19 Continuity

Continuous Functions

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

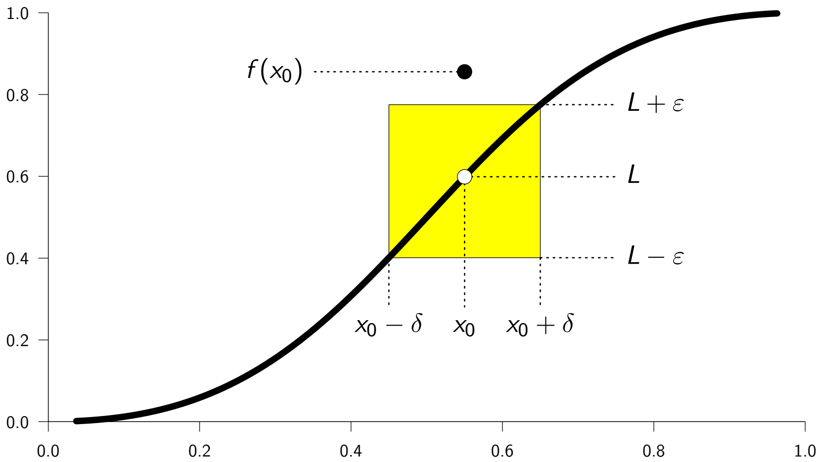
Instructor: David Earn

Lecture 18
Continuity
Monday 25 February 2019

Announcements

- A preliminary version of Assignment 4 has been posted on the course web site. More problems will be added soon.
  Due Friday 8 March 2019 at 1:25pm via crowdmark.
  BUT you should do it before Test #1.
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Limits of functions



Limits of functions

Definition (Limit of a function on an interval (a, b))

Let $a < x_0 < b$ and $f : (a, b) \to \mathbb{R}$. Then f is said to **approach the limit L as x approaches x_0**, often written "$f(x) \to L$ as $x \to x_0$" or

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \ni \ 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

Limits of functions

The function f need not be defined on an entire interval.
It is enough for f to be defined on a set with at least one accumulation point.

Definition (Limit of a function with domain $E \subseteq \mathbb{R}$)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then f is said to **approach the limit L as x approaches x_0**, *i.e.*,

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \ni \; \big(x \in E \;\wedge\; 0 < |x - x_0| < \delta\big) \implies |f(x) - L| < \varepsilon.$$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x\to 3}(2x+1) = 7\,.$$

(solution on board)

Proof that $2x+1 \to 7$ as $x \to 3$.

We must show that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $0 < |x-3| < \delta \implies |(2x+1)-7| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$|(2x+1)-7| < \varepsilon \iff |2x-6| < \varepsilon \iff 2\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{2}$$

Therefore, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Then $|x-3| < \delta \implies |(2x+1)-7| = |2x-6| = 2\,|x-3| < 2\frac{\varepsilon}{2} = \varepsilon$, as required. $\square$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 2} x^2 = 4\,.$$

(solution on board)

(and on next slide)

Limits of functions

Proof that $x^2 \to 4$ as $x \to 2$.

We must show that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $0 < |x-2| < \delta \implies |x^2 - 4| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$\left|x^2 - 4\right| < \varepsilon \iff |(x-2)(x+2)| < \varepsilon \iff |x-2|\,|x+2| < \varepsilon.$$

We can make $|x-2|$ as small as we like by choosing δ sufficiently small. Moreover, if x is close to 2 then $x+2$ will be close to 4, so we should be able to ensure that $|x+2| < 5$. To see how, note that

$$\begin{aligned} |x+2| < 5 \iff -5 < x+2 < 5 &\iff -9 < x-2 < 1 \\ &\Longleftarrow -1 < x-2 < 1 \iff |x-2| < 1\,. \end{aligned}$$

Therefore, given $\varepsilon > 0$, let $\delta = \min(1, \frac{\varepsilon}{5})$. Then $\left|x^2 - 4\right| = |(x-2)(x+2)| = |x-2|\,|x+2| < \frac{\varepsilon}{5}5 = \varepsilon$. $\square$