Mathematics 3A03 Real Analysis I
2017 Final Exam Review Sheet

The following are suggested problems to focus your study for the Final Exam. To keep this list of problems at a reasonable length, we have elected to focus only on a subset of the topics we have covered in this course. In particular, the distribution of problems on this list does not necessarily reflect the distribution of problems appearing on the Final Exam. However, at least one of the problems below will appear on the Final Exam.

1. Prove that $\{\sin(n)\}$ has a convergent subsequence.

2. Prove that the interval $(2, 4)$ is open.

3. Let $f : A \to B$ and $g : B \to A$ be functions so that $(f \circ g)(b) = b$ for all $b \in B$.
   (a) Prove that f is surjective and g is injective.
   (b) Is f necessarily injective? Is g necessarily surjective?

4. Suppose that $F_1, F_2, \ldots$ are closed and non-empty sets so that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$. Show that if one of the F_n is bounded, then $\cap_{n=1}^{\infty} F_n$ is non-empty.

5. Suppose $f : [0, 1] \to [0, 1]$ is a continuous function. Show that there is some point $x \in [0, 1]$ so that $f(x) = x$. *Hint: Consider the function $g(x) = f(x) - x$ and show that g has a zero.*

6. Suppose $f : [0, 1] \to [0, \infty)$ is a function so that for every $M > 0$, the set $f^{-1}([M, \infty))$ is finite. Show that f is integrable on $[0, 1]$ and $\int_0^1 f = 0$.

7. Show that a sequence of functions $\{f_n\}$ converges uniformly on a domain $D \subseteq \mathbb{R}$ if and only if $\{f_n\}$ is uniformly Cauchy on D: $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$\forall n, m \geq N, \quad \sup_{x \in D} |f_n(x) - f_m(x)| < \epsilon.$$

8. Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $D \subseteq \mathbb{R}$ to a function f. Show that if each f_n is uniformly continuous, then f is uniformly continuous.

9. Show that the following functions are integrable on $[-1, 1]$.
   (a) $f(x) = x^2 \sin(1/x)$
   (b) $f(x) = \sum_{n=1}^{\infty} 3^{-n} e^{nx^2}$

10. Suppose $\{x_n\}$ is a sequence of real numbers and $x_n \geq 0$ for all n. Show that $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums is bounded.

11. Recall from class that we defined a real number to be a subset $\alpha \subseteq \mathbb{Q}$ satisfying the following four properties:

    1. $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$;
    2. $\alpha \neq \varnothing$
    3. $\alpha \neq \mathbb{Q}$;
    4. there is no greatest element in α: $\forall x \in \alpha$, $\exists y \in \alpha$ so that $y > x$.

    Given a real number α, define its ***negative*** to be the set

    $$-\alpha = \{x \in \mathbb{Q} : \exists a \in \mathbb{Q} \setminus \alpha \text{ such that } x < -a\}$$

    Show that $-\alpha$ is a real number (i.e., that it satisfies the four properties above).