Math 3A03 - Tutorial 5 Questions - Winter 2019

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Problem 1. *Which of the following sets are countable:*

(a) $\mathbb{R} \setminus \mathbb{Q}$.

(b) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(c) $A \times B = \{(a, b) : a \in A, b \in B\}$, *where* A *and* B *are countable.*

(d) $\mathbb{Q} \cup \left(\sqrt{2}\mathbb{Q}\right)$, *where* $\sqrt{2}\mathbb{Q} = \left\{\sqrt{2}q : q \in \mathbb{Q}\right\}$.

Solution. (a) is not countable, the remaining sets are countable.

Problem 2. *Let* A, B *be countable sets, and* $\mathcal{C} = \{a + bi : a \in A, b \in B\}$, *where* $i^2 = -1$, *prove that* $\mathcal{C}$ *is countable.*

Solution. Since A, B are countable, then we have two functions $f_A(x) : A \to \mathbb{N}$, $f_B(x) : B \to \mathbb{N}$, which are both bijective. Suppose we have an element $a + bi \in \mathcal{C}$, define $f(a + bi) = 2^{f_A(a)}3^{f_B(b)}$, which is a function mapping $\mathcal{C}$ to $\mathbb{N}$. We'd like to show this function is injective and surjective.

First let's show injectivity, suppose that $a+bi, c+di \in \mathcal{C}$, and $f(a+bi) = f(c+di) \implies 2^{f_A(a)}3^{f_B(b)} = 2^{f_A(c)}3^{f_B(d)}$. Since $\gcd(2, 3) = 1$ we have that $f_A(a) = f_A(c)$ and $f_B(b) = f_B(d)$, further the functions f_A and f_B are bijective so they are also injective, and so $a = c$ and $b = d$, which shows injectivity for f.

Surjectivity will be a tad tricky, because f *isn't* surjective on $\mathbb{N}$, it is however surjective on a subset of $\mathbb{N}$ which we will define in a minute. Just recall that (as you will prove in your assignment), a subset of a countable set is also countable, so having a bijection to a subset of the natural numbers is enough. How do we define this subset? Recall a surjection roughly means that you map to the entirety of set, so we'd like to use $\mathcal{I} = f(\mathcal{C}) = \{f(a+bi) : a \in A, b \in B\}$, i.e. the image of the set $\mathcal{C}$ under the function f.

The function is still injective. For surjectivity, suppose that we have a $z \in \mathcal{I}$, we need to find an a, b such that $z = f(a+bi)$, but by definition each element of the set is $f(a+bi)$ for some $a \in A$ and $b \in B$. Hence f is injective and surjective as a map $f : \mathcal{C} \to \mathcal{I} \subset \mathbb{N}$, so it is a bijective map to $\mathcal{I}$. Since $\mathcal{I}$ is a subset of a countable set, by a problem in assignment 3 we have that $\mathcal{I}$ is countable. We can now conclude that $\mathcal{C}$ is countable, to be a bit more complete we can say that since $\mathcal{I}$ is countable there exists a bijective function $g : \mathcal{I} \to \mathbb{N}$, notice that $g \circ f : \mathcal{C} \to \mathbb{N}$ is a composition of bijective functions, which is also bijective, and hence there is a bijective function from $\mathcal{C}$ to $\mathbb{N}$, and hence $\mathcal{C}$ is countable.

Problem 3. *Show that every interior point is an accumulation point.*

Solution. Suppose that x is an interior point of some set $E \subseteq \mathbb{R}$, then there is a $c > 0$ such that $(x-c, x+c) \subseteq E$. Now suppose that $d > 0$ is an arbitrary positive number. We'd like to show that $(E \setminus \{x\}) \cap (x-d, x+d)$ is nonempty. Notice however that since $(x-c, x+c) \subseteq E$ then

$$(x-d, x+d) \cap (x-c, x+c) \subseteq (x-d, x+d) \cap E.$$

Further if we let $a = \min(c, d)$ then $(x-d, x+d) \cap (x-c, x+c) = (x-a, x+a)$, removing the point x we end up with the union of two intervals, $(x-a, x)$ and $(x, x+a)$, hence

$$(x-a, x) \cup (x, x+a) \subseteq (x-d, x+d) \cap (E \setminus x).$$

Since the intervals are nonempty then $(x-d, x+d) \cap (E \setminus x)$ contains at least one point.

Problem 4. *Write the open interval $(0, 2)$ as a union of closed sets. Can it be expressed as an intersection of closed sets?*

Solution. $(0,2) = \cup_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$. To justify this note that whenever $0 < x < 2$ there is a natural number n, sufficiently large, such that $x \geq \frac{1}{n}$ and $x \leq 2 - \frac{1}{n}$, and hence $x \in [\frac{1}{n}, 2 - \frac{1}{n}]$ for some n. To show that the boundary points aren't in the set, note that for every $n \in \mathbb{N}$ we have that $0 < \frac{1}{n}$, similarly $2 > 2 - \frac{1}{n}$.

This cannot be expressed as an intersection of closed sets as any intersection of closed sets is also closed (see class notes).

Problem 5. *(a) Find the closure, interior points, accumulation points, and boundary points of the set* $S = [0, \sqrt{5}] \cap \mathbb{Q}$. *Is this set open? Is it closed?*

(b) Show that $O = (0, 3)$ *is open.*

(c) Show that $C = [0, 3]$ *is closed.*

Solution. Next tutorial!