McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 12

Classification of Planar Linear Systems

2 October 2013

Announcements

- **Graduate Studies Information Session**
  TODAY: Wednesday, 2 October 2013, 4:30pm
  Hamilton Hall 217 and Math Café

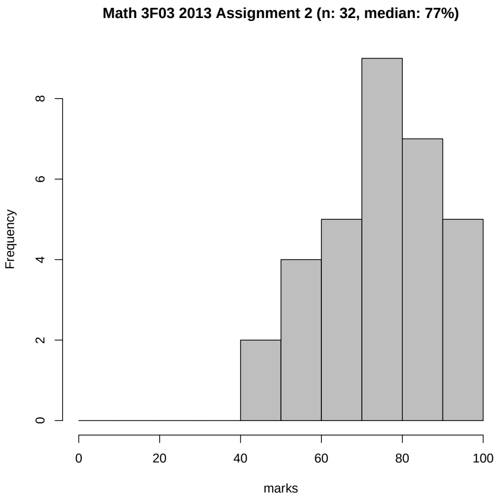
  Agenda:

  1. Information about graduate programs at Mac (Mathematics, Statistics, PhiMac, Computational Science and Engineering) and also Teachers College
  2. Questions from students
  3. Mac graduate students talk about their experiences and plans for future
  4. Pizza, refreshments and socializing

Announcements

- Assignment 2 due this Friday 4 October 2013.
- Assignment 1 Results. . .

Marks Distribution for Assignment 1



- 5% will be added to your mark.
- Check TA comments posted on course wiki.
- Plagiarism is not OK!

Improper cases ($\lambda_1 = \lambda_2$; unique eigendirection)

- Quantitative solution derived in instructor's solutions to 2012 Assignment 2 (Exercise 7, page 58).

$$X(t) = x_0 e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix} + y_0 e^{\lambda t}\begin{pmatrix}t\\1\end{pmatrix} = e^{\lambda t}\left(x_0\begin{pmatrix}1\\0\end{pmatrix} + y_0 t\begin{pmatrix}1\\\frac{1}{t}\end{pmatrix}\right)$$

- If $\lambda = 0$ then

$$X(t) = \begin{pmatrix}x_0 + y_0 t\\ y_0\end{pmatrix} \qquad \text{“Improper Line”}$$

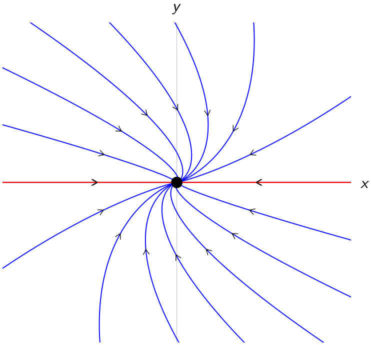
- If $\lambda \neq 0$ then asymptotic behaviour (as $t \to \infty$) is

$$X(t) \sim (x_0 + y_0 t)e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix}$$

i.e., motion is asymptotically tangent to the (unique) eigendirection (“Improper Sink or Source”).

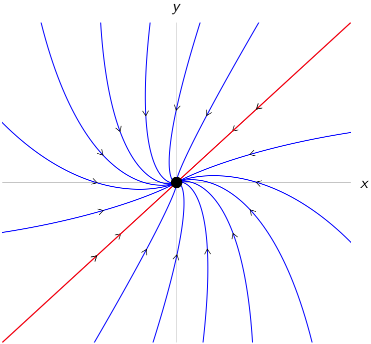
Case $\lambda_1 = \lambda_2 < 0$, $\exists!$ eigendirection ("Improper Sink")

Canonical coordinates:



$$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

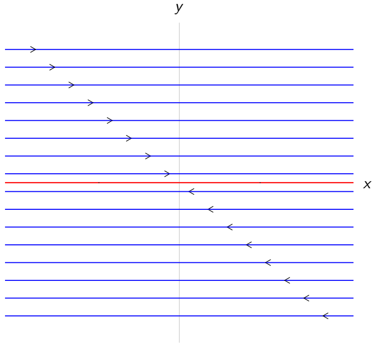
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

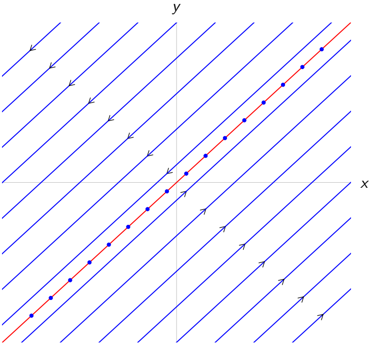
Case $\lambda_1 = \lambda_2 = 0$, $\exists!$ eigendirection ("Improper line")

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Classification of Planar Linear Systems

General linear (homogenous) planar system:

$$X' = AX\,, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}\,, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \qquad (*)$$

- We have considered all possible *eigensystem types*.
- Do we have the whole story for planar linear systems?
  Is there a *qualitative* difference between phase portraits of systems with the same eigensystem type?
- Let's first summarize what we've established so far.

Real Jordan Canonical forms of real 2×2 matrices

Any real 2×2 matrix A can be converted, via a change of basis, to one of the following three forms:

$$\text{(i)} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda_1, \lambda_2, \alpha, \beta, \lambda \in \mathbb{R}$, $\beta \neq 0$.

Eigenvalues and Eigenvectors in each case:

(i) $\lambda_1, \lambda_2 \in \mathbb{R}$ (not necessarily distinct)
$\exists$ basis of *real* eigenvectors

(ii) $\lambda_\pm = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ (distinct, conjugates)
$\exists$ basis of *complex* eigenvectors; $\nexists$ real eigenvectors

(iii) $\lambda \in \mathbb{R}$ (repeated eigenvalue)
$\exists!$ eigendirection (alg$(\lambda) = 2$, geom$(\lambda) = 1$)

Real Jordan Canonical forms of real 2×2 matrices

Any real 2×2 matrix A can be converted, via a change of basis, to one of the following three forms:

$$\text{(i)} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda_1, \lambda_2, \alpha, \beta, \lambda \in \mathbb{R}$, $\beta \neq 0$.

Transforming from original to canonical coordinates:

$$J = T^{-1}AT$$

(i) $T = (V_1\ V_2)$, where $AV_i = \lambda_i V_i$ for $i = 1, 2$

(ii) $T = (V_{\Re}\ V_{\Im})$, where $V = V_{\Re} + iV_{\Im}$ and $AV = (\alpha + i\beta)V$

(iii) $T = (V\ U)$, where $AV = \lambda V$ and $AU = \lambda U + V$
(U is called a **generalized eigenvector**)

Example: Calculation of Generalized Eigenvector

$$A_{\text{orig}} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$

- $p(\lambda) = \det(A_{\text{orig}} - \lambda I) = (\lambda + 2)^2 \implies \lambda = -2, \ \text{alg}(\lambda) = 2.$
- $A_{\text{orig}} V = -2V \implies V \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ($\exists!$ eigenvector)
- Find *generalized eigenvector* to complete basis with V.
- $A_{\text{orig}} U = -2U + V \implies U \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Take $V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and define $T = (V \ U)$.
- Then $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $T^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

  and $A = T^{-1} A_{\text{orig}} T = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$

Phase Portrait Types for $X' = AX$ in the Plane

- We can certainly restrict attention to matrices in Real Jordan Canonical Form
  - because all cases are equivalent to these after linear coordinate transformation.
- We considered cases defined by the signs of the real parts and existence of imaginary parts of the eigenvalues.
  - Is this completely comprehensive?
  - For example, is there a qualitative difference in the dynamics of $X' = AX$ for these possible A:

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

  - In fact, these are all equivalent *topologically* (which we'll discuss in greater detail soon)
  - Meanwhile, let's consider another classification...

Classification in the trace-determinant plane

Consider $X = AX$ with arbitrary real 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Characteristic polynomial:

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - (\mathrm{tr}A)\lambda + \det A \end{aligned}$$

$\therefore$ **Eigenvalues can be written**

$$\lambda_\pm = \frac{1}{2}\left(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}\right),$$

Classification in the trace-determinant plane

$$\lambda_\pm = \frac{1}{2}\left(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}\right)$$
$$\implies \lambda_+ + \lambda_- = \mathrm{tr}A$$
$$\lambda_+ \times \lambda_- = \frac{1}{4}\left((\mathrm{tr}A)^2 - [(\mathrm{tr}A)^2 - 4\det A]\right) = \det A$$

This generalizes to $\mathbb{R}^n$:

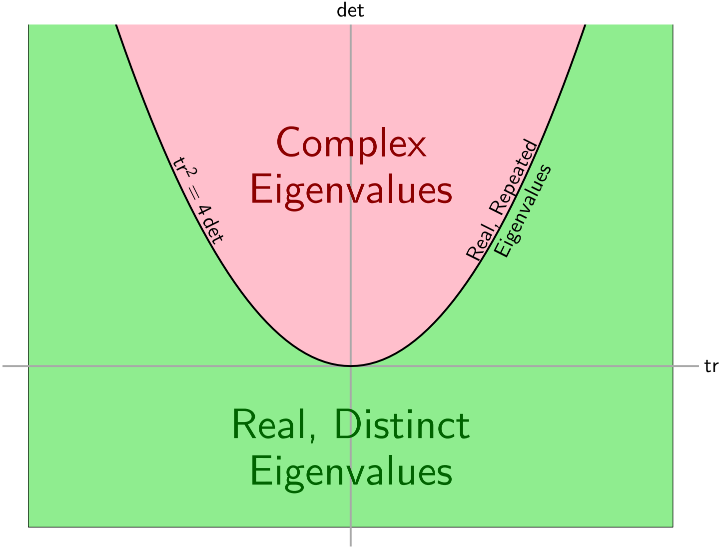
- $\mathrm{tr}A =$ sum of eigenvalues
- $\det A =$ product of eigenvalues

$\therefore$ Lots of qualitative info from trace and determinant.

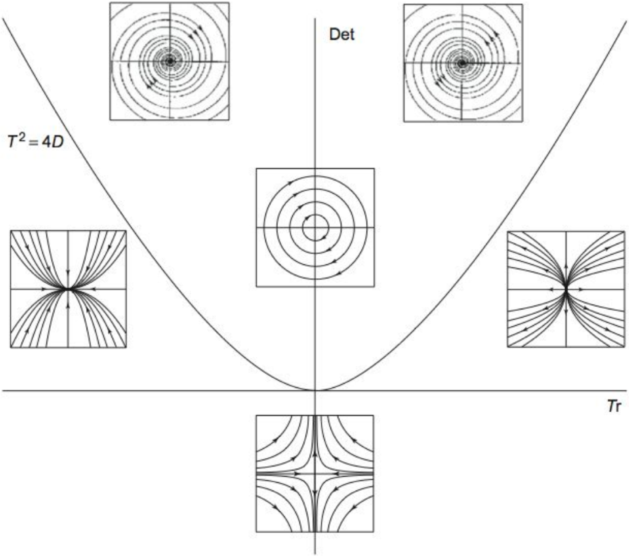
Examples in the plane:

- $\det A < 0 \implies \lambda_- < 0 < \lambda_+ \implies$ saddle
- $\det A > 0$ & $\mathrm{tr}A = 0 \implies \lambda_\pm = i\sqrt{|\det A|} \implies$ centre
- $\det A > 0$ & $(\mathrm{tr}A)^2 > 4\det A \implies 0 < \lambda_- < \lambda_+ \implies$ source

Classification in the trace-determinant plane



Classification in the trace-determinant plane



What the trace-determinant plane hides

- Trace-Determinant plane shows a *two-parameter* bifurcation diagram for planar linear system (parameters are $\mathrm{tr} = a + d$ and $\det = ad - bc$).
- But A has four parameters (a, b, c, d, or $\mathrm{tr}, \det$ and two others).
- In fact, tr-det plane does not tell the whole story of possible dynamics of linear systems.
- What are we missing in the tr-det plane?

What the trace-determinant plane hides

- tr-det plane suppresses speed of attraction to (or repulsion from) equilibrium.
- tr-det plane does not represent sense of rotation.
- Consider $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$.
  - This has same tr and det $\forall b \in \mathbb{R}$.
  - $\therefore$ Lies in same position on tr-det plane for any $b \in \mathbb{R}$.
  - But $\exists$ bifurcation as b crosses 0 (*e.g.*, number of invariant lines changes from 1 to ∞ to 1).
  - This is a special type of bifurcation, since another condition ($\lambda_1 = \lambda_2$ in this case) must already be satisfied.
    - *e.g.*, Go from $(\lambda_1, \lambda_2) = (1, 1 - \varepsilon)$ to $(\lambda_1, \lambda_2) = (1, 1 + \varepsilon)$, crossing $\text{tr}^2 = 4\,\text{det}$ curve at $(\lambda_1, \lambda_2) = (1, 1)$.
    - bifurcation occurs by varying the top-right entry of A while fixing $(\lambda_1, \lambda_2) = (1, 1)$ [*i.e.*, fixing $(\text{tr}, \text{det}) = (2, 1)$].

Determining sense of rotation from 2×2 matrix entries

$$X' = AX\,, \qquad X(0) = X_0\,, \qquad A = \begin{pmatrix} a_{12} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

- Suppose A has complex eigenvalues $\implies$ spiral or centre.
- Is rotation clockwise or counter-clockwise?
- Eigenvalues (or tr and det) do *not* tell us.
- Consider fate of initial condition on x-axis, $X_0 = (1, 0)$.
- Direction of flow at X_0 is given by the vector field:

$$AX_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}.$$

- AX_0 points up if $a_{21} > 0$ and down if $a_{21} < 0$.
- $\text{sign}(a_{21})$ determines sense of rotation.
- What does a_{11} determine?
- See problem 3 on 2012 Assignment 2 (Exercise 9, page 59).