McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 24

Introduction to Equilibria of Nonlinear ODEs

Wednesday 6 November 2013

Announcements

- **Assignment 4 delayed:**
  - Will be posted at the end of this week.
  - Due Friday 15 Nov 2013.
- **Midterm Test #1:**
  - Marking still in progress.
- **Dora will not be holding office hours today.**

Nonlinear change of coordinates

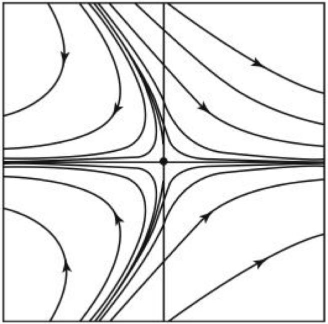


Figure 8.1 The phase plane for $x' = x + y^2$, $y' = -y$. Note the stable curve tangent to the y-axis.

- Saddle equilibrium: different from linear case?

- x-axis is invariant
- y-axis is not invariant
- $\exists$ nonlinear invariant curve through origin
- close to origin, linear approximation is excellent: standard saddle:
$$X' \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X$$
- turns out: can change variables to make system exactly linear: let
$$u = x + y^2/3,$$
$$v = y$$

Nonlinear change of coordinates

- Sometimes a nonlinear change of variables can convert a nonlinear system to a linear system.
- Then we can solve the system exactly and use all our results on linear systems to study the system **for all time**.
- Usually impossible to find such a change of variables, even if one exists. (Unless you cook up the example by hand, as the authors did for the saddle example on the previous slide.)
- In general, no such change of variables exists: far from equilibrium, nonlinear systems usually do not behave like linear systems.

Source AND Sink

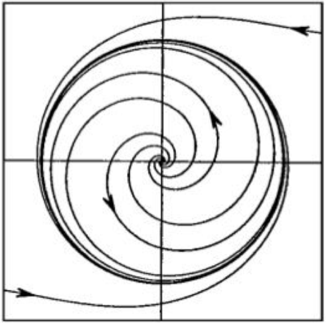


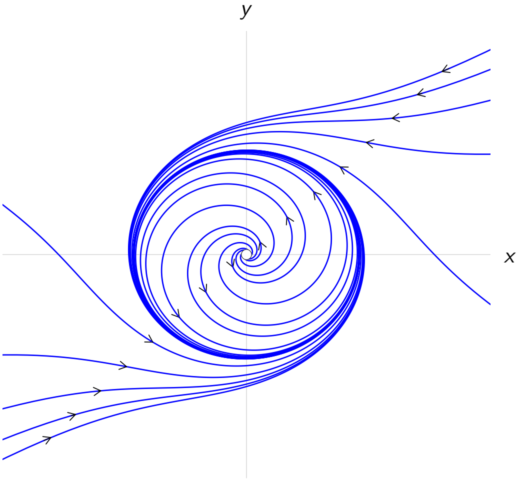
Figure 8.2 The phase plane for $r' = \frac{1}{2}(r - r^3), \theta' = 1$.

- Spiral source near origin
- Spiral sink far from origin

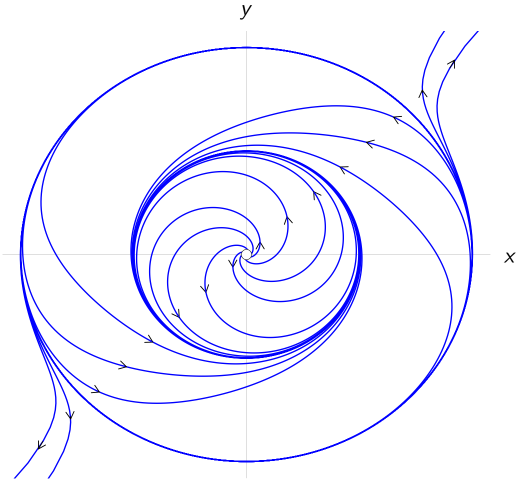
- Is this possible in a linear system?
- No way. We classified all linear systems, and didn't find such a phase portrait.
- What is there between the source and the sink in this example?
- A *periodic orbit*
- Easy to generalize this, e.g.,

$$r' = r(1 - r^2)[1 - (r/2)^2]$$
$$\theta' = 1$$

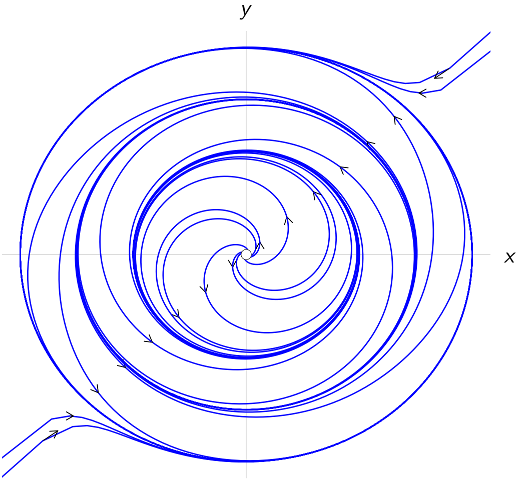
Phase Portrait for $r' = \frac{1}{2}r(1-r^2)$, $\theta' = 1$ (Figure 8.2)



Phase Portrait for $r' = r(1-r^2)(1-(r/2)^2)$, $\theta' = 1$



Phase Portrait for $r' = r(1-r^2)(1-\frac{4r^2}{9})(1-\frac{r^2}{4})$, $\theta' = 1$



Linearization may provide no useful info about solutions

Consider:

$$x' = x(x^2 + y^2)$$
$$y' = y(x^2 + y^2)$$

- Linearization is the zero matrix: every point is fixed.
- But full nonlinear system is a source!
- To see this, consider direction of vector field in each quadrant.
- Can also get exactly solution by converting to polar coordinates. . .

Linearization may provide no useful info about solutions

Coordinate relationships:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \theta &= \arctan(y/x) \end{aligned}$$

Time derivatives:

$$\begin{aligned} r' &= (xx' + yy')/r \\ \theta' &= \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}x' + \frac{y'}{x}\right) \\ &= (-yx' + xy')/r^2 \end{aligned}$$

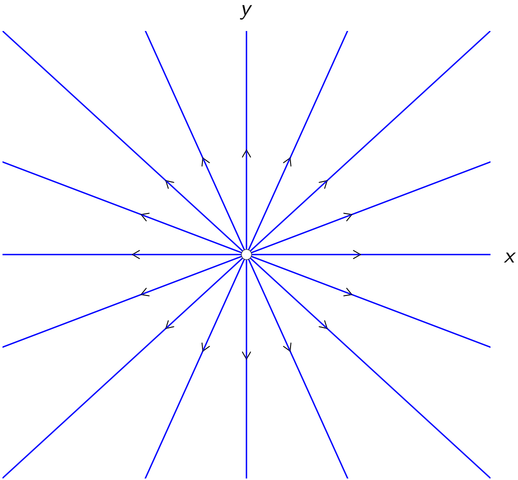
$X' = F(X)$ in Cartesian coordinates:

$$\begin{aligned} x' &= x(x^2 + y^2) \\ y' &= y(x^2 + y^2) \end{aligned}$$

In polar coordinates:

$$\begin{aligned} r' &= r^3 \\ \theta' &= 0 \end{aligned}$$

Phase Portrait for $r' = r^3$, $\theta' = 0$



Converting between Cartesian and polar coordinates

Polar to Cartesian:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Time derivatives:

$$x' = r'\cos\theta - r(\sin\theta)\theta'$$
$$y' = r'\sin\theta + r(\cos\theta)\theta'$$

Cartesian to polar:

$$r^2 = x^2 + y^2$$
$$\theta = \arctan(y/x)$$

Time derivatives:

$$r' = (xx' + yy')/r$$
$$\theta' = (xy' - yx')/r^2$$