Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 1
Introduction
Tuesday 3 September 2019

Where to find course information

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- The course web site: http://ms.mcmaster.ca/earn/3A03

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- Let's have a look now. . .

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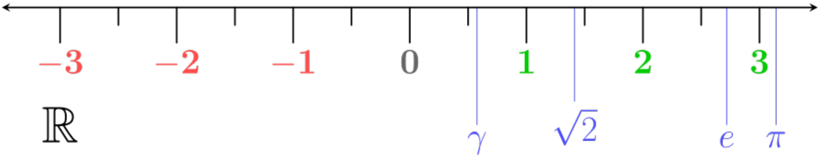

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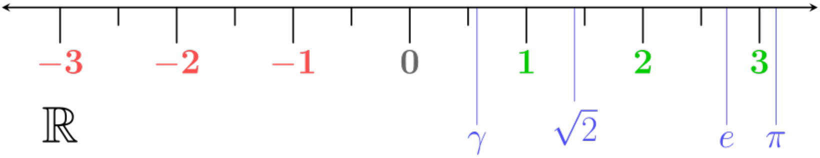

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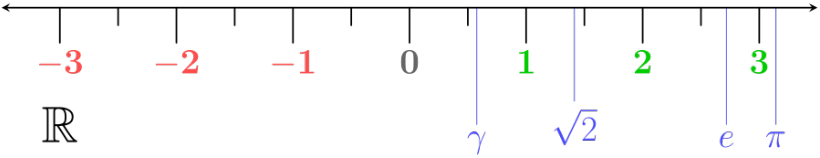

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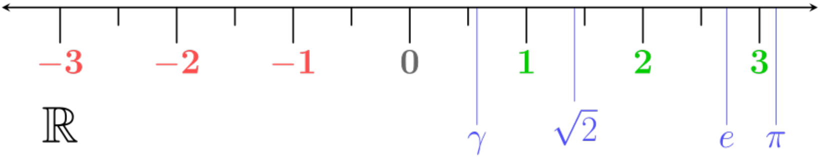

- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
  Approach: “Proof by contradiction.”

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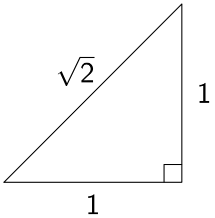

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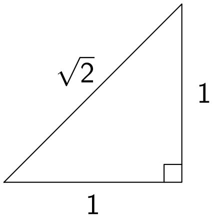

- So irrational numbers are "real".

Poll on rationality

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- Let's Deactivate the poll and View Results

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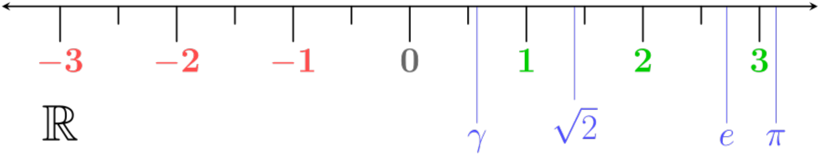

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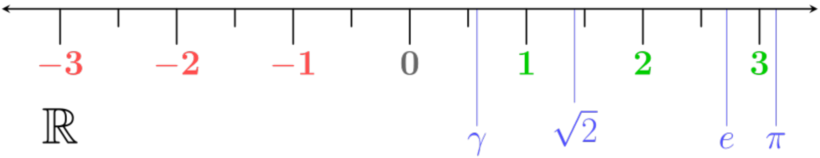

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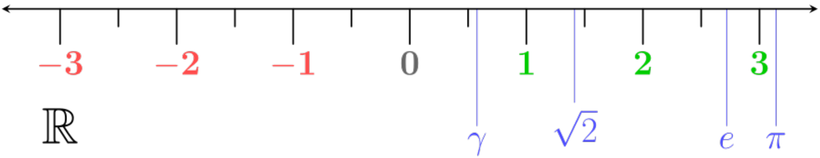

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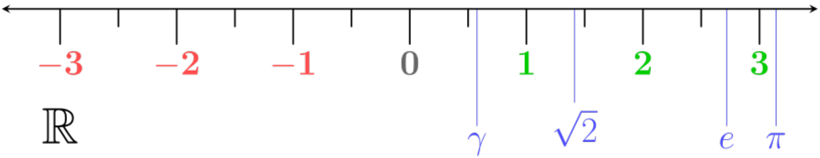

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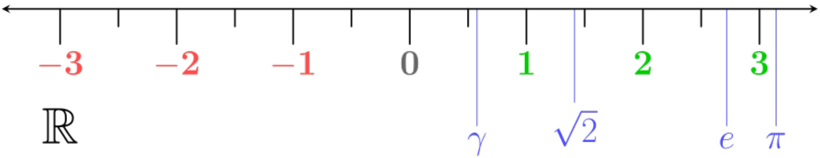

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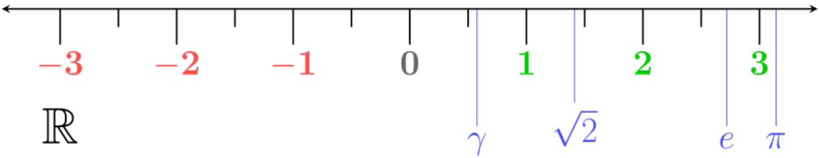

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- Do we need to *construct* integers first?
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- But <u>what</u> exactly are we supposed to *construct* numbers <u>from</u>?

Informal introduction to construction of numbers ($\mathbb{N}$)

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- Thus, n is defined to be a set containing n elements.

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- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Properties of $\mathbb{R}$
Thursday 5 September 2019

Where to find course information

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- The course web site: http://ms.mcmaster.ca/earn/3A03

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- **Assignment 1:** You should have received an e-mail from crowdmark. If not, please e-mail earn@math.mcmaster.ca ASAP stating your full name, student number, and when you registered in the course.

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Poll

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- Submit.

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The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

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Addition and multiplication together:

AM1 *Distributive law.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y)z = xz + yz$$

is true.

The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

Note: M3 ensures $0 \neq 1$ to exclude the uninteresting case of a field with only one element.

Standard Mathematical Shorthand

Standard Mathematical Shorthand

Quantifiers

Standard Mathematical Shorthand

Quantifiers

$\forall$

Standard Mathematical Shorthand

Quantifiers

$\forall$ for all

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
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Logical operands

$\wedge$

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and
$\vee$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and
$\vee$	logical or

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	

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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
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Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not
$\veebar$	

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$\forall$	for all
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Logical operands

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Note: $A \veebar B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

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Other shorthand

Standard Mathematical Shorthand

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Other shorthand

$\therefore$

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Other shorthand

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Other shorthand

$\therefore$	therefore
$\ni$	

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
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Logical operands

$\wedge$	logical and
$\vee$	logical or
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Other shorthand

$\therefore$	therefore
$\ni$	such that

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Other shorthand

$\therefore$	therefore
$\ni$	such that
$\equiv$	

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Other shorthand

$\therefore$	therefore
$\ni$	such that
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Other shorthand

$\therefore$	therefore
$\ni$	such that
$\equiv$	equivalent
$\because$	because
$\Longleftrightarrow$	

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent		

Standard Mathematical Shorthand

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	

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Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	contradiction

The field axioms (in mathematical shorthand) for field $\mathbb{F}$

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Addition axioms

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A1 *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(x+y) \in \mathbb{F} \;\wedge\; (x+y) = (y+x)$.

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Distribution axiom

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Multiplication axioms

M *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(xy) \in \mathbb{F} \;\wedge\; (xy) = (yx)$.

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M *Identity.* $\exists!\; 1 \in \mathbb{F} \setminus \{0\} \ni \forall x \in \mathbb{F}$, $x1 = 1x = x$.

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Distribution axiom

AI *Distribution.* $\forall x, y, z \in \mathbb{F}$, $(x+y)z = xz + yz$.

The field axioms (in mathematical shorthand) for field $\mathbb{F}$

Addition axioms

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Distribution axiom

- **AM** *Distribution.* $\forall x, y, z \in \mathbb{F},\ (x+y)z = xz + yz$.

Any collection $\mathbb{F}$ of mathematical objects is called a *field* if it satisfies these 9 algebraic properties.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are Fields?**
- Submit.

The integers modulo 3 ($\mathbb{Z}_3$)

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Imagine a clock that repeats after 3 hours rather than 12 hours.

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$+$	0	1	2
0	0	1	2
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$\cdot$	0	1	2
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Examples of fields

Examples of fields

Set	Field?	Why?

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)		

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)		

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
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reals ($\mathbb{R}$)		

Examples of fields

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Examples of fields

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complexes ($\mathbb{C}$)		

Examples of fields

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Examples of fields

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rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses
reals ($\mathbb{R}$)	**YES**	
complexes ($\mathbb{C}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**YES**	$2^{-1} = 2$

Ordered fields

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A field $\mathbb{F}$ is said to be ***ordered*** if the following properties hold:

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$\forall x, y \in \mathbb{F},\ \big((x = y) \wedge \neg(x < y) \wedge \neg(y < x)\big) \veebar \big((x \neq y) \wedge [(x < y) \veebar (y < x)]\big)$

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O2 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $y < z$ is true, then $x < z$ is true

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are ORDERED Fields?**
- $\boxed{\texttt{Submit}}$.

Examples of ordered fields

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Field	Ordered?	Why?

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rationals ($\mathbb{Q}$)		

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complexes ($\mathbb{C}$)	**NO**	Extra Challenge Problem: *Prove the field $\mathbb{C}$ cannot be ordered.*

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Food for thought: Is it possible for any finite field be ordered?

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  It is related to *upper and lower bounds*...

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 3
Properties of $\mathbb{R}$ II
Friday 6 September 2019

Putnam Competition

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- The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. It is organized by the Mathematical Association of America and is taken by over 4,000 participants at more than 500 colleges and universities. More information can be found at

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  Usually a slightly redundant set of axioms is stated to emphasize all the key properties.

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- To prove that $\mathbb{C}$ is not an ordered field, it is not sufficient to prove that the standard order on $\mathbb{R}$ cannot be extended to $\mathbb{C}$. You must show that it is not possible to define *any* order on $\mathbb{C}$ that makes it an ordered field.

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  - These videos go at a slower pace than we do, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

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Bounds

Definition (Upper Bound)

Let $E \subseteq \mathbb{R}$. A number M is said to be an ***upper bound*** for E if $x \leq M$ for all $x \in E$.

A set that has an upper bound is said to be ***bounded above***.

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A set that is bounded above and below is said to be ***bounded***.

Maxima and Minima

Maxima and Minima

Definition (Maximum)

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Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 3: bounded sets**
- $\boxed{\texttt{Submit}}$.

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$					

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1, 1]$	**YES**				

Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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Bounds, maxima and minima

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$\varnothing$	**YES**	**YES**	**YES**	$\nexists$	$\nexists$

Least upper bounds

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What sets have least upper bounds?

Least upper bounds

Least upper bounds

Example

Least upper bounds

Example

Set	bounded above	sup

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$		

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1

Least upper bounds

Example

Set	bounded above	sup
$[-1,1]$	**YES**	1
$[-1,1)$		

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	

Least upper bounds

Example

Set	bounded above	sup
$[-1,1]$	**YES**	1
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Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
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- Are there any other complete ordered fields?
- Extra Challenge Problem:
  Prove that $\mathbb{R}$ is the <u>only</u> complete ordered field.

Greatest lower bounds

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If m is the greatest lower bound of E then we write $m = \inf E$.

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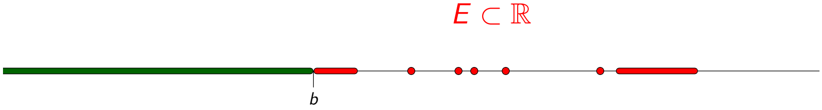

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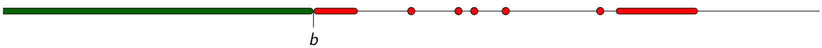
$E \subset \mathbb{R}$



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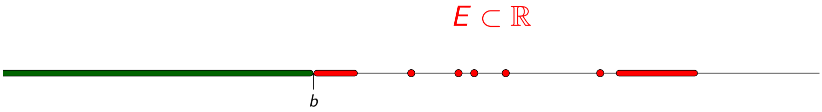

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Comment on least upper bounds and greatest lower bounds

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- The proof above shows that:

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- Similarly:

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Some notational abuse concerning sup and inf

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If you are asked "*What is the least upper bound of* $\mathbb{R}$*?*" how should you answer?

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Correct answer:

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If you are asked "*What is the least upper bound of* $\mathbb{R}$*?*" how should you answer?
Correct answer: "$\mathbb{R}$ is not bounded above so it <u>does not have</u> a least upper bound."

Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Archimedean property)

The set of natural numbers $\mathbb{N}$ has no upper bound.

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Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$.

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Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$.

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Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$,

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Theorem (Equivalences of the Archimedean property)

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3. *Given any $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

   i.e., Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in \mathbb{N}$).

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4. *Given any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

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4. *Given any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

   i.e., Given any positive number x, no matter how small, one can always find a fraction $1/n$ that is smaller than x.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 4
Properties of $\mathbb{R}$ III
Tuesday 10 September 2019

Comments arising...

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- Remember Assignment 1 is due Tuesday 17 Sep 2019 @ 2:25pm via crowdmark.

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- Last time we ended with some equivalent conditions relating $\mathbb{R}$ and $\mathbb{N}$.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 4: Theorem or Axiom?**
- $\boxed{\texttt{Submit}}$.

Consequences of the real number axioms (§§1.7–1.9)

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Theorem (Well-Ordering Property)

Every nonempty subset of $\mathbb{N}$ *has a smallest element.*

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Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$.

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Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below

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Suppose $b \notin S$.

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Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$

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Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound (in $\mathbb{R}$). Let $b = \inf S$. If $b \in S$ then $b = \min S$ and we are done.

Suppose $b \notin S$. Then $\exists n \in S$ such that $n < b + 1$ (otherwise $b + 1$ would be a lower bound for S that is greater than b) and, moreover, $n > b$ (since $b \notin S$). $\therefore$ $n \in S \cap (b, b+1)$. But just as $b + 1$ cannot be a lower bound for S, n cannot be a lower bound for S (since it too would be a lower bound greater than $b = \inf S$). $\therefore \exists m \in S \cap (b, n)$. But we now have $b < m < n < b + 1$, which is impossible because m and n are both integers. $\Rightarrow\Leftarrow$

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Theorem (Well-Ordering Property)

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Proof.

Let $S \subseteq \mathbb{N}$, $S \neq \varnothing$. Then S is a non-empty set of real numbers that is bounded below (for instance by 0), and hence has a greatest lower bound (in $\mathbb{R}$). Let $b = \inf S$. If $b \in S$ then $b = \min S$ and we are done.

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Consequences of the real number axioms (§§1.7–1.9)

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Corollary

Every nonempty subset of $\mathbb{Z}$ that is bounded below (in $\mathbb{R}$) has a smallest element.

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Every nonempty subset of $\mathbb{Z}$ that is bounded below (in $\mathbb{R}$) has a smallest element.

Proof.

The proof is identical to the proof of the well-ordering property for $\mathbb{N}$ except that we start with a set of integers that is bounded below, rather than having to first identify a lower bound for the set. □

Consequences of the real number axioms (§§1.7–1.9)

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Let $S \subseteq \mathbb{N}$. Suppose that $1 \in S$ and, for every $n \in \mathbb{N}$, if $n \in S$ then $n + 1 \in S$. Then $S = \mathbb{N}$.

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Proof.

Let $E = \mathbb{N} \setminus S$ and suppose $E \neq \varnothing$. Since $E \subset \mathbb{N}$ and $E \neq \varnothing$, the well-ordering property implies E has a smallest element, say m.

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Consequences of the real number axioms (§§1.7–1.9)

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Given $x \in \mathbb{R}$, consider the interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ for $n \in \mathbb{N}$.

The metric structure of $\mathbb{R}$ (§1.10)

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Definition (Absolute Value function)

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For any $x \in \mathbb{R}$,

$$|x| \stackrel{\textbf{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The metric structure of $\mathbb{R}$ (§1.10)

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4. $|x| - |y| \leq |x-y|$.

The metric structure of $\mathbb{R}$ (§1.10)

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Definition (Distance function or metric)

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3. $d(x,y) = d(y,x)$ — *distance is symmetric*
4. $d(x,y) \leq d(x,z) + d(z,y)$ — *the triangle inequality*

Note: Any function satisfying these properties can be considered a "distance" or "metric".

The metric structure of $\mathbb{R}$ (§1.10)

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Given $d(x, y) = |x - y|$, the properties of the distance function are equivalent to:

The metric structure of $\mathbb{R}$ (§1.10)

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Theorem (Metric properties of the absolute value function)

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□

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$$f(x) = \frac{|x|}{1+|x|},$$
and define
$$d(x,y) = f(x-y).$$
Prove that $d(x,y)$ can be interpreted as a distance between x and y because it satisfies all the properties of a metric.