McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 19
More on Matrix Exponentials
Monday 21 October 2013

Announcements

Test #1

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

Location: T29 / 101

- Further info may be posted on the course wiki closer to the test date.

Announcements

- **Assignment 3** due THIS Friday 25 Oct 2013 @ 1:30pm.
- Slides for Lecture 12 updated
  (extra slide on sense of rotation).
- **BASIC NOTIONS SEMINAR:**
  - Presented by: Youzhou Zhou
  - Title: **"Coin Flipping"**
  - When: Thursday, 24 October 2013, 5:30–6:30pm
  - Where: HH/312

Abstract

Coin flipping is the simplest model in probability theory. Simple as it may be, it can generate almost all the important theorems in probability theory. The model is quite straightforward: one can flip a coin independently and repeatedly. For each flipping, a head can show up with probability p; therefore, a tail shows up with probability $1 - p$. If we look at different aspects of this model, we can find different distributions. Almost all the distributions in probability theory can be obtained from this model.

In this talk, based on this model, I will talk about the law of large numbers, the Monte-Carlo method and large deviation principle. Also the Black-Scholes formula will be proved from the binomial tree model. Lastly, by considering the excursions of random walk, the Poisson-Dirichlet distribution may be mentioned. Interestingly, the number factorization can also be described by the Poisson-Dirichlet distribution, which now has found its many applications in finance, spin glass, machine learning and nonparametric Bayesian statistics.

The Matrix Exponential: Convergence

Definition (Matrix Exponential e^A)

If A is an $n \times n$ matrix then

$$e^A = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} A^k \,,$$

where $A^0 \equiv I_n$, the $n \times n$ identity matrix.

Proposition (Convergence of matrix series defining e^A)

If A is any real $n \times n$ matrix then the power series that defines e^A converges, i.e., the limit exists and is a real $n \times n$ matrix.

The Matrix Exponential: Convergence

Lemma (The comparison test for series of real numbers)

Suppose that

$$|\alpha_k| \leq \beta_k \quad \textit{for all } k \in \mathbb{N}.$$

Then if $\sum_{k=1}^{\infty} \beta_k$ converges, so does $\sum_{k=1}^{\infty} |\alpha_k|$, i.e., the sequence $\{\alpha_k\}$ is absolutely summable.

- How can this help us prove convergence of e^A?
- It will help if we can bound each entry of $\frac{1}{k!}A^k$ by a summable sequence of real numbers.

The Matrix Exponential: Convergence

Proof of convergence of matrix series defining e^A.

Write $A = (a_{ij})$ and $A^k = (a_{ij}(k))$.
Let $a = \max_{ij} |a_{ij}|$, the maximum magnitude of entries of A.

$$\begin{aligned}
|a_{ij}(2)| &= \left| ij \text{ entry of } A^2 \right| \\
&= \left| \sum_{\ell=1}^{n} a_{i\ell} a_{\ell j} \right| \\
&\le \sum_{\ell=1}^{n} |a_{i\ell} a_{\ell j}| && (\triangle \text{ inequality}) \\
&\le \sum_{\ell=1}^{n} a^2 && (\text{definition of } a) \\
&= na^2 && \textit{i.e.,} \quad |a_{ij}(2)| \le na^2
\end{aligned}$$

The Matrix Exponential: Convergence

Proof of convergence of matrix series defining e^A (CONTINUED).

$$
\begin{aligned}
|a_{ij}(3)| &= \left|ij \text{ entry of } A^3\right| &&= \left|ij \text{ entry of } A^2 \cdot A\right| \\
&= \left|\sum_{\ell=1}^{n} a_{i\ell}(2) a_{\ell j}\right| &&\leq \sum_{\ell=1}^{n} |a_{i\ell}(2)| \cdot |a_{\ell j}| \\
&\leq \sum_{\ell=1}^{n} na^2 \, |a_{\ell j}| &&= na^2 \sum_{\ell=1}^{n} |a_{\ell j}| \\
&\leq na^2 \sum_{\ell=1}^{n} a &&= na^2 \cdot na \\
&= n^2 a^3
\end{aligned}
$$

$\square$

The Matrix Exponential: Convergence

Proof of convergence of matrix series defining e^A (CONTINUED).

Similarly, we have $\quad |a_{ij}(k)| \le n^{k-1}a^k \le (na)^k.$

$$\therefore \left|\frac{1}{k!}a_{ij}(k)\right| \le \frac{1}{k!}\big(na\big)^k$$

But $$\sum_{k=0}^{N}\frac{1}{k!}\big(na\big)^k \to e^{na} \quad \forall n \in \mathbb{N}, \quad \forall a \in \mathbb{R}\,,$$

so the comparison test implies that

$$\sum_{k=0}^{N}\frac{1}{k!}a_{ij}(k) \quad \text{converges absolutely} \quad \forall i,j \in \{1,\ldots,n\}.$$

We call the resulting well-defined matrix e^A. $\square$

Important Properties of Matrix Exponentials

Proposition

For $n \times n$ matrices A, B, T, where T is invertible:

(i) $e^{T^{-1}AT} = T^{-1}e^{A}T$

(ii) $AB = BA \implies e^{A+B} = e^{A}e^{B}$

(iii) e^{A} *is invertible and* $(e^{A})^{-1} = e^{-A}$

Proof that (iii) follows from (ii).

$$e^{A}e^{-A} = e^{A+(-A)} = e^{0_n} = I_n\,,$$

because the zero matrix is a diagonal matrix:

$$0_n = \operatorname{diag}(0,\ldots,0) \implies e^{0_n} = \operatorname{diag}(e^{0},\ldots,e^{0}) = I_n\,.$$

$\therefore\ (e^{A})^{-1} = e^{-A}$. $\square$

Important Properties of Matrix Exponentials

Proof of (i) $e^{T^{-1}AT} = T^{-1}e^{A}T$.

$$
\begin{aligned}
e^{T^{-1}AT} &= \lim_{N\to\infty} \sum_{k=0}^{N} \frac{1}{k!} \left(T^{-1}AT\right)^{k} \\
&= \lim_{N\to\infty} \sum_{k=0}^{N} \frac{1}{k!} \big(\underbrace{T^{-1}AT\,T^{-1}AT \cdots T^{-1}AT}_{k \text{ times}}\big) \\
&= \lim_{N\to\infty} \sum_{k=0}^{N} \frac{1}{k!} T^{-1}A^{k}T \\
&= T^{-1}\left[\lim_{N\to\infty} \sum_{k=0}^{N} \frac{1}{k!} A^{k}\right] T \\
&= T^{-1}e^{A}T
\end{aligned}
$$

$\square$

Important Properties of Matrix Exponentials

(ii) $AB = BA \implies e^{A+B} = e^A e^B$

- Proof of (ii) is non-trivial.
- Let's just verify that the content of the result is non-trivial.

Important Properties of Matrix Exponentials

Example

If $AB \neq BA$ then it is possible that $e^{A+B} \neq e^A e^B$.
(Exercise 13, page 137.)

Try to find the simplest possible example.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad AB \neq BA\,.$$

$$A^2 = B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad A^k = B^k = 0 \quad \forall k \geq 2.$$

$$e^A = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e^B = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$e^A e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Important Properties of Matrix Exponentials

Example (CONTINUED)

If $AB \neq BA$ then it is possible that $e^{A+B} \neq e^A e^B$.

$$A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(A+B)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$(A+B)^3 = A + B$$

$$(A+B)^4 = (A+B)^2 = I_2$$

$$(A+B)^k = \begin{cases} I_2, & k \text{ even} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k \text{ odd} \end{cases}$$

Important Properties of Matrix Exponentials

Example (CONTINUED)

If $AB \neq BA$ then it is possible that $e^{A+B} \neq e^A e^B$.

$$\begin{aligned}
\therefore\ e^{A+B} &= \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k \\
&= \sum_{k \text{ even}} \frac{1}{k!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k \text{ odd}} \frac{1}{k!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh 1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sinh 1 \\
&= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}
\end{aligned}$$

Important Properties of Matrix Exponentials

Example (CONTINUED)

If $AB \neq BA$ then it is possible that $e^{A+B} \neq e^A e^B$.

Summarizing, with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we have

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad e^B e^A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$e^{A+B} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

$$\implies e^{A+B} \neq e^A e^B \neq e^B e^A \neq e^{A+B} \qquad \square$$

Important Properties of Matrix Exponentials

In fact, any of the following is possible if $AB \neq BA$.

(a) $e^A e^B = e^B e^A = e^{A+B}$
(b) $e^A e^B = e^B e^A \neq e^{A+B}$
(c) $e^A e^B \neq e^B e^A = e^{A+B}$
(d) $e^A e^B \neq e^B e^A \neq e^{A+B}$

Is there a (non-trivial) condition on the *entries* of A and B that guarantees that $e^{A+B} = e^A e^B$? How about $e^A e^B = e^B e^A$?

Proposition (Condition for $e^A e^B = e^B e^A$)

If A and B are $n \times n$ matrices of **algebraic numbers** *then* $e^A e^B = e^B e^A \iff AB = BA.$

For some hints. . .

See Horn and Johnson "Topics in Matrix Analysis", page 437. □