Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$ I
Monday 4 February 2019

Announcements

- Assignment 3 was posted on Saturday.
  Due Friday 15 Feb 2019 at 1:25pm.
  IMPORTANT CHANGE:
  - For the remainder of the term, assignments must be submitted **electronically**, not as a hardcopy.
  - You should have received a link for Assignment 3 via e-mail from crowdmark. If you have not received such an e-mail, please e-mail earn@math.mcmaster.ca.
  - If you write your solutions by hand, you will need to scan or photograph them to submit them via the online system.
  - If you use LaTeX to create a pdf file, you will need to separate your solutions for each question.
  - Marked assignments will be available online, rather than being returned in tutorial.
- Today: "How big is $\mathbb{R}$?" (see last few slides for Lecture 12) and intro to "Topology of $\mathbb{R}$"

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

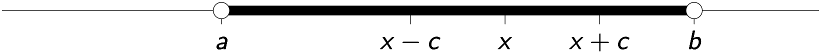
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



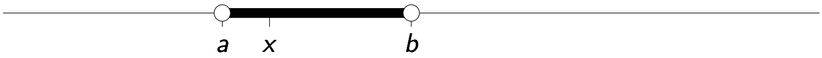
Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an **interior point** of E if x lies in an open interval that is contained in E, *i.e.*, $\exists c > 0$ such that $(x-c, x+c) \subset E$.

Interior point examples

Set E	**Interior points?**
$(-1,1)$	Every point
$[0,1]$	Every point *except the endpoints*
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	Every point
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	Every point *except 1*
$(-1,1)\setminus\{\frac{1}{2}\}$	Every point

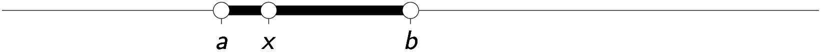
Neighbourhood



Definition (Neighbourhood)

A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

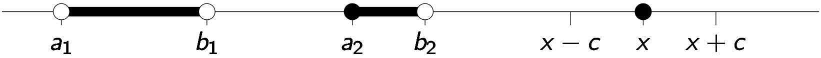


Definition (Deleted neighbourhood)

A **deleted neighbourhood** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

$$(a, b) \setminus \{x\}$$

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

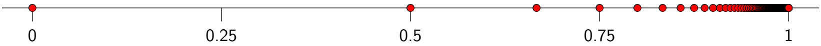
Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an **isolated point** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \cap E = \{x\}$.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	$\nexists$
$[0,1]$	$\nexists$
$\mathbb{N}$	Every point
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	$\nexists$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\nexists$

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point)

If $E \subseteq \mathbb{R}$ then x is an **accumulation point** or **limit point** of E if every neighbourhood of x contains infinitely many points of E,

$$\textit{i.e.,} \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing \,.$$

Notes:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Accumulation point examples

Set E	Accumulation points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$ II
Friday 8 February 2019

Announcements

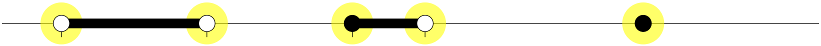
- Assignment 3 was posted on Saturday.
  Due Friday 15 Feb 2019 at 1:25pm
  via crowdmark

- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	$[-1,1]$
$[0,1]$	$[0,1]$
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$[-1,1]$
$(-1,1)\setminus\{\frac{1}{2}\}$	$[-1,1]$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\{1\}$

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a **boundary point** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing \\ \wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing \,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the **boundary** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	Boundary points?
$(-1,1)$	$\{-1,1\}$
$[0,1]$	$\{0,1\}$
$\mathbb{N}$	$\mathbb{N}$
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}\cup\{1\}$

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is **closed** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then $\overline{E} = E \cup E'$ is the **closure** of E.

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is **open** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the **interior** of E, denoted E° or E°, is the set of all interior points of E.

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$	NO	YES	$[-1,1]$	E	$\{-1,1\}$
$[0,1]$	YES	NO	E	$(0,1)$	$\{0,1\}$
$\mathbb{N}$	YES	NO	$\mathbb{N}$	$\varnothing$	$\mathbb{N}$
$\mathbb{R}$	YES	YES	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$
$\varnothing$	YES	YES	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbb{Q}$	NO	NO	$\mathbb{R}$	$\varnothing$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	NO	NO	$[-1,1]$	$(-1,1)$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	NO	YES	$[-1,1]$	E	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	NO	NO	$E\cup\{1\}$	$\varnothing$	$E\cup\{1\}$

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of $\mathbb{R}$ III
Monday 11 February 2019

Announcements

- Assignment 3 is **Due Friday 15 Feb 2019 at 1:25pm via crowdmark**
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Concepts covered recently

- Countable set
- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots ,$$

$$\textit{i.e.,} \qquad G = \bigcup_{n=1}^{\infty} (a_n, b_n) .$$

The open intervals (a_n, b_n) are said to be the **component intervals** *of G.*

(Textbook (TBB) Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x$ is an interior point of $G \implies$
  - some neighbourhood of x is contained in G, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subseteq G$
  - $\exists$ a largest neighbourhood of x that is contained in G: this "**component of G**" is $I_x = (\alpha, \beta)$, where

$$\alpha = \inf\{a : (a, x] \subset G\}, \qquad \beta = \sup\{b : [x, b) \subset G\}$$

  - I_x contains a rational number, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$
- $\therefore$ We can index all the intervals I_x by rational numbers
- $\therefore$ There are are most countably many intervals that make up G (*i.e.*, G is the union of a sequence of intervals)
- We can choose a disjoint subsequence of these intervals whose union is all of G (see proof in textbook for details).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the **complement** of E is the set

$$E^{c} = \{x \in \mathbb{R} : x \notin E\}.$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^{c} is closed.

(Textbook (TBB) Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(Textbook (TBB) Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(Textbook (TBB) Theorem 4.18)

Local vs. Global properties

Definition (Bounded function)

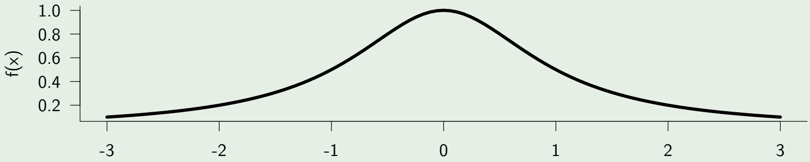
A real-valued function f is **bounded** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

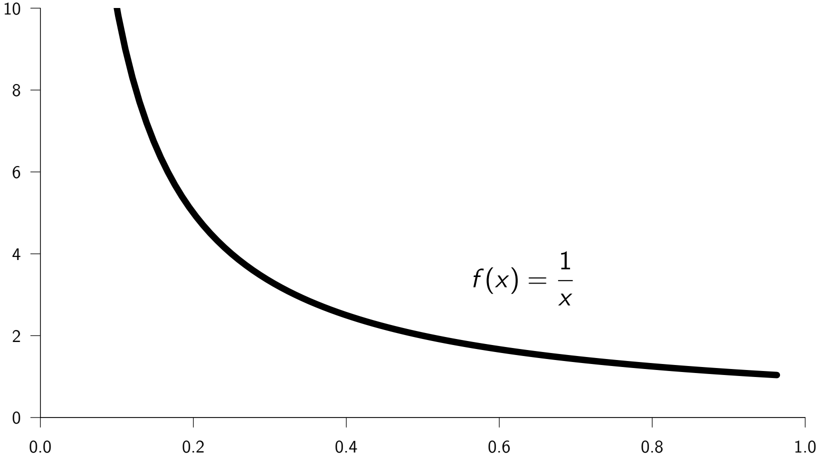
Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

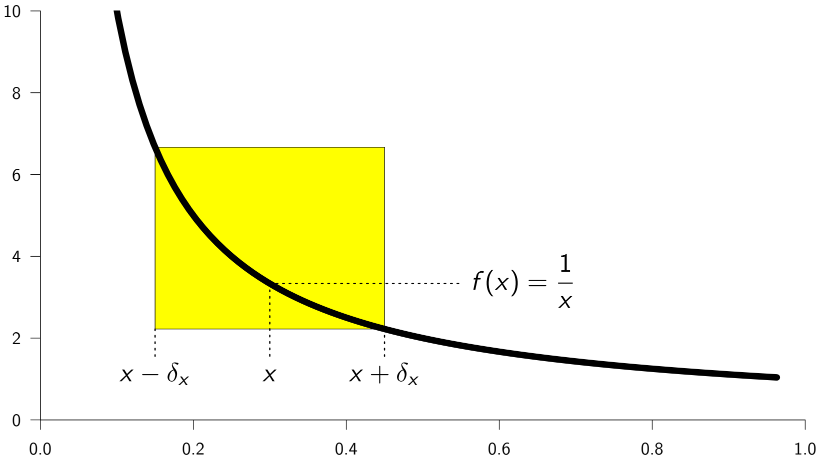


Local vs. Global properties



$f(x) = 1/x$ is <u>not</u> bounded on the interval $E = (0, 1)$.

Local vs. Global properties



$f(x) = 1/x$ is **locally bounded** on the interval $E = (0, 1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \leq M_x \ \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is **locally bounded** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is **locally bounded** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is not locally bounded on $(0, 1)$.

(solution on board)

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

(solution on board)

Extra Challenge Problem: Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?