Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 3

This assignment is **due** on **Tuesday 22 October 2019 at 2:25pm**.
PLEASE NOTE that you must **submit online** via crowdmark.
You will receive an e-mail from crowdmark with the required link.
Do **NOT** submit a hardcopy of this assignment.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

THIS IS A DRAFT VERSION OF THE ASSIGNMENT. THE FINAL VERSION OF THE ASSIGNMENT WILL BE POSTED AS SOON AS IT IS READY. – DE

1. Consider the sequence $\{a_n\}$ defined by

$$a_1 = 0.1,\ a_2 = 0.12,\ a_3 = 0.123,\ \ldots,\ a_{12} = 0.123456789101112,\ \ldots$$

Prove that $\{a_n\}$ converges.

2. Suppose $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences and let $c_n = |a_n - b_n|$ for all n. Prove that $\{c_n\}$ is Cauchy.

3. Suppose $\{a_n\}$ is a sequence of real numbers. The following statement looks similar to the Cauchy criterion:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_{n+1} - a_n| < \varepsilon.$$

Prove that there is a sequence $\{a_n\}$ that satisfies this criterion and yet is not Cauchy.

4. Give examples of functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
   (a) f is one-to-one but not onto;
   (b) f is onto but not one-to-one;
   (c) f is a bijection that is not the identity.

5. Prove or disprove: There exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that
   (a) f is one-to-one but not onto, g is onto but not one-to-one, and $f \circ g$ is a bijection;
   (b) f is onto but not one-to-one, g is one-to-one but not onto, and $f \circ g$ is a bijection.

6. Let U be an uncountable subset of $\mathbb{R}$, and let $U_n = U \cap [-n, n]$ for each $n \in \mathbb{N}$.
   (a) Prove that for some $k \in \mathbb{N}$, U_k is uncountable.
   (b) Prove that there is a convergent sequence $\{a_n\}$ such that $a_n \in U$ for all n and $a_n \neq a_m$ whenever $n \neq m$.