Continuous Functions

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

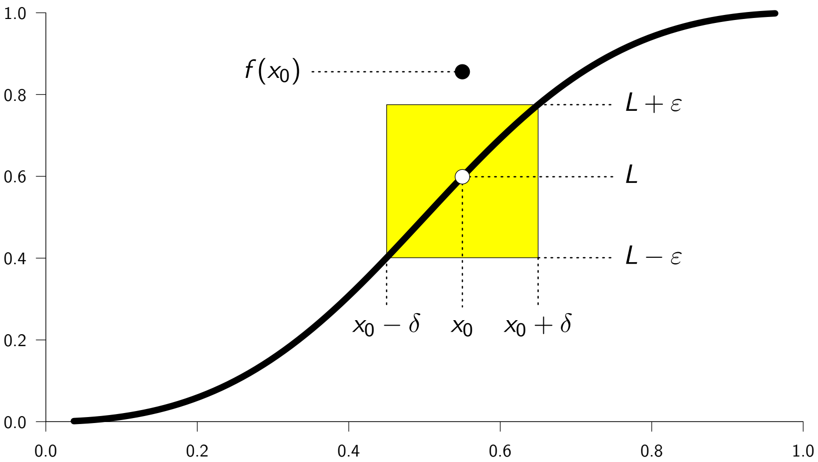
Instructor: David Earn

Lecture 18
Continuity
Monday 25 February 2019

Announcements

- A preliminary version of Assignment 4 has been posted on the course web site. More problems will be added soon.
  Due Friday 8 March 2019 at 1:25pm via crowdmark.
  BUT you should do it before Test #1.
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Limits of functions



Limits of functions

Definition (Limit of a function on an interval (a, b))

Let $a < x_0 < b$ and $f : (a, b) \to \mathbb{R}$. Then f is said to **approach the limit L as x approaches x_0**, often written "$f(x) \to L$ as $x \to x_0$" or

$$\lim_{x \to x_0} f(x) = L \,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \ni 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

Limits of functions

The function f need not be defined on an entire interval.
It is enough for f to be defined on a set with at least one accumulation point.

Definition (Limit of a function with domain $E \subseteq \mathbb{R}$)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then f is said to **approach the limit L as x approaches x_0**, *i.e.*,

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \ni \left(x \in E \;\wedge\; 0 < |x - x_0| < \delta \right) \implies |f(x) - L| < \varepsilon.$$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x\to 3}(2x+1) = 7\,.$$

(solution on board)

Proof that $2x+1 \to 7$ as $x \to 3$.

We must show that $\forall\varepsilon > 0\ \exists\delta > 0$ such that $0 < |x-3| < \delta \implies |(2x+1)-7| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$|(2x+1)-7| < \varepsilon \iff |2x-6| < \varepsilon \iff 2\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{2}$$

Therefore, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Then $|x-3| < \delta \implies |(2x+1)-7| = |2x-6| = 2\,|x-3| < 2\frac{\varepsilon}{2} = \varepsilon$, as required. $\square$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 2} x^2 = 4\,.$$

(solution on board)

(and on next slide)

Limits of functions

Proof that $x^2 \to 4$ as $x \to 2$.

We must show that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $0 < |x-2| < \delta \implies \left|x^2-4\right| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$\left|x^2-4\right| < \varepsilon \iff |(x-2)(x+2)| < \varepsilon \iff |x-2|\,|x+2| < \varepsilon.$$

We can make $|x-2|$ as small as we like by choosing δ sufficiently small. Moreover, if x is close to 2 then $x+2$ will be close to 4, so we should be able to ensure that $|x+2| < 5$. To see how, note that

$$\begin{aligned} |x+2| < 5 \iff & -5 < x+2 < 5 \iff -9 < x-2 < 1 \\ \impliedby & -1 < x-2 < 1 \iff |x-2| < 1\,. \end{aligned}$$

Therefore, given $\varepsilon > 0$, let $\delta = \min(1, \frac{\varepsilon}{5})$. Then $\left|x^2-4\right| = |(x-2)(x+2)| = |x-2|\,|x+2| < \frac{\varepsilon}{5}5 = \varepsilon.$ $\square$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Continuity II
Wednesday 27 February 2019

Announcements

- A preliminary version of Assignment 4 has been posted on the course web site. More problems will be added soon.
  Due Friday 8 March 2019 at 1:25pm via crowdmark.
  BUT you should do it before Test #1.

- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)
  - Test will cover everything up to the end of the topology section.

- Niky Hristov will hold extra office hours this Friday 1 March 2019, 11:30am–12:30pm and immediately before class on the day of the test, *i.e.*, Monday 4 March 2019, 10:30–11:30am.

- Solutions to $\lim_{x\to 3}(2x+1) = 7$ and $\lim_{x\to 2} x^2 = 4$ are now in the slides for the previous lecture.

Limits of functions

Rather than the ε-δ definition, we can exploit our experience with sequences to define "$f(x) \to L$ as $x \to x_0$".

Definition (Limit of a function via sequences)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for every sequence $\{e_n\}$ of points in $E \setminus \{x_0\}$,

$$\lim_{n \to \infty} e_n = x_0 \quad \Longrightarrow \quad \lim_{n \to \infty} f(e_n) = L\,.$$

Limits of functions

Lemma (Equivalence of limit definitions)

The ε-δ definition of limits and the sequence definition of limits are equivalent.

(solution on board)

Note: The definition of a limit via sequences is sometimes easier to use than the ε-δ definition.

Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (ε-$\delta \implies$ seq).

Suppose the ε-δ definition holds and $\{e_n\}$ is a sequence in $E \setminus \{x_0\}$ that converges to x_0. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$. But since $e_n \to x_0$, given $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|e_n - x_0| < \delta$. This means that if $n \geq N$ then $x = e_n$ satisfies $0 < |x - x_0| < \delta$, implying that we can put $x = e_n$ in the statement $|f(x) - L| < \varepsilon$. Hence, for all $n \geq N$, $|f(e_n) - L| < \varepsilon$. Thus,

$$e_n \to x_0 \implies f(e_n) \to L\,,$$

as required. $\square$

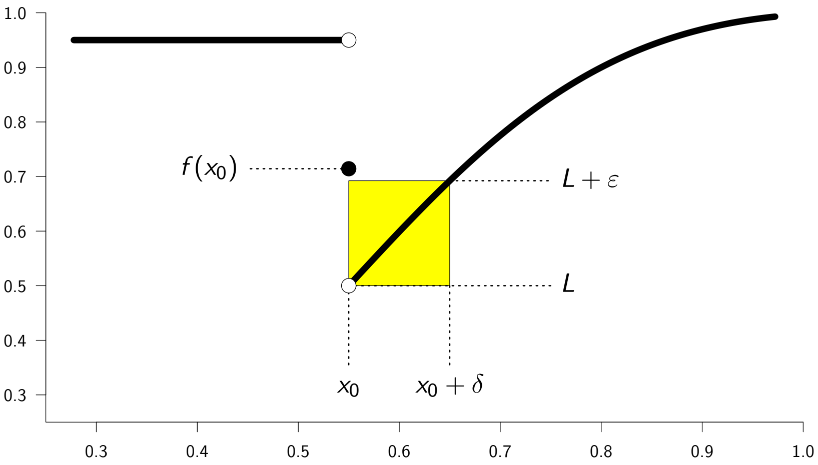
Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (seq $\implies$ ε-δ) via contrapositive.

Suppose that as $x \to x_0$, $f(x) \not\to L$ according to the ε-δ definition. We must show that $f(x) \not\to L$ according to the sequence definition.

Since the ε-δ criterion does not hold, $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there is some $x_\delta \in E$ for which $0 < |x_\delta - x_0| < \delta$ and yet $|f(x_\delta) - L| \geq \varepsilon$. This is true, in particular, for $\delta = 1/n$, where n is any natural number. Thus, $\exists \varepsilon > 0$ such that: $\forall n \in \mathbb{N}$, there exists $x_n \in E$ such that $0 < |x_n - x_0| < 1/n$ and yet $|f(x_n) - L| \geq \varepsilon$. This demonstrates that there is a sequence $\{x_n\}$ in $E \setminus \{x_0\}$ for which $x_n \to x_0$ and yet $f(x_n) \not\to L$. Hence, $f(x) \not\to L$ as $x \to x_0$ according to the sequence criterion, as required. $\square$

One-sided limits



One-sided limits

Definition (Right-Hand Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

One-sided limits

One-sided limits can also be expressed in terms of sequence convergence.

Definition (Right-Hand Limit – sequence version)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every decreasing sequence $\{e_n\}$ of points of E with $e_n > x_0$ and $e_n \to x_0$ as $n \to \infty$,

$$\lim_{n \to \infty} f(e_n) = L\,.$$

Infinite limits

Definition (Right-Hand Infinite Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = \infty$$

if for every $M > 0$ there is a $\delta > 0$ such that $f(x) \geq M$ whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

Properties of limits

There are theorems for limits of functions of a real variable that correspond (and have similar proofs) to the various results we proved for limits of sequences:

- Uniqueness of limits
- Algebra of limits
- Order properties of limits
- Limits of absolute values
- Limits of Max/Min

See Chapter 5 of textbook for details.