Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

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Lecture 25
Nullclines
Wednesday 6 November 2013

Announcements

- **Assignment 4 delayed:**
  - Will be posted at the end of this week.
  - Due Friday 15 Nov 2013.
- **Midterm Test #1:**
  - Marking still in progress.
- **Dora will not be holding an office hour today.**

Equilibria

Consider a general autonomous ODE,

$$X' = F(X), \qquad X \in \mathbb{R}^n. \tag{$\heartsuit$}$$

Write out component-wise:

$$\begin{aligned} x_1' &= f_1(x_1, \ldots, x_n) \\ x_2' &= f_2(x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, \ldots, x_n) \end{aligned}$$

Definition (Equilibrium)

An **equilibrium** of ($\heartsuit$) is a point $X^* \in \mathbb{R}^n$ where $F(X^*) = 0$, *i.e.*, $x_1' = x_2' = \cdots = x_n' = 0$.

Nullclines

Definition (Nullcline)

A **nullcline** of ($\heartsuit$) is a curve (or more generally a hypersurface) where ONE component of the vector field vanishes, *i.e.*, $\exists j \in \{1, \ldots, n\}$ such that $x_j' = 0$, *i.e.*, $f_j(x_1, \ldots, x_n) = 0$.

- In simple cases, we can solve for x_j in terms of the other x_ks to get an explicit formula for the nullcline. (In general, we just get an implicit algebraic relationship that may not be possible to solve for x_j.)
- **Points where nullclines $\forall j$ intersect are equilibria.**
- The vector field is $\perp$ x_j-axis along an x_j nullcline.
- Nullclines are very helpful for constructing phase portraits of nonlinear systems.

Nullclines of planar systems

In the plane, $X' = F(X)$ can be written

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

- x nullcines: $f(x,y) = 0$
  - $x' = 0 \implies$ vector field is strictly $\uparrow$ or $\downarrow$.
  - Vector field is strictly $\uparrow$ or $\downarrow$ ONLY on x nullclines.
  - x nullclines divide the plane into regions where the vector field points left or right.
- y nullcines: $g(x,y) = 0$
  - $y' = 0 \implies$ vector field is strictly $\leftarrow$ or $\rightarrow$.
  - Vector field is strictly $\leftarrow$ or $\rightarrow$ ONLY on y nullclines.
  - y nullclines divide the plane into regions where the vector field points up or down.

Nullclines of planar systems

$\therefore$ If we draw all nullclines of a planar system (*i.e.*, all x nullclines and all y nullclines) then we divide the plane into **basic regions** in which:

- The vector field is never vertical or horizontal.
- The vector field points into ONE quadrant throughout the region (*i.e.*, NE, NW, SE or SW).

Example (1)

$$x' = y - x^3$$
$$y' = x - 2$$