Mathematics 3F03
Advanced Differential Equations

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Lecture 8

More Linear Algebra Review
and More Planar Dynamics

20 September 2013

Announcements

- Assignment 1 **was due today**, 20 Sep 2013, before class, in the appropriately labelled locker next to HH-105.

Eigensystems

- Find **eigenvalues** by solving for zeros of **characteristic polynomial**,

$$p(\lambda) = \det(A - \lambda I)\,.$$

- For each eigenvalue, find associated **eigenvectors** by solving for V in

$$\begin{aligned} AV &= \lambda V \\ \textit{i.e.,} \quad (A - \lambda I)V &= 0\,. \end{aligned}$$

- An **eigendirection** is a direction along which there is an eigenvector. (Any vector along such a direction is an eigenvector.)

Multiplicities of Eigenvalues

Definition (Algebraic Multiplicity: $\text{alg}(\lambda_i)$)

The **algebraic multiplicity** of an eigenvalue λ_i is the number of times the factor $(\lambda - \lambda_i)$ appears in the characteristic polynomial.

- *i.e.*, $\text{alg}(\lambda_i)$ is the multiplicity of λ_i as a root of $p(\lambda)$.

Definition (Geometric Multiplicity: $\text{geom}(\lambda_i)$)

The **geometric multiplicity** of an eigenvalue λ_i is the number linearly independent eigenvectors associated with λ_i.

- Equivalently, $\text{geom}(\lambda_i) = \dim(\text{Ker}\,(A - \lambda_i I))$, *i.e.*, the dimension of the subspace that $A - \lambda_i I$ annihilates.

Multiplicities of Eigenvalues

Proposition

For any eigenvalue λ_i,

$$geom(\lambda_i) \leq alg(\lambda_i)\,.$$

Equilibria of Linear Systems of ODEs

$$X' = AX, \qquad A \in \mathbb{R}^{n\times n}, \quad X \in \mathbb{R}^n$$

- There is always at least one equilibrium:
  $X = 0 \in \mathbb{R}^n$ (the origin).
- If A^{-1} exists then this equilibrium is *unique*.
  ($AX = 0 \implies X = A^{-1}0 = 0$.)
- If $A = 0$ (the zero matrix) then every solution is an equilibrium.
  ($AX = 0$ for any X.)
- If $A \neq 0$ but A^{-1} does not exist ($\det A = 0$) then $\exists$ infinitely many equilibria (on a line through the origin).

Equilibria of Affine (Inhomogeneous) Systems of ODEs

$$X' = AX + B, \qquad A \in \mathbb{R}^{n\times n}, \quad X, B \in \mathbb{R}^n$$

- If $B = 0$ then this is the linear case (previous slide).
- If $B \neq 0$ then origin is *not* an equilibrium.
- If $B \neq 0$ and A^{-1} exists then
  $\exists!$ equilibrium at $X_* = -A^{-1}B \neq 0$.
- If $B \neq 0$ and $\det A = 0$ then
  - Maybe no equilibria
  - Maybe infinitely many equilibria

Tangential Remarks: Concepts of Homogeneity

Definition (Homogeneous in time)

A vector field F is **homogeneous in time** if it does not depend on time, *i.e.*, $F(t, X) = F(X)\ \forall t$. Such a vector field yields an autonomous ODE.

Definition (Homogeneous in space)

A vector field F is **homogeneous in space** if it does not depend on space, *i.e.*, $F(t, X) = F(t)\ \forall X$.

A vector field F that is homogeneous in both time and space is constant, *i.e.*, $\exists C \in \mathbb{R}^n$ such that $F(t, X) = C\ \forall t, X$.

Tangential Remarks: Concepts of Homogeneity

Definition (Homogeneous of degree k)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **homogeneous of degree** k if, for any $\alpha \in \mathbb{R}$,

$$f(\alpha X) = \alpha^k f(X)\,.$$

A function that is homogeneous of degree 1 often said to be simply **homogeneous**.

Examples:

- $f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.
- $f(x, y) = x + y$ is homogeneous.
- $f(x, y) = x + y + 1$ is *not* homogeneous.

Connection to linear systems of ODEs:

- The ODE $X' = AX$ is said to be homogeneous because each component of the RHS is homogeneous of degree 1.

General solutions of linear systems of ODEs

Equilibria are important but certainly not the only solutions!

$$X' = AX, \qquad A \in \mathbb{R}^{n\times n}, \quad X \in \mathbb{R}^n \qquad (*)$$

Lemma

If λ is an eigenvalue of A with associated eigenvector V_0 then $X(t) = e^{\lambda t} V_0$ is a solution of $(*)$.

Proof.

$$X'(t) = \frac{d}{dt}\left(e^{\lambda t} V_0\right) = \begin{pmatrix} \frac{d}{dt}(x_0 e^{\lambda t}) \\ \frac{d}{dt}(y_0 e^{\lambda t}) \end{pmatrix} = \begin{pmatrix} x_0 \lambda e^{\lambda t} \\ y_0 \lambda e^{\lambda t} \end{pmatrix} = e^{\lambda t} \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= e^{\lambda t} \lambda V_0 = A(e^{\lambda t} V_0) = AX(t)\,.$$

$\square$

General solutions of linear systems of ODEs

Theorem

Suppose a 2×2 real matrix A has distinct (real) eigenvalues, λ_1 and λ_2, and associated eigenvectors, V_1 and V_2. Consider the initial value problem

$$X' = AX, \qquad X(0) = X_0 \,. \tag{$**$}$$

Then there exist $\alpha_1^0, \alpha_2^0 \in \mathbb{R}$ such that $X_0 = \alpha_1^0 V_1 + \alpha_2^0 V_2$ and

$$X(t) = \alpha_1^0 e^{\lambda_1 t} V_1 + \alpha_2^0 e^{\lambda_2 t} V_2$$

is the unique solution to $(**)$.

Proof.

On the blackboard... □