Math 3A03 - Tutorial 7 Questions - Winter 2019

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Problem 1. *Prove that* $\lim_{x \to a} x^3 = a^3$ *for every* $a \in \mathbb{R}$.

Solution. Goal: Given a $a \in \mathbb{R}$ and an $\varepsilon > 0$ we need to find a $\delta > 0$ so that if $|x - a| < \delta$ then $|x^3 - a^3| < \varepsilon$. Notice that our δ will depend on ε, also we should somehow manipulate $|x^3 - a^3|$ to isolate $|x - a|$. We can start by factoring using the fact we have a difference of cubes:

$$|x^3 - a^3| = |x - a||x^2 - xa + a^2|.$$

Using the triangle inequality we get

$$|x - a||x^2 - xa + a^2| \leq |x - a|(|x|^2 + |ax| + |a|^2).$$

We'd like to get an upper bound on this second term, for this we'd need to remember that we have a choice in δ, and since $|x - a| < \delta$ it means we can restrict x to be near a. Let's say $\delta \leq 1$, then $|x-a| < 1$, i.e. $x \in (a-1, a+1)$, meaning $|x| < |a| + 1$ hence

$$|x^3 - a^3| < |x - a|(2|a|^2 + 3|a| + 1).$$

Since $|x - a| < \delta$ if we choose $\delta \leq \frac{\varepsilon}{2|a|^2+3|a|+1}$, we get $|x^3 - a^3| < \varepsilon$.

To clean this up a bit: Given a $a \in \mathbb{R}$ and an $\varepsilon > 0$ let $\delta = \min(1, \frac{\varepsilon}{2|a|^2+3|a|+1})$, if $|x - a| < \delta$ then (from the inequalities above) we have

$$|x^3 - a^3| < |x - a|(2|a|^2 + 3|a| + 1) < \varepsilon,$$

and hence $\lim_{x \to a} x^3 = a^3$ for every $a \in \mathbb{R}$ by the formal definition of the limit.

As a small remark notice that our δ is a function not only of ε, but of the point a as well, a choice for δ at one point does not necessarily hold at another!

Problem 2. *Suppose that $f(x)$ and $g(x)$ are continuous on $\mathbb{R}$, with $f(x) = g(x)$ on E, a dense subset of $\mathbb{R}$. Prove that $f(x) = g(x)$ on $\mathbb{R}$.*

Solution. Recall that in a previous tutorial we showed that if a set E is dense in $\mathbb{R}$ then given any $y \in \mathbb{R}$ there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset E$ such that $\lim_{n\to\infty} y_n = y$. Recall also that using the sequential definition of a limit we may rewrite the condition for continuity of f as given a convergent sequence of real numbers x_n with limit x, $\lim_{n\to\infty} f(x_n) = f(x)$.

Suppose that $x \in \mathbb{R}$, then since E is a dense subset we have a sequence of points $\{x_n\}_{n=1}^{\infty}$ in E such that $\lim_{n\to\infty} x_n = x$, and for each $n \in \mathbb{N}$ $f(x_n) = g(x_n)$. Taking the limit on both sides we get $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n)$, since the two functions are continuous we have that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$