13 Topology of $\mathbb{R}$ I

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$ I
Monday 4 February 2019

Announcements

- Assignment 3 was posted on Saturday.
  Due Friday 15 Feb 2019 at 1:25pm.
  IMPORTANT CHANGE:
  - For the remainder of the term, assignments must be submitted **electronically**, not as a hardcopy.
  - You should have received a link for Assignment 3 via e-mail from crowdmark. If you have not received such an e-mail, please e-mail earn@math.mcmaster.ca.
  - If you write your solutions by hand, you will need to scan or photograph them to submit them via the online system.
  - If you use LaTeX to create a pdf file, you will need to separate your solutions for each question.
  - Marked assignments will be available online, rather than being returned in tutorial.
- Today: "How big is $\mathbb{R}$?" (see last few slides for Lecture 12) and intro to "Topology of $\mathbb{R}$"

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

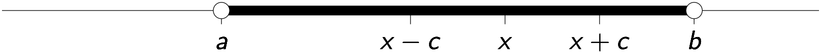
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



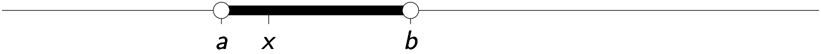
Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an **interior point** of E if x lies in an open interval that is contained in E, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subset E$.

Interior point examples

Set E	**Interior points?**
$(-1,1)$	Every point
$[0,1]$	Every point *except the endpoints*
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	Every point
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	Every point *except 1*
$(-1,1)\setminus\{\frac{1}{2}\}$	Every point

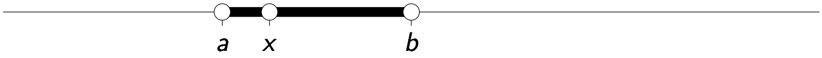
Neighbourhood



Definition (Neighbourhood)

A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

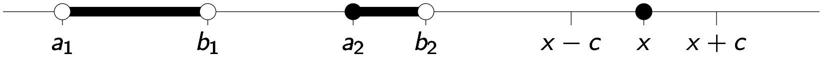


Definition (Deleted neighbourhood)

A **deleted neighbourhood** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

$$(a, b) \setminus \{x\}$$

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

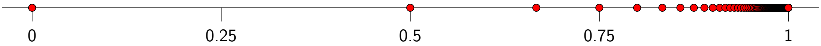
Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an **isolated point** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \cap E = \{x\}$.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	$\nexists$
$[0,1]$	$\nexists$
$\mathbb{N}$	Every point
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	$\nexists$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\nexists$

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point)

If $E \subseteq \mathbb{R}$ then x is an **accumulation point** or **limit point** of E if every neighbourhood of x contains infinitely many points of E,

$$\textit{i.e.,} \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing\,.$$

Notes:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1) \cup [0,1]$	
$(-1,1) \setminus \{\frac{1}{2}\}$	
$\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$	