Math 3A03 - Tutorial 10 Questions - Winter 2019

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Problem 1. *Let*

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ x - 2 & \text{if } 1 < x \leq 2 \end{cases}$$

$$P_n = \left\{ \frac{i}{n} \right\}_{i=0}^{2n} = \left\{ 0, \frac{1}{2}, \frac{2}{n}, \ldots, \frac{i}{n}, \ldots, \frac{2n-1}{n}, 2 \right\}.$$

(a) *Compute* $U(f, P_n)$ *and* $L(f, P_n)$*, you may use the fact that* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$*, then show that given* $\varepsilon > 0$ *there is a* P_N *such that* $U(f, P_N) - L(f, P_N) < \varepsilon$*. Conclude that* f *is integrable on* $[0, 2]$.

(b) *Modify the above argument slightly, and conclude* f *is integrable directly from the definition. i.e., What can you say about* $\sup L(f, P)$ *and* $\inf U(f, P)$*?*

Problem 2. (a) *Let* f *be continuous on* (a, b) *and bounded on* $[a, b]$*. Prove that* f *is integrable on* $[a, b]$.

(b) *Let* f *be piecewise continuous on* $[a, b]$ *(meaning there are* k *points* $a = x_1 < x_2 < \ldots < x_j < \ldots < x_k = b$ *with* f *continuous on* (x_j, x_{j+1}) *and* f *bounded on* $[x_j, x_{j+1}]$*, which is to say* f *makes finitely many finite jumps), prove that* f *is integrable on* $[a, b]$.

Problem 3. *Let*

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Prove that $f(x)$ *is not integrable on* $[0, 1]$.