Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 4
Models of Harvesting
11 September 2013

Announcements

- NO CLASS this Friday 13 Sep 2013.
- Assignment 1 will be posted on the course wiki on Friday.
- **Putnam Mathematical Competition:** There will be an organizational meeting on Thursday 19 Sep 2013 in HH 410.

  The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. More information can be found at `http://math.scu.edu/putnam` or on the undergraduate page of the Math department's website. This year's competition will occur on 7 Dec 2013. If you are interested in participating or learning more, drop by on Thursday 19 Sep and/or send email to Prof. Bradd Hart (`hartb@mcmaster.ca`).

Constant rate harvesting

Motivation: Daily Quota, *e.g.*, maximum number of fish per day

$$\frac{dx}{dt} = f_h(x) = rx\left(1 - \frac{x}{K}\right) - h\,, \qquad x(0) = x_0$$

- h is the number of fish caught per unit time (assumed strictly constant).
- RHS is C^1 everywhere $\implies$ $\exists!$ solution $\forall x_0 \in \mathbb{R}$.
- But note: $\quad \max\left(rx\left(1 - \frac{x}{K}\right)\right) = \frac{rK}{4}$ (which occurs for $x = K/2$).
- $\therefore$ If $h > \frac{rK}{4}$ then $x \to -\infty$ as $t \to \infty$ (for any x_0).
- Whoops!

A general principle

A population model should not have any solutions that start positive and become negative!

- To be biologically sensible, the model must satisfy

$$\left.\frac{dx}{dt}\right|_{x=0} \geq 0\,.$$

- But in the constant rate harvesting model,

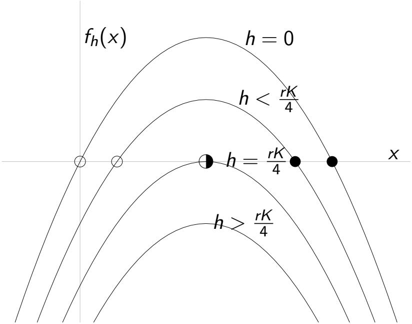
$$\left.\frac{dx}{dt}\right|_{x=0} = f_h(0) = -h < 0 \quad \forall h > 0\,.$$

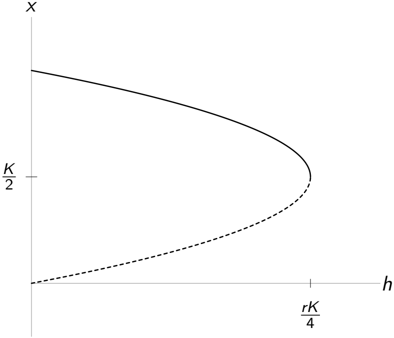
- So for any $h > 0$ this model can display absurd dynamics!
- But the model is perfectly sensible mathematically, and useful to analyze qualitatively as a simple example.

Constant rate harvesting

Equilibria occur at $x_{\pm} = \frac{1}{2}\left(K \pm \sqrt{K(K-(4h/r))}\right)$.

Graphs of $f_h(x)$ $\Longrightarrow$ Bifurcation Diagram wrt h





Note: Curves in bifurcation diagram are $x_-(h)$ (dashed) and $x_+(h)$ (solid). Equivalently: $h(x) = rx(1 - x/K)$.

Constant Effort Harvesting

$$\frac{dx}{dt} = f_E(x) = rx\left(1 - \frac{x}{K}\right) - Ex\,, \qquad x(0) = x_0$$

- Instead of constant rate h, here we have constant *per capita* rate E.
- Mathematically sensible: $f_E(x)$ is C^1 everywhere $\implies \exists!$ solution $\forall x_0 \in \mathbb{R}$.
- Biologically well-posed: population cannot become negative $\because f_E(0) = 0$.
- What happens?