Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 26
Equilibria, Nullclines and Linearization Theorem
Friday 8 November 2013

Announcements

- **Assignment 4 delayed:**
  - Will be posted before midnight tonight.
  - Due Friday 15 Nov 2013.
- **Midterm Test #1:**
  - Marking still in progress.
- **Infectious disease slides and animations from last class:**
  - See my TEDx talk posted at `http://www.math.mcmaster.ca/`

Equilibria

Consider a general autonomous ODE,

$$X' = F(X), \qquad X \in \mathbb{R}^n. \tag{$\heartsuit$}$$

Write out component-wise:

$$\begin{aligned} x_1' &= f_1(x_1, \ldots, x_n) \\ x_2' &= f_2(x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, \ldots, x_n) \end{aligned}$$

Definition (Equilibrium)

An **equilibrium** of ($\heartsuit$) is a point $X^* \in \mathbb{R}^n$ where $F(X^*) = 0$, *i.e.*, $x_1' = x_2' = \cdots = x_n' = 0$.

Nullclines

Definition (Nullcline)

A **nullcline** of ($\heartsuit$) is a curve (or more generally a hypersurface) where ONE component of the vector field vanishes, *i.e.*, $\exists j \in \{1, \ldots, n\}$ such that $x_j' = 0$, *i.e.*, $f_j(x_1, \ldots, x_n) = 0$.

- In simple cases, we can solve for x_j in terms of the other x_ks to get an explicit formula for the nullcline. (In general, we just get an implicit algebraic relationship that may not be possible to solve for x_j.)
- **Points where nullclines $\forall j$ intersect are equilibria.**
- The vector field is $\perp$ x_j-axis along an x_j nullcline.
- Nullclines are very helpful for constructing phase portraits of nonlinear systems.

Nullclines of planar systems

In the plane, $X' = F(X)$ can be written

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

- x nullcines: $f(x,y) = 0$
  - $x' = 0 \implies$ vector field is strictly $\uparrow$ or $\downarrow$.
  - Vector field is strictly $\uparrow$ or $\downarrow$ ONLY on x nullclines.
  - x nullclines divide the plane into regions where the vector field points left or right.
- y nullcines: $g(x,y) = 0$
  - $y' = 0 \implies$ vector field is strictly $\leftarrow$ or $\rightarrow$.
  - Vector field is strictly $\leftarrow$ or $\rightarrow$ ONLY on y nullclines.
  - y nullclines divide the plane into regions where the vector field points up or down.

Nullclines of planar systems

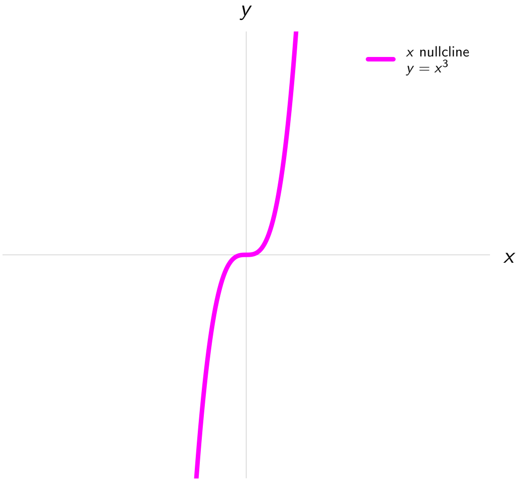
$\therefore$ If we draw all nullclines of a planar system (*i.e.*, all x nullclines and all y nullclines) ten we divide the plan into **basic regions** in which:

- The vector field is never vertical or horizontal.
- The vector field points into ONE quadrant throughout the region (*i.e.*, NE, NW, SE or SW).

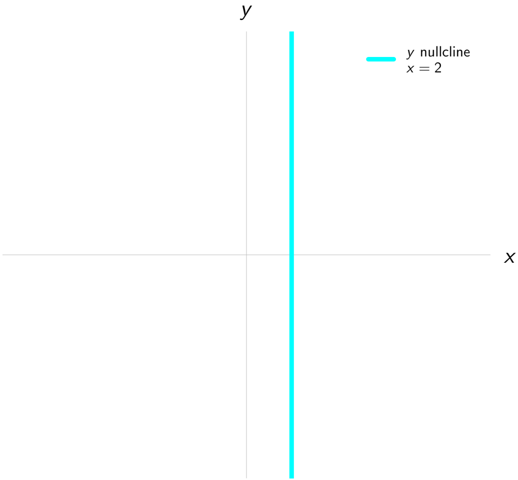
Example (1)

$$x' = y - x^3$$
$$y' = x - 2$$

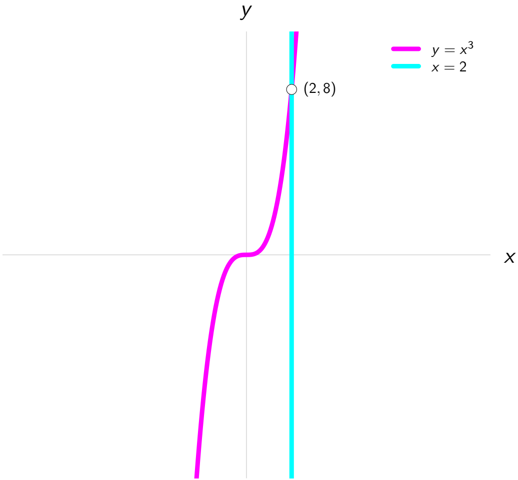
Example (1) $\quad x' = y - x^3$, $\; y' = x - 2 \quad$ x nullcline



Example (1) $\quad x' = y - x^3 \quad y' = x - 2 \quad$ y nullcline



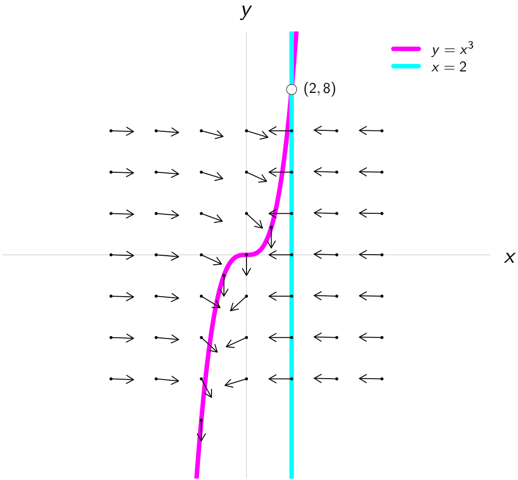
Example (1) $\quad \begin{aligned} x' &= y - x^3 \\ y' &= x - 2 \end{aligned} \quad$ all nullclines



Example (1) $\quad \begin{aligned} x' &= y - x^3 \\ y' &= x - 2 \end{aligned}$ Flow in basic regions

- The vector field points into ONE quadrant throughout a basic region (*i.e.*, NE, NW, SE or SW).
- Pick at least one point in each region (say X_i in region i).
- At each X_i, plot direction of $F(X_i)$, *i.e.*, $\frac{F(X_i)}{\|F(X_i)\|}$.
- Plotting many such arrows gives the direction field.

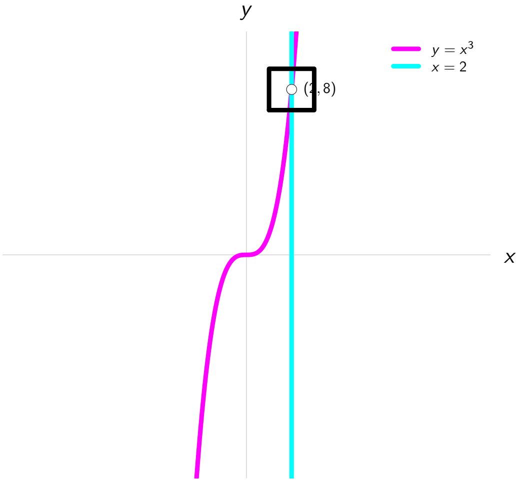
Example (1) $\quad x' = y - x^3 \quad y' = x - 2 \quad$ nullclines & direction field



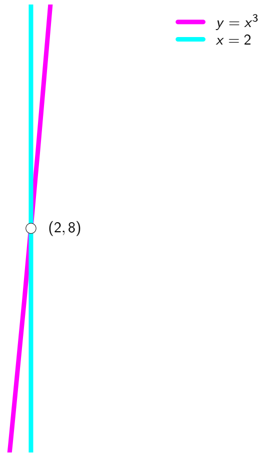
Example (1)

$$x' = y - x^3$$
$$y' = x - 2$$

zoom region



Example (1) $\begin{aligned} x' &= y - x^3 \\ y' &= x - 2 \end{aligned}$ zoomed in



Example (1) $\quad \begin{aligned} x' &= y - x^3 \\ y' &= x - 2 \end{aligned}$ $\quad$ equilibrium analysis

- Linearization about X_*: $\quad (X - X_*)' = X' = DF_{X_*}(X - X_*)$.

$$X_* = \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \qquad DF_{(x,y)} = \begin{pmatrix} -3x^2 & 1 \\ 1 & 0 \end{pmatrix}, \qquad DF_{(2,8)} = \begin{pmatrix} -12 & 1 \\ 1 & 0 \end{pmatrix}$$

- $\det\left(DF_{(2,8)}\right) = -1 < 0$
  $\implies$ eigenvalues have opposite signs $\implies$ saddle.

- Eigenvalues: $\lambda_\pm = -6 \pm \sqrt{37} \simeq \{-12.083, 0.083\}$
  N.B. $\quad |\lambda_-| > 100 \times |\lambda_+|$.

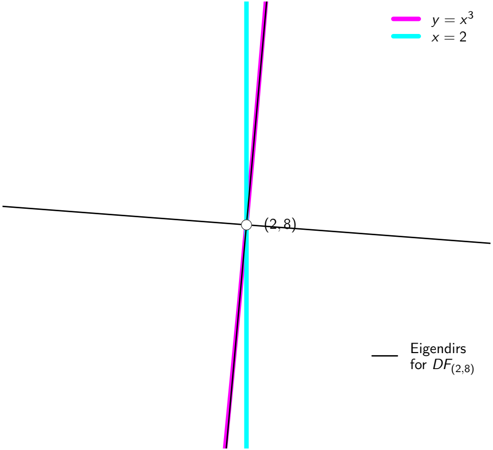
- Eigenvectors:
  $$V_\pm = \begin{pmatrix} -6 \pm \sqrt{37} \\ 1 \end{pmatrix} \simeq \left\{ \begin{pmatrix} 0.083 \\ 1 \end{pmatrix}, \begin{pmatrix} -12.083 \\ 1 \end{pmatrix} \right\}$$

- Eigendirections are orthogonal: $V_+ \cdot V_- = 0$

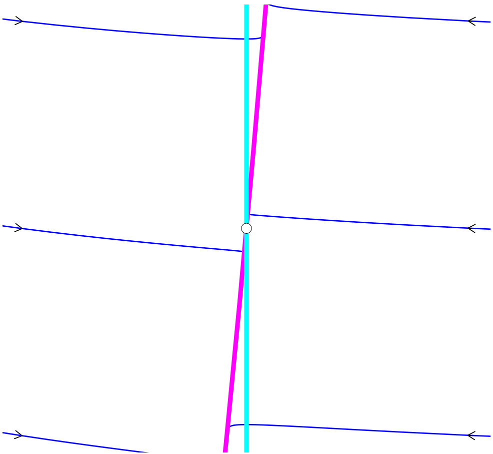
Example (1)

$$x' = y - x^3$$
$$y' = x - 2$$

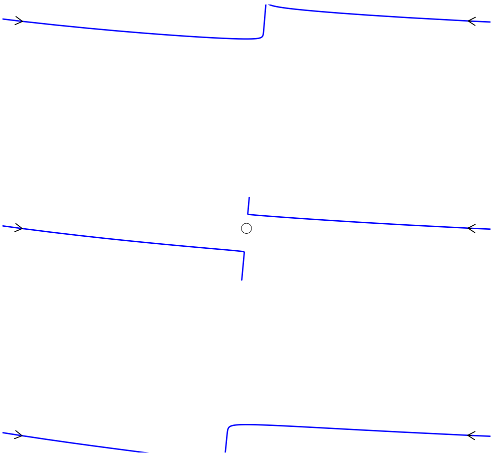
"eigenzoom"



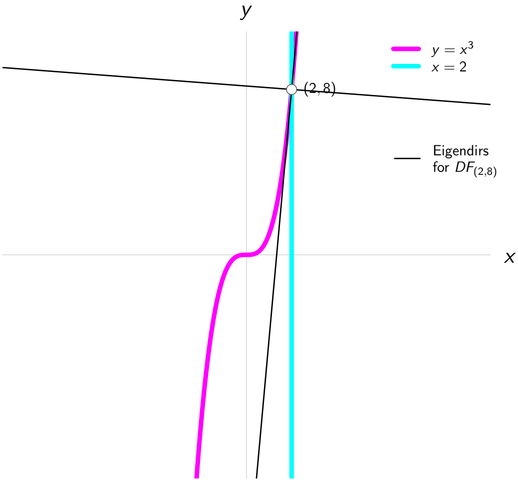
Example (1) $\quad x' = y - x^3 \quad y' = x - 2 \quad$ nullclines & phase portrait



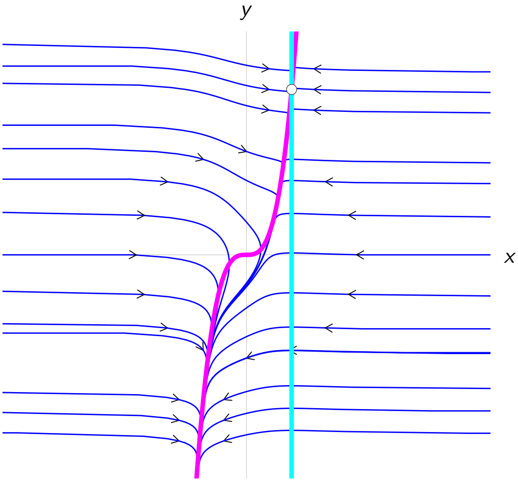
Example (1) $\begin{aligned} x' &= y - x^3 \\ y' &= x - 2 \end{aligned}$ phase portrait only



Example (1) $\quad \begin{aligned} x' &= y - x^3 \\ y' &= x - 2 \end{aligned} \quad$ zoomed back out



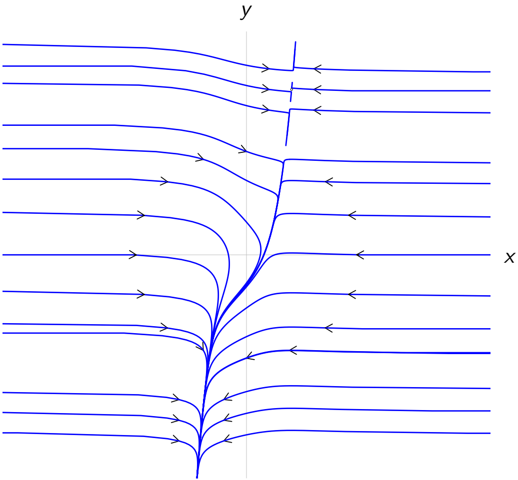
Example (1) $x' = y - x^3$, $y' = x - 2$ nullclines & phase portrait



Example (1)

$$x' = y - x^3$$
$$y' = x - 2$$

phase portrait only



Nullclines of planar systems

- Is there always one x nullcline and one y nullcline?

Example (2)

$$x' = (y - x^3)(y - \sin x)$$
$$y' = (x - 2)(x + 2)$$

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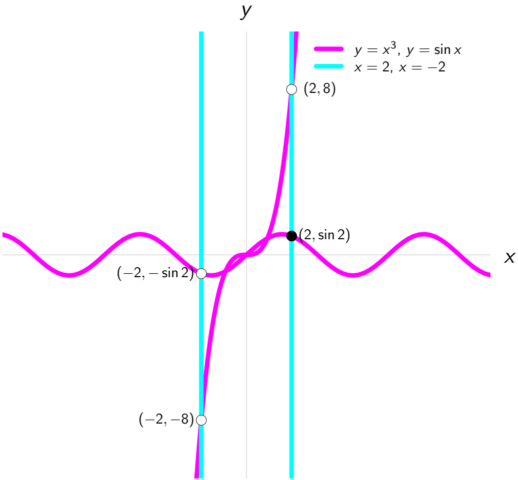
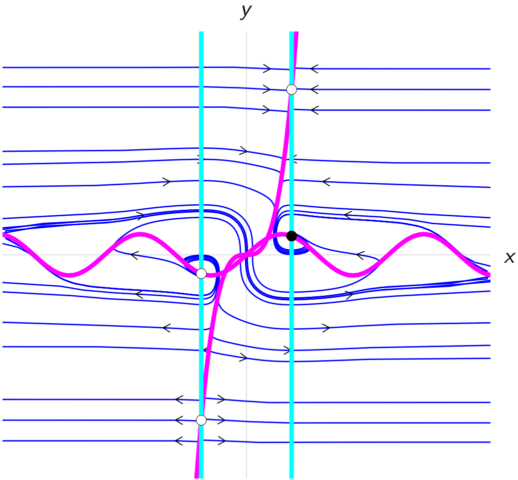
$$x' = (y - x^3)(y - \sin x)$$
$$y' = (x - 2)(x + 2)$$

nullclines



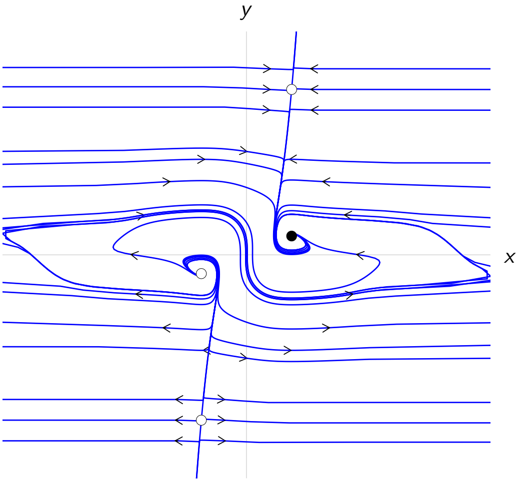
Example (2) $\begin{aligned} x' &= (y - x^3)(y - \sin x) \\ y' &= (x-2)(x+2) \end{aligned}$ & phase portrait



Example (2)

$$x' = (y - x^3)(y - \sin x)$$
$$y' = (x - 2)(x + 2)$$

phase portrait



Linearization Theorem

- Under what circumstances is linearization about an equilibrium enough to characterize the phase portrait near the equilibrium?

Definition (Hyperbolic equilibrium of a nonlinear ODE)

An equilibrium X_* of $X' = F(X)$ is **hyperbolic** if all the eigenvalues of DF_{X_*} have non-zero real parts.

Theorem (Hartman-Grobman)

If $F \in C^\infty(\mathbb{R}^n)$ and $X' = F(X)$ has a hyperbolic equilibrium at X_* then the nonlinear flow is topologically conjugate to the flow of the linearized system in a sufficiently small open ball about X_*.

Linearization Theorem

Proof of Hartman-Grobman (linearization) theorem.

Requires analysis... beyond the scope of this course... but see textbook §8.2 for discussion of cases of hyperbolic sinks and sources with distinct eigenvalues. Note that *similar ideas*:

- work for nonlinear sinks and sources with repeated eigenvalues;
- work for nonlinear spiral sinks and spiral sources;
- **do NOT work for nonlinear saddles**.

□

Nonlinear saddles in the plane

Imagine a linear saddle drawn on pizza dough, and manipulate the dough however you like *without* cutting or puncturing.

For a generic nonlinear saddle in the plane:

- $\nexists$ stable invariant line
- $\nexists$ unstable invariant line

However:

- $\exists$ stable invariant *curve*:
  $W^{\mathrm{s}}(X_*) = \{X : \phi_t(X) \to X_* \text{ as } t \to \infty\}$
- $\exists$ unstable invariant *curve*
  $W^{\mathrm{u}}(X_*) = \{X : \phi_t(X) \to X_* \text{ as } t \to -\infty\}$
- The stable and unstable invariant curves meet at the equilibrium point X_*.
- As $t \to \infty$, all points NOT on the stable invariant curve $\to \infty$.

Nonlinear saddles in higher dimensions

Near hyperbolic equilibria:

- $\exists$ stable and unstable *manifolds*.
- For example:
  - Start with 3D linear system in canonical coordinates with $\lambda_1 < \lambda_2 < 0 < \lambda_3$;
  - Bend *xy*-plane down to some nonlinear surface: this will be the stable manifold.
  - Bend *z*-axis in some way: this will be the unstable manifold (a curve in this example).