Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 17
Genericity and 3D Phase Portraits of Linear Systems
Wednesday 16 October 2013

Announcements

Test #1

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

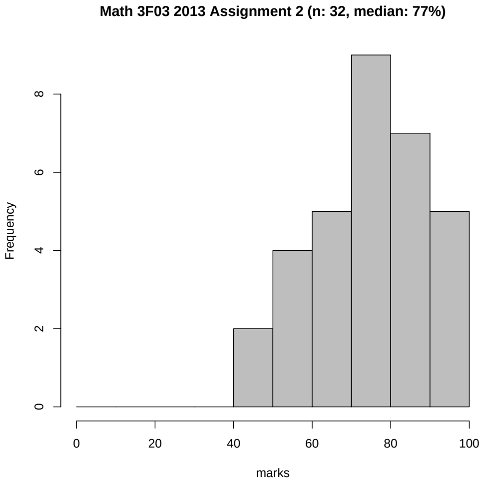
Location: T29 / 101

- Further info may be posted on the course wiki closer to the test date.

Announcements

- **Assignment 2** results much better than Assignment 1.
- Make sure you hand in your entire solution!
- **Assignment 3** due on Friday 25 Oct 2013.
  - **Check course wiki for updates!**
  - **More questions will be posted by the end of the week!**
- Tutorial this Friday, 18 Oct 2013.

Marks Distribution for Assignment 2



- Carefully read instructor's solutions posted on course wiki.
- Check TA comments posted on course wiki.

Genericity

Intuitive notion of "generic" property of an object

- "almost all" objects in a class possess the property
- parameter values chosen at random yield the property

Formal definition of “generic”

Definition (Open Ball in $\mathbb{R}^n$)

The *open ball* of radius ϵ about $X \in \mathbb{R}^n$ is the set

$$B_\epsilon(X) = \{Y \in \mathbb{R}^n : |Y - X| < \epsilon\}.$$

Definition (Open Set in $\mathbb{R}^n$)

A set $U \subset \mathbb{R}^n$ is *open* if for any point $X \in U$ there is an open ball B such that $X \in B \subset U$.

Formal definition of "generic"

Definition (Dense Set)

A set $D \subset S$ is *dense* in S if there are points in D arbitrarily close to each point in S.

Examples:

- $\mathbb{Q}$ is dense in $\mathbb{R}$.
- $\mathbb{Q}^c$ is dense in $\mathbb{R}$.
- $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$.
- $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.
- $\mathbb{Z}^n$ is NOT dense in $\mathbb{Q}^n$.

Formal definition of “generic”

Proposition (Finite intersections of open, dense sets)

If $U_1, \ldots, U_k$ *are each open and dense in* S *then* $U = U_1 \cap \cdots \cap U_k$ *is also open and dense in* S.

Definition (Generic Property)

A property of a mathematical object is *generic* if it is satisfied in an open, dense subset of the space in which the object lives.

Theorem (Having distinct eigenvalues is a generic property)

The set of real $n \times n$ *matrices with distinct eigenvalues is open and dense in the space of all real* $n \times n$ *matrices.*

Qualitative Dynamics of 3D Linear Systems

3D Linear Systems

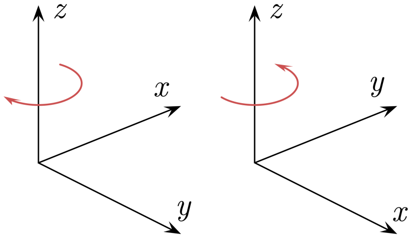
- Real 3×3 matrix A.
- Original coordinates: $X' = AX$
- Canonical coordinates: $Y' = JY$ with $J = T^{-1}AT$.
- Three eigenvalues: λ_1, λ_2, λ_3.
- **Either** $\lambda_j \in \mathbb{R}\ \forall j$ **OR** $\lambda_1 \in \mathbb{R}$, $\mathrm{Im}(\lambda_2) \neq 0$, $\lambda_3 = \overline{\lambda_2}$.
- Variety of cases...

3D Linear Systems

Eigenvalue types and their effects on phase portraits

- $\lambda_j < 0 \implies \exists$ stable direction
- $\lambda_j > 0 \implies \exists$ unstable direction
- $\lambda_j < 0$ for $j = 1, 2 \implies$ stable (planar) subspace
- $\lambda_j > 0$ for $j = 1, 2 \implies$ unstable (planar) subspace
- $\mathrm{Im}(\lambda_j) \neq 0 \implies$ oscillation
- Generalizes to higher dimensions, but harder to visualize...

Handedness of Coordinate Systems



Left-handed Right-handed

- Always assume RIGHT-HANDED coordinate system.
- Viewed from above ($z > 0$), right-handed coordinate systems have the usual x-y orientation.

Distinct Eigenvalues: Saddles

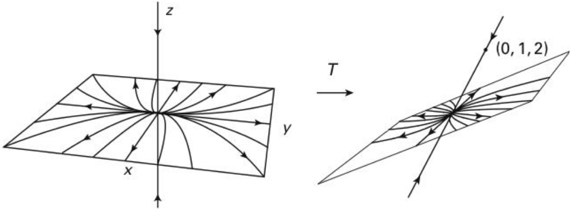


Figure 6.1 The stable and unstable subspaces of a saddle in dimension 3. On the left, the system is in canonical form.

- $\lambda_3 < 0 < \lambda_2 < \lambda_1$ $\qquad V_3 = (0,1,2)^{\mathsf{T}}$
- How do we know $\lambda_2 < \lambda_1$?
- What happens as $x \to 0$, *i.e.*, as $t \to -\infty$?
  Look at x-y plane from above... $\qquad |y/x| \to \infty$
- $\left|\frac{y(t)}{x(t)}\right| \propto \frac{e^{\lambda_2 t}}{e^{\lambda_1 t}} = e^{(\lambda_2-\lambda_1)t} \to \infty$ as $t \to -\infty \implies \lambda_2 < \lambda_1$

Distinct Eigenvalues: Sinks

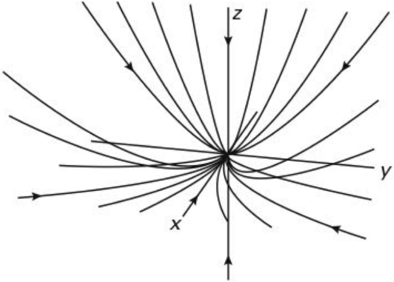


Figure 6.2 A sink in three dimensions.

(in canonical coordinates)

- $\lambda_j < 0$ for $j = 1, 2, 3$
- As $t \to \infty$, $y/x \to 0$, $z/y \to 0$, $\quad \Longrightarrow \quad \lambda_3 < \lambda_2 < \lambda_1$

Distinct Eigenvalues: Spiral Centre

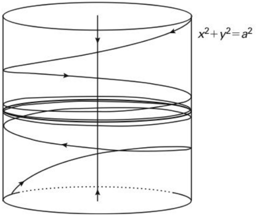


Figure 6.3 The phase portrait for a spiral center.

- $\lambda_3 < 0$
- $\lambda_{1,2} = \pm i\beta \;\; (\beta \neq 0)$
- Rotation is clockwise viewed from above, *i.e.*, in the x-y plane
- Sign of β?

- Jordan block associated with $\lambda_{1,2}$ is $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$
- Given clockwise rotation, does it follow that bottom left entry $(-\beta)$ is negative, so $\beta > 0$? **No.**
- $\lambda_{1,2}$ associated with x-y plane, not x or y specifically
  - Better to call them $\lambda_\pm$.
  - No meaning to sign(β).
  - Bottom left entry in *original coordinates* determines sense of rotation.
  - Eigenvalues do not determine sense of rotation.

Distinct Eigenvalues: Spiral Saddle

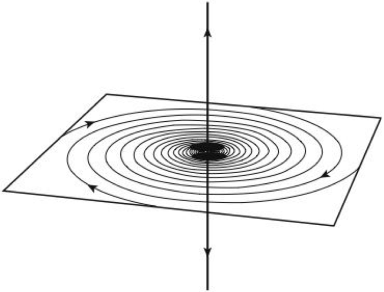


Figure 6.4 A spiral saddle in canonical form.

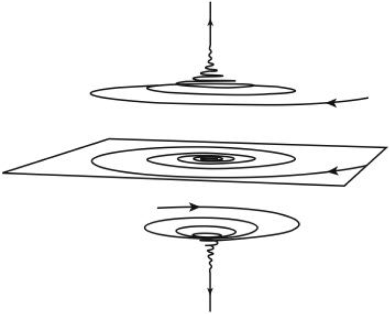


Figure 6.5 Typical solutions of the spiral saddle tend to spiral toward the unstable line.

- $\lambda_3 > 0, \quad \lambda_{1,2} = \alpha \pm i\beta$
- $\alpha < 0$, $\beta \neq 0$, sign(β) undetermined
- What would happen if $\lambda_3 = 0$? $\lambda_3 < 0$? $\alpha = 0$? $\alpha > 0$?

Repeated Eigenvalues: "Improper" Sink

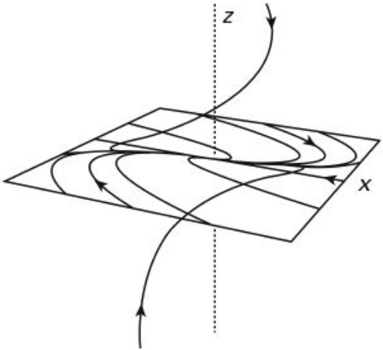


Figure 6.9 The phase portrait for repeated real eigenvalues.

- $\lambda_{1,2,3} = \lambda < 0$
- $\exists!$ invariant line (x axis)
- $\exists!$ invariant plane (x-y plane)
- Hard to infer $\exists!$ invariant plane without knowing the possible forms of solutions with triply-repeated eigenvalue...

Repeated Eigenvalues: "Improper" Sink

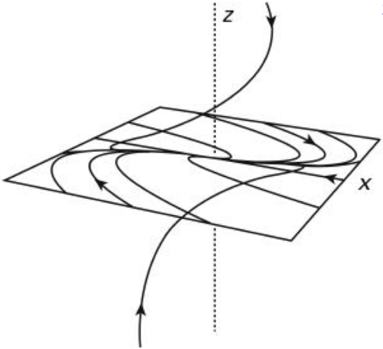


Figure 6.9 The phase portrait for repeated real eigenvalues.

$\exists!$ invariant line $\Longrightarrow A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

$$\Longrightarrow X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix}$$

$\Longrightarrow$ If $X(0)$ in x-y plane then $X(t)$ in x-y plane $\forall t$

Repeated Eigenvalues: "Improper" Sink

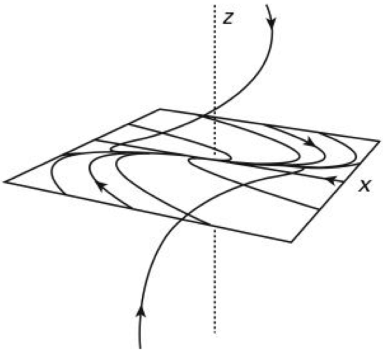


Figure 6.9 The phase portrait for repeated real eigenvalues.

How would the phase portrait for

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

differ from the phase portrait shown here?

What about

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} ?$$

How would its phase portrait differ?