# Mathematics 3F03
# Advanced Differential Equations

Instructor: David Earn

Lecture 14

*Topological Classification of Planar Linear Systems (continued)*

*7 October 2013*

# Announcements

- Assignment 3 will be posted at the end of the week and will be due on Friday 18 Oct 2013.
- TA office hours on Wednesday afternoon will probably be held in the corridor outside HH-403, not in HH-403 itself. Look for Dora in the vicinity of the office if she is not at her desk.

# Hyperbolicity and Topological Equivalence

## Definition (Hyperbolic Linear System, Hyperbolic Matrix)

Linear systems $X' = AX$ in which all eigenvalues of $A$ have non-zero real parts are called **hyperbolic**. The matrix $A$ is also said to be hyperbolic in this case.

## Theorem (Hyperbolicity and Conjugacy of Planar Linear Systems)

*If two $2 \times 2$ matrices $A_1$ and $A_2$ are hyperbolic then the associated planar linear systems $X' = A_i X$ $(i = 1, 2)$ are conjugate if and only if $A_1$ and $A_2$ have the same number of eigenvalues with negative real part.*

In order to prove this theorem (*cf.* §4.2 of textbook), we need to express the pizza dough idea in precise mathematical terms.

# Recall Important Properties of Functions

## Definition (One-to-one or injective)

A function $h : M \to N$ is **one-to-one** if no two points in the domain ($M$) are mapped to the same point in the range ($N$), *i.e.*,

$$h(X) = h(Y) \implies X = Y.$$

## Definition (Onto or surjective)

A function $h : M \to N$ is **onto** if every point in the range ($N$) is mapped to from at least one point in the domain ($M$), *i.e.*,

$$\forall Y \in N,\ \exists X \in M \text{ such that } h(X) = Y.$$

# Recall Important Properties of Functions

## Definition (Continuous)

A function $h : M \to N$ is **continuous at the point** $X_0 \in M$ if

$$\lim_{X \to X_0} h(X) = h(X_0)\,.$$

$h$ is **continuous** if it is continuous at every point in its domain, *i.e.*,

$$\lim_{X \to X_0} h(X) = h(X_0) \qquad \forall X_0 \in M.$$

# Homeomorphism

## Definition

A function $h : \mathbb{R}^n \to \mathbb{R}^n$ that is one-to-one, onto, continuous, and has a continuous inverse $h^{-1}$ is said to be a **homeomorphism** of $\mathbb{R}^n$.

**Question:** *Do we need to state that $h^{-1}$ must also be continuous?*

**Answer:** *No, continuity of $h^{-1}$ follows automatically*
*in this particular context.*

## Theorem (Brouwer's Domain Invariance Theorem)

*If $f : \mathbb{R}^n \to \mathbb{R}^n$ is 1:1 and continuous then $f^{-1}$ is continuous.*

## Theorem (More General Domain Invariance Theorem)

*If $f : M \to N$ is a one-to-one and continuous map between n-manifolds without boundary, then $f$ is an open map.*

# Flow of a differential equation

If the initial value problem $X' = F(X,t)$, $X(0) = X_0$, has a unique solution $\forall X_0 \in \mathbb{R}^n$ (true for $F(X,t) = AX$) then we can define:

## Definition (Flow of a differential equation)

The flow of $X' = F(X,t)$ is the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ for which $\phi(t, X_0)$ (thought of as a function of $t$ for given $X_0$) gives the solution curve that goes through $X_0$ at time $t = 0$.

## Definition (Time-$t$ map)

For each time $t \in \mathbb{R}$, we define the time-$t$ map $\phi_t$ via

$$\phi_t(X_0) = \phi(t, X_0).$$

For each point $X_0 \in \mathbb{R}^n$, $\phi_t$ specifies where $X_0$ moves to, a time $t$ in the future (or the past if $t < 0$).

# Conjugacy

We can now make the pizza dough idea precise:
Two flows are conjugate if there is a homeomorphism between them that preserves solutions. More precisely:

## Definition (Conjugacy)

Flows $\phi^A$ and $\phi^B$ in $\mathbb{R}^n$ are conjugate if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)) \quad \forall X_0 \in \mathbb{R}^n. \qquad (*)$$

We then call $h$ a conjugacy.

## Proposition

*If $h$ is a conjugacy then so is $h^{-1}$.*

# $h$ is a conjugacy $\iff$ $h^{-1}$ is a conjugacy

**Proof.**

If $h$ is a conjugacy then

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)) \quad \forall X_0 \in \mathbb{R}^n. \qquad (*)$$

Applying $h^{-1}$ to both sides, we have

$$h^{-1}(\phi^B(t, h(X_0))) = \phi^A(t, X_0) \quad \forall X_0 \in \mathbb{R}^n.$$

For each $X_0 \in \mathbb{R}^n$, let $Y_0 = h(X_0)$. Then $X_0 = h^{-1}(Y_0)$ and

$$h^{-1}(\phi^B(t, Y_0)) = \phi^A(t, h^{-1}(Y_0)) \quad \forall Y_0 \in \mathbb{R}^n, \qquad (**)$$

i.e., $h^{-1}$ is a conjugacy. $\square$

# Conjugacy is an equivalence relation

**Theorem (Conjugacy of flows of differential equations is an *equivalence relation*)**

*Suppose $\phi^A$, $\phi^B$ and $\phi^C$ are flows in $\mathbb{R}^n$ and write $\phi^A \sim \phi^B$ to mean "the flow $\phi^A$ is conjugate to the flow $\phi^B$". Then*

(i) $\phi^A \sim \phi^A$ *(reflexivity)*,

(ii) $\phi^A \sim \phi^B \implies \phi^B \sim \phi^A$ *(symmetry)*,

(iii) *if* $\phi^A \sim \phi^B$ *and* $\phi^B \sim \phi^C$ *then* $\phi^A \sim \phi^C$ *(transitivity)*.

**Proof.**

See instructor's solutions to Problem 1 on 2012 Assignment 3. □

# Topological classification of phase portraits

Now recall theorem stated earlier...

**Theorem (Hyperbolicity and Conjugacy of Planar Linear Systems)**

*If two $2 \times 2$ matrices $A_1$ and $A_2$ are hyperbolic then the associated planar linear systems $X' = A_i X$ $(i = 1, 2)$ are conjugate if and only if $A_1$ and $A_2$ have the same number of eigenvalues with negative real part.*

**Proof.**

See §4.2 in the textbook. □

However, let's sketch a different proof in the "difficult" case of conjagacy of sinks and spiral sinks...

# Recall (?) definitions from analysis

Let $\{f_n\}$ be a sequence of functions defined on a set $\mathcal{S}$, and let $f$ be a function that is also defined on $\mathcal{S}$.

## Definition (Pointwise Convergence)

If, for all $x \in \mathcal{S}$, $\quad f(x) = \lim_{n\to\infty} f_n(x)\,, \quad$ then we say that $\{f_n\}$ **converges pointwise to $f$ on $\mathcal{S}$**.

## Definition (Uniform Convergence)

$f$ is called the **uniform limit of $\{f_n\}$ on $\mathcal{S}$** if for every $\varepsilon > 0$ there is some $N$ such that for all $x \in \mathcal{S}$,

$$n > N \implies \|f(x) - f_n(x)\| < \varepsilon.$$

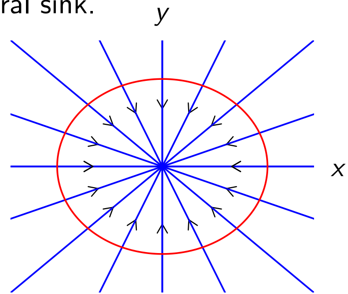
We also say that $\{f_n\}$ **converges uniformly to $f$ on $\mathcal{S}$** or that $f_n$ **approaches $f$ uniformly on $\mathcal{S}$**.

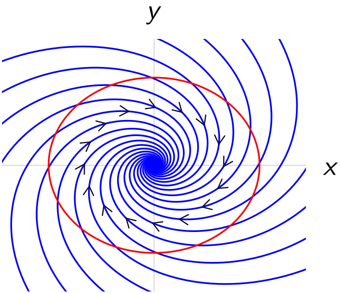
# Recall (?) theorem from analysis

**Theorem (Uniform convergence of continuous functions)**

*Suppose that $\{f_n\}$ is a sequence of functions that are continuous on a compact set $\mathcal{S}$ (such as a closed interval $[a, b] \subset \mathbb{R}$), and suppose that $\{f_n\}$ converges uniformly on $\mathcal{S}$ to $f$. Then $f$ is also continuous on $\mathcal{S}$.*

Now consider a closed disk around the origin of a stable star and a spiral sink.



# Conjugacy of stable star and spiral sink

## Sketch of proof (formalizing twisting pizza dough).

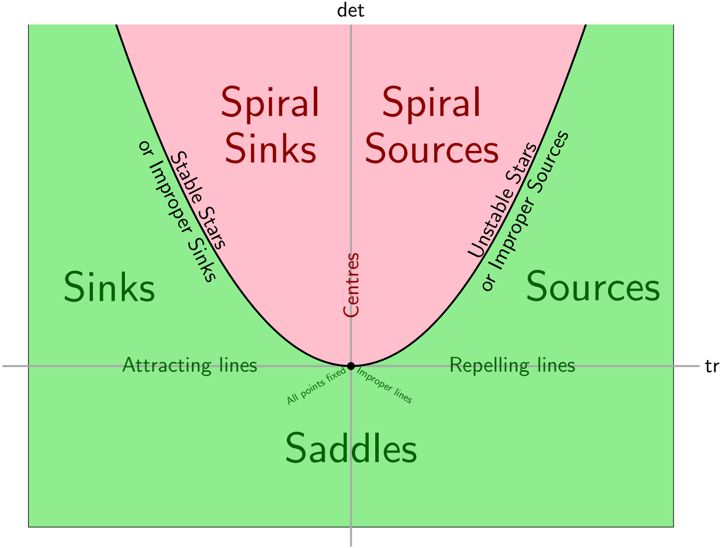
The challenge is to show that there is a continuous mapping between the star and spiral inside the red (unit) circle.

Consider a sequence of radii $r_n = 1/2^n$ and note that $r_n \to 0$ as $n \to \infty$.

Define a homeomorphism $h_n$ to twist the star to the spiral exactly between $r_n$ and $r_{n+1}$, preserving what was done for smaller $n$, and define the function $h$ via $h(X) = \lim_{n\to\infty} h_n(X)\ \forall X \in \mathbb{R}^2$.

This sequence of functions $h_n$ converges *uniformly* to $h$, since for any $\varepsilon > 0$ there exists $N$ such that for all $n > N$, $\|h - h_n\| < \epsilon$ (just choose $N$ such that $1/2^N < \epsilon$). □

# Colours $\Longrightarrow$ real (green) or complex (red) eigenvalues



# Colours $\Longrightarrow$ conjugacy classes