Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 13

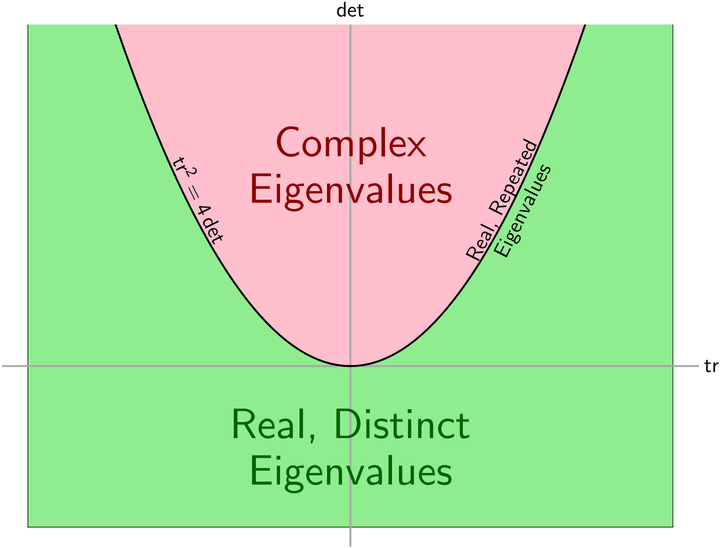
Topological Classification of Planar Linear Systems

4 October 2013

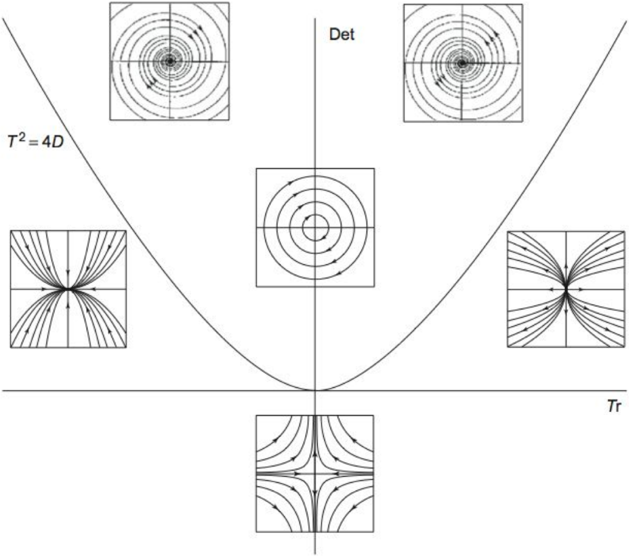
Announcements

- Assignment 2 was due TODAY, Friday 4 October 2013.
- TA office hours on Wednesday afternoon will probably be held in the corridor outside HH-403, not in HH-403 itself. Look for Dora in the vicinity of the office if she is not at her desk.

Classification in the trace-determinant plane



Classification in the trace-determinant plane



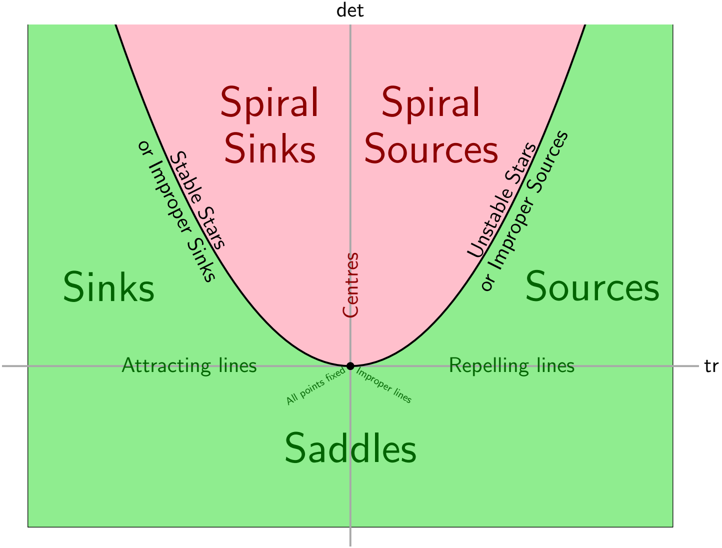
What the trace-determinant plane hides

- Trace-Determinant plane shows a *two-parameter* bifurcation diagram for planar linear system (parameters are $\mathrm{tr} = a + d$ and $\det = ad - bc$).
- But A has four parameters (a, b, c, d, or $\mathrm{tr}, \det$ and two others).
- In fact, tr-det plane does not tell the whole story of possible dynamics of linear systems.
- What are we missing in the tr-det plane?

What the trace-determinant plane hides

- tr-det plane suppresses speed of attraction to (or repulsion from) equilibrium.
- tr-det plane does not represent sense of rotation.
  - See problem 3 on 2012 Assignment 2 (Exercise 9, page 59).
- Consider $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$.
  - This has same tr and det $\forall b \in \mathbb{R}$.
  - $\therefore$ Lies in same position on tr-det plane for any $b \in \mathbb{R}$.
  - But $\exists$ bifurcation as b crosses 0 (*e.g.*, number of invariant lines changes from 1 to ∞ to 1).
  - This is a special type of bifurcation, since another condition ($\lambda_1 = \lambda_2$ in this case) must already be satisfied.
    - *e.g.*, Go from $(\lambda_1, \lambda_2) = (1, 1-\varepsilon)$ to $(\lambda_1, \lambda_2) = (1, 1+\varepsilon)$, crossing $\text{tr}^2 = 4\,\text{det}$ curve at $(\lambda_1, \lambda_2) = (1, 1)$.
    - bifurcation occurs by varying the top-right entry of A while fixing $(\lambda_1, \lambda_2) = (1, 1)$ [*i.e.*, fixing $(\text{tr}, \text{det}) = (2, 1)$].

Classification in the trace-determinant plane



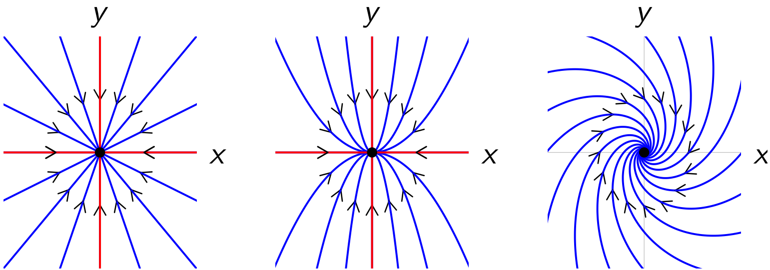
Topological equivalence of phase portraits

Intuitive idea:

- Two phase portraits in the phase plane are **topologically equivalent** or **conjugate** if it is possible to deform either one into the other via a continuous transformation of the phase plane.

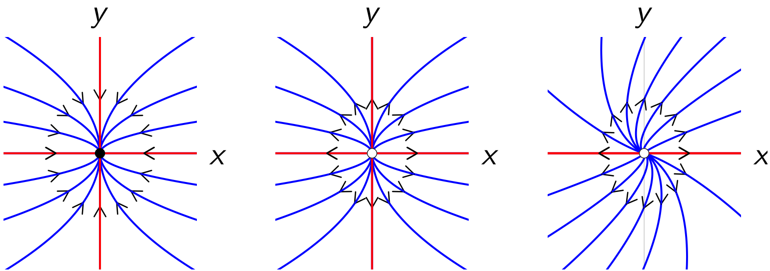
- Imagine comparing two phase portraits drawn on (raw) pizza dough.
  - Can you make the second phase portrait look like the first by stretching and squeezing the dough without tearing or puncturing?

Example: Are these phase portraits conjugate?



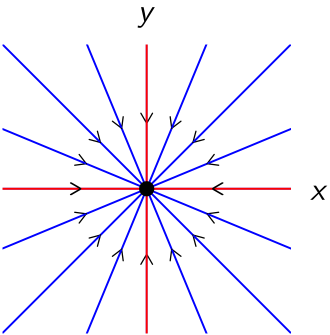
- These three sinks are, in fact, topologically conjugate!
- We say they are in the same **conjugacy class**.

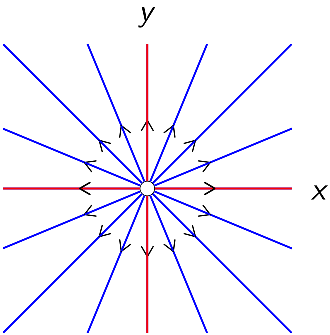
Example: Are these phase portraits conjugate?



- The sink (leftmost) is *not* conjugate to either source.
- But the normal and improper source *are* in the same conjugacy class.

Example: Why are these phase portraits not conjugate?





- Not possible to transform from one to the other *continuously*.
- Must puncture at the origin and then turn the infinite annulus inside out (and swap the origin and "the point at ∞").

Conjuacy classes for phase portraits in the plane

- The main conjugacy classes are:
  - sinks
  - sources
  - saddles
- What other conjugacy classes are there?
  - attracting lines
  - repelling lines
  - improper lines
  - all points fixed
  - centres

Eigenvalue characterization of conjuacy classes

- The main conjugacy classes are:
  - sinks $(\Re(\lambda_i) < 0,\ i = 1, 2)$
  - sources $(\Re(\lambda_i) > 0,\ i = 1, 2)$
  - saddles $(\lambda_1 < 0 < \lambda_2)$
- What other conjugacy classes are there?
  - attracting lines $(\lambda_1 = 0,\ \lambda_2 < 0)$
  - repelling lines $(\lambda_1 = 0,\ \lambda_2 > 0)$
  - improper lines $(\lambda_1 = \lambda_2 = 0,\ A \neq 0)$
  - all points fixed $(\lambda_1 = \lambda_2 = 0,\ A = 0)$
  - centres $(\Re(\lambda_i) = 0,\ \Im(\lambda_i) \neq 0,\ i = 1, 2)$
- What disinguishes the second list from the first?
  - At least one eigenvalue with a zero real part.

Hyperbolicity and Topological Equivalence

Definition (Hyperbolic Linear System, Hyperbolic Matrix)

Linear systems $X' = AX$ in which all eigenvalues of A have non-zero real parts are called **hyperbolic**. The matrix A is also said to be hyperbolic in this case.

Theorem (Hyperbolicity and Conjugacy of Planar Linear Systems)

If two 2×2 matrices A_1 and A_2 are hyperbolic then the associated planar linear systems $X' = A_i X$ $(i = 1, 2)$ are conjugate if and only if A_1 and A_2 have the same number of eigenvalues with negative real part.

In order to prove this theorem (*cf.* §4.2 of textbook), we need to express the pizza dough idea in precise mathematical terms.