

Mathematics and Statistics

$$\int_{M} d\omega = \int_{\partial M} \omega$$

Mathematics 3F03 Advanced Differential Equations

Instructor: David Earn

Lecture 22 More Foundations of Nonlinear Systems Monday 28 October 2013

Announcements

Test #1 THIS WEDNESDAY!

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

Location: T29 / 101

Dynamical Systems

Definition

A *smooth dynamical system* on \mathbb{R}^n is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t, X) = \phi_t(X)$ satisfies

- **1** $\phi_0: \mathbb{R}^n \to \mathbb{R}^n$ is the identity function: $\phi_0(X) = X$;
- **2** The composition $\phi_t \circ \phi_s = \phi_{t+s} \ \forall t, s \in \mathbb{R}$.
 - Differential equations that yield unique solutions define smooth dynamical systems: we will see why today.

Existence/Uniqueness Simplified Summary

IVP:
$$X' = F(t, X), \quad X(t_0) = X_0, \quad (t, X) \in \mathbb{R} \times \mathbb{R}^n$$

- F continuous on an open rectangle $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$ $\Longrightarrow \exists$ solution through any $(t_0, X_0) \in \mathcal{R}$ Peano Existence Theorem
- F C^1 on $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$ $\Longrightarrow \exists !$ solution through any $(t_0, X_0) \in \mathcal{R}$ Fundamental Existence and Uniqueness Theorem
- F C^1 and $\frac{\partial F}{\partial X}$ uniformly bounded on strip $S = [t_1, t_2] \times \mathbb{R}^n$ $\implies \exists !$ solution throughout the time interval in $[t_1, t_2]$ Global Picard Theorem

Note: The summary statements above are simplified, compared to the full statements discussed in the previous lecture. Rather than "F C^1 ", the full statements give the weaker hypothesis that "F is C^0 in t and C^1 in X". Even this is a stronger hypothesis than necessary. We discussed **Lipschitz continuity** in the previous lecture.

Continuous Dependence on Initial Conditions

Let's now restrict attention to autonomous systems, X' = F(X)...

Theorem

Suppose F is C^1 and consider the system X' = F(X). The flow of the system, $\phi(t, X)$, is a continuous function of X.

- This means that solutions starting at nearby initial conditions remain close (at least for a short time).
- Separation of solutions happens no faster than exponential (see theorem, p.147).
- Where does exponential separation come from?

Continuous Dependence on Parameters

Theorem

Let $X' = F_a(X)$ be a one-parameter family of differential equations for which F_a is C^1 as a function of X and as a function of the parameter A. Then the flow of this system is also a continuous function of A.

■ Easy proof (based on a very useful trick): Expand the system by one dimension, letting the additional variable be a with associated equation a' = 0. Then just apply the previous theorem!

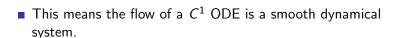
Smooth Dependence on Initial Conditions

Theorem

Suppose F is C^1 and consider the system X' = F(X). The flow of the system, $\phi(t, X)$, is a C^1 function of t and X.

Proof.

See §17.6, p. 405.



An important type of NON-AUTONOMOUS system

Theorem

Let A(t) be an $n \times n$ matrix function of t, with entries $a_{ij}(t)$ that are continuous functions for $t \in [t_1, t_2]$. Then the IVP

$$X'=A(t)X, \qquad X(t_0)=X_0,$$

has a unique solution that is defined on the entire interval $[t_1, t_2]$.

- Note that it is enough for A(t) to be continuous. It does not need to be C^1 to guarantee uniqueness.
 - This is a special case of the Global Picard Theorem, where F C^0 in t and C^1 and uniformly bounded in X was enough. Proof of uniform boundedness: The time interval $[t_1, t_2]$ is closed \implies each continuous function $|a_{ij}(t)|$ has a maximum value on this interval. Let $K_{ij} = \max\{|a_{ij}(t)| : t_1 \le t \le t_2\}$ and let $K = \max_{ij} K_{ij}$. Then $\|\partial F/\partial x_i\| = \|(a_{1i}(t), \dots, a_{ni}(t))\| = [a_{1i}^2(t) + \dots + a_{ni}^2(t)]^{1/2} \le nK$.
- Easy to see continuity is enough in the one-dimensional case: Solve x' = a(t)x, $x(0) = x_0$.

Why do we care about X' = A(t)X?

Consider the general autonomous (non-linear) IVP

$$X' = F(X), \qquad X(t_0) = X_0, \qquad X, X_0 \in \mathbb{R}^n, \qquad (\heartsuit)$$

where $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1 . Is there some A(t) such that X' = A(t)X can help us understand the behaviour of (\heartsuit) ?

... Think about it over the break...