McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 6: Sequence convergence**
- $\boxed{\texttt{Submit}}$.

Announcements

- Assignment 1 is due via crowdmark 5 minutes before class on Monday.
- Consider writing the Putnam competition.

Sequences

- A ***sequence*** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

A lot of different notation is common for sequences:

$f(1),\ f(2),\ f(3),\ \ldots$	$\{f(n)\}_{n=1}^{\infty}$
$f_1,\ f_2,\ f_3,\ \ldots$	$\{f(n)\}$
$\{f(n) : n = 1, 2, 3, \ldots\}$	$\{f_n\}_{n=1}^{\infty}$
$\{f(n) : n \in \mathbb{N}\}$	$\{f_n\}$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

A lot of different notation is common for sequences:

$f(1),\ f(2),\ f(3),\ \ldots$	$\{f(n)\}_{n=1}^{\infty}$
$f_1,\ f_2,\ f_3,\ \ldots$	$\{f(n)\}$
$\{f(n) : n = 1, 2, 3, \ldots\}$	$\{f_n\}_{n=1}^{\infty}$
$\{f(n) : n \in \mathbb{N}\}$	$\{f_n\}$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

A lot of different notation is common for sequences:

$$\begin{array}{ll}
f(1),\ f(2),\ f(3),\ \ldots & \{f(n)\}_{n=1}^{\infty} \\
f_1,\ f_2,\ f_3,\ \ldots & \{f(n)\} \\
\{f(n) : n = 1, 2, 3, \ldots\} & \{f_n\}_{n=1}^{\infty} \\
\{f(n) : n \in \mathbb{N}\} & \{f_n\}
\end{array}$$

Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to ***iterate***. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \ldots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \ldots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

Specifying sequences

Example (geometric progression with common ratio r)

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is:

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Direct formula:

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Direct formula: $x_n = f(n) =$

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula:

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \dots$

Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \dots$

Recursive formula: $x_1 = c$, $f(x) =$

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c, \ cr, \ cr^2, \ cr^3, \ \ldots$

Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$,

Specifying sequences

Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Specifying sequences

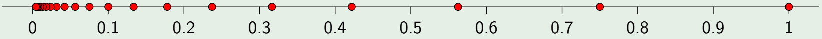
Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Specifying sequences

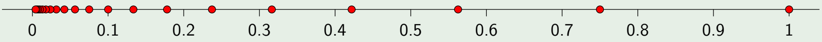
Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

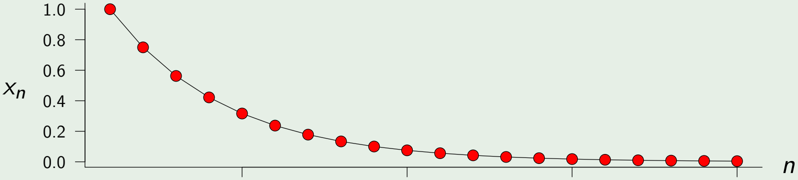
Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Graph of $f(n)$:



Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is:

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: $2,\ \frac{5}{4},\ \frac{10}{9}\ \frac{17}{16},\ \ldots$

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, ...

Direct formula:

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: $2,\ \frac{5}{4},\ \frac{10}{9}\ \frac{17}{16},\ \ldots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, ...

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

Recursive formula:

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, ...

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = 2, \quad f(x) =$

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: $2,\ \frac{5}{4},\ \frac{10}{9}\ \frac{17}{16},\ \ldots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, ...

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: $2,\ \frac{5}{4},\ \frac{10}{9}\ \frac{17}{16},\ \dots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \dots$

Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

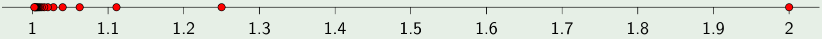
Sequence is: $2,\ \frac{5}{4},\ \frac{10}{9}\ \frac{17}{16},\ \ldots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2},\ n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:



Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

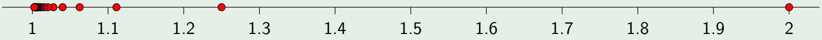
Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, $\ldots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

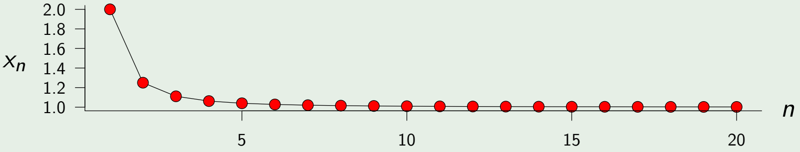
Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:



Graph of $f(n)$:



Convergence of sequences

Convergence of sequences

We know from previous experience that:

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition:

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition:

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*,

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε,

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** mathematically rigorous?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L smaller than ε

Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L smaller than ε by choosing n big enough".

Convergence of sequences

Convergence of sequences

Definition (Limit of a sequence)

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to** L* if,

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to** L* if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to L*** if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim\limits_{n\to\infty} s_n = L$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to** L* if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim\limits_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim\limits_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L,

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim\limits_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$.

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε.

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to** L* if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\displaystyle\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=}$$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall\varepsilon > 0$$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon \,.$$

In this case, we write $\displaystyle\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \overset{\text{def}}{=} \quad \forall\varepsilon > 0 \quad \exists N \in \mathbb{N}$$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to L*** if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\displaystyle\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies$$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Convergence of sequences

Convergence terminology:

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be ***convergent***.

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be ***convergent***.
- A sequence that is not convergent is said to be ***divergent***.

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be ***convergent***.
- A sequence that is not convergent is said to be ***divergent***.

Remark (Sequences in spaces other than $\mathbb{R}$)

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be ***convergent***.
- A sequence that is not convergent is said to be ***divergent***.

Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

Convergence of sequences

Example

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note:

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was "rough work".

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was "rough work". Only after this rough work did we have enough information to be able to write down a rigorous proof.

Convergence of sequences

Convergence of sequences

Example

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty \,.$$

(solution on board)

Note:

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty \, .$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$. If that bound is less than ε then so is $|s_n - L|$.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Tuesday 17 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 7: Sequence divergence**
- $\boxed{\texttt{Submit}}$.

Announcements

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to https://sas.mcmaster.ca/volunteer-notetaking/.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to https://sas.mcmaster.ca/volunteer-notetaking/.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to https://sas.mcmaster.ca/volunteer-notetaking/.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted. *Always due 5 minutes before class on the due date.*

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted. *Always due 5 minutes before class on the due date.*
- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements continued. . .

Announcements continued. . .

- Remember that solutions to assignments and tests from previous years are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)

Uniqueness of limits

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

So, we are justified in referring to “the” limit of a convergent sequence.

Divergence of sequences

Divergence of sequences

Divergence is the logical opposite

Divergence of sequences

Divergence is the logical opposite (negation)

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R},$$

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \quad \exists \varepsilon > 0$$

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0$ such that:

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N}$$

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; \not)$$

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; \ni \;\; |s_n - L| \geq \varepsilon.$$

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \ \ \exists \varepsilon > 0 \ \text{ such that: } \forall N \in \mathbb{N} \ \ \exists n \geq N \ \ \ni \ \ |s_n - L| \geq \varepsilon.$$

Notes:

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \ \ \exists \varepsilon > 0 \ \text{ such that: } \forall N \in \mathbb{N} \ \ \exists n \geq N \ \ \ni \ \ |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L, ε and N.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N} \;\; \exists n \geq N \;\ni\; |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \ \ \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N} \ \exists n \geq N \ \ni \ \ |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

Divergence to $\pm\infty$

Definition (Divergence to ∞)

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if,

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N such that

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to ∞*** if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow$$

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \geq M$$

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \geq M \,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \geq M \,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} s_n = \infty$.

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to ∞*** if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \geq M\,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} s_n = \infty$.

Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers ***diverges to $-\infty$*** if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \leq M\,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{ \frac{n^3 - 1}{n + 1} \right\} \quad \text{diverges to } \infty \, .$$

(solution on board)

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{\frac{n^3-1}{n+1}\right\} \quad \text{diverges to } \infty\,.$$

(solution on board)

Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{ \dfrac{n^3 - 1}{n + 1} \right\}$ diverges to ∞.

Clean proof.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{ \dfrac{n^3 - 1}{n+1} \right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$,

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N,\ n - 1 \geq M$.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 =$$

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} =$$

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n-1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq$$

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore \forall n \geq N$ we have

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore\ \forall n \geq N,\ n-1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n-1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore\ \forall n \geq N$ we have

$$\frac{n^3-1}{n+1} \geq M\,,$$

as required. □

Sequences of partial sums (a.k.a. Series)

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$,

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the ***sequence of partial sums of*** $\{x_n\}$

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the ***sequence of partial sums of*** $\{x_n\}$ to be $\{s_n\}$, where

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the ***sequence of partial sums of*** $\{x_n\}$ to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \,.$$

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the ***sequence of partial sums of*** $\{x_n\}$ to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \,.$$

Note: We can start from any integer, not necessarily $k = 1$.

Boundedness of sequences

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \le M$.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L$$

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \;\wedge\; \lim_{n\to\infty} s_n = L \quad \Longrightarrow$$

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni$$

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \le M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \le M$$

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \land \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \;\wedge\; \lim_{n\to\infty} s_n = L \quad\implies\quad \exists M > 0 \;\ni\; |s_n| \leq M \;\forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof?

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \le M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ni |s_n| \le M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof? Find a counterexample

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \;\wedge\; \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \;\ni\; |s_n| \leq M \;\forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof? Find a counterexample,

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof? Find a counterexample, *e.g.*, $\{(-1)^n\}$.

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Thursday 19 September 2019

What we've done so far on sequences

What we've done so far on sequences

- Definition of convergence.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Series: sequence of partial sums.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Series: sequence of partial sums.
- Bounded sequences.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Series: sequence of partial sums.
- Bounded sequences.
- Examples.

Boundedness of sequences

Boundedness of sequences

Corollary (Unbounded sequences diverge)

Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the ***harmonic series*** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

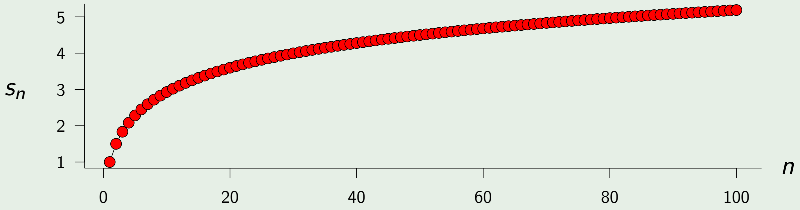
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the ***harmonic series*** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



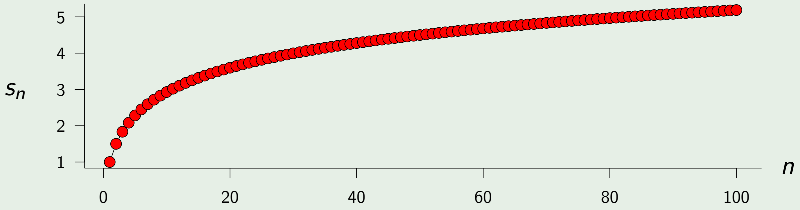
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the ***harmonic series*** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



Prove that s_n diverges to ∞.

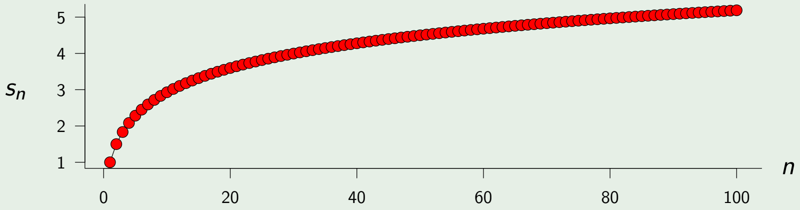
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the ***harmonic series*** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



Prove that s_n diverges to ∞.

(solution on board)

Harmonic series – idea for proof of divergence

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\ 1\times\frac{1}{2}}}_{s_2\ >\ 1\times\frac{1}{2}} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\ 2\times\frac{1}{4}}}_{s_4\ >\ 2\times\frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\ 4\times\frac{1}{8}}}_{s_8\ >\ 3\times\frac{1}{2}} + \cdots$$

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\,1\times\frac{1}{2}}}_{s_2\,>\,1\times\frac{1}{2}}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\,2\times\frac{1}{4}}}_{s_4\,>\,2\times\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\,4\times\frac{1}{8}}}_{s_8\,>\,3\times\frac{1}{2}}+\cdots$$

$$\Longrightarrow \qquad s_{2^n} > n\times\frac{1}{2}$$

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\ 1\times\frac{1}{2}}}_{s_2\ >\ 1\times\frac{1}{2}}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\ 2\times\frac{1}{4}}}_{s_4\ >\ 2\times\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\ 4\times\frac{1}{8}}}_{s_8\ >\ 3\times\frac{1}{2}}+\cdots$$

$$\Longrightarrow \qquad s_{2^n} \ > \ n\times\frac{1}{2}$$

Note: These sorts calculations are just “rough work”, not a formal proof.

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\ 1\times\frac{1}{2}}}_{s_2\ >\ 1\times\frac{1}{2}}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\ 2\times\frac{1}{4}}}_{s_4\ >\ 2\times\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\ 4\times\frac{1}{8}}}_{s_8\ >\ 3\times\frac{1}{2}}+\cdots$$

$$\implies \qquad s_{2^n} \ > \ n\times\frac{1}{2}$$

Note: These sorts calculations are just “rough work”, not a formal proof.
A proof must be a clearly presented coherent argument from beginning to end.

Harmonic series – clean proof of divergence

Proof.

Harmonic series – clean proof of divergence

Proof.

Part (i).

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}.$

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii).

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \ \forall n \in \mathbb{N}$.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}.$

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \ \forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \; \forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N$

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \ \forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\widetilde{N}}}$

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0$ $\forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\widetilde{N}}} > \widetilde{N}/2$

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \le 0$ then note that $s_n > 0\ \forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \ge N$, we have $s_n \ge s_N = s_{2^{\widetilde{N}}} > \widetilde{N}/2 = \lceil M \rceil$

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0$ $\forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\widetilde{N}}} > \widetilde{N}/2 = \lceil M \rceil \geq M$

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \ \forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\widetilde{N}}} > \widetilde{N}/2 = \lceil M \rceil \geq M$, as required.

$\square$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 8: Harmonic series of primes**
- Submit.

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim_{n\to\infty} C\, s_n = C\big(\lim_{n\to\infty} s_n\big)$;

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ and $\{t_n\}$ are convergent sequences and $C \in \mathbb{R}$.

1. $\displaystyle\lim_{n\to\infty} C\, s_n = C\Big(\lim_{n\to\infty} s_n\Big)$;
2. $\displaystyle\lim_{n\to\infty} (s_n + t_n) = \Big(\lim_{n\to\infty} s_n\Big) + \Big(\lim_{n\to\infty} t_n\Big)$;

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$*.*

1. $\lim_{n\to\infty} C\, s_n = C\big(\lim_{n\to\infty} s_n\big)$;
2. $\lim_{n\to\infty} (s_n + t_n) = \big(\lim_{n\to\infty} s_n\big) + \big(\lim_{n\to\infty} t_n\big)$;
3. $\lim_{n\to\infty} (s_n - t_n) = \big(\lim_{n\to\infty} s_n\big) - \big(\lim_{n\to\infty} t_n\big)$;

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$*.*

1. $\displaystyle\lim_{n\to\infty} C\, s_n = C\big(\lim_{n\to\infty} s_n\big)$;
2. $\displaystyle\lim_{n\to\infty} (s_n + t_n) = \big(\lim_{n\to\infty} s_n\big) + \big(\lim_{n\to\infty} t_n\big)$;
3. $\displaystyle\lim_{n\to\infty} (s_n - t_n) = \big(\lim_{n\to\infty} s_n\big) - \big(\lim_{n\to\infty} t_n\big)$;
4. $\displaystyle\lim_{n\to\infty} (s_n t_n) = \big(\lim_{n\to\infty} s_n\big)\big(\lim_{n\to\infty} t_n\big)$;

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim\limits_{n\to\infty} C\, s_n = C\big(\lim\limits_{n\to\infty} s_n\big)$;
2. $\lim\limits_{n\to\infty}(s_n + t_n) = \big(\lim\limits_{n\to\infty} s_n\big) + \big(\lim\limits_{n\to\infty} t_n\big)$;
3. $\lim\limits_{n\to\infty}(s_n - t_n) = \big(\lim\limits_{n\to\infty} s_n\big) - \big(\lim\limits_{n\to\infty} t_n\big)$;
4. $\lim\limits_{n\to\infty}(s_n t_n) = \big(\lim\limits_{n\to\infty} s_n\big)\big(\lim\limits_{n\to\infty} t_n\big)$;
5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}\,.$$

(solution on board)

Revisit example

Revisit example

Example (previously proved directly from definition)

Revisit example

Example (previously proved directly from definition)

Use the algebraic properties of limits to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Revisit example

Example (previously proved directly from definition)

Use the algebraic properties of limits to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Friday 20 September 2019

Announcements

Announcements

- Assignment 2 is posted.

Announcements

- Assignment 2 is posted.
  Due 1 Oct 2019, at 2:25pm.

What we've done so far on sequences

What we've done so far on sequences

- Definition of convergence.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.
- Harmonic series diverges.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (more today).

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$, $\quad |s_n t_n - ST| =$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$, $\quad |s_n t_n - ST| = |s_n t_n - ST + s_n T - s_n T|$

$$=$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\leq \end{aligned}$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\leq |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\leq |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$,

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \; \forall n \in \mathbb{N}$.

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M$ $\forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$,

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| <$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M}$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\leq |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \leq M \; \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| <$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\leq |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \leq M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)}\,.$$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1+|T|)} .$$

Then $|s_n t_n - ST| <$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)} .$$

Then $|s_n t_n - ST| < \varepsilon/2 + \varepsilon/2$

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \ \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)}\,.$$

Then $|s_n t_n - ST| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. $\square$

Quotient Rule for Limits

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0 \; \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$,

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0$ $\forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right|$

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ *and* $t_n \to T \neq 0$ *then* $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right|$

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0$ $\forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\displaystyle \left|\frac{1}{t_n} - \frac{1}{T}\right| = \left|\frac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|}$.

Since $\{t_n\}$ converges,

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$,

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide)

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0 \; \forall n$ *and* $t_n \to T \neq 0$ *then* $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0 \ \forall n$ *and* $t_n \to T \neq 0$ *then* $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.

Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon |T|^2/2$.

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.
Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.
Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon |T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| =$$

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0$ $\forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\frac{1}{t_n} - \frac{1}{T}\right| = \left|\frac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.

Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon|T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} <$$

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\frac{1}{t_n} - \frac{1}{T}\right| = \left|\frac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|}$.
Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.
Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon |T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} < \frac{\varepsilon |T|^2}{2} \cdot \frac{2}{|T|} \cdot \frac{1}{|T|}$$

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\frac{1}{t_n} - \frac{1}{T}\right| = \left|\frac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|}$.
Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.
Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon |T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} < \frac{\varepsilon |T|^2}{2} \cdot \frac{2}{|T|} \cdot \frac{1}{|T|} = \varepsilon\,,$$

as required. $\square$

Quotient Rule for Limits

Details missing on previous slide:

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $$-\frac{|T|}{2} < t_n - T < \frac{|T|}{2}\,,$$

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

In either case, $\forall n \geq N_1$, we have

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

In either case, $\forall n \geq N_1$, we have $0 < \dfrac{|T|}{2} < |t_n|$.

Order properties of limits (§2.8)

Order properties of limits (§2.8)

Theorem (Limits retain order)

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences then*

$$s_n \leq t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n \,.$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$,

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n\,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ $\ni$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$.

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$S - T =$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \, .$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$S - T = S - T +$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n\,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$S - T = S - T + s_n - s_n +$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\le \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\le (S - s_n) + (t_n - T) \qquad\qquad (\because s_n - t_n \le 0) \\ &< \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \leq t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\leq (S - s_n) + (t_n - T) \qquad\qquad (\because s_n - t_n \leq 0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences then*

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\le (S - s_n) + (t_n - T) && (\because s_n - t_n \le 0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\le (S - s_n) + (t_n - T) && (\because s_n - t_n \le 0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $S - T \le 0$

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\le (S - s_n) + (t_n - T) \qquad\qquad (\because s_n - t_n \le 0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $S - T \le 0$, *i.e.*, $S \le T$. $\square$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 9: Order property of limits**
- $\boxed{\texttt{Submit}}$.

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Theorem (Limits retain bounds)

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Theorem (Limits retain bounds)

If $\{s_n\}$ is a convergent sequence then

$$\alpha \le s_n \le \beta \quad \forall n \in \mathbb{N} \quad \implies \quad \alpha \le \lim_{n\to\infty} s_n \le \beta\,.$$

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Theorem (Limits retain bounds)

If $\{s_n\}$ is a convergent sequence then

$$\alpha \le s_n \le \beta \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \alpha \le \lim_{n\to\infty} s_n \le \beta \,.$$

Proof.

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Theorem (Limits retain bounds)

If $\{s_n\}$ is a convergent sequence then

$$\alpha \le s_n \le \beta \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \alpha \le \lim_{n\to\infty} s_n \le \beta\,.$$

Proof.

Apply previous theorem with $\alpha_n = \alpha \;\forall n$ and $\beta_n = \beta \;\forall n$. $\square$

Order properties of limits (§2.8)

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Proof? (What's **WRONG**?).

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L.$ *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L.$

Proof? (What's **WRONG**?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge.

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Proof? (What's **WRONG**?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$).

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Proof? (What's WRONG?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \to X$.

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Proof? (What's **WRONG**?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \to X$. Hence, by order retension, $L \le X \le L \implies X = L$. □

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \geq N$, $|s_n - L| < \varepsilon$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon$$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon \qquad \text{and}$$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.,*

$$-\varepsilon < s_n - L < \varepsilon \qquad \textit{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon \qquad \textit{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

But $s_n \le x_n \le t_n \implies$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences such that*

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ *is convergent and* $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \geq N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon \qquad \text{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

But $s_n \leq x_n \leq t_n \implies \quad s_n - L \leq x_n - L \leq t_n - L$

$\implies$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon \qquad \textit{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

But $s_n \le x_n \le t_n$

$$\begin{aligned} &\implies \quad s_n - L \le x_n - L \le t_n - L \\ &\implies -\varepsilon < s_n - L \le x_n - L \le t_n - L < \varepsilon \\ &\implies \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.,*

$$-\varepsilon < s_n - L < \varepsilon \qquad \text{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

But $s_n \le x_n \le t_n$

$$\begin{aligned} &\implies \quad s_n - L \le x_n - L \le t_n - L \\ &\implies -\varepsilon < s_n - L \le x_n - L \le t_n - L < \varepsilon \\ &\implies |x_n - L| < \varepsilon, \end{aligned}$$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon \qquad \textit{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

But $s_n \le x_n \le t_n$

$$\begin{aligned} &\implies \quad s_n - L \le x_n - L \le t_n - L \\ &\implies -\varepsilon < s_n - L \le x_n - L \le t_n - L < \varepsilon \\ &\implies |x_n - L| < \varepsilon, \end{aligned}$$

as required. $\square$

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ *converges then so does* $\{|s_n|\}$, *and*

$$\lim_{n\to\infty} |s_n| = \left| \lim_{n\to\infty} s_n \right| .$$

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ converges then so does $\{|s_n|\}$, and

$$\lim_{n\to\infty} |s_n| = \left|\lim_{n\to\infty} s_n\right| \,.$$

Proof.

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ converges then so does $\{|s_n|\}$, and

$$\lim_{n\to\infty} |s_n| = \left|\lim_{n\to\infty} s_n\right| .$$

Proof.

See Assignment 2! □

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\} ,$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\} .$$

Idea for proof:

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n, \ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n, \ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} =$$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

$$\forall x, y \in \mathbb{R} \quad \min\{x, y\} =$$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

$$\forall x, y \in \mathbb{R} \quad \min\{x, y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

$$\forall x, y \in \mathbb{R} \quad \min\{x, y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$$

Prove these facts, then use theorems on sums and absolute values of limits.

Monotone convergence (§2.9)

Monotone convergence (§2.9)

Definition (Monotonic sequence)

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is ***monotonic*** iff it satisfies any of the following conditions:

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is ***monotonic*** iff it satisfies any of the following conditions:

(i) ***Increasing***: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is ***monotonic*** iff it satisfies any of the following conditions:

(i) ***Increasing***: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

(ii) ***Decreasing***: $s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \cdots$;

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is ***monotonic*** iff it satisfies any of the following conditions:

(i) ***Increasing***: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

(ii) ***Decreasing***: $s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \cdots$;

(iii) ***Non-decreasing***: $s_1 \le s_2 \le s_3 \le \cdots \le s_n \le s_{n+1} \le \cdots$;

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is ***monotonic*** iff it satisfies any of the following conditions:

(i) ***Increasing***: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

(ii) ***Decreasing***: $s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \cdots$;

(iii) ***Non-decreasing***: $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$;

(iv) ***Non-increasing***: $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$.

Monotone convergence (§2.9)

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;

(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;

(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;

(iv) $\{s_n\}$ *non-increasing and bounded* $\implies s_n \to \inf\{s_n\}$.

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded. In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;

(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;

(iv) $\{s_n\}$ *non-increasing and bounded* $\implies s_n \to \inf\{s_n\}$.