6 Sequences

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 6: Sequence convergence**
- Submit.

Announcements

- Assignment 1 is due via crowdmark 5 minutes before class on Monday.
- Consider writing the Putnam competition.

Announcements for week of 21–25 January 2019

- Office hour on Monday 21 Jan 2019 will be at **3:30pm** (rather than the usual 1:30pm).
- Wednesday's lecture will be given by Niky Hristov.

Sequences

- A ***sequence*** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R} .$$

A lot of different notation is common for sequences:

$f(1),\ f(2),\ f(3),\ \ldots$	$\{f(n)\}_{n=1}^{\infty}$
$f_1,\ f_2,\ f_3,\ \ldots$	$\{f(n)\}$
$\{f(n) : n = 1, 2, 3, \ldots\}$	$\{f_n\}_{n=1}^{\infty}$
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Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to ***iterate***. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \ldots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \ldots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

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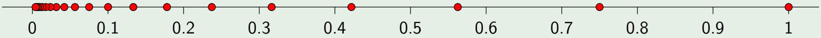
Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



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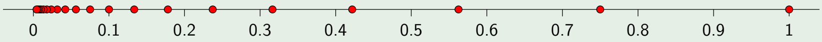
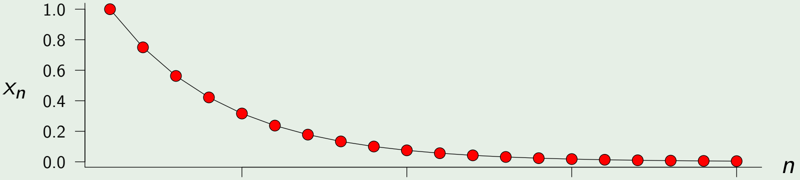


Graph of $f(n)$:



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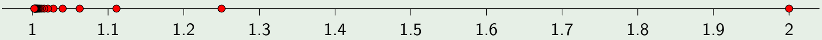

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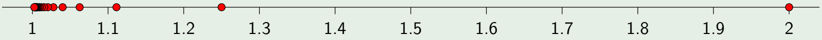

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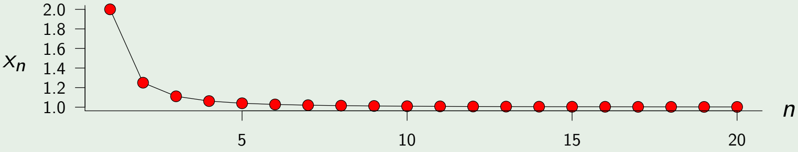

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Shorthand:

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Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

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Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

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Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

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So, we are justified in referring to "the" limit of a convergent sequence.

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Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

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Definition (Divergence to ∞)

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Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers ***diverges to*** $-\infty$ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \leq M\,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{\frac{n^3-1}{n+1}\right\} \quad \text{diverges to } \infty\,.$$

(solution on board)

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Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

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Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

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as required. $\square$