March 2019

Announcements

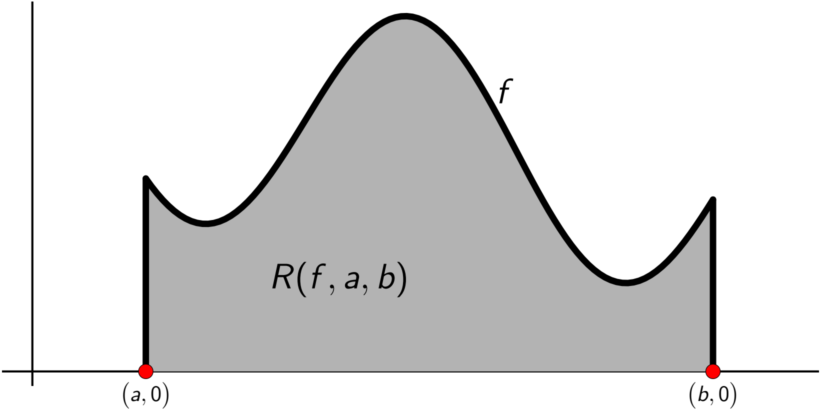
- Assignment 5 is due on **Monday 25 March 2019 @ 11:30am via crowdmark**.
- Test 2 is on **Monday 1 April 2019, 7:00pm–8:30pm in MDCL 1110.**
- Assignment 6 will be due on **Monday 8 April 2019 @ 11:30am via crowdmark**.
- Final exam on **Monday 15 April 2019 @ 4:00pm in IWC/2**.
- NY Times article by Steven Stogatz in honour of Pi Day.
  - Great example of mathematical science writing for the general public.

Last time...

- Proved Mean Value Theorem.
- Proved Darboux's Theorem.
- Sketched proof of Inverse Function Theorem.

Integration

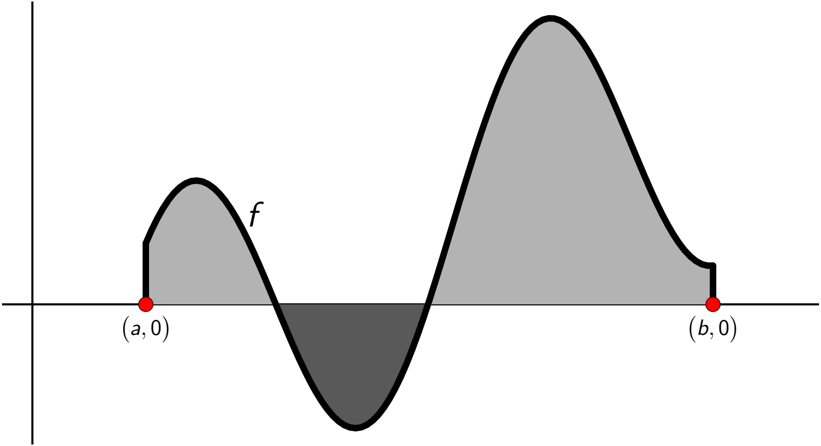
Integration



- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it.
- Textbook presentation of integral is different (but equivalent).

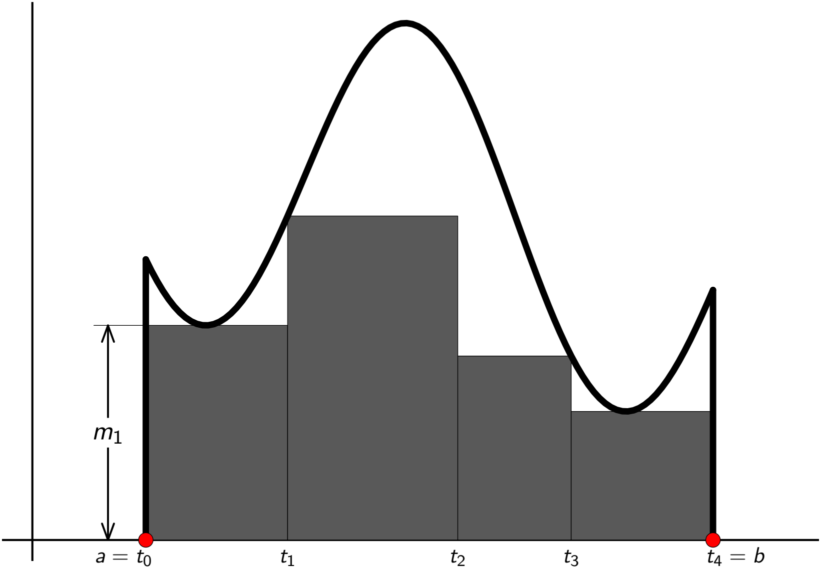
Our treatment is closer to that in M. Spivak "Calculus" (2008).

Integration

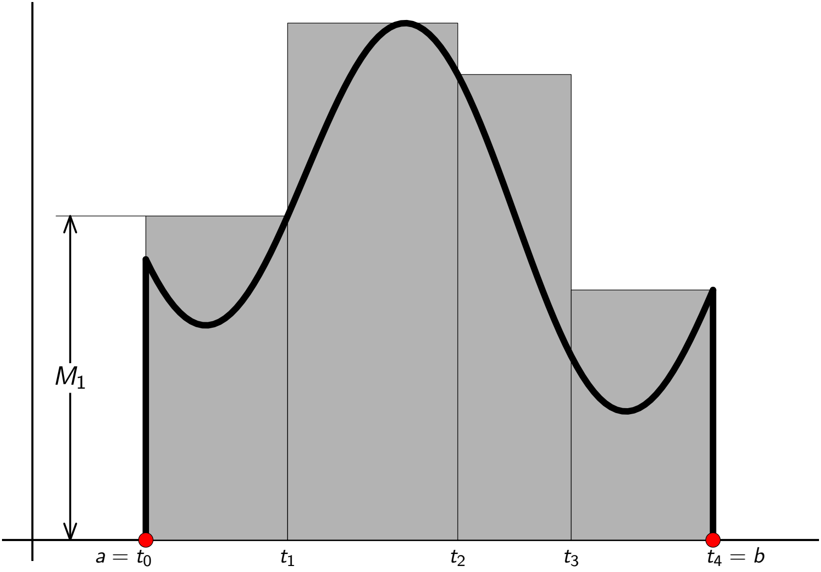


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

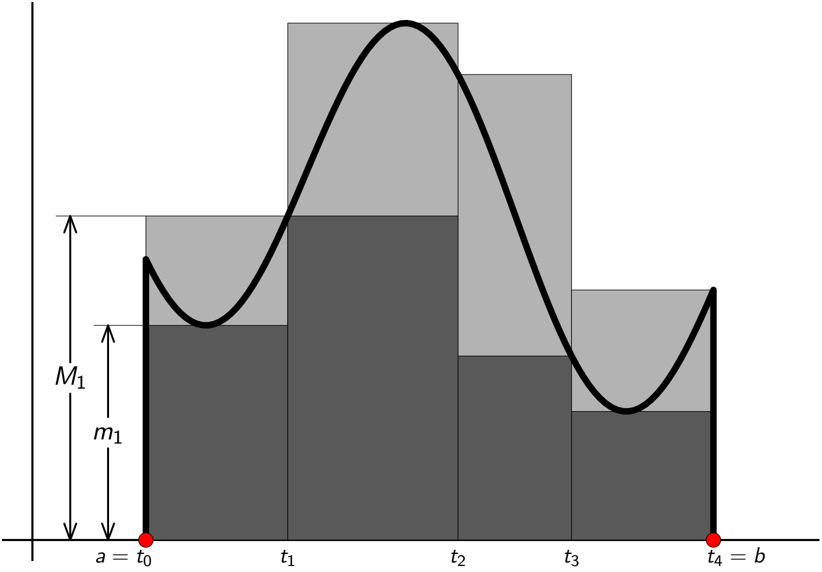
Lower sum



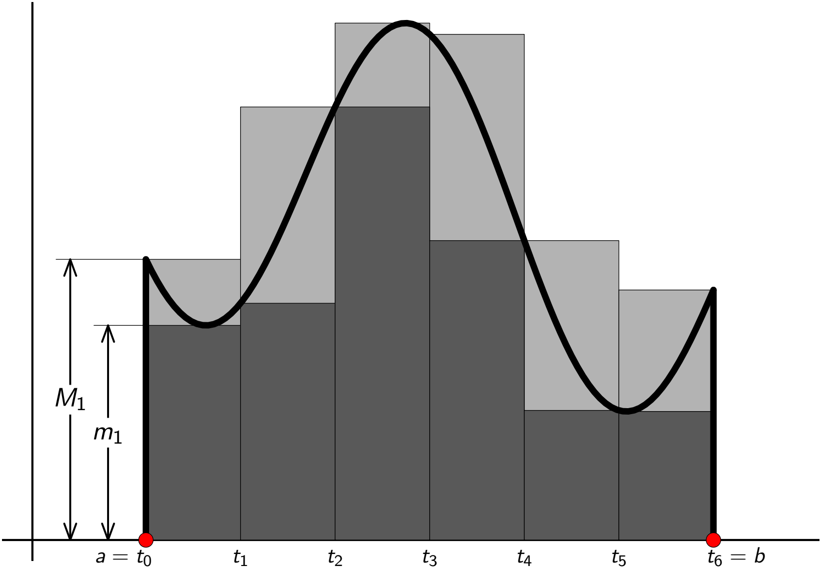
Upper sum



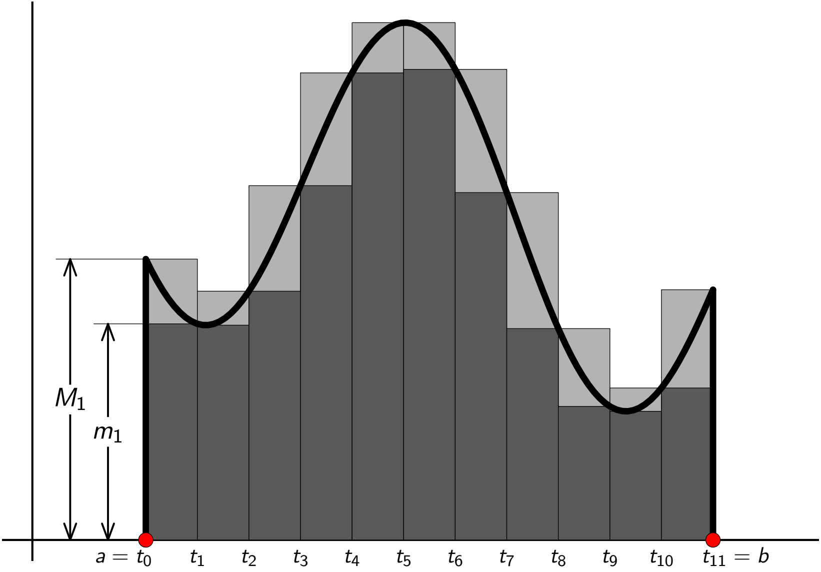
Lower and upper sums



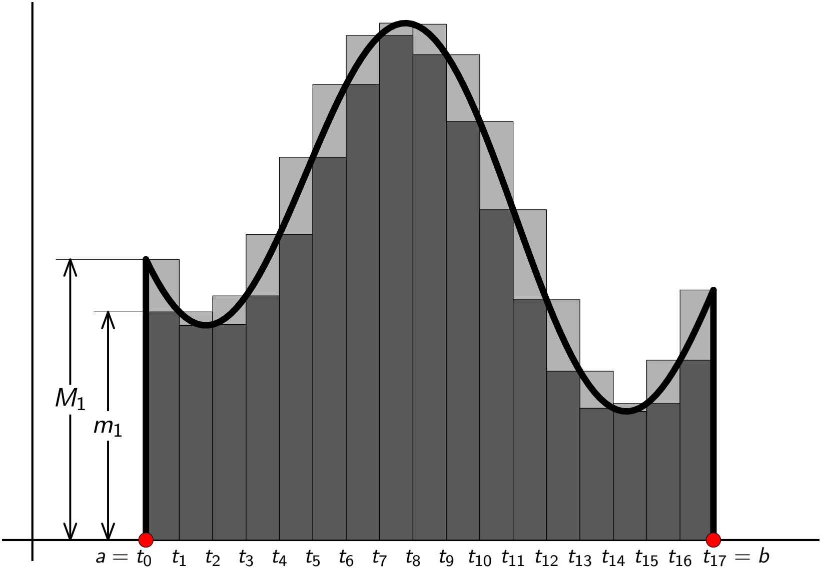
Lower and upper sums



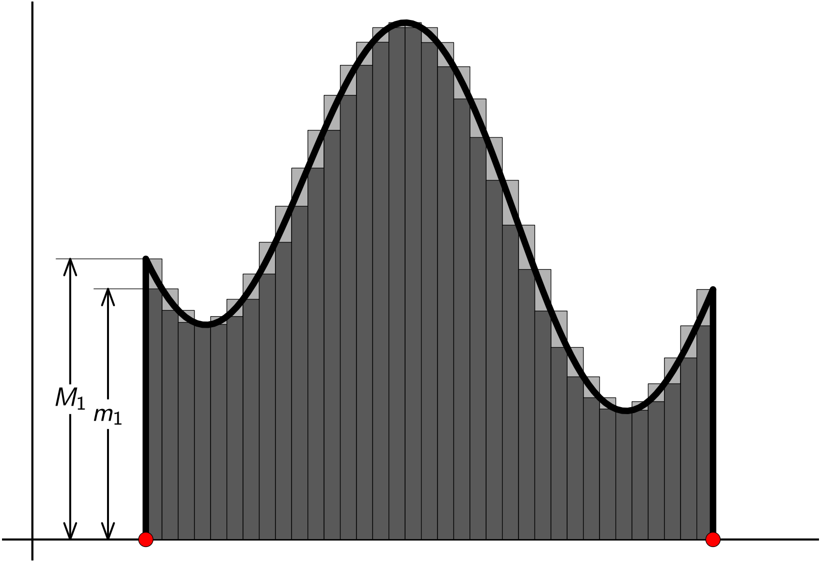
Lower and upper sums



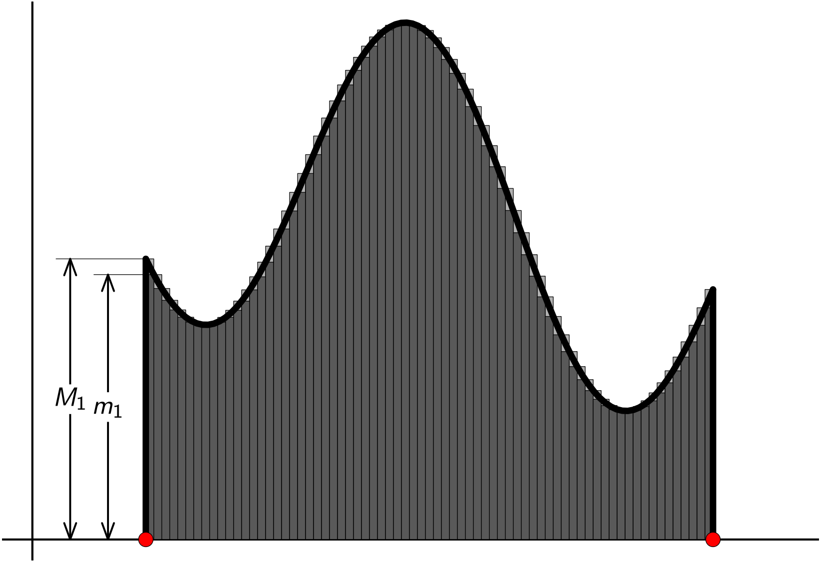
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

so the ith subinterval in the partition is

$$[t_{i-1}, t_i]\,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.

- However, these sums have been defined precisely without any appeal to a concept of "area".

- The requirement that f be bounded on $[a, b]$ is essential in order that all the m_i and M_i be well-defined.

- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \leq M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1})\,. \qquad i = 1, \ldots, n.$$

$\therefore$ For <u>any</u> partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})\,,$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})\,.$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are <u>any</u> two partitions of $[a, b]$, it <u>*ought*</u> to be true that

$$L(f, P_1) \leq U(f, P_2),$$

  because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

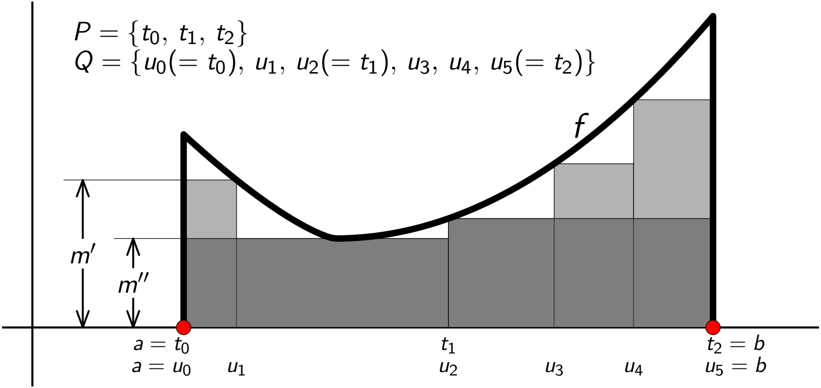
- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can define "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions P_1, P_2 ...

Rigorous development of the integral

Lemma

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $\quad L(f,P) \le L(f,Q) \quad$ *and* $\quad U(f,P) \ge U(f,Q)$.



Rigorous development of the integral

Proof of Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$P = \{t_0, \ldots, t_n\},$$
$$Q = \{t_0, \ldots, t_{k-1}, u, t_k, \ldots, t_n\},$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b.$$

Let

$$m' = \inf\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\},$$
$$m'' = \inf\{\, f(x) \,:\, x \in [u, t_k] \,\}.$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

Then $$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

and $$L(f,Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) + m''(t_k - u) + \sum_{i=k+1}^{n} m_i(t_i - t_{i-1}).$$

$\therefore$ To prove $L(f,P) \leq L(f,Q)$, it is enough to show

$$m_k(t_k - t_{k-1}) \leq m'(u - t_{k-1}) + m''(t_k - u).$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\},$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k \quad &= \quad \inf \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\le \quad \inf \{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m' \,. \end{aligned}$$

Thus, $m_k \le m'$, and, similarly, $m_k \le m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) \quad &= \quad m_k(t_k - u + u - t_{k-1}) \\ &= \quad m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\le \quad m'(u - t_{k-1}) + m''(t_k - u)\,, \end{aligned}$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \leq L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point that P_j. Then

$$L(f,P) = L(f,P_0) \leq L(f,P_1) \leq \cdots \leq L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \leq L(f,Q)$.

(Proving $U(f,P) \geq U(f,Q)$ is similar: check!) $\square$

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \le U(f, P_2)\,.$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)\,.$$

$\square$

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 27
Integration II
Monday 18 March 2019

Announcements

- Part of Assignment 5 is posted on the course web site (*more to come*). It is due on **Monday 25 March 2019 @ 11:30am via crowdmark**.
- Test 2 is on **Monday 1 April 2019, 7:00pm–8:30pm in MDCL 1110.**
- Assignment 6 will be due on **Monday 8 April 2019 @ 11:30am via crowdmark**.
- Final exam on **Monday 15 April 2019 @ 4:00pm in IWC/2**.

Rigorous development of the integral

Important inferences that follow from the partition theorem:

- For <u>any</u> partition P', the upper sum $U(f,P')$ is an upper bound for the set of <u>all</u> lower sums $L(f,P)$.

$$\therefore \quad \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \le U(f,P') \qquad \forall P'$$

$$\therefore \quad \sup\{L(f,P)\} \le \inf\{U(f,P)\}$$

$\therefore$ For <u>any</u> partition P',

$$L(f,P') \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le U(f,P')$$

- If $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ then we can define "area of $R(f,a,b)$" to be this number.
    - Is it possible that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$?

Rigorous development of the integral

Example

$\exists?\ f : [a, b] \to \mathbb{R}$ such that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$

Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b], \\ 0 & x \in \mathbb{Q}^c \cap [a,b]. \end{cases}$$

If $P = \{t_0, \dots, t_n\}$ then $m_i = 0 \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q}^c \neq \varnothing)$,
and $M_i = 1 \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q} \neq \varnothing)$.

$\therefore \quad L(f,P) = 0$ and $U(f,P) = b - a$ for <u>any</u> partition P.

$\therefore \quad \sup\{L(f,P)\} = 0 < b - a = \inf\{U(f,P)\}$. $\square$

Can we define "area of $R(f,a,b)$" for such a weird function?

Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \to \mathbb{R}$ is said to be **integrable** on $[a, b]$ if it is <u>bounded</u> on $[a, b]$ <u>and</u>

$$\sup\big\{L(f, P) : P \text{ a partition of } [a, b]\big\} = \inf\big\{U(f, P) : P \text{ a partition of } [a, b]\big\}.$$

In this case, this common number is called the **integral** of f on $[a, b]$ and is denoted

$$\int_a^b f$$

<u>Note:</u> If f is integrable then for <u>any</u> partition P we have

$$L(f, P) \le \int_a^b f \le U(f, P),$$

and $\int_a^b f$ is the <u>unique</u> number with this property.

Rigorous development of the integral

- *Notation:*

$$\int_a^b f(x)\,dx \qquad \text{means precisely the same as} \qquad \int_a^b f\,.$$

  - The symbol "dx" has no meaning in isolation just as "$x \to$" has no meaning except in $\lim_{x\to a} f(x)$.

- It is not clear from the definition which functions are integrable.

- The definition of the integral does not itself indicate how to compute the integral of any given integrable function. So far, without a lot more effort we can't say much more than these two things:
  1. If $f(x) \equiv c$ then f is integrable on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
  2. The weird example function is not integrable.

Rigorous development of the integral

- A function that is integrable according to our definition is usually said to be **Riemann integrable**, to distinguish this definition from other definitions of integrability.
- In Math 4A03 you will define "Lebesgue integrable", a more subtle concept that makes it possible to attach meaning to "area of $R(f, a, b)$" for the weird example function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent condition for integrability)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

Proof.

2016 Assignment 5. □

Note: This theorem is just a restatement of the definition of integrability. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.

Integral theorems

Theorem

If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Rough work to prepare for proof:

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

Given $\varepsilon > 0$, choose a partition P that is so fine that $M_i - m_i < \varepsilon$ for all i. Then

$$U(f, P) - L(f, P) < \varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) = \varepsilon(b - a)\,.$$

Not quite what we want. So choose the partition P such that $M_i - m_i < \varepsilon/(b - a)$ for all i. To get that, choose P such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \text{if } |x - y| < \max_{1 \le i \le n}(t_i - t_{i-1}),$$

which we can do because f is uniformly continuous on $[a, b]$.

Integral theorems

Proof that continuous $\implies$ integrable

Since f is continuous on the compact set $[a, b]$, it is bounded on $[a, b]$ (which is the first requirement to be integrable on $[a, b]$).

Also, since f is continuous on the compact set $[a, b]$, it is uniformly continuous on $[a, b]$. $\therefore \forall \varepsilon > 0 \; \exists \delta > 0$ such that $\forall x, y \in [a, b]$,

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \,.$$

Now choose a partition of $[a, b]$ such that the length of each subinterval $[t_{i-1}, t_i]$ is less than δ, *i.e.*, $t_i - t_{i-1} < \delta$. Then, for any $x, y \in [t_{i-1}, t_i]$ we have $|x - y| < \delta$ and therefore

. . . continued. . .

Integral theorems

Proof that continuous $\implies$ integrable (cont.)

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \forall x, y \in [t_{i-1}, t_i]\,.$$

$$\therefore \qquad M_i - m_i \le \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a} \qquad i = 1, \ldots, n.$$

Since this is true for all i, it follows that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon\,. \end{aligned}$$

$\square$

Properties of the integral

Theorem (Integral segmentation)

Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$. Finally, if f is integrable on $[a, b]$ then

$$\int_a^b f = \int_a^c f + \int_c^b f\,. \qquad (\heartsuit)$$

(a good exercise)

This theorem motivates these definitions:

$$\int_a^a f = 0 \qquad \text{and} \qquad \int_a^b f = -\int_b^a f \quad \text{if } a > b.$$

Then $(\heartsuit)$ holds for any $a, b, c \in \mathbb{R}$.

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 28
Integration III
Wednesday 20 March 2019

Announcements

- Assignment 5 is due on **Monday 25 March 2019 @ 11:30am via crowdmark**.
- Test 2 is on **Monday 1 April 2019, 7:00pm–8:30pm in MDCL 1110.**
- Assignment 6 will be due on **Monday 8 April 2019 @ 11:30am via crowdmark**.
- Final exam on **Monday 15 April 2019 @ 4:00pm in IWC/2**.

Last time. . .

Rigorous development of integral:

- Definition: integrable.
- Example: non-integrable function.
- Theorem: Equivalent "ε-P" definition of integrable.
- Theorem: continuous $\Longrightarrow$ integrable.
- Theorem: Integral segmentation.

Properties of the integral

Theorem (Algebra of integrals – a.k.a. $\int_a^b$ is a linear operator)

If f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$ then $f + g$ and cf are integrable on $[a, b]$ and

1. $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g;$
2. $\displaystyle\int_a^b cf = c\int_a^b f.$

(proofs are relatively easy; good exercises)

Theorem (Integral of a product)

If f and g are integrable on $[a, b]$ then fg is integrable on $[a, b]$.

(proof is much harder; tough exercise)

Properties of the integral

Lemma (Integral bounds)

Suppose f is integrable on $[a, b]$. If $m \le f(x) \le M$ for all $x \in [a, b]$ then

$$m(b-a) \le \int_a^b f \le M(b-a)\,.$$

Proof.

For any partition P, we must have $m \le m_i \ \forall i$ and $M \ge M_i \ \forall i$.

$$\therefore \qquad m(b-a) \le L(f,P) \le U(f,P) \le M(b-a) \qquad \forall P$$

$$\therefore \quad m(b-a) \le \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \le M(b-a)\,.$$

□

Properties of the integral

Theorem (Integrals are continuous)

If f is integrable on $[a, b]$ and F is defined on $[a, b]$ by

$$F(x) = \int_a^x f \,,$$

then F is continuous on $[a, b]$.

Proof

Let's first consider $x_0 \in [a, b)$ and show F is continuous from above at x_0, *i.e.*, $\lim_{x \to x_0^+} F(x) = F(x_0)$. If $x \in (x_0, b]$ then

$$(\heartsuit) \quad \implies \quad F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f = \int_{x_0}^x f \,. \qquad (*)$$

...continued...

Properties of the integral

Proof (cont.)

Since f is integrable on $[a, b]$, it is bounded on $[a, b]$, so $\exists M > 0$ such that

$$-M \le f(x) \le M \qquad \forall x \in [a, b],$$

from which the integral bounds lemma implies

$$-M(x - x_0) \le \int_{x_0}^{x} f \le M(x - x_0)\,,$$

$$\therefore \quad (*) \implies -M(x - x_0) \le F(x) - F(x_0) \le M(x - x_0)\,.$$

$\therefore$ For any $\varepsilon > 0$ we can ensure $|F(x) - F(x_0)| < \varepsilon$ by requiring $0 \le x - x_0 < \varepsilon/M$, which proves $\lim_{x \to x_0^+} F(x) = F(x_0)$.

A similar argument starting from $x_0 \in (a, b]$ and $x \in [a, x_0)$ yields $\lim_{x \to x_0^-} F(x) = F(x_0)$. Thus, "integrals are continuous". $\square$

Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus)

Let f be integrable on $[a, b]$, and define F on $[a, b]$ by

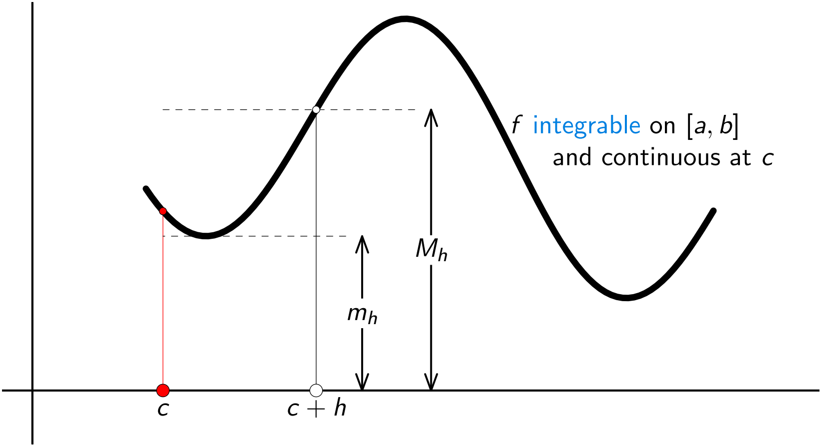
$$F(x) = \int_a^x f\,.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c, and

$$F'(c) = f(c)\,.$$

Note: If $c = a$ or b, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.

Fundamental Theorem of Calculus



$$F(c+h) - F(c) \simeq f(c+h)\cdot h$$

and $\lim_{h\to 0} f(c+h) = f(c)$

$$\implies \quad \lim_{h\to 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus

Suppose $c \in [a, b)$, and $0 < h \leq b - c$. Then the integral segmentation theorem implies

$$F(c+h) - F(c) = \int_c^{c+h} f \,.$$

Motivated by the sketch, define

$$m_h = \inf \left\{ f(x) \,:\, x \in [c, c+h] \right\},$$
$$M_h = \sup \left\{ f(x) \,:\, x \in [c, c+h] \right\}.$$

Then the integral bounds lemma implies

$$m_h \cdot h \leq \int_c^{c+h} f \leq M_h \cdot h \,,$$

...continued...

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus (cont.)

and hence

$$m_h \le \frac{F(c+h)-F(c)}{h} \le M_h\,.$$

This inequality is true for any integrable function. However, because f is continuous at c, we have

$$\lim_{h\to 0^+} m_h = \lim_{h\to 0^+} M_h = f(c)\,,$$

so the squeeze theorem implies

$$F'_+(c) = \lim_{h\to 0^+} \frac{F(c+h)-F(c)}{h} = f(c)\,.$$

A similar argument for $c \in (a,b]$ and $c-a \le h < 0$ yields $F'_-(c) = f(c)$. □

Fundamental Theorem of Calculus

Corollary

If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

Proof.

Let $F(x) = \int_a^x f$. Then throughout $[a, b]$ we have $F' = f = g'$.

$\therefore\ \exists c \in \mathbb{R}$ such that $F = g + c$ (2016 Assignment 5).

$\therefore\ F(a) = g(a) + c$.

But $F(a) = \int_a^a f = 0$, so $c = -g(a)$.

$\therefore\ F(x) = g(x) - g(a)$.

This is true, in particular, for $x = b$, so $\int_a^b f = g(b) - g(a)$. $\square$

Fundamental Theorem of Calculus

Theorem (Second Fundamental Theorem of Calculus)

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

Notes:

- This looks like the corollary to the first fundamental theorem, except that f is only assumed integrable, not continuous.
- Recall from Darboux's theorem that if $f = g'$ for some g then f has the intermediate value property, but f need not be continuous.
- The proof of the second fundamental theorem is completely different from the corollary to the first, because we cannot use the first fundamental theorem (which assumed f is continuous).

Fundamental Theorem of Calculus

Proof of Second Fundamental Theorem of Calculus

Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each $i = 1, \dots, n$, $\exists\, x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1})\,.$$

Define m_i and M_i as usual. Then $m_i \le f(x_i) \le M_i\ \forall i$, so

$$m_i(t_i - t_{i-1}) \le f(x_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})\,,$$

$$\text{i.e.,}\quad m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1})\,.$$

$$\therefore\quad \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \le \sum_{i=1}^{n} \Big(g(t_i) - g(t_{i-1})\Big) \le \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

$$\text{i.e.,}\qquad L(f,P) \le \qquad g(b) - g(a) \qquad \le U(f,P)$$

for <u>any</u> partition P. $\quad\therefore\quad g(b) - g(a) = \int_a^b f\,.$ $\square$