31 Sequences of Functions

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 31
Sequences of Functions
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Limits of Functions

We know from calculus that it can be useful to represent functions as limits of other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1+\frac{x}{1!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}, \qquad \cdots$$

Our goal is to give meaning to the phrase "*limit of functions*", and discuss how functions behave under limits.

Pointwise Convergence

- There are multiple inequivalent ways to define the limit of a sequence of functions.
- $\therefore$ There are multiple different notions of what it means for a sequence of functions to converge.
- Some convergence notions are better behaved than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges pointwise on D to f** if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

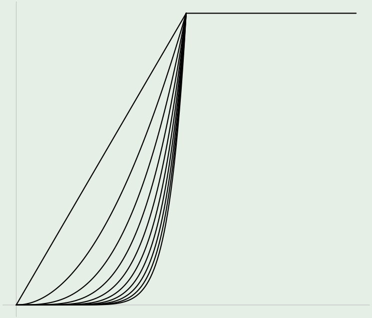
Unfortunately, *pointwise convergence does not preserve many useful properties of functions*.

Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



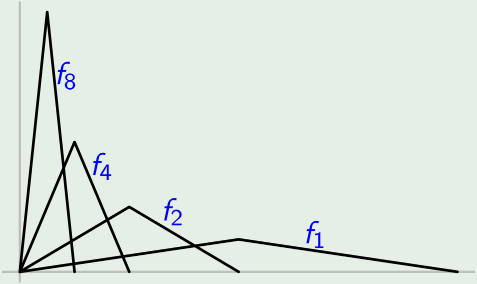
- Limit of sequence (of continuous functions) is not continuous.
- By smoothing the corner at $x = 1$, we get a sequence of differentiable functions that converge to a function that is not even continuous.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n}. \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges uniformly on D to f** if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for all $x \in D$,

$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

uniform convergence $\overset{\Longrightarrow}{\not\Longleftarrow}$ pointwise convergence

Uniform Convergence

The following theorems illustrate the sense in which uniform convergence is better behaved than pointwise convergence in relation to common constructions in analysis.

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \,.$$

(Textbook (TBB) §9.5.2, p. 571ff)

The proof that f is integrable is rather involved. We will skip it.

Uniform Convergence

Proof that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$ given that f is integrable.

Given that f is integrable, to prove the equality, we will show that

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \left| \int_a^b f - \int_a^b f_n \right| < \varepsilon \qquad \forall n \geq N.$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f - f_n) \right| &\leq \int_a^b |f - f_n| \qquad \text{"triangle inequality"} \\ &\leq U\big(|f - f_n|\,,\,\{a, b\}\big) = \Big(\sup_{x\in[a,b]} \big|f(x) - f_n(x)\big|\Big)(b - a)\,. \end{aligned}$$

But f_n converges uniformly to f, which means that

$$\exists N \in \mathbb{N} \quad \text{such that} \quad \sup_{x\in[a,b]} |f(x) - f_n(x)| < \frac{\varepsilon}{b - a} \qquad \forall n \geq N.$$

For such n, we have $\left|\int_a^b f - \int_a^b f_n\right| < \varepsilon$, as required. $\square$

Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.

Proof.

Fix $x \in [a, b]$ and $\varepsilon > 0$. We must show $\exists \delta > 0$ such that if $y \in [a, b]$ and $|y - x| < \delta$ then $|f(y) - f(x)| < \varepsilon$.

Since the f_n uniformly converge to f, there is some $N \in \mathbb{N}$ so that $|f_N(y) - f(y)| < \frac{\varepsilon}{3}$ for all $y \in [a, b]$. Fix such an N.

Since f_N is continuous, there is some $\delta > 0$ so that if $y \in [a, b]$ satisfies $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$. For such y, we then have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

as required. □