

1 Introduction

2 Properties of \mathbb{R}

3 Properties of \mathbb{R} II



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 1
Introduction
Monday 7 January 2019

Where to find course information

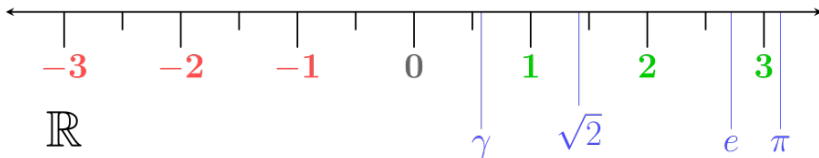
- The course web site:
<http://www.math.mcmaster.ca/earn/3A03>
- Click on [Course information](#) to download course information as pdf file. *You are expected to read and pay attention to every word of this file.*
- Let's have a look now. . .

What is a “real” number?



What is a “real” number?

- The “Reals” (\mathbb{R}) are all the numbers that are needed to fill in the “number line” (so it has no “gaps” or “holes”).
- Why aren’t the rational numbers (\mathbb{Q}) sufficient?



- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
Approach: “Proof by contradiction.”

$\sqrt{2}$ is irrational

Theorem

$$\sqrt{2} \notin \mathbb{Q}.$$

Proof.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m, n) = 1$ such that $m/n = \sqrt{2}$.

$$\therefore \left(\frac{m}{n}\right)^2 = (\sqrt{2})^2 \implies \frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.$$

$\therefore m^2$ is even $\implies m$ is even (\because odd numbers have odd squares).

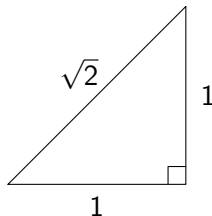
$\therefore m = 2k$ for some $k \in \mathbb{N}$.

$$\therefore 4k^2 = m^2 = 2n^2 \implies 2k^2 = n^2 \implies n \text{ is even.}$$

$\therefore 2$ is a factor of both m and n . **Contradiction!** $\therefore \sqrt{2} \notin \mathbb{Q}$. \square

Does $\sqrt{2}$ exist?

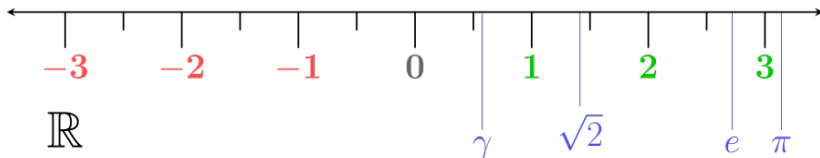
- We have established that $\sqrt{2}$ is not rational.
- But do we really know it exists?
- Can we do without it?
- No. Objects with side length $\sqrt{2}$ exist!



- So irrational numbers are “real”.

What exactly *are* non-rational real numbers?

- We have solid intuition for what rational numbers are. (Ratios of integers.)
- The number line contains numbers that are not rational.



- Can we *construct* irrational numbers? (Just as we construct rationals as ratios of integers?)
- Do we need to *construct* integers first?
- Maybe we should start with 0, 1, 2, ...
- But what exactly are we supposed to *construct* numbers from?

Informal introduction to construction of numbers (\mathbb{N})

- Assume we know what a **set** is.
- Define $0 \equiv \emptyset = \{\}$ (the empty set)
- Define $1 \equiv \{0\} = \{\emptyset\} = \{\{\}\}$
- Define $2 \equiv \{0, 1\} = \{\{\}, \{\{\}\}\}$
- Define $n + 1 \equiv n \cup \{n\}$ (successor function)
- Define **natural numbers** $\mathbb{N} = \{1, 2, 3, \dots\}$
 - Some books define $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \dots\}$.
 - It is more common to define \mathbb{N} to start with 1.
- Thus, n is defined to be a set containing n elements.

Informal introduction to construction of numbers (\mathbb{N})

Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as “the set of all sets containing n elements”.
- The earlier definition possibly better captures our intuitive notion of what n “really is”, but such “sets” are unweildy and create serious challenges for development of mathematical foundations.

Informal introduction to construction of numbers (\mathbb{N})

Order of natural numbers:

- Natural numbers defined as above have the right order:

$$m \leq n \iff m \subseteq n$$

Note: we define " \leq " on natural numbers via " \subseteq " on sets.

Addition and multiplication of natural numbers:

- Still possible to define in terms of sets, but trickier.
- We'll defer this for later, after gaining more experience with rigorous mathematical concepts.
- If you can't wait, see this free e-book:

"Transition to Higher Mathematics"
<http://openscholarship.wustl.edu/books/10/>.

Informal introduction to construction of numbers (\mathbb{Z})

Integers:

- Need additive inverses for all natural numbers.
 - Need to define $\cdot, +, -$, for all pairs of integers.
 - Again, possible to define everything via set theory.
 - Again, we'll defer this for later.
-
- For now, we'll assume we "know" what the naturals \mathbb{N} and the integers \mathbb{Z} "are".
 - We can then *construct* the rationals \mathbb{Q} ...



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Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 2
Properties of \mathbb{R}
Wednesday 9 January 2019

Where to find course information

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<http://www.math.mcmaster.ca/earn/3A03>
- Click on [Course information](#) to download pdf file.
 - **Read it!!**
- Check the course web site regularly!

What we did last class

- The “Reals” (\mathbb{R}) are all the numbers that are needed to fill in the “number line” (so it has no “gaps” or “holes”).
- The rationals (\mathbb{Q}) have “holes”, e.g., $\sqrt{2}$.
- Numbers can be constructed using sets. We will discuss this *informally*. A more formal approach is taken in Math 4L03 (Mathematical Logic) or in [this online e-book](#).
 - The naturals ($\mathbb{N} = \{1, 2, 3, \dots\}$) can be constructed from \emptyset :
 $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, \dots , $n + 1 = n \cup \{n\}$.
 - The integers (\mathbb{Z}), and operations on them ($+$, $-$, \cdot), can also be constructed from sets and set operations (but we deferred that for later).
 - Given \mathbb{N} and \mathbb{Z} , we can construct \mathbb{Q} ...

Informal introduction to construction of numbers (\mathbb{Q})

Rationals:

- *Idea:* Associate \mathbb{Q} with $\mathbb{Z} \times \mathbb{N}$
- Use notation $\frac{a}{b} \in \mathbb{Q}$ if $(a, b) \in \mathbb{Z} \times \mathbb{N}$.
- Define equivalence of rational numbers:

$$\frac{a}{b} = \frac{c}{d} \stackrel{\text{def}}{=} a \cdot d = b \cdot c$$

- Define order for rational numbers:

$$\frac{a}{b} \leq \frac{c}{d} \stackrel{\text{def}}{=} a \cdot d \leq b \cdot c$$

Informal introduction to construction of numbers (\mathbb{Q})

Rationals, continued:

- Define operations on rational numbers:

$$\frac{a}{b} + \frac{c}{d} \stackrel{\text{def}}{=} \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} \stackrel{\text{def}}{=} \frac{a \cdot c}{b \cdot d}$$

- Constructed in this way (ultimately from the empty set), \mathbb{Q} satisfies all the standard properties we associate with the rational numbers.
- Formally, \mathbb{Q} is a set of **equivalence classes** of $\mathbb{Z} \times \mathbb{N}$.
Extra Challenge Problem: Are “+” and “·” well-defined on \mathbb{Q} ?

Properties of the rational numbers (\mathbb{Q})

Addition:

A1 *Closed and commutative under addition.* For any $x, y \in \mathbb{Q}$ there is a number $x + y \in \mathbb{Q}$ and $x + y = y + x$.

A2 *Associative under addition.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y) + z = x + (y + z)$$

is true.

A3 *Existence and uniqueness of additive identity.* There is a unique number $0 \in \mathbb{Q}$ such that, for all $x \in \mathbb{Q}$,

$$x + 0 = 0 + x = x.$$

A4 *Existence of additive inverses.* For any number $x \in \mathbb{Q}$ there is a corresponding number denoted by $-x$ with the property that

$$x + (-x) = 0.$$

Properties of the rational numbers (\mathbb{Q})

Multiplication:

- M1** *Closed and commutative under multiplication.* For any $x, y \in \mathbb{Q}$ there is a number $xy \in \mathbb{Q}$ and $xy = yx$.
- M2** *Associative under multiplication.* For any $x, y, z \in \mathbb{Q}$ the identity $(xy)z = x(yz)$ is true.
- M3** *Existence and uniqueness of multiplicative identity.* There is a unique number $1 \in \mathbb{Q} \setminus \{0\}$ such that, for all $x \in \mathbb{Q}$, $x1 = 1x = x$.
- M4** *Existence of multiplicative inverses.* For any non-zero number $x \in \mathbb{Q}$ there is a corresponding number denoted by x^{-1} with the property that $xx^{-1} = 1$.

Properties of the rational numbers (\mathbb{Q})

Addition and multiplication together:

AM1 Distributive law. For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y)z = xz + yz$$

is true.

The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers \mathbb{Q} a **field**.

Note: M3 ensures $0 \neq 1$ to exclude the uninteresting case of a field with only one element.

Standard Mathematical Shorthand

Quantifiers

\forall	for all
\exists	there exists
\nexists	there does not exist
$\exists!$	there exists a unique

Logical operands

\wedge	logical and
\vee	logical or
\neg	logical not
$\underline{\vee}$	logical exclusive or

Note: $A \underline{\vee} B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

Other shorthand

\therefore	therefore	\because	because
$\}$	such that	\iff	if and only if
\equiv	equivalent	$\Rightarrow \Leftarrow$	contradiction

The field axioms (in mathematical shorthand) for field \mathbb{F}

Addition axioms

A1 *Closed, commutative.* $\forall x, y \in \mathbb{F},$
 $\exists (x+y) \in \mathbb{F} \wedge (x+y) = (y+x).$

A2 *Associative.* $\forall x, y, z \in \mathbb{F},$
 $(x+y) + z = x + (y+z).$

A3 *Identity.* $\exists! 0 \in \mathbb{F} \nmid \forall x \in \mathbb{F},$
 $x + 0 = 0 + x = x.$

A4 *Inverses.* $\forall x \in \mathbb{F}, \exists (-x) \in \mathbb{F} \nmid$
 $x + (-x) = 0.$

Multiplication axioms

M1 *Closed, commutative.* $\forall x, y \in \mathbb{F},$
 $\exists (xy) \in \mathbb{F} \wedge (xy) = (yx).$

M2 *Associative.* $\forall x, y, z \in \mathbb{F},$
 $(xy)z = x(yz).$

M3 *Identity.* $\exists! 1 \in \mathbb{F} \setminus \{0\} \nmid$
 $\forall x \in \mathbb{F}, x1 = 1x = x.$

M4 *Inverses.* $\forall x \in \mathbb{F} \setminus \{0\},$
 $\exists x^{-1} \in \mathbb{F} \nmid xx^{-1} = 1.$

Distribution axiom

AM1 *Distribution.* $\forall x, y, z \in \mathbb{F}, (x+y)z = xz + yz.$

Any collection \mathbb{F} of mathematical objects is called a *field* if it satisfies these 9 algebraic properties.

Examples of fields

Set	Field?	Why?
rational numbers (\mathbb{Q})	YES	
integers (\mathbb{Z})	NO	no multiplicative inverses
reals (\mathbb{R})	YES	
complexes (\mathbb{C})	YES	
integers modulo 3 (\mathbb{Z}_3)	YES	$2^{-1} = 2$

The integers modulo 3 (\mathbb{Z}_3)

Imagine a clock that repeats after 3 hours rather than 12 hours.

\mathbb{Z}_3 contains the three elements $\{0, 1, 2\}$, with addition and multiplication defined as follows:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Ordered fields

A field \mathbb{F} is said to be **ordered** if the following properties hold:

Order axioms

- 01** For any $x, y \in \mathbb{F}$, exactly one of the statements $x = y$, $x < y$ or $y < x$ is true (“**trichotomy**”), *i.e.*,
 $\forall x, y \in \mathbb{F}, ((x = y) \wedge \neg(x < y) \wedge \neg(y < x)) \vee ((x < y) \wedge \neg(x = y) \wedge \neg(y < x)) \vee ((y < x) \wedge \neg(x = y) \wedge \neg(x < y))$
- 02** For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $y < z$ is true, then $x < z$ is true, *i.e.*, $\forall x, y, z \in \mathbb{F}, (x < y) \wedge (y < z) \implies (x < z)$
- 03** For any $x, y \in \mathbb{F}$, if $x < y$ is true, then $x + z < y + z$ is also true for any $z \in \mathbb{F}$, *i.e.*, $\forall x, y \in \mathbb{F}, (x < y) \implies x + z < y + z, \forall z \in \mathbb{F}$
- 04** For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $z > 0$ is true, then $xz < yz$ is also true,
i.e., $\forall x, y, z \in \mathbb{F}, (x < y) \wedge (0 < z) \implies (xz < yz)$

Examples of ordered fields

Field	Ordered?	Why?
rationals (\mathbb{Q})	YES	
reals (\mathbb{R})	YES	
integers modulo 3 (\mathbb{Z}_3)	NO	Next slide. . .
complexes (\mathbb{C})	NO	Extra Challenge Problem: <i>Prove the field \mathbb{C} cannot be ordered.</i>

The field of integers modulo 3 cannot be ordered

Proposition

\mathbb{Z}_3 is not an *ordered field*.

Proof.

Approach: proof by contradiction.

If \mathbb{Z}_3 is ordered, then O1 (trichotomy) implies that either $0 < 1$ or $1 < 0$ (and not both).

Suppose $0 < 1$ and $1 \not< 0$. Then O3 $\implies 0 + 1 < 1 + 1$,
i.e., $1 < 2$. \therefore O2 (transitivity) $\implies 0 < 2$.

Using O3 again, we have $0 + 1 < 2 + 1$, i.e., $1 < 0$. $\Rightarrow \Leftarrow$

Now suppose $1 < 0$. Similarly reach a contradiction (check!).
 $\therefore \mathbb{Z}_3$ cannot be ordered. □

Food for thought: Is it possible for any finite field be ordered?

What other properties does \mathbb{R} have?

- \mathbb{R} is an **ordered field**.
- \mathbb{R} includes numbers that are not in \mathbb{Q} , e.g., $\sqrt{2}$.
- What additional properties does \mathbb{R} have?
- Only one more property is required to fully characterize \mathbb{R} . . .
It is related to *upper and lower bounds*. . .



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Mathematics 3A03 Real Analysis I

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Lecture 3
Properties of \mathbb{R} II
Thursday 10 January 2019

Announcements and comments arising from Lecture 2

- My office hours will be Mondays 1:30pm–2:20pm going forward. (Or by appointment.)
- Questions for next week's tutorials and some “Logic Notes” are posted on the [Tutorials page](#) of the course web site.
- Field Axiom [M3](#) was corrected before posting slides for Lecture 2.
- No claim is being made that the [field axioms](#) as stated are absolutely minimal (*i.e.*, that there are no redundancies). In fact, we don't need to assume:
 - Identities are unique.
 - Inverses are unique.
 - Commutivity under addition (!).

Usually a slightly redundant set of axioms is stated to emphasize all the key properties.

An additional online resource

A sequence of 15 short (3–7 minute) videos covering the very basics of mathematical logic and theorem proving has been posted associated with a course at the University of Toronto:

- Go to <http://uoft.me/MAT137>, click on the **Videos** tab and then on **Playlist 1**.

These videos go at a slower pace than we do, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

More comments arising from Lecture 2

- The property that completes the specification of \mathbb{R} has to somehow fill in all the “holes” in \mathbb{Q} .
- It is true that if $x, y \in \mathbb{Q}$ then $\exists r \in \mathbb{R} \setminus \mathbb{Q}$ with $x < r < y$. But this property is not sufficient to characterize \mathbb{R} , because it is satisfied by subsets of \mathbb{R} .

Bounds

Definition (Upper Bound)

Let $E \subseteq \mathbb{R}$. A number M is said to be an **upper bound** for E if $x \leq M$ for all $x \in E$.

A set that has an upper bound is said to be **bounded above**.

Definition (Lower Bound)

Let $E \subseteq \mathbb{R}$. A number m is said to be a **lower bound** for E if $m \leq x$ for all $x \in E$.

A set that has a lower bound is said to be **bounded below**.

A set that is bounded above and below is said to be **bounded**.

Maxima and Minima

Definition (Maximum)

Let $E \subseteq \mathbb{R}$. A number M is said to be **the maximum** of E if M is an **upper bound** for E and $M \in E$. If such an M exists we write $M = \max E$.

Definition (Minimum)

Let $E \subseteq \mathbb{R}$. A number m is said to be **the minimum** of E if m is a **lower bound** for E and $m \in E$. If such an m exists we write $m = \min E$.

We refer to “the” maximum and “the” minimum of E because there cannot be more than one of each. (*Proof?*)

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$	YES	YES	YES	-1	1
$[-1, 1)$	YES	YES	YES	-1	\nexists
$[-1, \infty)$	YES	NO	NO	-1	\nexists
$[-1, -\frac{1}{4}) \cup (\frac{1}{2}, 1]$	YES	YES	YES	-1	1
\mathbb{N}	YES	NO	NO	1	\nexists
\mathbb{R}	NO	NO	NO	\nexists	\nexists
\emptyset	YES	YES	YES	\nexists	\nexists

Least upper bounds

Definition (Least Upper Bound/Supremum)

A number M is said to be the **least upper bound** or **supremum** of a set E if

- (i) M is an upper bound of E , and
- (ii) if \tilde{M} is an upper bound of E then $M \leq \tilde{M}$.

If M is the least upper bound of E then we write $M = \sup E$.

Note: We can refer to “the” least upper bound of E because there cannot be more than one. (Proof?)

What sets have least upper bounds?

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	YES	1
$[-1, 1)$	YES	1
\emptyset	YES	\nexists
$\{x \in \mathbb{R} : x^2 < 2\}$	YES	$\sqrt{2}$
$\{x \in \mathbb{Q} : x^2 < 2\}$	YES	$\notin \mathbb{Q}$

Least upper bounds

The property that any set that is bounded above has a least upper bound is what distinguishes the real numbers \mathbb{R} from the rational numbers \mathbb{Q} .

Does this realization allow us to finish constructing \mathbb{R} ?

YES, but we will delay the construction until later in the course.

For now, we will simply annoint the least upper bound property as an axiom:

Completeness Axiom

If $E \subseteq \mathbb{R}$, $E \neq \emptyset$, and E is bounded above, then E has a **least upper bound** (i.e., $\sup E$ exists and $\sup E \in \mathbb{R}$).

\mathbb{R} is a complete ordered field

- Any field \mathbb{F} that satisfies the order axioms and the completeness axiom is said to be a **complete ordered field**.
- \mathbb{R} is a complete ordered field.
- Are there any other complete ordered fields?
- **Extra Challenge Problem:**
Prove that \mathbb{R} is the only complete ordered field.

Greatest lower bounds

Definition (Greatest Lower Bound/Infimum)

A number m is said to be the **greatest lower bound** or **infimum** of a set E if

- (i) m is a lower bound of E , and
- (ii) if \tilde{m} is a lower bound of E then $\tilde{m} \leq m$.

If m is the greatest lower bound of E then we write $m = \inf E$.

Greatest lower bounds

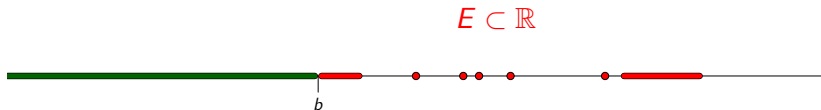
- The existence of **least upper bounds** was taken as an axiom.
- The existence of **greatest lower bounds** then follows.

Theorem

If $E \subseteq \mathbb{R}$, $E \neq \emptyset$, and E is bounded below, then E has a **greatest lower bound** (i.e., $\inf E$ exists and $\inf E \in \mathbb{R}$).

Proof?

Idea of proof:



$$L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound of } E\}$$

Greatest lower bounds

Theorem

If $E \subseteq \mathbb{R}$, $E \neq \emptyset$, and E is bounded below, then E has a **greatest lower bound** (i.e., $\inf E$ exists and $\inf E \in \mathbb{R}$).

Proof.

Recall graphical idea of proof.

Let $L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound of } E\}$. Then:

- $L \neq \emptyset$ ($\because E$ is **bounded below**).
- L is **bounded above** ($\because x \in E \implies x$ an **upper bound** for L).
- $\therefore L$ has a **least upper bound**, say $b = \sup L$.

Now show $b = \inf E$. First show $b \in L$ (i.e., $x \in E \implies b \leq x$). Suppose $x \in E$ and $b \not\leq x$; then by **O1 (trichotomy)**, we must have $b > x$. Now $b = \sup L$ and $x < b$, so x is not an upper bound of L , i.e., there is some $\ell \in L$ such that $x < \ell$. But then ℓ is not a lower bound of E . $\Rightarrow \Leftarrow \therefore b \in L$ and b is also $\max L$, i.e., $b = \inf E$. \square

Comment on least upper bounds and greatest lower bounds

- The proof above shows that:

$$\inf E = \sup\{x \in \mathbb{R} : x \text{ is a lower bound of } E\}$$

- Similarly:

$$\sup E = \inf\{x \in \mathbb{R} : x \text{ is an upper bound of } E\}$$

Some notational abuse concerning sup and inf

By convention, for convenience, we (and your textbook) sometimes write:

$$\begin{aligned}\inf \mathbb{R} &= -\infty \\ \sup \mathbb{R} &= \infty \\ \inf \emptyset &= \infty \\ \sup \emptyset &= -\infty\end{aligned}$$

This is an **abuse of notation**, since \emptyset and \mathbb{R} do not have **least upper** or **greatest lower** bounds in \mathbb{R} . ∞ is not a real number.

If you are asked “What is the **least upper bound** of \mathbb{R} ?” how should you answer?

Correct answer: “ \mathbb{R} is not bounded above so it does not have a least upper bound.”

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Archimedean property)

The set of natural numbers \mathbb{N} has no upper bound.

Proof.

Suppose \mathbb{N} is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$, hence $n + 1 \leq B$ for all $n \in \mathbb{N}$, i.e., $n \leq B - 1$ for all $n \in \mathbb{N}$. Thus, $B - 1$ is an upper bound for \mathbb{N} , contradicting B being the least upper bound. □

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Equivalences of the Archimedean property)

1 *The set of natural numbers \mathbb{N} has no upper bound.*

2 *Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

i.e., No matter how large a real number x is, there is always a natural number n that is larger.

3 *Given any $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

i.e., Given any positive number y , no matter how large, and any positive number x , no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in \mathbb{N}$).

4 *Given any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

i.e., Given any positive number x , no matter how small, one can always find a fraction $1/n$ that is smaller than x .