McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

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Lecture 22
More Foundations of Nonlinear Systems
Monday 28 October 2013

Announcements

Test #1 THIS WEDNESDAY!

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

Location: T29 / 101

Dynamical Systems

Definition

A *smooth dynamical system* on $\mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t, X) = \phi_t(X)$ satisfies

1. $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function: $\phi_0(X) = X$;
2. The composition $\phi_t \circ \phi_s = \phi_{t+s}$ $\forall t, s \in \mathbb{R}$.

- Differential equations that yield unique solutions define smooth dynamical systems: we will see why today.

Existence/Uniqueness Simplified Summary

IVP: $\quad X' = F(t, X), \quad X(t_0) = X_0, \quad (t, X) \in \mathbb{R} \times \mathbb{R}^n$

- F continuous on an open rectangle $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$
  $\implies \exists$ solution through any $(t_0, X_0) \in \mathcal{R}$
  Peano Existence Theorem
- F C^1 on $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$
  $\implies \exists!$ solution through any $(t_0, X_0) \in \mathcal{R}$
  Fundamental Existence and Uniqueness Theorem
- F C^1 and $\frac{\partial F}{\partial X}$ uniformly bounded on strip $\mathcal{S} = [t_1, t_2] \times \mathbb{R}^n$
  $\implies \exists!$ solution throughout the time interval in $[t_1, t_2]$
  Global Picard Theorem

Note: The summary statements above are simplified, compared to the full statements discussed in the previous lecture. Rather than "F C^1", the full statements give the weaker hypothesis that "F is C^0 in t and C^1 in X". Even this is a stronger hypothesis than necessary. We discussed **Lipschitz continuity** in the previous lecture.

Continuous Dependence on Initial Conditions

Let's now restrict attention to autonomous systems, $X' = F(X)$...

Theorem

Suppose F is C^1 and consider the system $X' = F(X)$. The flow of the system, $\phi(t, X)$, is a continuous function of X.

- This means that solutions starting at nearby initial conditions remain close (at least for a short time).
- Separation of solutions happens no faster than exponential (see theorem, p.147).
- Where does exponential separation come from?

Continuous Dependence on Parameters

Theorem

Let $X' = F_a(X)$ be a one-parameter family of differential equations for which F_a is C^1 as a function of X and as a function of the parameter a. Then the flow of this system is also a continuous function of a.

- Easy proof (based on a very useful trick): Expand the system by one dimension, letting the additional variable be a with associated equation $a' = 0$. Then just apply the previous theorem!

Smooth Dependence on Initial Conditions

Theorem

Suppose F is C^1 and consider the system $X' = F(X)$. The flow of the system, $\phi(t, X)$, is a C^1 function of t and X.

Proof.

See §17.6, p. 405. □

- This means the flow of a C^1 ODE is a smooth dynamical system.

An important type of NON-AUTONOMOUS system

Theorem

Let $A(t)$ be an $n \times n$ matrix function of t, with entries $a_{ij}(t)$ that are continuous functions for $t \in [t_1, t_2]$. Then the IVP

$$X' = A(t)X\,, \qquad X(t_0) = X_0\,,$$

has a unique solution that is defined on the entire interval $[t_1, t_2]$.

- Note that it is enough for $A(t)$ to be continuous. It does not need to be C^1 to guarantee uniqueness.
  - This is a special case of the Global Picard Theorem, where F C^0 in t and C^1 and uniformly bounded in X was enough.
    Proof of uniform boundedness: The time interval $[t_1, t_2]$ is closed $\implies$ each continuous function $|a_{ij}(t)|$ has a maximum value on this interval. Let $K_{ij} = \max\{|a_{ij}(t)| : t_1 \le t \le t_2\}$ and let $K = \max_{ij} K_{ij}$. Then $\|\partial F/\partial x_i\| = \|(a_{1i}(t), \ldots, a_{ni}(t))\| = [a_{1i}^2(t) + \cdots + a_{ni}^2(t)]^{1/2} \le nK$.
- Easy to see continuity is enough in the one-dimensional case: Solve $x' = a(t)x$, $x(0) = x_0$.

Why do we care about $X' = A(t)X$?

Consider the general autonomous (**non-linear**) IVP

$$X' = F(X)\,, \qquad X(t_0) = X_0\,, \qquad X, X_0 \in \mathbb{R}^n, \qquad (\heartsuit)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Is there some $A(t)$ such that $X' = A(t)X$ can help us understand the behaviour of $(\heartsuit)$?

... Think about it over the break...