McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 21
Foundations of Nonlinear Systems
Wednesday 23 October 2013

Announcements

Test #1

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

Location: T29 / 101

- Further info may be posted on the course wiki closer to the test date.

Announcements

- **Assignment 3** due THIS Friday 25 Oct 2013 @ 1:30pm.
- TUTORIAL this Friday, 25 Oct 2013.
- **BASIC NOTIONS SEMINAR:**
  - Presented by: Youzhou Zhou
  - Title: **"Coin Flipping"**
  - When: Thursday, 24 October 2013, 5:30–6:30pm
  - Where: HH/312

Abstract

Coin flipping is the simplest model in probability theory. Simple as it may be, it can generate almost all the important theorems in probability theory. The model is quite straightforward: one can flip a coin independently and repeatedly. For each flipping, a head can show up with probability p; therefore, a tail shows up with probability $1 - p$. If we look at different aspects of this model, we can find different distributions. Almost all the distributions in probability theory can be obtained from this model.

In this talk, based on this model, I will talk about the law of large numbers, the Monte-Carlo method and large deviation principle. Also the Black-Scholes formula will be proved from the binomial tree model. Lastly, by considering the excursions of random walk, the Poisson-Dirichlet distribution may be mentioned. Interestingly, the number factorization can also be described by the Poisson-Dirichlet distribution, which now has found its many applications in finance, spin glass, machine learning and nonparametric Bayesian statistics.

Dynamical Systems

Definition

A *smooth dynamical system* on $\mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t, X) = \phi_t(X)$ satisfies

1. $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function: $\phi_0(X) = X$;
2. The composition $\phi_t \circ \phi_s = \phi_{t+s}\ \forall t, s \in \mathbb{R}$.

- Note that the definition implies that, for any t, ϕ_t has an inverse, namely ϕ_{-t} (just set $s = -t$ in condition 2).
- Differential equations that yield unique solutions define dynamical systems: we will see why today.
- Discrete-time maps (*e.g.*, $x_{t+1} = f(x_t)$) can also define dynamical systems and are also commonly used in science and engineering. However, we then define $\phi : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$.

Peano Existence Theorem

Theorem

Consider the initial value problem (IVP)

$$X' = F(t, X)\,, \qquad X(t_0) = X_0\,,$$

where $t \in \mathbb{R}$ *and* $X \in \mathbb{R}^n$.

Suppose F **is continuous** *in some "open rectangle"* $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$, *i.e., in a set*

$$\mathcal{R} = \{(t, x_1, \ldots, x_n) : \quad t_1 < t < t_2, \quad a_i < x_i < b_i, \quad i = 1, \ldots, n\}$$

that contains the point (t_0, X_0). *Then* **there exists a solution** *to the IVP, defined for times* $t \in [t_0 - h, t_0 + h]$, **for some** $h > 0$.

Fundamental Existence and Uniqueness Theorem

Theorem

Consider the IVP

$$X' = F(t, X)\,, \qquad X(t_0) = X_0\,,$$

where $t \in \mathbb{R}$ *and* $X \in \mathbb{R}^n$.

Suppose F **AND** $\partial F/\partial x_i$**,** $i = 1, \ldots, n$**, are continuous** *in some open rectangle* $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$ *that contains the point* (t_0, X_0). *Then* **there exists a UNIQUE solution** *to the IVP, defined for all* t *in* **some** *closed interval* $[t_0 - h, t_0 + h]$, *where* h *is a positive constant.*

Global Picard Theorem

Theorem

Consider the IVP $\quad X' = F(t, X),\ X(t_0) = X_0, \quad (t, X) \in \mathbb{R} \times \mathbb{R}^n.$

Suppose $t_0 \in [t_1, t_2]$ *and* F **and** $\partial F / \partial x_i$**,** $i = 1, \ldots, n$**, are continuous ON THE INFINITE STRIP**

$$\mathcal{S} = \{(t, X) : \ t_1 \leq t \leq t_2,\ X \in \mathbb{R}^n\}$$

AND there exists a positive constant L**, such that**

$$\left| \frac{\partial F}{\partial x_i}(t, X) \right| \leq L, \quad i = 1, \ldots, n, \quad \forall (t, X) \in \mathcal{S}\,.$$

Then **there exists a unique solution** *to the IVP, defined* **throughout the entire time interval**, *i.e.,* $\forall t \in [t_1, t_2]$.

Existence/Uniqueness Simplified Summary

IVP: $X' = F(t, X), \quad X(t_0) = X_0, \quad (t, X) \in \mathbb{R} \times \mathbb{R}^n$

- F continuous on an open rectangle $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$
  $\implies \exists$ solution through any $(t_0, X_0) \in \mathcal{R}$
- F C^1 on $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$
  $\implies \exists!$ solution through any $(t_0, X_0) \in \mathcal{R}$
- F C^1 and $\frac{\partial F}{\partial X}$ uniformly bounded on strip $\mathcal{S} = [t_1, t_2] \times \mathbb{R}^n$
  $\implies \exists!$ solution throughout the time interval in $[t_1, t_2]$

Note: The summary statements above are simplified, compared to the full statements on previous slides. Rather than "F C^1", the full statements give the weaker hypothesis that "F is C^0 in t and C^1 in X". Even this is a stronger hypothesis than necessary...

Lipschitz continuity

Definition (Lipschitz)

Consider an open set $\mathcal{S} \subset \mathbb{R} \times \mathbb{R}^n$. A function $F : \mathcal{S} \to \mathbb{R}^n$ is **Lipschitz continuous** or simply **Lipschitz** on $\mathcal{S}$ if there exists a constant K such that

$$\|F(t,Y) - F(t,X)\| \leq K \, \|Y - X\| \ ,$$

for all (t,X) and $(t,Y) \in \mathcal{S}$. We call K a **Lipschitz constant** for F on $\mathcal{S}$.

Definition (Locally Lipschitz)

We say F is **locally Lipschitz** on $\mathcal{S}$ if, for each $(t,X) \in \mathcal{S}$ there is an open set $\tilde{\mathcal{S}} \subset \mathcal{S}$ such that $(t,X) \in \tilde{\mathcal{S}}$ and F is Lipschitz on $\tilde{\mathcal{S}}$ (the Lipschitz constant might depend on $\tilde{\mathcal{S}}$).

Existence/Uniqueness Additional Comments

Theorem (on an open set $\mathcal{S}$)

$C^1 \implies$ locally Lipschitz $\implies C^0$.

Theorem (Strengthened Fundamental Theorem)

Consider the IVP $\quad X' = F(t,X),\ X(t_0) = X_0, \quad (t,X) \in \mathbb{R} \times \mathbb{R}^n.$

If F is continuous in t and locally Lipschitz in X, on an open set $\mathcal{S} \subset \mathbb{R} \times \mathbb{R}^n$, then $\exists!$ solution through any $(t_0, X_0) \in \mathcal{S}$.

Extra Credit Challenge

Show that

$$C^0 \;\not\Rightarrow\; \text{locally Lipschitz} \;\not\Rightarrow\; \text{Lipschitz} \;\not\Rightarrow\; C^1$$

and also that

$$C^1 \;\not\Rightarrow\; \text{Lipschitz}$$

Also show that "locally Lipschitz" is not necessary for uniqueness of solutions, *i.e.*, find an (autonomous, one-dimensional) ODE that has a unique solution but is not locally Lipschitz.