# Mathematics 3F03
# Advanced Differential Equations

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Lecture 1
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# Classes

- Four meetings per week by default, but sometimes fewer.
- Tutorials will not always occur in the same time slot and *will not necessarily occur every week.*
- Tutorial style will be adopted in some classes (or parts of classes).
- If slides are presented in class they will be posted on the course wiki, BUT *slides never contain the full content of lectures!*
- Please tell me if you notice errors in slides (or anything else).
- Take notes on anything not in the slides!
- Feedback always welcome: help make the course better.

# Where to find course information

- The course wiki:
  `http://www.math.mcmaster.ca/earn/3F03`
- Click on "Course information".
- Download pdf or read online.
- Let's have a look now...

# Assigned Reading

- Read Chapter 1 of the textbook.

# What is a differential equation?

- An equation involving an *unknown function* and its derivatives.
- Solution of the equation is a function.

# ODEs vs PDEs

- An *Ordinary Differential Equation* (ODE) involves an unknown function of one variable, e.g.,

$$\frac{dx}{dt} = ax + 5 \tag{1}$$

  Here, the unknown function is $x(t)$.

- A *Partial Differential Equation* (PDE) involves an unknown function of several variables, e.g.,

$$\frac{\partial f}{\partial t} = ax^2 \frac{\partial^2 f}{\partial x^2} + 5 \tag{2}$$

  Here, the unknown function is $f(x,t)$.

- This course: ODEs.
- Math 3FF3: PDEs.

# Autonomous vs non-autonomous

- Autonomous: no explicit time-dependence, e.g.,

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} = \sin x + x^2 + x \tag{3}$$

- Non-autonomous: explicit time-dependence

$$\frac{dx}{dt} = \sin\left(x - vt\right) + x^2 + tx + t + 1 \tag{4}$$

# Order

- The order of the highest order derivative that occurs in the equation.
- First order example:

$$\frac{dx}{dt} = \sin(x^2 + 3e^{\alpha t}) \tag{5}$$

- Second order example:

$$\frac{d^2x}{dt^2} = ax^2 + bx - c\frac{dx}{dt} \tag{6}$$

# Linear vs Nonlinear

- Consider functions $x(t)$ and $y(t)$. If any linear combination of $x(t)$ and $y(t)$ is a solution of an equation whenever both $x(t)$ and $y(t)$ are solutions, then the equation is linear.
- **In operator notation:** If $\mathcal{L}[x] = 0$ and $\mathcal{L}[y] = 0$ implies $\mathcal{L}[ax + by] = 0$ for any $a, b \in \mathbb{R}$ then the equation $\mathcal{L}[x] = 0$ is linear. Otherwise, it is nonlinear.
- Linear ODE example:

$$\frac{d^2x}{dt^2} + t^2 \frac{dx}{dt} = (\sin \omega t)x \tag{7}$$

- Nonlinear ODE example:

$$\frac{d^2x}{dt^2} + tx \frac{dx}{dt} = t \sin \omega x \tag{8}$$

# Why study differential equations?

- Models of dynamical processes often expressed as differential equations.
- Important in science, engineering, social science, finance, ...
- In fact, is there anything of importance in life that can't be described with a differential equation?

# Malthusian population growth

- time $t$, population size (or density) $x$.
- Suppose reproductive rate proportional to population size (or density):

$$\frac{dx}{dt} = ax \tag{9}$$

  Parameter $a$ is called the *intrinsic reproductive rate*.
  ($a$ has units of inverse time, $[t^{-1}]$.)
- Easy to solve by separation of variables (or by guessing!):

$$x(t) = x(0)e^{at} \,. \tag{10}$$

- The "guessing" method turns out to generalize more easily to higher dimensional systems.

# Malthusian population growth

- $x(t) = x(0)e^{at}$
- Exponential population growth if $a > 0\dots$
- Eventual disaster certain...
- If $x(0) < 0$ then $x$ does not represent a biological population, but equation is still mathematically sensible (solutions tend to $-\infty$).
- Figure also shows the **phase line**

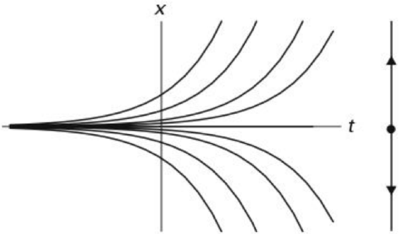
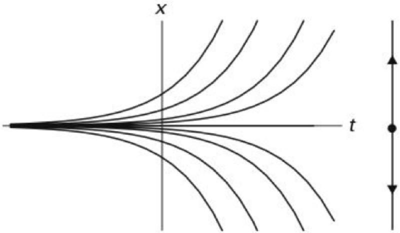



Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

# Qualitative aspects of solutions

- Figures 1.1 and 1.2 show solutions for the full **one-parameter family of differential equations** $\left\{\frac{dx}{dt} = ax : a \in \mathbb{R}\right\}$
- As $a$ decreases through 0, there is a **bifurcation** from **unstable** to **stable equilibrium** (fixed point) at $x = 0$.

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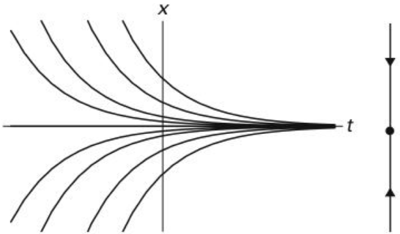


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

# Quantitative Theory: What you learned in Math 2C03 or equivalent

- Exact analytical solutions, e.g., formula for solution function expressed in terms of common functions, series or integrals.
- Possible for linear differential equations.
- Rarely possible for non-linear differential equations.
- Most differential equations that arise in applications are non-linear. So what can we learn from them?

# Qualitative Theory: Emphasis of this course

- Determine behaviour of solutions without obtaining exact solutions.
- Does a solution exist? If so, is it unique?
- Where does the solution go if we wait a long time? Does this depend on initial conditions?
- What kinds of solutions are possible?
  - Steady states (equilibria)
  - Periodic orbits
  - Chaotic trajectories
- Stability
- Bifurcations