26 Integration

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 26
Integration
Friday 15 March 2019

Announcements

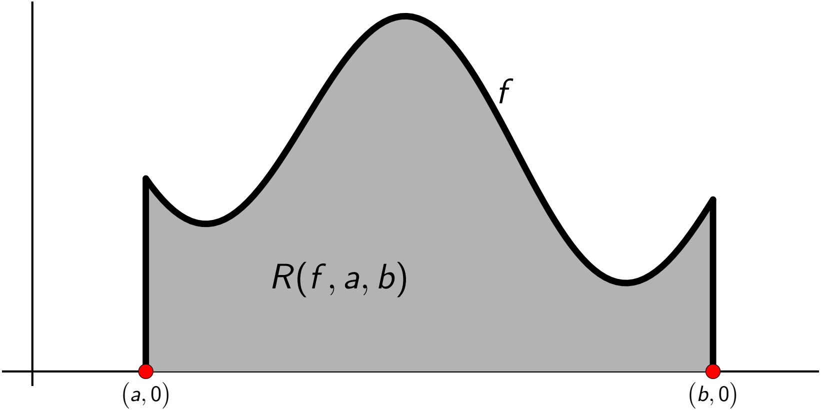
- Part of Assignment 5 is posted on the course web site (*more to come*). It is due on **Monday 25 March 2019 @ 11:30am via crowdmark**.
- Test 2 is on **Monday 1 April 2019, 7:00pm–8:30pm in MDCL 1110.**
- Assignment 6 will be due on **Monday 8 April 2019 @ 11:30am via crowdmark**.
- Final exam on **Monday 15 April 2019 @ 4:00pm in IWC/2**.
- NY Times article by Steven Stogatz in honour of Pi Day.
  - Great example of mathematical science writing for the general public.

Last time. . .

- Proved Mean Value Theorem.
- Proved Darboux's Theorem.
- Sketched proof of Inverse Function Theorem.

Integration

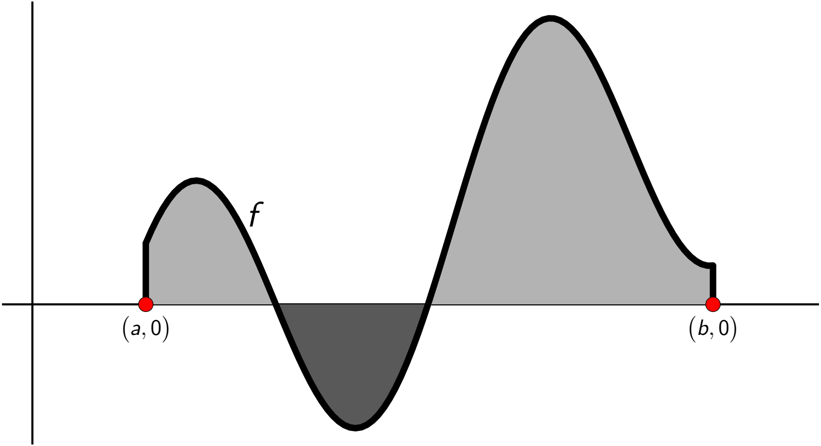
Integration



- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it.
- Textbook presentation of integral is different (but equivalent).

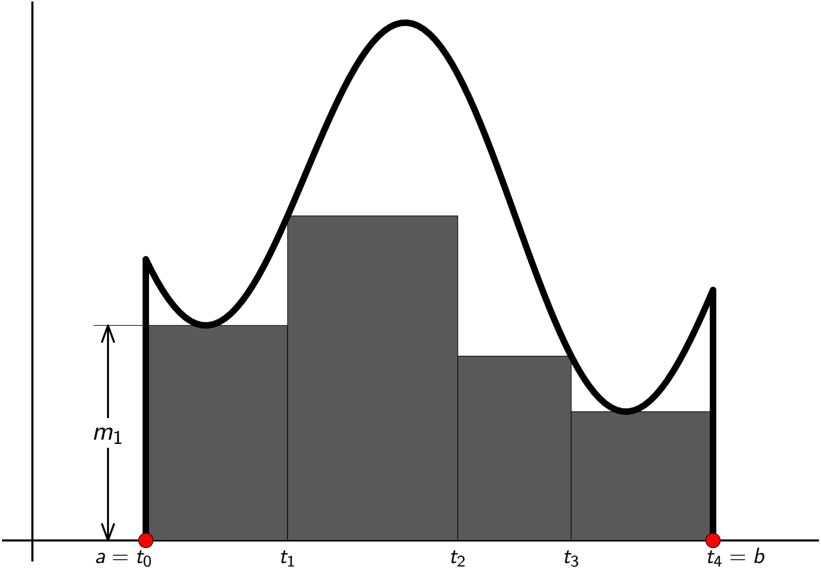
Our treatment is closer to that in M. Spivak "Calculus" (2008).

Integration

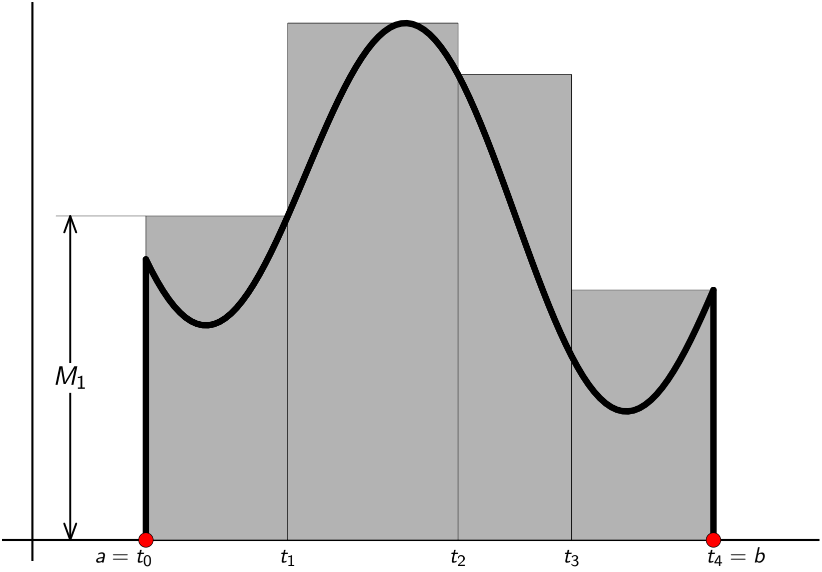


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

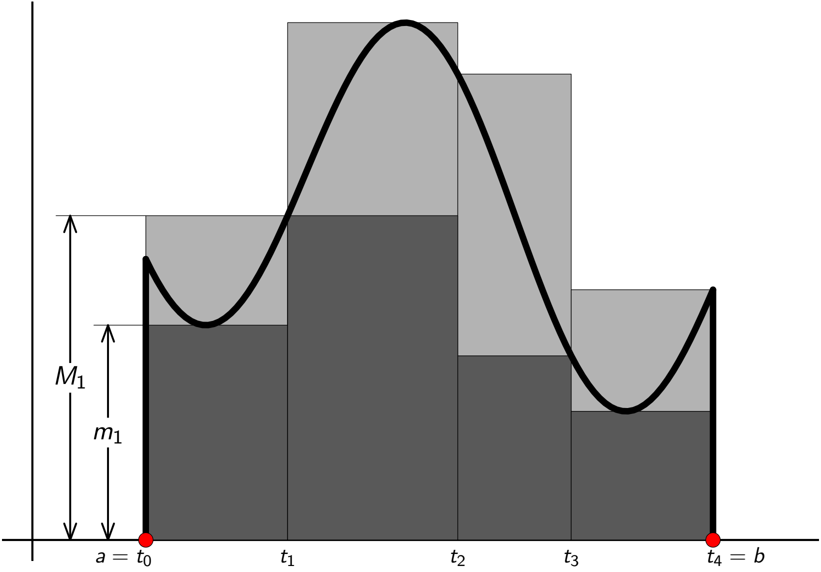
Lower sum



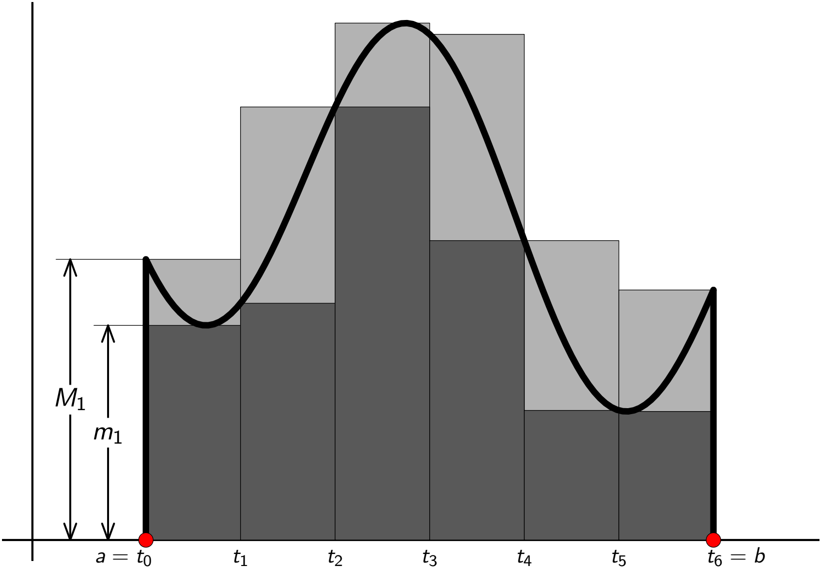
Upper sum



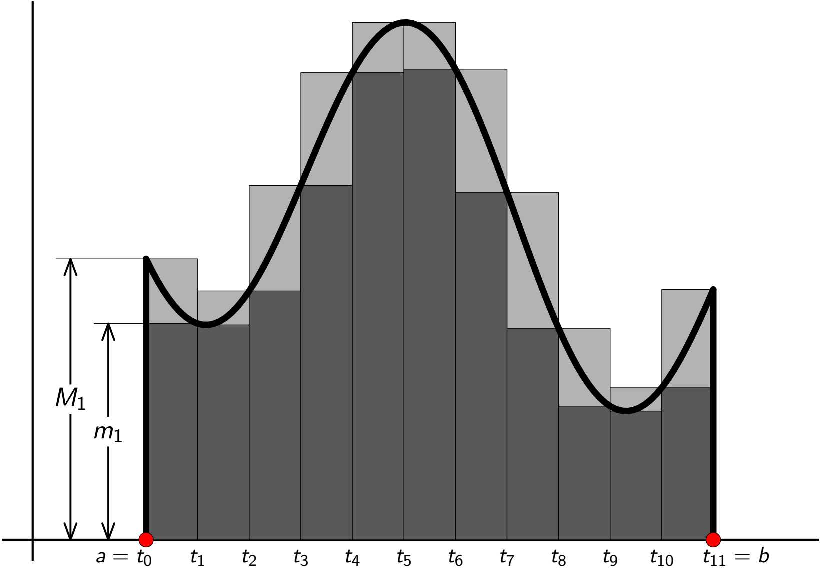
Lower and upper sums



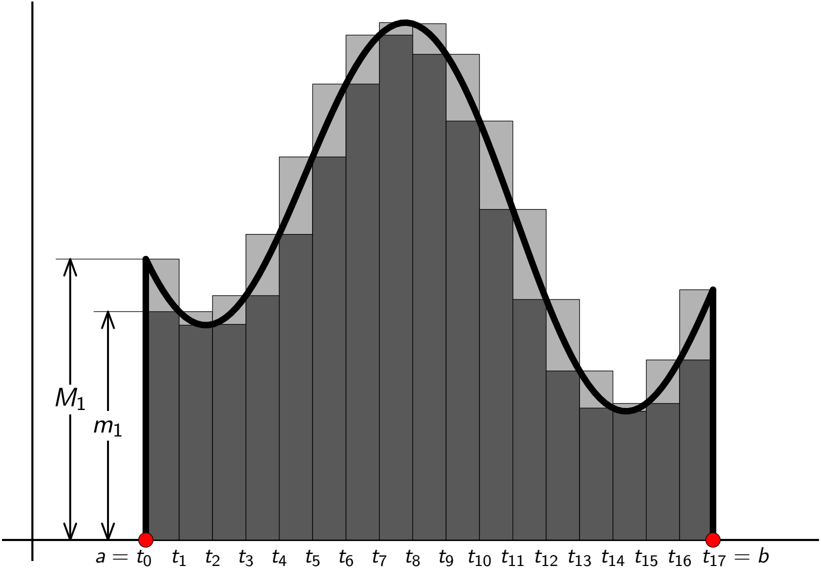
Lower and upper sums



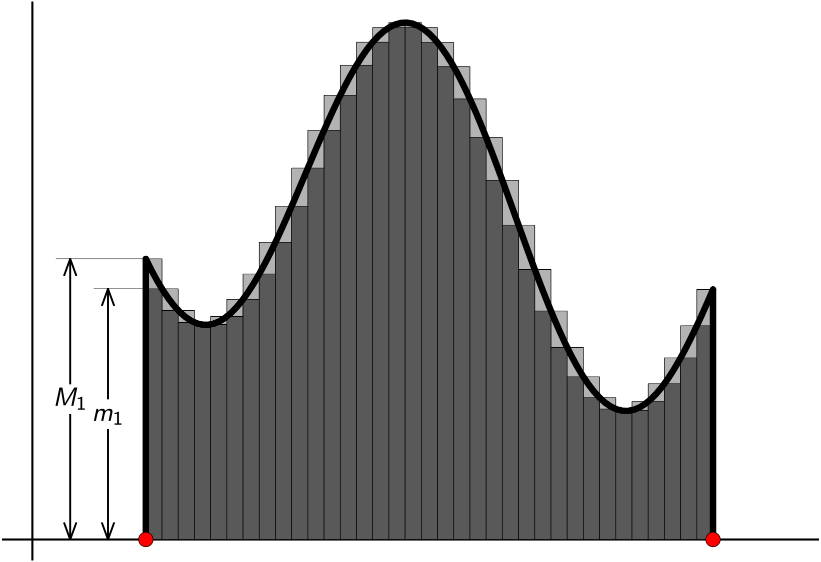
Lower and upper sums



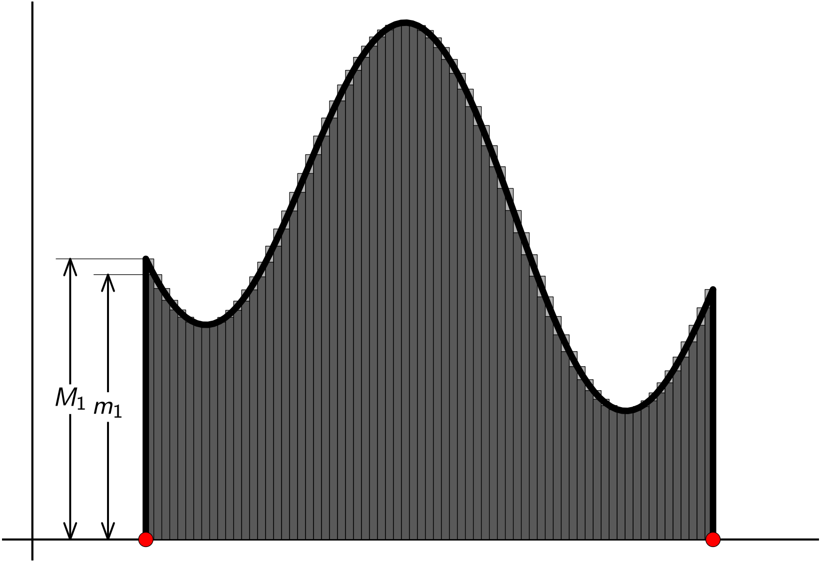
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

so the ith subinterval in the partition is

$$[t_{i-1}, t_i]\,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely without any appeal to a concept of "area".
- The requirement that f be bounded on $[a, b]$ is essential in order that all the m_i and M_i be well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \leq M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}) . \qquad i = 1, \ldots, n.$$

$\therefore$ For any partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) ,$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) .$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it *ought* to be true that

$$L(f, P_1) \leq U(f, P_2),$$

  because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

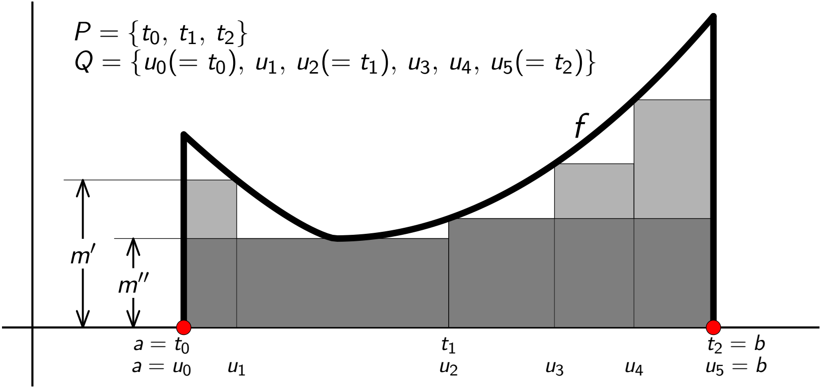
- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can define "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions P_1, P_2 ...

Rigorous development of the integral

Lemma

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $L(f,P) \leq L(f,Q)$ *and* $U(f,P) \geq U(f,Q)$.



Rigorous development of the integral

Proof of Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$P = \{t_0, \ldots, t_n\},$$
$$Q = \{t_0, \ldots, t_{k-1}, u, t_k, \ldots, t_n\},$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\,.$$

Let

$$m' = \inf\{\, f(x) \;:\; x \in [t_{k-1}, u] \,\},$$
$$m'' = \inf\{\, f(x) \;:\; x \in [u, t_k] \,\}.$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

Then $$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

and $$L(f,Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) + m''(t_k - u) + \sum_{i=k+1}^{n} m_i(t_i - t_{i-1}).$$

$\therefore$ To prove $L(f,P) \le L(f,Q)$, it is enough to show

$$m_k(t_k - t_{k-1}) \le m'(u - t_{k-1}) + m''(t_k - u).$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\},$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k \quad &= \quad \inf\{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\le \quad \inf\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m' \,. \end{aligned}$$

Thus, $m_k \le m'$, and, similarly, $m_k \le m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) \quad &= \quad m_k(t_k - u + u - t_{k-1}) \\ &= \quad m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\le \quad m'(u - t_{k-1}) + m''(t_k - u)\,, \end{aligned}$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \leq L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point that P_j. Then

$$L(f,P) = L(f,P_0) \leq L(f,P_1) \leq \cdots \leq L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \leq L(f,Q)$.

(Proving $U(f,P) \geq U(f,Q)$ is similar: check!) $\square$

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2)\,.$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)\,.$$

□