# 6 Sequences

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

# Mathematics 3A03
# Real Analysis I

Instructor: David Earn

Lecture 6

Sequences

Friday 18 January 2019

# Announcements

- Solutions to Assignment 1 will be posted later today.
  **Study them!**
- Assignment 2: check the course web page over the weekend.
- Remember that solutions to assignments and tests from the 2016 and 2017 versions of the course are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)
- No late submission of assignments. No exceptions.
  However, best 5 of 6 assignments will be counted.
  *Always due 5 minutes before class on the due date.*
- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

# Announcements for week of 21–25 January 2019

- Office hour on Monday 21 Jan 2019 will be at **3:30pm** (rather than the usual 1:30pm).
- Wednesday's lecture will be given by Niky Hristov.

# Sequences

- A **sequence** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

# Formal definition of a sequence

## Definition (Sequence of Real Numbers)

A **sequence of real numbers** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

*A lot of different notation is common for sequences:*

$$f(1),\ f(2),\ f(3),\ \ldots \qquad \{f(n)\}_{n=1}^{\infty}$$

$$f_1,\ f_2,\ f_3,\ \ldots \qquad \{f(n)\}$$

$$\{f(n) : n = 1, 2, 3, \ldots\} \qquad \{f_n\}_{n=1}^{\infty}$$

$$\{f(n) : n \in \mathbb{N}\} \qquad \{f_n\}$$

# Specifying sequences

There are two main ways to specify a sequence:

**1. Direct formula.**

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

**Example (arithmetic progression with common difference $d$)**

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

# Specifying sequences

## 2. Recursive formula.

Specify first term and function $f(x)$ to **iterate**. □

*i.e.*, Given $x_1$ and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \ldots$$

### Example (arithmetic progression with common difference $d$)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \ldots$$

*Note:* $f$ is the most typical function name for both the direct and recursive specifications. The correct interpretation of $f$ should be clear from context.

# Specifying sequences

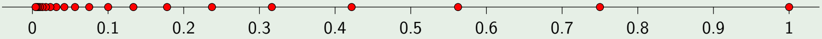
## Example (geometric progression with common ratio $r$)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

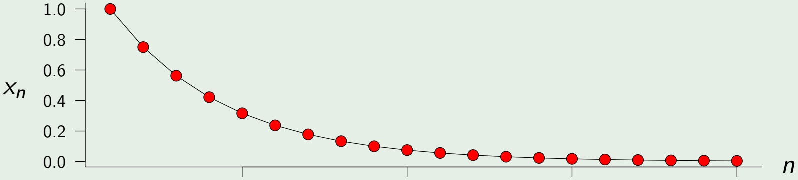
Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:




Graph of $f(n)$:



# Specifying sequences

## Example ($f(n) = 1 + \frac{1}{n^2}$)

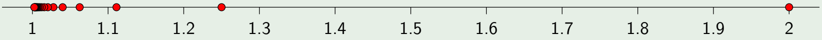
Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, ...

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

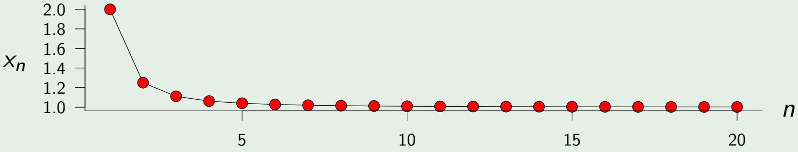
Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for $n$ in terms of $x$ in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:




Graph of $f(n)$:



# Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of **convergence** *mathematically rigorous*?

*Informal definition:* "$x_n \to L$ as $n \to \infty$" means "we can make the difference between $x_n$ and $L$ as small as we like by choosing $n$ big enough".

*More careful informal definition:* "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say $\varepsilon$, we can make the distance between $x_n$ and $L$ smaller than $\varepsilon$ by choosing $n$ big enough".

# Convergence of sequences

## Definition (Limit of a sequence)

A sequence $\{s_n\}$ **converges to $L$** if, given any $\varepsilon > 0$ there is some integer $N$ such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\displaystyle\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that $L$ is the **limit** of the sequence $\{s_n\}$.

*Note:* To use this definition to prove that the limit of a sequence is $L$, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable $N$, which will depend on $\varepsilon$. This means that *the $N$ that we find will be a function of $\varepsilon$*.

*Shorthand:*

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

# Convergence of sequences

**Convergence terminology:**

- A sequence that converges is said to be **convergent**.
- A sequence that is not convergent is said to be **divergent**.

**Remark (Sequences in spaces other than $\mathbb{R}$)**

*The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.*

# Convergence of sequences

## Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

*Note:* Our strategy here was to solve for $n$ in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big $N$ has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was "rough work". Only after this rough work did we have enough information to be able to write down a rigorous proof.