Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 6: Sequence convergence**
- $\boxed{\texttt{Submit}}$.

Announcements

- Assignment 1 is due via crowdmark 5 minutes before class on Monday.
- Consider writing the Putnam competition.

Sequences

- A ***sequence*** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

A lot of different notation is common for sequences:

$$\begin{array}{ll}
f(1),\ f(2),\ f(3),\ \ldots & \{f(n)\}_{n=1}^{\infty} \\
f_1,\ f_2,\ f_3,\ \ldots & \{f(n)\} \\
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Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,} \quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to ***iterate***. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \ldots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \ldots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

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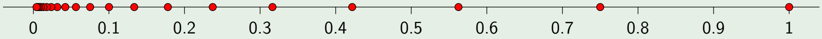
Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



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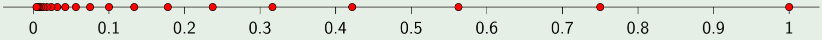

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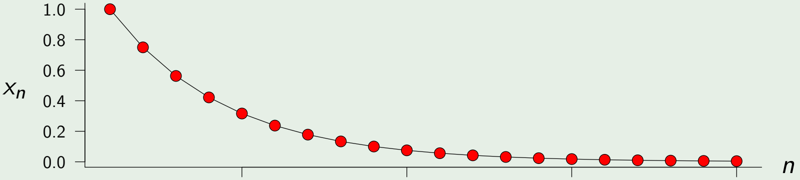

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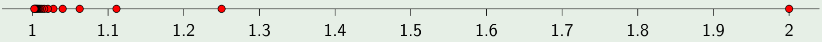

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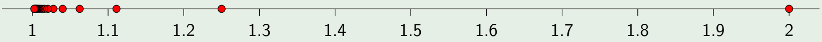

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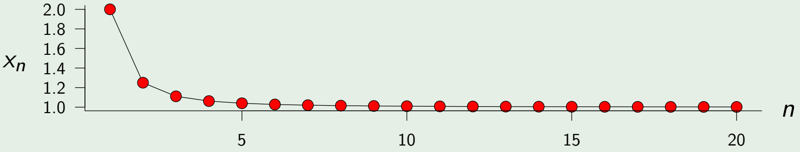

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In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

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Convergence of sequences

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Convergence of sequences

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Remark (Sequences in spaces other than $\mathbb{R}$)

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Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ *with* $d(s_n, L)$.

Convergence of sequences

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Use the formal definition of a limit of a sequence to prove that

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Tuesday 17 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 7: Sequence divergence**
- $\boxed{\texttt{Submit}}$.

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- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements continued. . .

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- Remember that solutions to assignments and tests from previous years are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)

Uniqueness of limits

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Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

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So, we are justified in referring to "the" limit of a convergent sequence.

Divergence of sequences

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- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

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Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers ***diverges to*** $-\infty$ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \leq M\,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{ \frac{n^3 - 1}{n + 1} \right\} \quad \text{diverges to } \infty \, .$$

(solution on board)

Divergence to ∞

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Use the formal definition to prove that

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Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

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$$\forall n \in \mathbb{N}, \quad n - 1 =$$

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$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{ \dfrac{n^3 - 1}{n+1} \right\}$ diverges to ∞.

Clean proof.

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$$\frac{n^3-1}{n+1} \geq M\,,$$

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Note: We can start from any integer, not necessarily $k = 1$.

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Proof? Find a counterexample, *e.g.*, $\{(-1)^n\}$.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Thursday 19 September 2019

What we've done so far on sequences

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- Definition of convergence.

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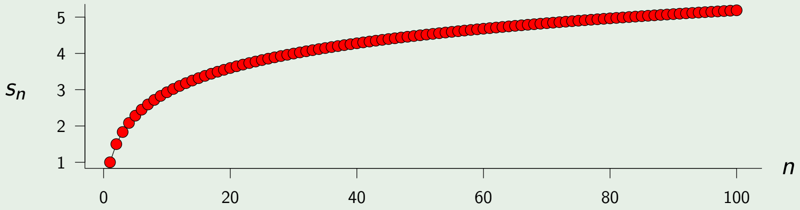

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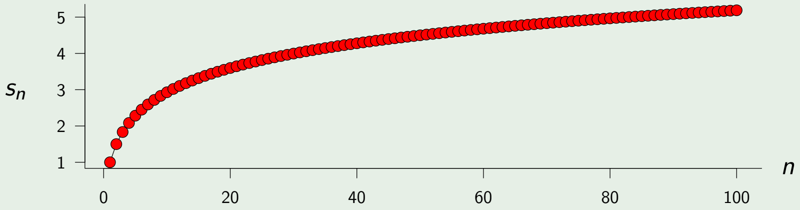

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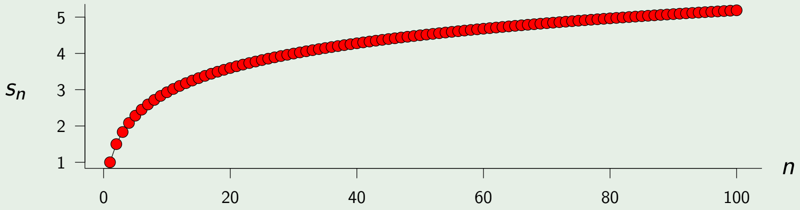

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A proof must be a clearly presented coherent argument from beginning to end.

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$\square$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 8: Harmonic series of primes**
- $\boxed{\texttt{Submit}}$.

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

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5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
$$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} \,.$$

(solution on board)

Revisit example

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Example (previously proved directly from definition)

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Mathematics
and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Friday 20 September 2019

Announcements

Announcements

- Assignment 2 is posted.

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  Due 1 Oct 2019, at 2:25pm.

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- Every convergent sequence is bounded.

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What we've done so far on sequences

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Details missing on previous slide:

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In either case, $\forall n \geq N_1$, we have $0 < \dfrac{|T|}{2} < |t_n|$.

Order properties of limits (§2.8)

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Theorem (Limits retain order)

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If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \, .$$

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Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$

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If $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences then*

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Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

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If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

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Hence $S - T \leq 0$

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Hence $S - T \le 0$, *i.e.*, $S \le T$. □

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 9: Order property of limits**
- Submit.

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

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Proof.

Apply previous theorem with $\alpha_n = \alpha\ \forall n$ and $\beta_n = \beta\ \forall n$. □

Order properties of limits (§2.8)

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Theorem (Squeeze Theorem)

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Order properties of limits (§2.8)

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$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$).

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$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \to X$. Hence, by order retension, $L \le X \le L \implies X = L$. □

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 10
Sequences V
Tuesday 24 September 2019

Announcements

Announcements

- Assignment 2 is posted.

Announcements

- Assignment 2 is posted.
  Due 1 Oct 2019, at 2:25pm.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (sums, products, quotients).
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Today:

- Proof of Squeeze Theorem
- Absolute value and max/min of limits.
- Monotone convergence.

Order properties of limits (§2.8)

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Correct Proof.

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Correct Proof.

Given $\varepsilon > 0$, find $N \ni$

Order properties of limits (§2.8)

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Order properties of limits (§2.8)

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as required. $\square$

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ converges then so does $\{|s_n|\}$, and

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See Assignment 2! $\square$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

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Prove these facts, then use theorems on sums and absolute values of limits.

Monotone convergence (§2.9)

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 10: Monotone convergence**
- `Submit`.

Monotone convergence (§2.9)

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Proof.

. . . next slide. . . □

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Proof of [part (iii)] is similar to [part (i)].

Proof of [part (iv)] is similar to [part (ii)]. $\square$

Subsequences

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Definition (Subsequence)

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Let $\{s_1, s_2, s_3, \ldots\}$ be a sequence.

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Let $\{s_1, s_2, s_3, \ldots\}$ be a sequence. If $\{n_1, n_2, n_3, \ldots\}$ is an increasing sequence of natural numbers then

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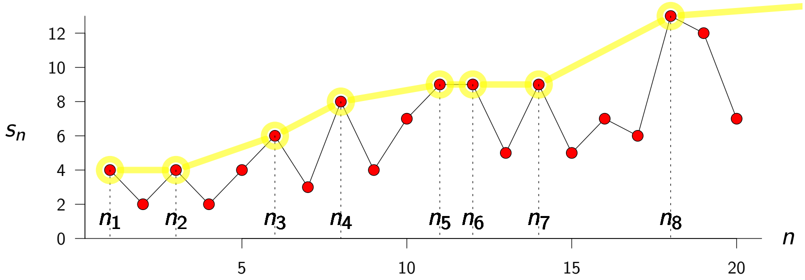

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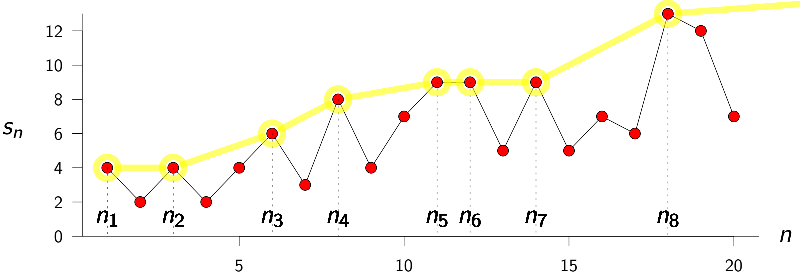

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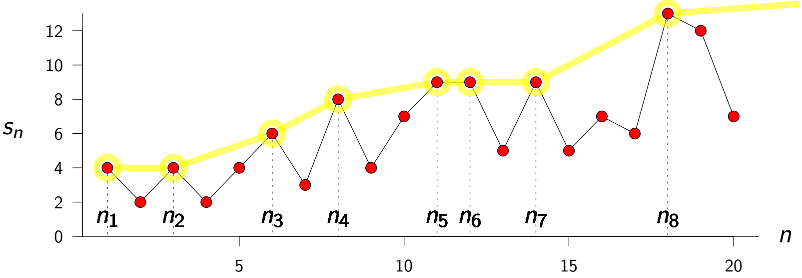

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 11
Sequences VI
Thursday 26 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 11: Point of no return**
- $\boxed{\texttt{Submit}}$.

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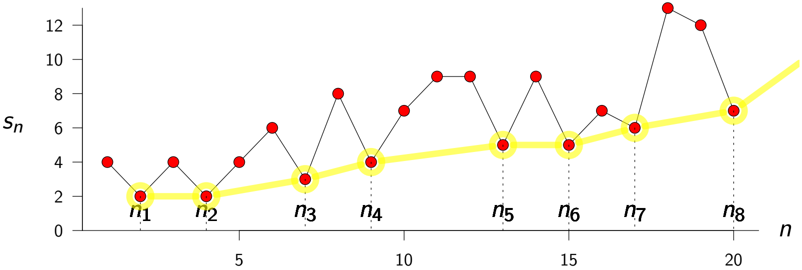

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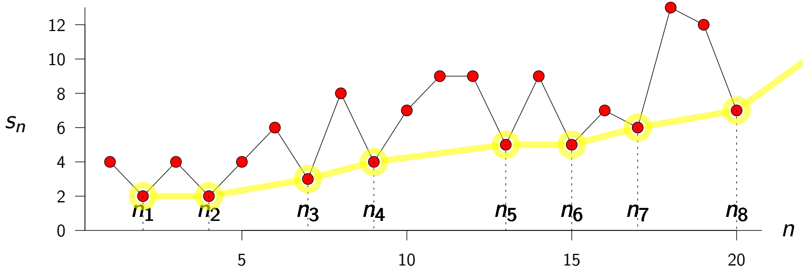

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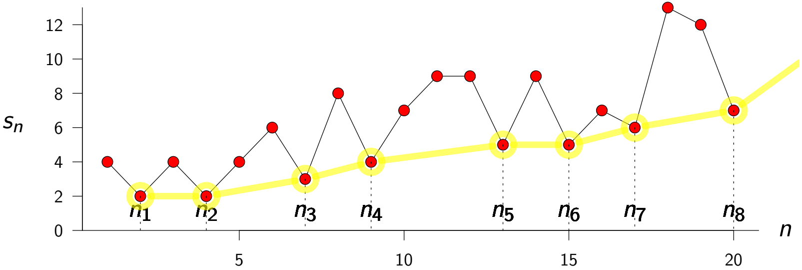

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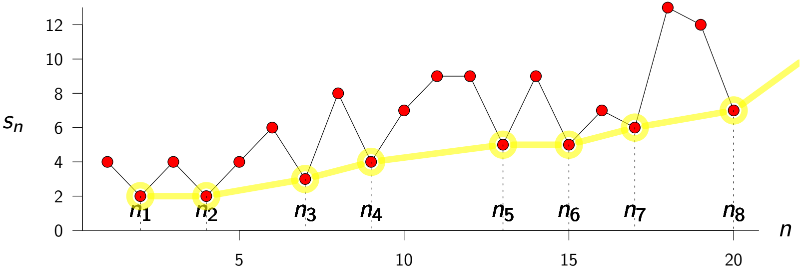

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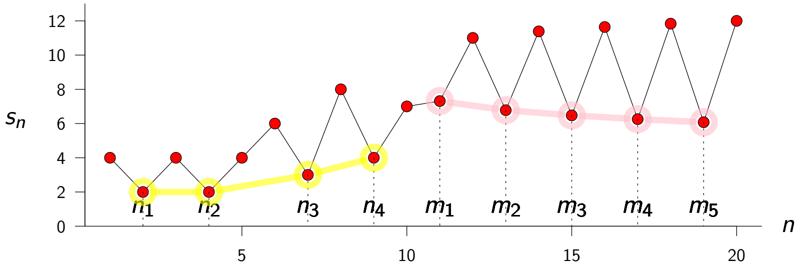

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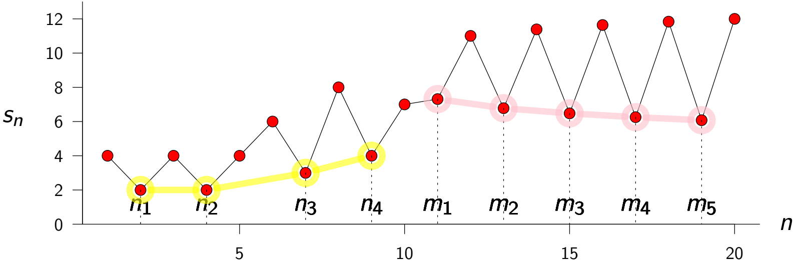

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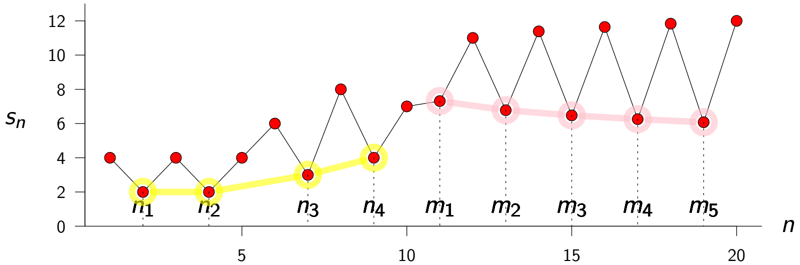

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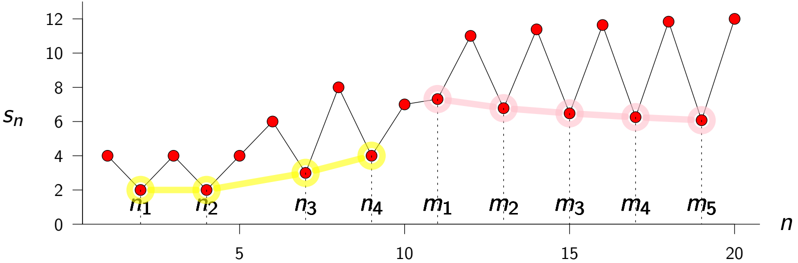

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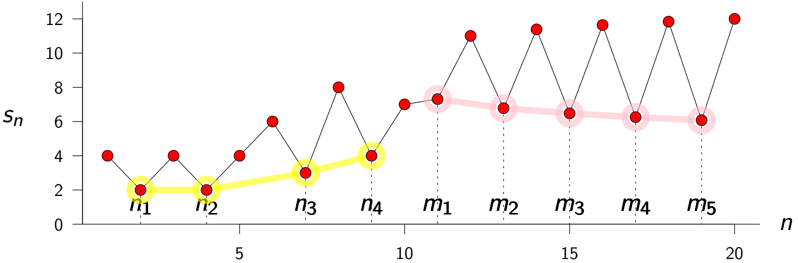

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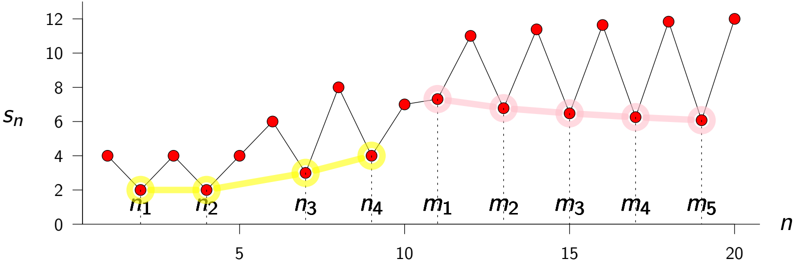

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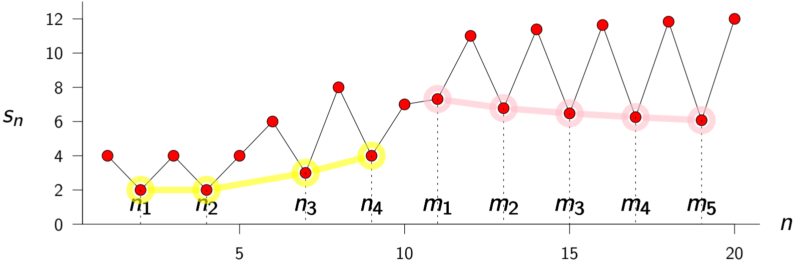

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Convergent subsequences

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Every bounded sequence of real numbers contains a convergent subsequence.

Proof.

Suppose $\{x_n\}$ is a bounded sequence. It follows from the previous theorem that $\{x_n\}$ contains a subsequence $\{x_{m_k}\}$ that is monotonic. Since $\{x_n\}$ is bounded, the subsequence $\{x_{m_k}\}$ is bounded as well (by the same bound). Thus, $\{x_{m_k}\}$ is a subsequence of $\{x_n\}$ that is both bounded and monotonic. Hence, it converges by the Monotone Convergence Theorem. □

Cauchy sequences

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Theorem (Cauchy criterion)

A sequence of real numbers $\{s_n\}$ is convergent iff it is a Cauchy sequence.

Remark: The proof of the "only if" direction is easy. The proof of the "if" direction contains only one tricky feature: showing that every Cauchy sequence $\{s_n\}$ is bounded.

Cauchy sequences

Proof of Cauchy criterion

Cauchy sequences

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"only if":

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then,

Cauchy sequences

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Cauchy sequences

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Cauchy sequences

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Cauchy sequences

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Cauchy sequences

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Cauchy sequences

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Cauchy sequences

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$$s_{N+1} - 1 < s_m < s_{N+1} + 1 \qquad \forall m > N.$$

Cauchy sequences

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Thus $\{s_m : m > N\}$ is bounded;

Cauchy sequences

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Thus $\{s_m : m > N\}$ is bounded; moreover, since there are only finitely many other s_i's, the whole sequence $\{s_n\}$ is bounded.

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

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Cauchy sequences

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"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

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Thus $\{s_m : m > N\}$ is bounded; moreover, since there are only finitely many other s_i's, the whole sequence $\{s_n\}$ is bounded. Hence, by the Bolzano-Weierstrass theorem, some subsequence of s_n converges;

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

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Thus $\{s_m : m > N\}$ is bounded; moreover, since there are only finitely many other s_i's, the whole sequence $\{s_n\}$ is bounded. Hence, by the Bolzano-Weierstrass theorem, some subsequence of s_n converges; let's write this subsequence as $\{s_{m_k}\}$,

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

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Thus $\{s_m : m > N\}$ is bounded; moreover, since there are only finitely many other s_i's, the whole sequence $\{s_n\}$ is bounded. Hence, by the Bolzano-Weierstrass theorem, some subsequence of s_n converges; let's write this subsequence as $\{s_{m_k}\}$, and its limit as L.

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

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. . . continued on next slide. . .

Cauchy sequences

Proof of Cauchy criterion (cont'd).

We will show that $\{s_n\}$ converges to L.

Cauchy sequences

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Cauchy sequences

Proof of Cauchy criterion (cont'd).

We will show that $\{s_n\}$ converges to L. To prove this, consider any $\varepsilon > 0$. Since the sequence $\{s_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

Cauchy sequences

Proof of Cauchy criterion (cont'd).

We will show that $\{s_n\}$ converges to L. To prove this, consider any $\varepsilon > 0$. Since the sequence $\{s_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \qquad \text{for all } n, m \geq N.$$

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Since the subsequence $\{s_{m_k}\}$ converges to L, there is some N' so that

$$|s_{m_k} - L| < \frac{\varepsilon}{2} \qquad \text{for all } k \geq N'.$$

Cauchy sequences

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Now fix an integer k satisfying $k \geq N'$ and $m_k \geq N$.

Cauchy sequences

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Since the subsequence $\{s_{m_k}\}$ converges to L, there is some N' so that

$$|s_{m_k} - L| < \frac{\varepsilon}{2} \qquad \text{for all } k \geq N'.$$

Now fix an integer k satisfying $k \geq N'$ and $m_k \geq N$. Then $\forall n \geq N$,

$$|s_n - L| \leq |s_n - s_{m_k} + s_{m_k} - L|$$

Cauchy sequences

Proof of Cauchy criterion (cont'd).

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Now fix an integer k satisfying $k \geq N'$ and $m_k \geq N$. Then $\forall n \geq N$,

$$\begin{aligned} |s_n - L| &\leq |s_n - s_{m_k} + s_{m_k} - L| \\ &\leq |s_n - s_{m_k}| + |s_{m_k} - L| \end{aligned}$$

Cauchy sequences

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Cauchy sequences

Proof of Cauchy criterion (cont'd).

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Now fix an integer k satisfying $k \geq N'$ and $m_k \geq N$. Then $\forall n \geq N$,

$$\begin{aligned} |s_n - L| &\leq |s_n - s_{m_k} + s_{m_k} - L| \\ &\leq |s_n - s_{m_k}| + |s_{m_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as required. □

Cauchy sequences

Cauchy sequences

Notes:

Cauchy sequences

Notes:

- The Cauchy criterion is sometimes easier to use in proofs than the original definition of convergence.

Cauchy sequences

Notes:

- The Cauchy criterion is sometimes easier to use in proofs than the original definition of convergence.
- Its significance is more evident in spaces other than $\mathbb{R}$, where Cauchy sequences do <u>not necessarily</u> converge.