McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 6: Sequence convergence**
- Submit.

Announcements

- Assignment 1 is due via crowdmark 5 minutes before class on Monday.
- Consider writing the Putnam competition.

Sequences

- A ***sequence*** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R} \,.$$

A lot of different notation is common for sequences:

$$\begin{array}{ll} f(1),\ f(2),\ f(3),\ \ldots & \{f(n)\}_{n=1}^{\infty} \\ f_1,\ f_2,\ f_3,\ \ldots & \{f(n)\} \\ \{f(n) : n = 1, 2, 3, \ldots\} & \{f_n\}_{n=1}^{\infty} \\ \{f(n) : n \in \mathbb{N}\} & \{f_n\} \end{array}$$

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Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to ***iterate***. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \dots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \dots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

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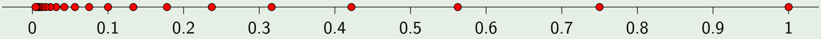
Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



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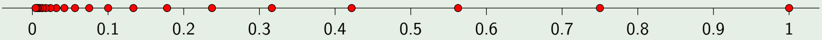
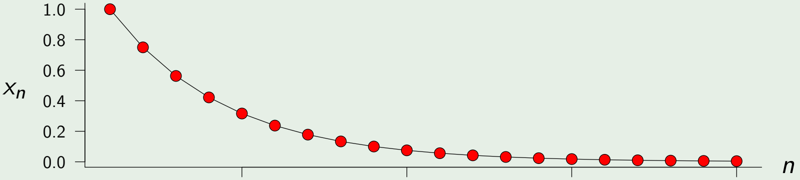


Graph of $f(n)$:



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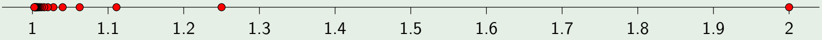

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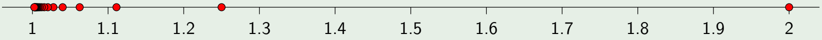

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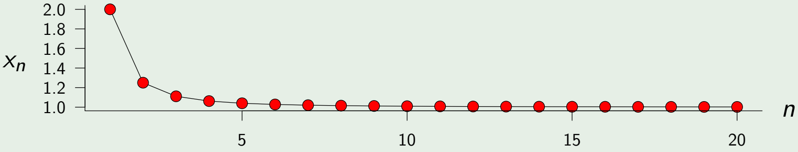

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Convergence of sequences

Definition (Limit of a sequence)

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Remark (Sequences in spaces other than $\mathbb{R}$)

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Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

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Use the formal definition of a limit of a sequence to prove that

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Convergence of sequences

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Tuesday 17 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 7: Sequence divergence**
- $\boxed{\texttt{Submit}}$.

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- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements continued...

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- Remember that solutions to assignments and tests from previous years are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)

Uniqueness of limits

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Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

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So, we are justified in referring to "the" limit of a convergent sequence.

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- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

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Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers ***diverges to*** $-\infty$ if, for every real number M there is an integer N such that

$$n \ge N \quad \Longrightarrow \quad s_n \le M\,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{\frac{n^3-1}{n+1}\right\} \quad \text{diverges to } \infty\,.$$

(solution on board)

Divergence to ∞

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Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

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Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$.

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Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.

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$$\forall n \in \mathbb{N}, \quad n - 1 =$$

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Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore \forall n \geq N$ we have

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$$\frac{n^3-1}{n+1} \geq M\,,$$

as required. □

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Note: We can start from any integer, not necessarily $k = 1$.

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Thursday 19 September 2019

What we've done so far on sequences

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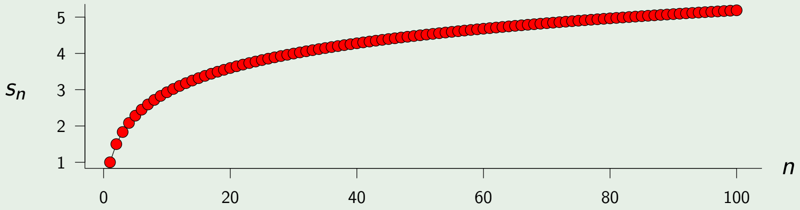

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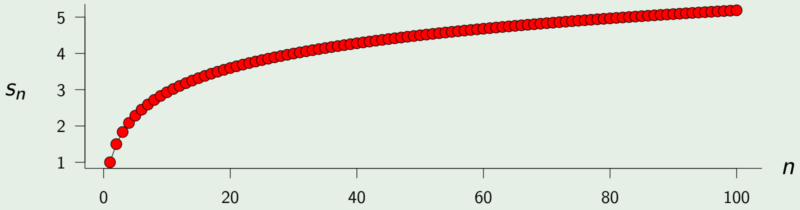

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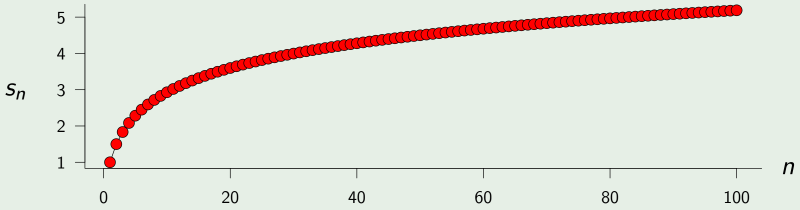

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Approach: Group terms and use the corollary above.

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Note: These sorts calculations are just “rough work”, not a formal proof.

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A proof must be a clearly presented coherent argument from beginning to end.

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$\square$

Poll

- Go to `https://www.childsmath.ca/childsa/forms/main_login.php`
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 8: Harmonic series of primes**
- `Submit`.

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

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5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
$$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} \,.$$

(solution on board)

Revisit example

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Example (previously proved directly from definition)

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Mathematics
and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Friday 20 September 2019

Announcements

Announcements

- Assignment 2 is posted.

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  Due 1 Oct 2019, at 2:25pm.

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- Harmonic series diverges.
- Algebra of limits (more today).

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Order properties of limits (§2.8)

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Theorem (Limits retain order)

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If $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences then*

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Hence $S - T \leq 0$

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Hence $S - T \le 0$, *i.e.*, $S \le T$. □

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 9: Order property of limits**
- Submit.

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

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Proof.

Apply previous theorem with $\alpha_n = \alpha \ \forall n$ and $\beta_n = \beta \ \forall n$. $\square$

Order properties of limits (§2.8)

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Theorem (Squeeze Theorem)

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$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$).

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 10
Sequences V
Tuesday 24 September 2019

Announcements

Announcements

- Assignment 2 is posted.

Announcements

- Assignment 2 is posted.
  Due 1 Oct 2019, at 2:25pm.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (sums, products, quotients).
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Today:

- Proof of Squeeze Theorem
- Absolute value and max/min of limits.
- Monotone convergence.

Order properties of limits (§2.8)

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Correct Proof.

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Given $\varepsilon > 0$, find $N \ni$

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Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$

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Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and

Order properties of limits (§2.8)

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Order properties of limits (§2.8)

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But $s_n \le x_n \le t_n \implies \qquad s_n - L \le x_n - L \le t_n - L$

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Order properties of limits (§2.8)

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But $s_n \le x_n \le t_n \implies \quad s_n - L \le x_n - L \le t_n - L$

$$\implies -\varepsilon < s_n - L \le x_n - L \le t_n - L < \varepsilon$$

$$\implies |x_n - L| < \varepsilon,$$

as required. □

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ *converges then so does* $\{|s_n|\}$, *and*

$$\lim_{n\to\infty} |s_n| = \left| \lim_{n\to\infty} s_n \right| .$$

Order properties of limits (§2.8)

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Order properties of limits (§2.8)

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See Assignment 2! $\square$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

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Order properties of limits (§2.8)

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Idea for proof:

Order properties of limits (§2.8)

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Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} =$$

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Order properties of limits (§2.8)

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Order properties of limits (§2.8)

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Prove these facts, then use theorems on sums and absolute values of limits.

Monotone convergence (§2.9)

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(iv) ***Non-increasing***: $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 10: Monotone convergence**
- $\boxed{\texttt{Submit}}$.

Monotone convergence (§2.9)

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Theorem (Monotone Convergence Theorem)

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Monotone convergence (§2.9)

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Proof.

. . . a few slides ahead. . . □

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What are the first few terms of these subsequences?

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- $\{s_n : n \text{ even}\}$ $\qquad \{2^2, 4^2, 6^2, \ldots\}$
- $\{s_n : n = 2k + 1,\ \exists k \in \mathbb{N}\}$ $\qquad \{3^2, 5^2, 7^2, \ldots\}$
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Subsequences

Definition (Subsequence)

Let $\{s_1, s_2, s_3, \ldots\}$ be a sequence. If $\{n_1, n_2, n_3, \ldots\}$ is an increasing sequence of natural numbers then $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ is a ***subsequence*** of $\{s_1, s_2, s_3, \ldots\}$.

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Let's draw some pictures to help us visualize how we might construct a proof...

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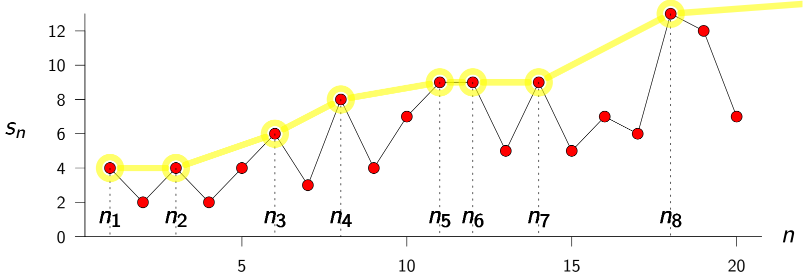

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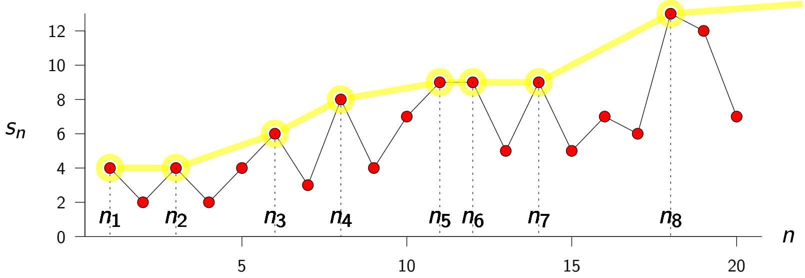

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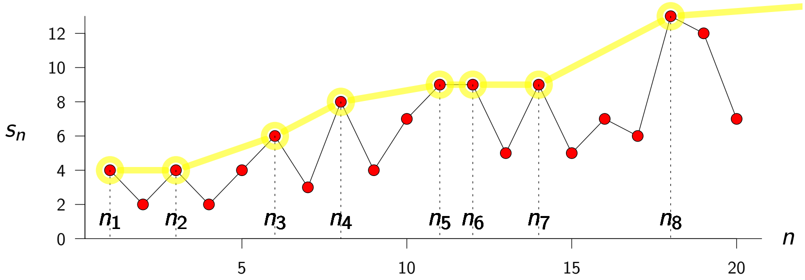

If this works indefinitely then we have a non-decreasing subsequence. But if we can find only finitely many such terms then we're stuck because our subsequence is defined using earlier terms.

Monotone convergence (§2.9)

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(solution on board)

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Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $s_N > L - \varepsilon$ (which is possible $\because$ L is the <u>least</u> upper bound of $\{s_n\}$). But $\{s_n\}$ is non-decreasing, so $\forall n \ge N$ we have $s_N \le s_n$

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Given a monotonic sequence $\{s_n\}$ we must show that

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$\Longrightarrow$ For <u>any</u> sequence (monotonic or not) convergent implies bounded.

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Proof of [part (iv)] is similar to [part (ii)]. $\square$

Monotonic subsequences

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Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \cdots$) by discarding any terms that are less than the running maximum:

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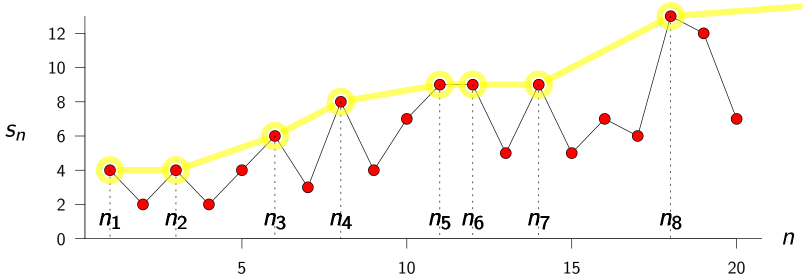

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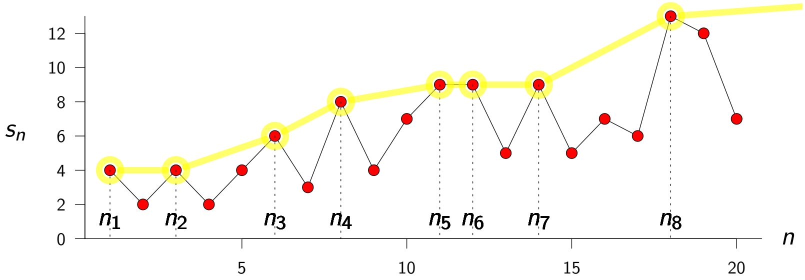

If this works indefinitely then we have a non-decreasing subsequence.

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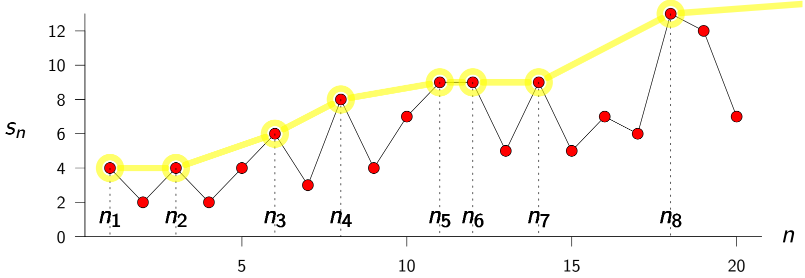

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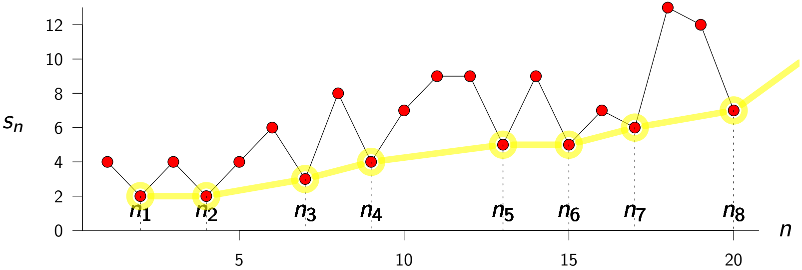

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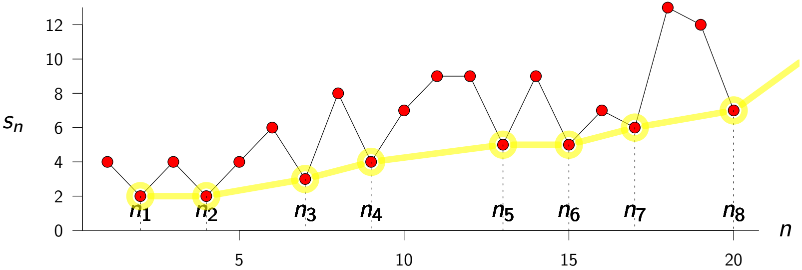

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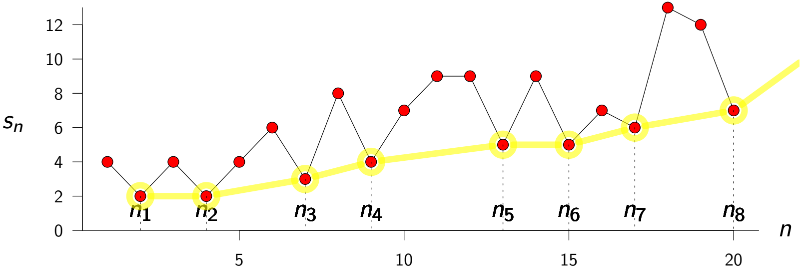

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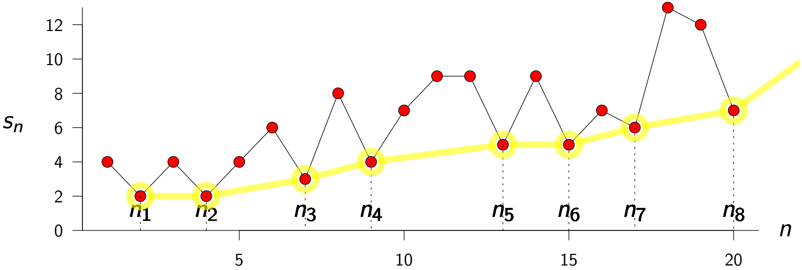

If this works indefinitely then we have a non-decreasing subsequence. What if there are only finitely many such terms? (There might not be any at all!)

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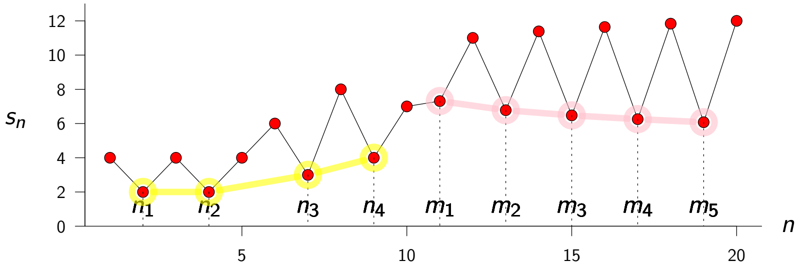
If there are only finitely many s_{n_i} such that $s_{n_i} \leq s_n \ \forall n > n_i \ \ldots$



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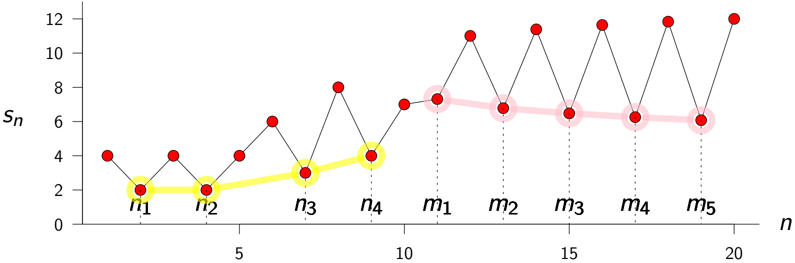

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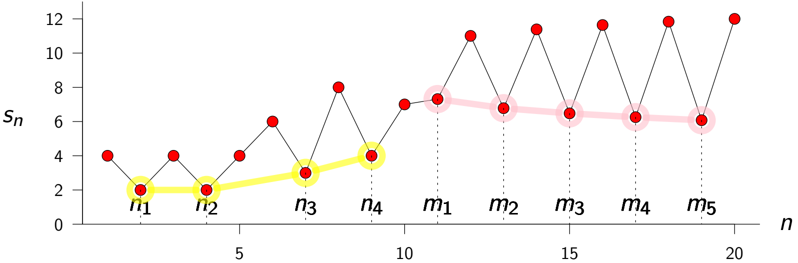

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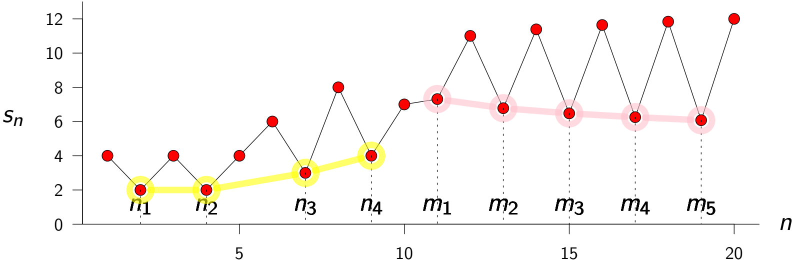

... then after the last "turn-back point" (s_{n_4} above) there must be some $m_1 > n_4$ such that s_{m_1} is **not** $\leq$ all later terms

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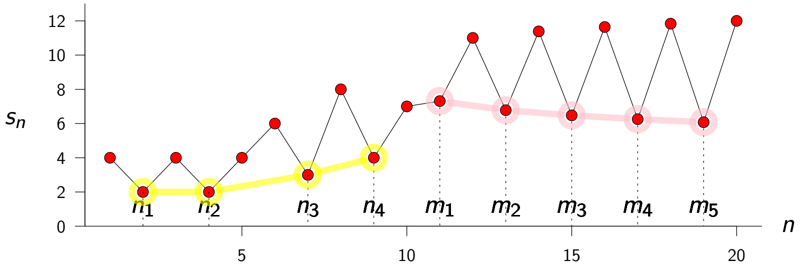

$\ldots$ then after the last "turn-back point" (s_{n_4} above) there must be some $m_1 > n_4$ such that s_{m_1} is **not** $\leq$ all later terms, *i.e.*, $\exists m_2 > m_1$ with $s_{m_2} < s_{m_1}$

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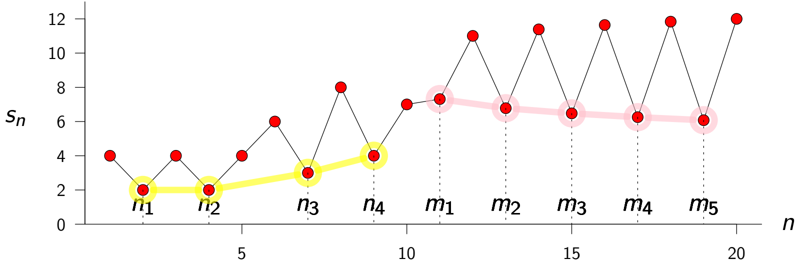

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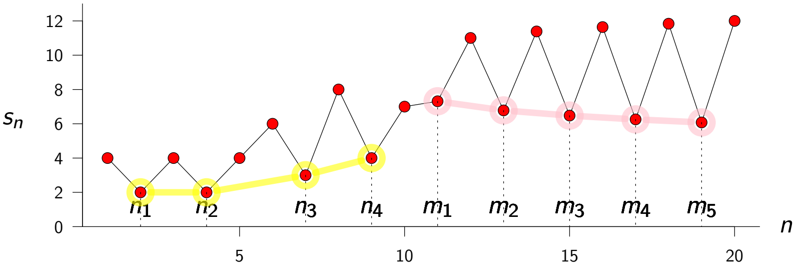

... then after the last "turn-back point" (s_{n_4} above) there must be some $m_1 > n_4$ such that s_{m_1} is **not** $\leq$ all later terms, *i.e.*, $\exists m_2 > m_1$ with $s_{m_2} < s_{m_1}$, and similarly for m_2, so there must be a decreasing subsequence $\quad s_{m_1} > s_{m_2} > s_{m_3} > \cdots$

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