McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 31
Sequences of Functions
Wednesday 27 March 2019

Limits of Functions

We know from calculus that it can be useful to represent functions as limits of other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1+\frac{x}{1!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}, \qquad \cdots$$

Our goal is to give meaning to the phrase "*limit of functions*", and discuss how functions behave under limits.

Pointwise Convergence

- There are multiple inequivalent ways to define the limit of a sequence of functions.
- $\therefore$ There are multiple different notions of what it means for a sequence of functions to converge.
- Some convergence notions are better behaved than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges pointwise on D to f** if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

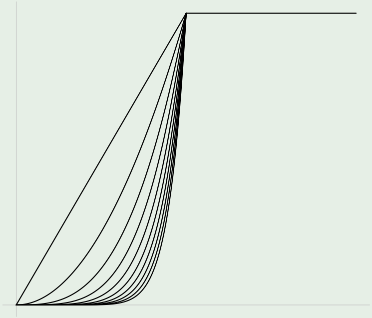
Unfortunately, *pointwise convergence does not preserve many useful properties of functions*.

Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



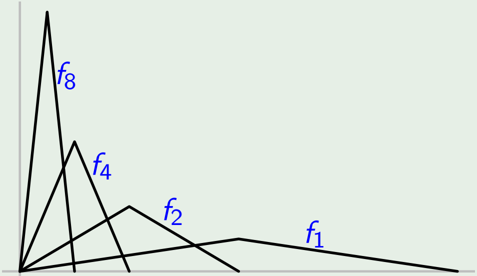
- Limit of sequence (of continuous functions) is not continuous.
- By smoothing the corner at $x = 1$, we get a sequence of differentiable functions that converge to a function that is not even continuous.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n}. \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges uniformly on D to f** if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, <u>for all $x \in D$</u>,

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

$$\text{uniform convergence} \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad \text{pointwise convergence}$$

Uniform Convergence

The following theorems illustrate the sense in which uniform convergence is better behaved than pointwise convergence in relation to common constructions in analysis.

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \,.$$

(Textbook (TBB) §9.5.2, p. 571ff)

The proof that f is integrable is rather involved. We will skip it.

Uniform Convergence

Proof that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$ given that f is integrable.

Given that f is integrable, to prove the equality, we will show that

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \left| \int_a^b f - \int_a^b f_n \right| < \varepsilon \qquad \forall n \geq N.$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \left| \int_a^b f - \int_a^b f_n \right| &= \left| \int_a^b (f - f_n) \right| \leq \int_a^b |f - f_n| \qquad \text{“triangle inequality”} \\ &\leq U\big(|f - f_n|\,,\,\{a,b\}\big) = \Big(\sup_{x\in[a,b]} \big|f(x) - f_n(x)\big| \Big)(b-a)\,. \end{aligned}$$

But f_n converges uniformly to f, which means that

$$\exists N \in \mathbb{N} \quad \text{such that} \quad \sup_{x\in[a,b]} |f(x) - f_n(x)| < \frac{\varepsilon}{b-a} \qquad \forall n \geq N.$$

For such n, we have $\left|\int_a^b f - \int_a^b f_n\right| < \varepsilon$, as required. $\square$

Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.

Proof.

Fix $x \in [a, b]$ and $\varepsilon > 0$. We must show $\exists \delta > 0$ such that if $y \in [a, b]$ and $|y - x| < \delta$ then $|f(y) - f(x)| < \varepsilon$.

Since the f_n uniformly converge to f, there is some $N \in \mathbb{N}$ so that $|f_N(y) - f(y)| < \frac{\varepsilon}{3}$ for all $y \in [a, b]$. Fix such an N.

Since f_N is continuous, there is some $\delta > 0$ so that if $y \in [a, b]$ satisfies $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$. For such y, we then have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\ &\le |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

as required. □

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Mathematics 3A03
Real Analysis I

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Lecture 32
Sequences of Functions II
Friday 29 March 2019

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey2/index.php/893454

Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.

–Dr. Marek Smieja (Infectious Diseases)

Last time...

Convergence of sequences of functions:

- Pointwise convergence
- Uniform convergence
- Theorem about integrability and uniform convergence
- Theorem about continuity and uniform convergence

Test 2 on Monday (1 April 2019), 7:00pm, MDCL 1110

- All material covered so far (not today's lecture).
- Emphasis on material since the first test,
  but the subject is cumulative.
- Let's look at the test.

Uniform Convergence

The interaction between uniform convergence and differentiability is more subtle.

Theorem (Differentiability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

1. *f_n' is continuous for each n,*
2. *the sequence $\{f_n'\}$ converges uniformly on $[a, b]$,*
3. *the sequence $\{f_n\}$ converges pointwise to a function f.*

Then f is differentiable and $\{f_n'\}$ converges uniformly to f'.

(Textbook (TBB) §9.6, p. 578ff)

Note: If we weaken the first condition to f_n' being integrable, but explicitly require in the second condition that the uniform limit is continuous, then the theorem is still true and no more difficult to prove.

Series of Real Numbers

Suppose $\{x_n\}$ is a sequence of real numbers. Recall that the **sequence of partial sums** is the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} x_n \,.$$

If the sequence of partial sums converges, then we write the limit as

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_n = \lim_{n\to\infty} s_n \,.$$

In this case, we call $\sum_{k=1}^{\infty} x_k$ a **convergent series**. A **divergent series** is a sequence of partial sums that diverges; we sometimes abuse notation and write $\sum_{k=1}^{\infty} x_k$ for divergent series as well.

A **series** is either a convergent series or a divergent series.

Our goal now is to extend this to sequences of <u>functions</u>.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a set $D \subseteq \mathbb{R}$. The **sequence of partial sums** is the sequence $\{S_n\}$ where S_n is the function defined on D by

$$S_n(x) = \sum_{k=1}^{n} f_k(x) \, .$$

When talking about limits of the S_n, we will write $\sum_{k=1}^{\infty} f_k$ and refer to this as a **series**.

Keep in mind that this is very informal, since the terminology does not specify any sense in which the S_n converge, nor does it assume that the S_n converge at all!

We will now make this more formal.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a domain D, and $\{S_n\}$ is its sequence of partial sums.

Definition (Convergence of Series)

If the sequence of partial sums $\{S_n\}$ converges pointwise on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ **converges pointwise on D to f**.

If the $\{S_n\}$ converge uniformly on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ **converges uniformly on D to f**.

In both cases, we will write $f = \sum_{k=1}^{\infty} f_k$ to denote that the **series converges to f**.

Series of Functions

The theorems on convergence of <u>sequences</u> of integrable, continuous and differentiable functions have several immediate implications for series of functions.

In the following, we assume that $\{f_n\}$ is a sequence of functions defined on an interval $[a, b]$.

Corollary (Integrals of Series)

Suppose the f_n are integrable and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is integrable and

$$\int_a^b f = \sum_{k=1}^{\infty} \int_a^b f_k \,.$$

Series of Functions

Corollary (Continuity of Series)

Suppose the f_n are continuous and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is continuous.

Corollary (Differentiability of Series)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a,b]$ such that

- *f_n' is continuous for each n,*
- *the series $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a,b]$,*
- *the series $\sum_{k=1}^{\infty} f_k$ converges pointwise to a function f.*

Then f is differentiable and $f' = \sum_{k=1}^{\infty} f_k'$.

Proving Uniform Convergence

We have just seen that several useful conclusions can be drawn when a series converges uniformly. The following gives a practical way of proving uniform convergence.

Theorem (Weierstrass M-test)

Let $\{f_n\}$ be a sequence of functions defined on $D \subseteq \mathbb{R}$, and suppose $\{M_n\}$ is a sequence of real numbers such that

$$|f_n(x)| \leq M_n, \qquad \forall x \in D,\ \forall n \in \mathbb{N}.$$

If $\sum_n M_n$ converges, then $\sum_{k=1}^{\infty} f_k$ converges uniformly.

Proving Uniform Convergence

Approach to proving the Weierstrass M-test:

- Let $S_n = \sum_{k=1}^{n} f_k$ be the nth partial sum.
- Show that for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that

$$\sup_{x \in D} |S_n(x) - S_m(x)| < \varepsilon\,, \qquad \forall n, m \geq N.$$

  This condition is called the **uniform Cauchy criterion**.
- Prove that the uniform Cauchy criterion implies uniform convergence.
  - This part is an excellent exercise for you.

    Note: The proof is similar to the proof of the Cauchy criterion for real numbers (in Lecture 12).

Proving Uniform Convergence

Proof of the Weierstrass M-test.

Let $\varepsilon > 0$. Suppose the series $\sum M_n$ converges. By the Cauchy criterion for real numbers, there is some integer N so that

$$\left| \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k \right| < \varepsilon\,, \qquad \forall n, m \geq N.$$

Without loss of generality, we can assume $m < n$, so the above can be written

$$M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon.$$

Note that we have $S_n - S_m = f_{m+1} + f_{m+2} + \cdots + f_n$, so the assumption that $|f_k| \leq M_k$ gives, for all $x \in D$,

$$|S_n(x) - S_m(x)| \leq M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon. \qquad \square$$

Proving Uniform Convergence

Example

Let $p > 1$, and consider the series $$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}.$$

This satisfies $$\left|\frac{\sin(kx)}{k^p}\right| \le \frac{1}{k^p} \quad \text{for all } x \in \mathbb{R}.$$

Since the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, it follows from the Weierstrass M-test that the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ converges uniformly.

Hence it is a continuous function.

In fact, if $p > 2$ then the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ is differentiable:

Let $f_k(x) = \frac{\sin(kx)}{k^p}$. The f_k' are continuous and another application of the Weierstrass M-test shows that $\sum_{k=1}^{\infty} f_k'$ converges uniformly. Hence the series is differentiable and the derivative is $\sum_{k=1}^{\infty} f_k'$.