McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

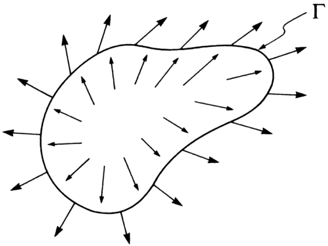
Lecture 32
Index Theory
Wednesday 20 November 2013

Announcements

- Assignment 5 due this Friday, 22 Nov 2013, at 1:30pm.
  - Also look at Assignment 5 from 2011: includes problem on Poincaré Bendixson theorem and other relevant problems.
- Test 2 next Wednesday, 27 Nov 2013, at 11:30am.
  - Emphasis is on material covered since Test 1, including what we cover this week.
  - Location: T29 / 101

Index Theory: Concept

Consider a simply connected domain $D \subset \mathbb{R}^2$ *and a closed loop* $\Gamma \subset D$ *that contains no fixed points (equilibria) of* $X' = F(X)$.



- Arrows represent values of the vector field F.

- As you slide all the way along Γ, the angle ϕ of the vector field changes by an integer number of full rotations: $\Delta\phi = 2k\pi$ $\exists k \in \mathbb{Z}$.
- k is the **index** of Γ.

Index Theory: Formal Definition

Definition (Index of a closed curve Γ)

Consider a smooth, planar vector field, $(x', y') = (f(x,y), g(x,y))$, defined in a simply connected domain $D \subset \mathbb{R}^2$. Suppose Γ is a closed loop in D ($\Gamma \subset D$) and that Γ contains no fixed points of the vector field (*i.e.*, there is no point $(x_*, y_*) \in \Gamma$ such that $f(x_*, y_*) = g(x_*, y_*) = 0$). Then the index of Γ is

$$k = \frac{1}{2\pi} \oint_\Gamma d\phi \,,$$

which can be calculated in Cartesian coordinates via

$$k = \frac{1}{2\pi} \oint_\Gamma d\left(\arctan \frac{g(x,y)}{f(x,y)}\right) = \frac{1}{2\pi} \oint_\Gamma \frac{f\,dg - g\,df}{f^2 + g^2} \,.$$

Index Theory: Meaning of ϕ in the definition of index k

$$k = \frac{1}{2\pi} \oint_\Gamma d\phi$$

- Here, ϕ refers to the angle that the vector field makes with horizontal *at the point in question.*
  - It does NOT refer to the azimuth of polar coordinates (r, θ) in the phase plane!
  - If the vector field is expressed in polar coordinates then it is *usually harder* to compute the index of a given curve:

$$\begin{aligned} k &= \frac{1}{2\pi} \oint_\Gamma d\left(\arctan \frac{y'}{x'}\right) \\ &= \frac{1}{2\pi} \oint_\Gamma d\left(\arctan \frac{r' \sin\theta + r(\cos\theta)\theta'}{r' \cos\theta - r(\sin\theta)\theta'}\right), \end{aligned}$$

where r and θ are the standard polar coordinates in the plane.

Index Theory: Properties

- The index is the same if Γ is smoothly deformed, as long as it is not deformed through some fixed point of the vector field.
- The index of a fixed point is defined to be the index of a closed curve that contains only this one fixed point, and where no fixed points are on the closed curve.
- Although the index of a given curve is often difficult to compute from the definition for a given vector field, some very useful general results can be proved.

Index Theory: Basic Theorem

Theorem (The Index Theorem)

The index of a

(i) *sink, source, or centre is* $+1$*;*

(ii) *hyperbolic saddle point is* -1*;*

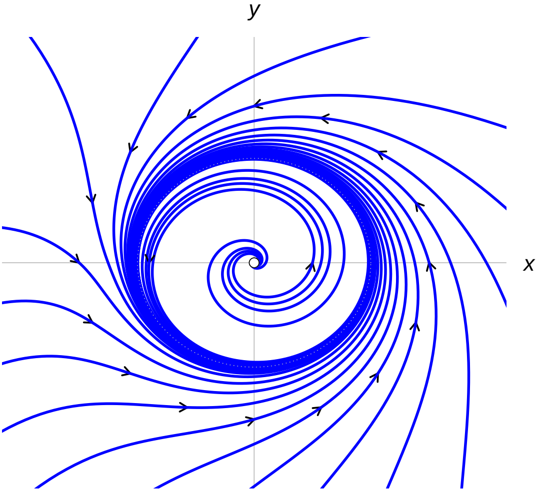
(iii) *periodic orbit is* $+1$*;*

(iv) *closed curve not enclosing any fixed points is* 0*;*

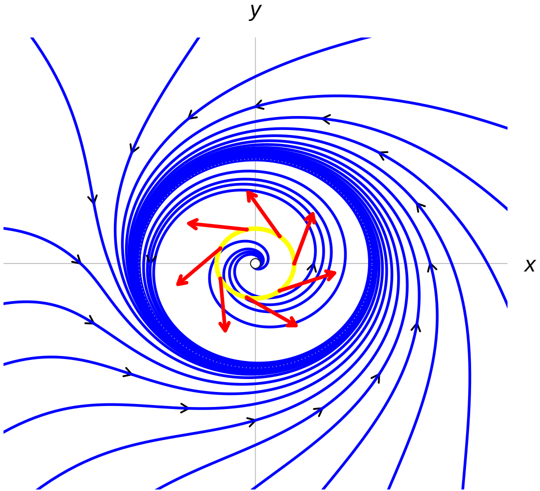
(v) *closed curve is equal to the sum of the indices of the fixed points enclosed within it;*

where "closed curve" refers to a closed loop on which there are no fixed points.

Example: periodic orbit

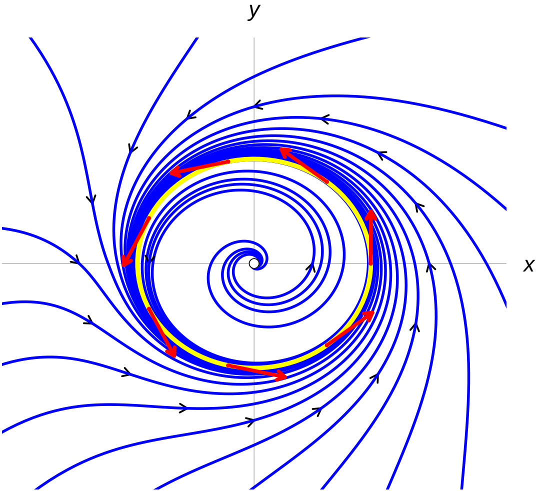


Index of equilibrium at origin



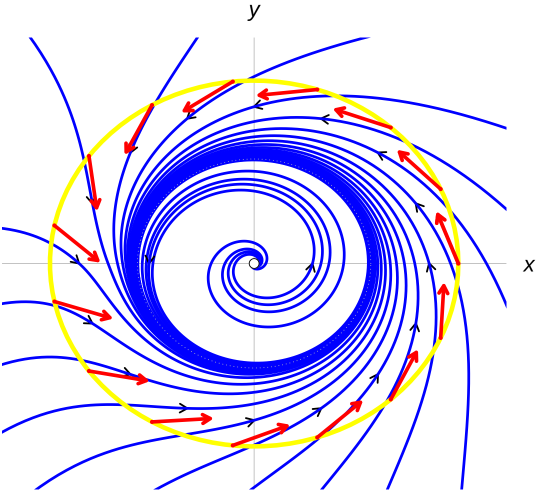
$k = 1$

Index of periodic orbit



$k = 1$

Index along closed curve outside periodic orbit



$k = 1$

Index Theory: Global Theorem Concerning Periodic Orbits

Corollary (Corollary to the Index Theorem)

(i) *Inside any periodic orbit γ there must be at least one fixed point.*

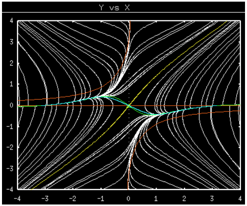
(ii) *If there is only one fixed point inside γ then it must be a sink, source, or centre.*

(iii) *If all the fixed points within γ are hyperbolic, then there must be an odd number, $2n+1$, of which n are saddles and $n+1$ are either sinks or sources.*

Example of use of Bendixson Negative Criterion

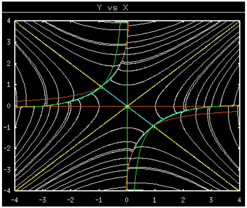
$$x' = xy^2 + \sin y$$
$$y' = x^2 y + \sin x + \alpha y$$

$\alpha = -1$



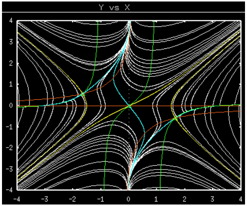
$\nabla \cdot F > 0$, if $\|X\| > \sqrt{|\alpha|}$

$\alpha = 0$



$\nabla \cdot F > 0$, $\forall X \neq (0,0)$

$\alpha = 1$



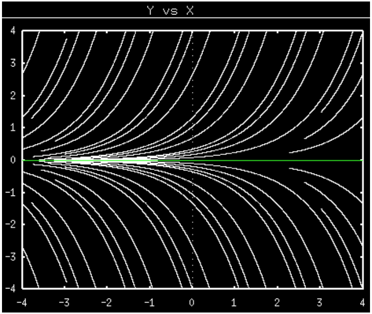
$\nabla \cdot F > 0$, $\forall X \in \mathbb{R}^2$

- Index theorem $\implies$ no periodic orbits enclosing these saddles.

Example of use of Bendixson-Dulac Negative Criterion

$$x' = \frac{1}{x^2 + y^2 + 1}$$

$$y' = \frac{y}{x^2 + y^2 + 1}$$



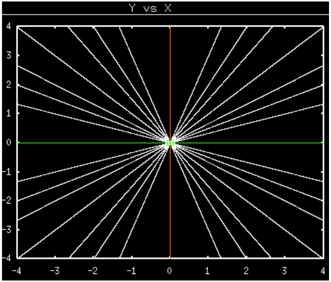
- $\nabla \cdot F$ changes sign in the plane.
- But Dulac function $h(x, y) = x^2 + y^2 + 1$ works: $\nabla \cdot (hF) > 0$ on $\mathbb{R}^2$, so no periodic orbits.
- Alternative argument: this system has no equilibria, hence index theorem implies there are no periodic orbits.

Example of use of Bendixson-Dulac Negative Criterion

Slightly different example:

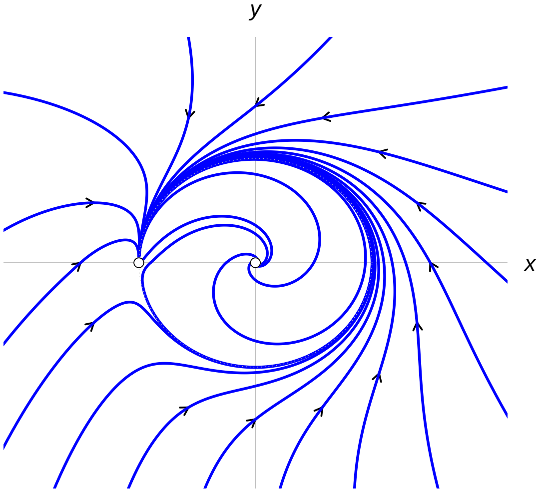
$$x' = \frac{x}{x^2 + y^2 + 1}$$

$$y' = \frac{y}{x^2 + y^2 + 1}$$

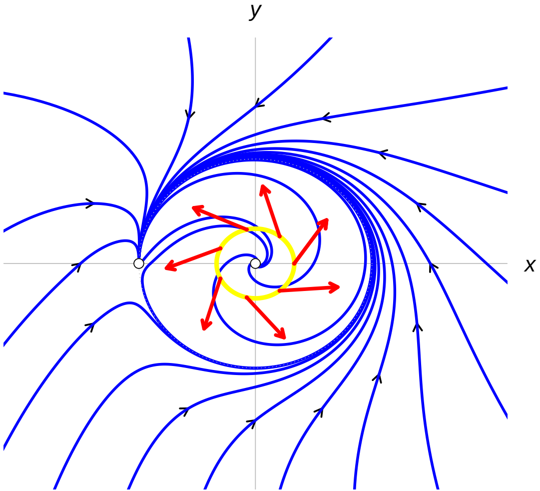


- Now there is an equilibrium, and it is a source, so index theorem does not rule out periodic orbits.
- $\nabla \cdot F$ changes sign in the plane again.
- But Dulac function $h(x,y) = x^2 + y^2 + 1$ works again: $\nabla \cdot (hF) > 0$ on $\mathbb{R}^2$, so no periodic orbits.

Example with homoclinic orbit

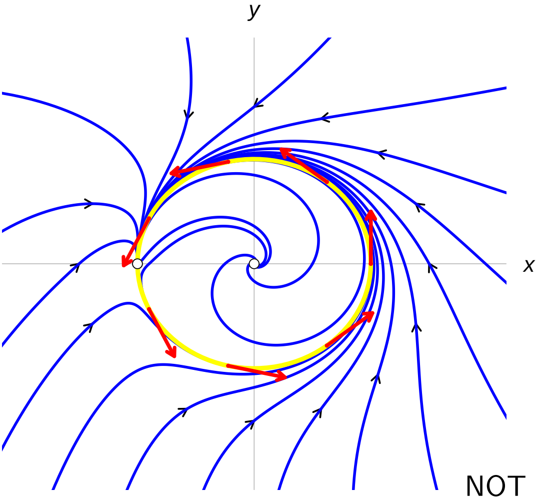


Index of equilibrium at origin



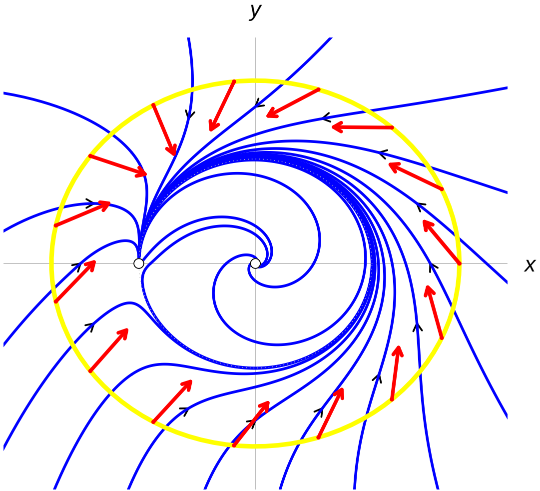
$k = 1$

Index of homoclinic orbit?



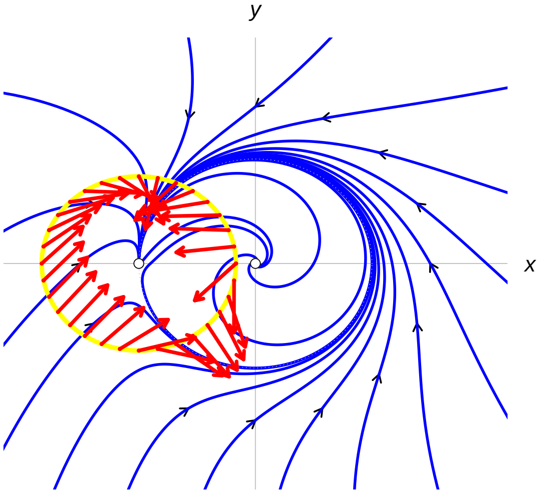
NOT defined!!

Index computed outside homoclinic orbit



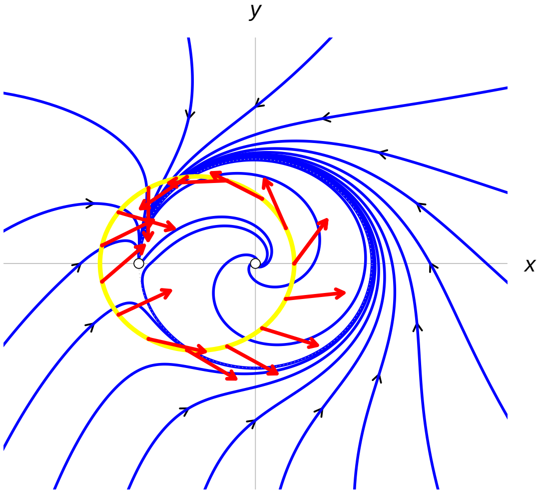
$k = 1$

Index of homoclinic point



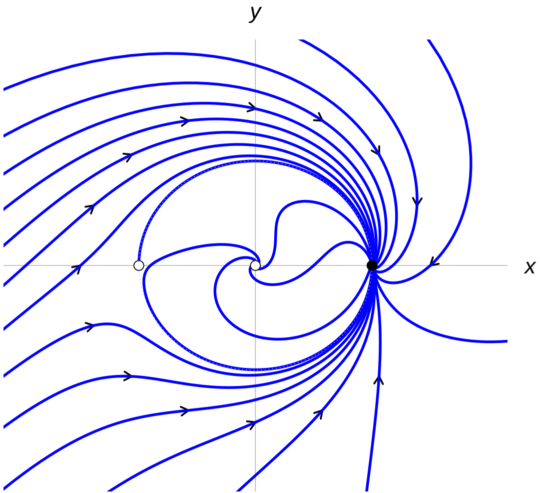
$k = 0$

Index along closed curve containing both equilibria

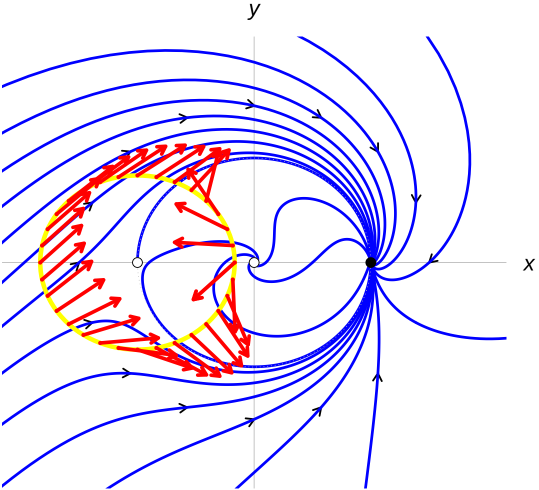


$k = 1$

Example with heteroclinic orbits

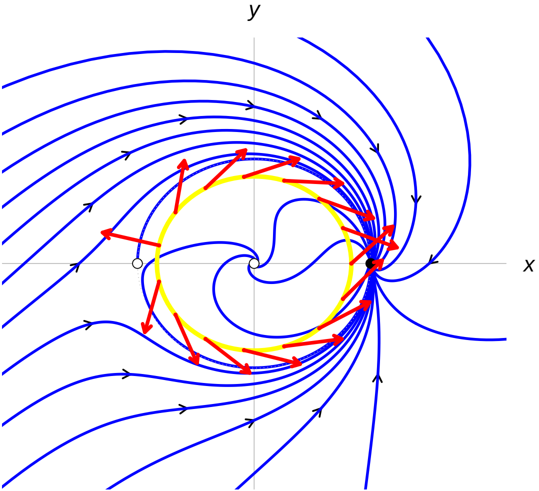


Index calculation



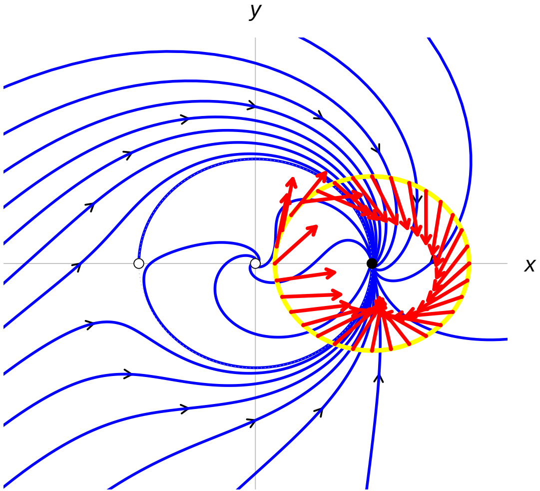
$k = -1$

Index calculation



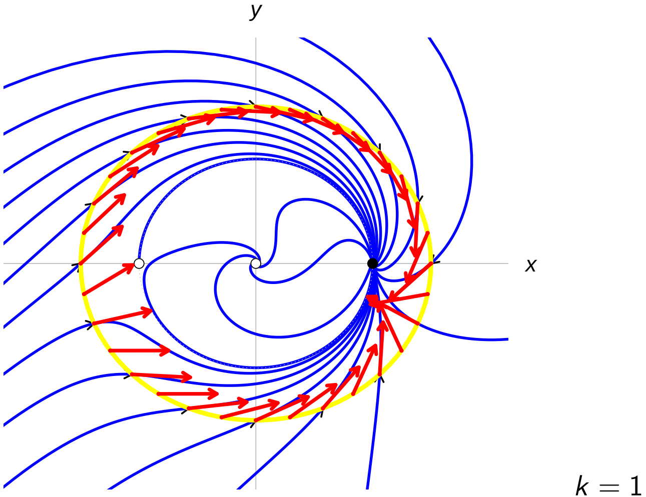
$k = 1$

Index calculation

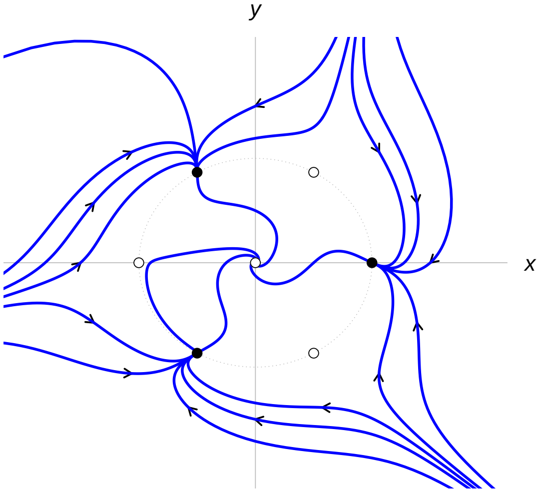


$k = 1$

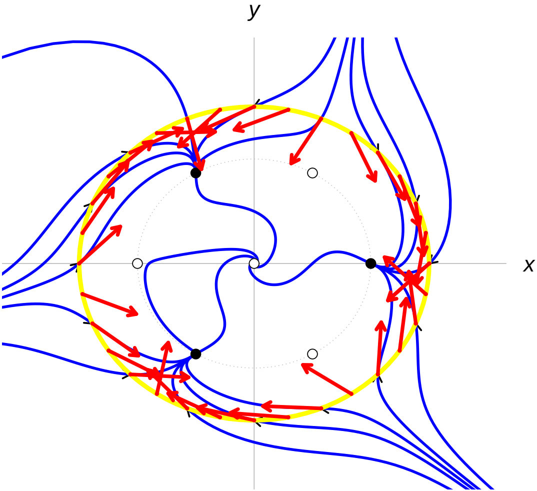
Index calculation



Example with more complex heteroclinic cycle

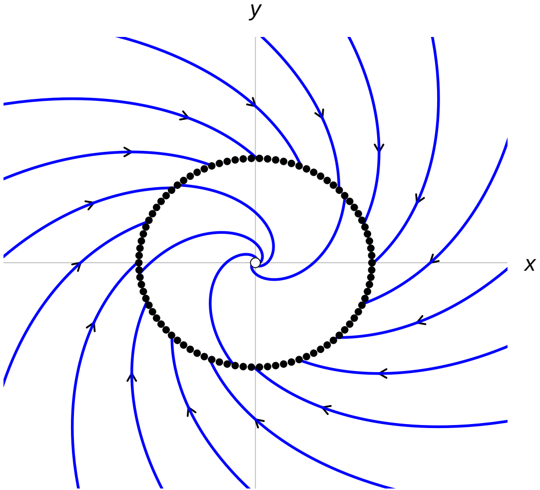


Index calculation

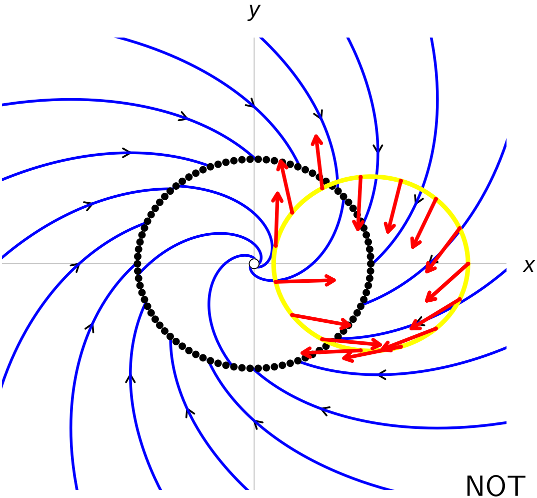


$k = 1$

Example: every point on unit circle is an equilibrium



Index calculation?



NOT defined!!

Creating examples with homoclinic or heteroclinic orbits

- No radial motion on unit circle.
- Equilibria on unit circle and origin only.
- n equilibria on unit circle
  - $n = 1 \implies$ homoclinic
  - $n \geq 2 \implies$ heteroclinic cycle

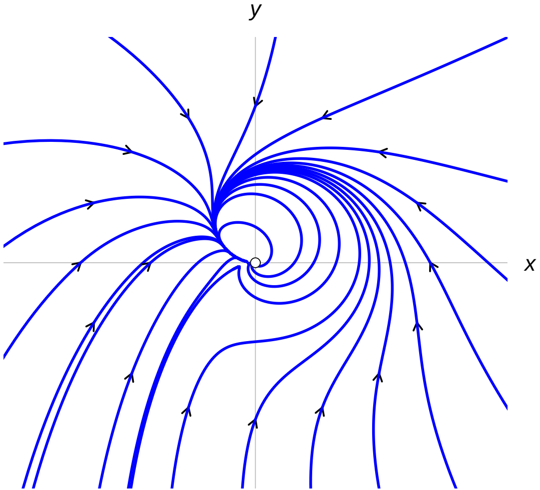
Example (Invariant unit circle containing n equilibria)

$$r' = r(1 - r)$$

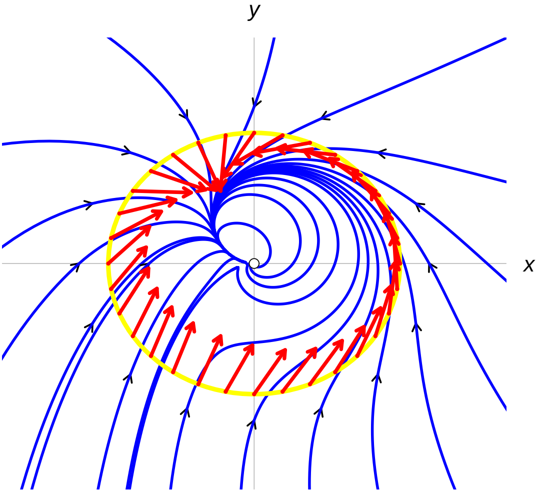
$$\theta' = r \sin\left(\frac{n\theta}{2}\right) + (1 - r)$$

- $n = 0 \implies$ all points on unit circle fixed.
- The vector field is C^∞ in this example $\implies$ such complications are possible even in a very smooth flow.

Example: homoclinic orbit without interior equilibrium

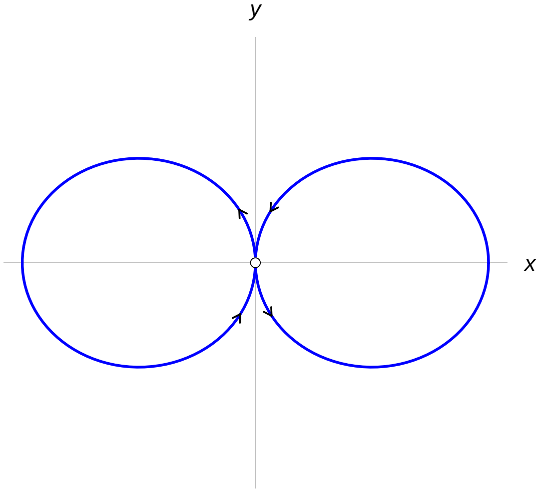


Index of homoclinic point

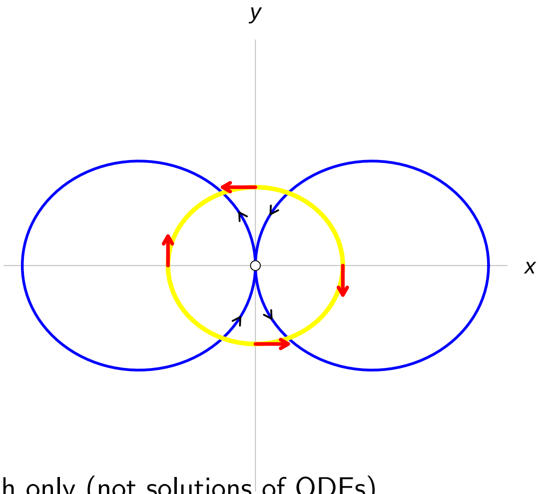


$k = 1$

Homoclinic points can be saddle points (even hyperbolic!)



Homoclinic points can be saddle points (even hyperbolic!)



This is a sketch only (not solutions of ODEs). $k = -1$

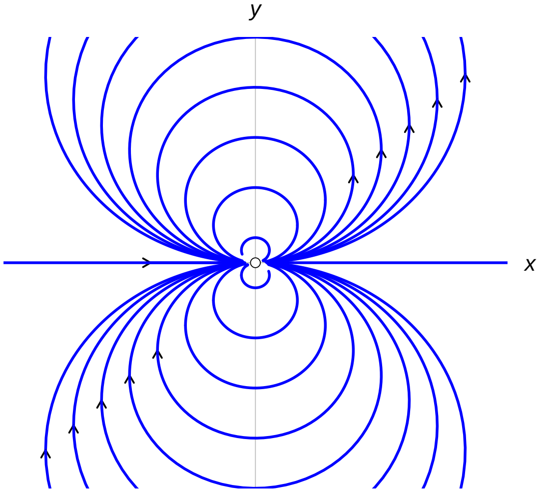
Index theory comments

- The sketches on previous slides are not rigorous.
- Analytical index calculation (when occasionally possible) will not miss any fast rotations of sliding vector.
- Remember there can be no equilibria on a closed curve used to calculate an index!
- Homoclinic orbits are not like periodic orbits: they do not necessarily enclose equilibria.
- We have seen examples with homoclinic points that have index $k = 0$, 1 or -1. *Being a homoclinic point is a non-local phenomenon.*
- There is an analogue of index theory in higher dimensions (*degree theory*).

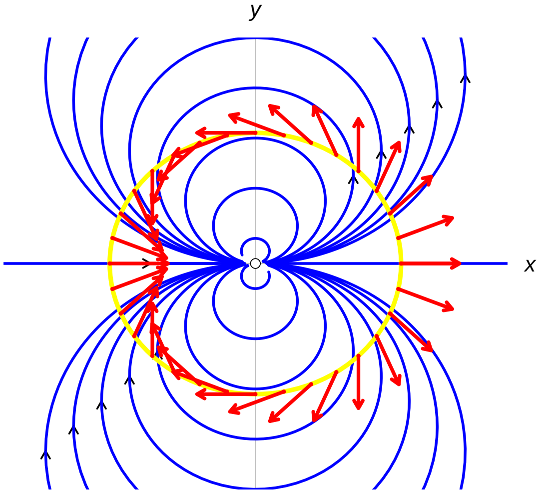
Question

- Is it possible for an equilibrium point to have an index other than 0 or ± 1? *i.e.*, other than the cases we've seen?

Example: all orbits (except one) homoclinic to origin



Example: all orbits (except one) homoclinic to origin



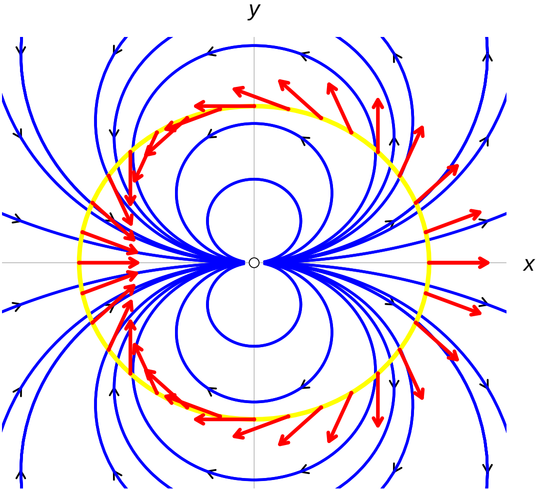
$k = 2$

Example: all orbits (except one) homoclinic to origin

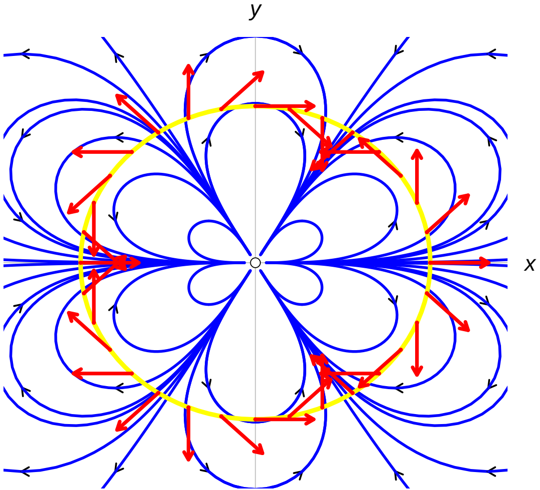
$$x' = x^2 - y^2$$
$$y' = 2xy$$

- Non-hyperbolic
- Can we construct an example where ALL orbits are homoclinic to a given point?
- Yes, on the sphere:
  - Just join the two orbits on the x axis to itself "at infinity" and squash everything onto the surface of a sphere.
  - This transformation is NOT a topological conjugacy.

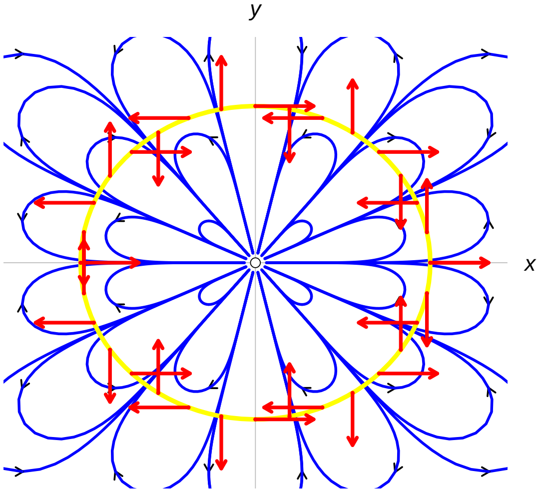
Example: index 2



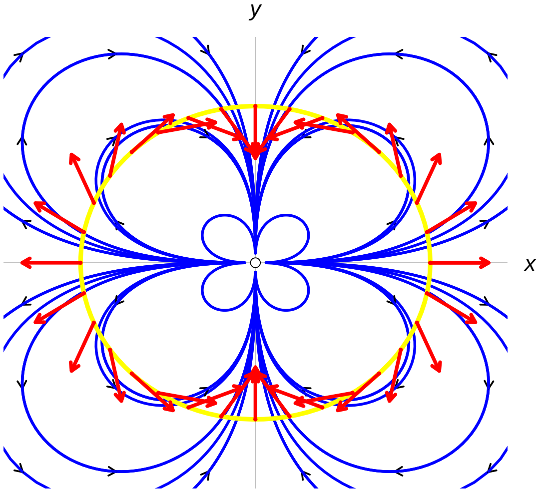
Example: index 4



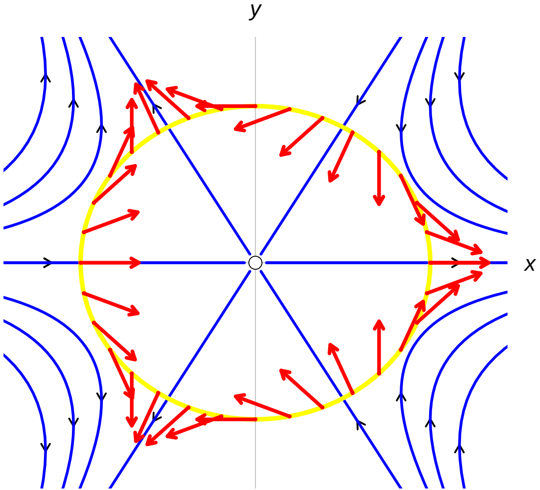
Example: index 8



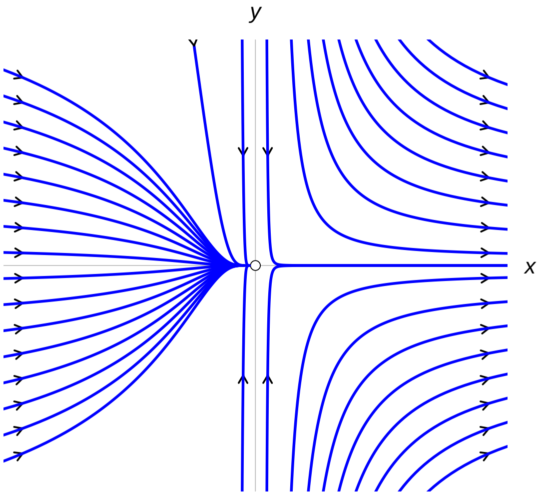
Example: index 3



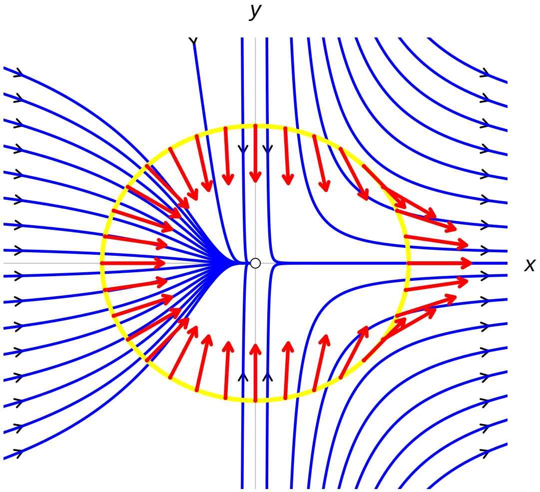
Example: index -2



saddle-node example



saddle-node example



$k = 0$

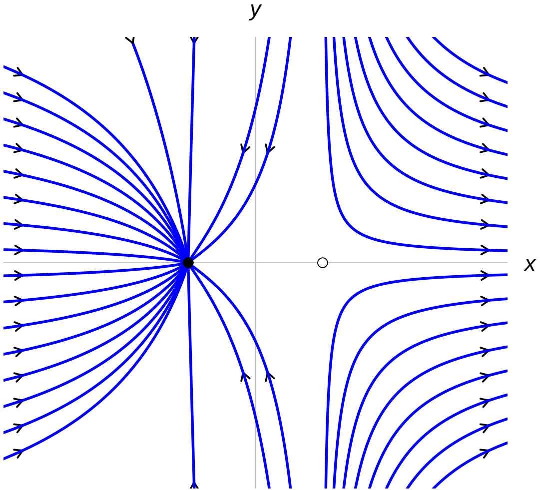
saddle-node example

$$x' = x^2$$
$$y' = -y$$

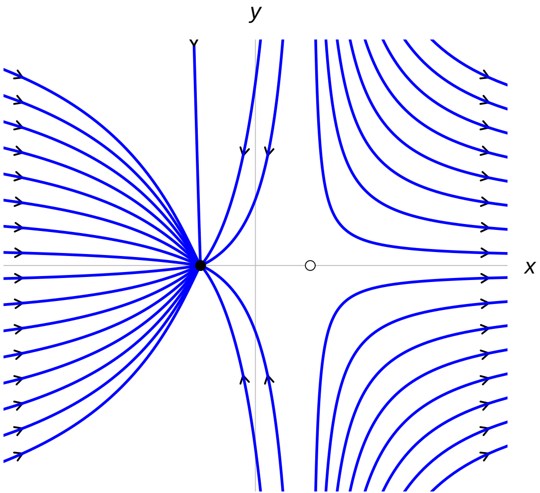
- Non-hyperbolic.
- This arises when a saddle and a node collide (a **saddle-node bifurcation**).
- A **node** is a general term for a source or sink. A source is an **unstable node** and a sink is a **stable node**.
- The bifurcation occurs as α passes through 0 in the system

$$x' = x^2 - \alpha$$
$$y' = -y$$

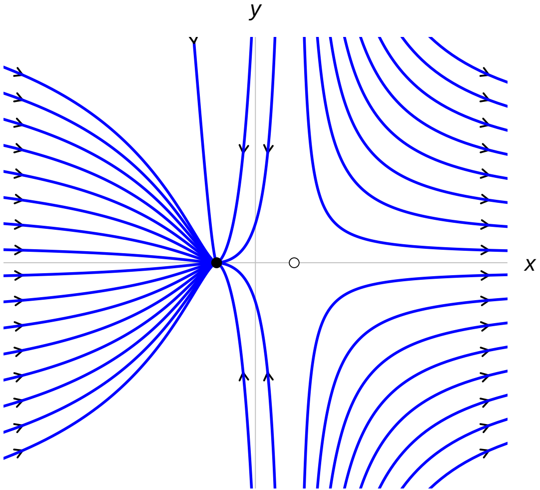
saddle-node bifurcation animation $\alpha = 0.3$



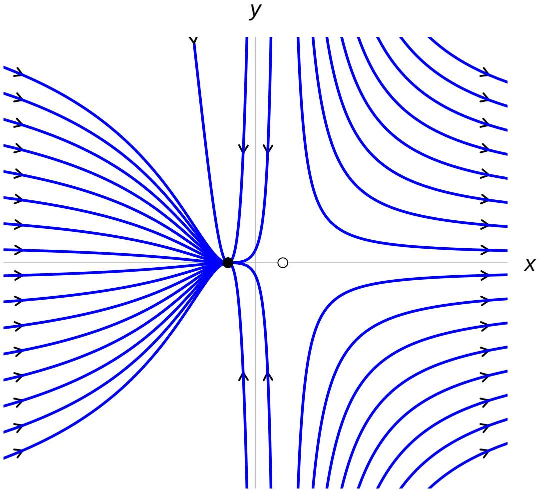
saddle-node bifurcation animation $\alpha = 0.2$



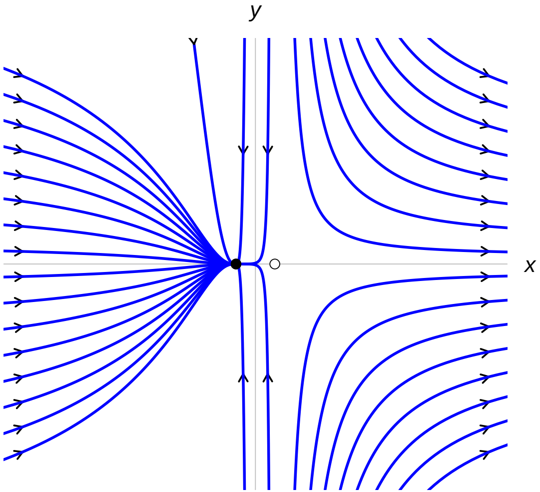
saddle-node bifurcation animation $\alpha = 0.1$



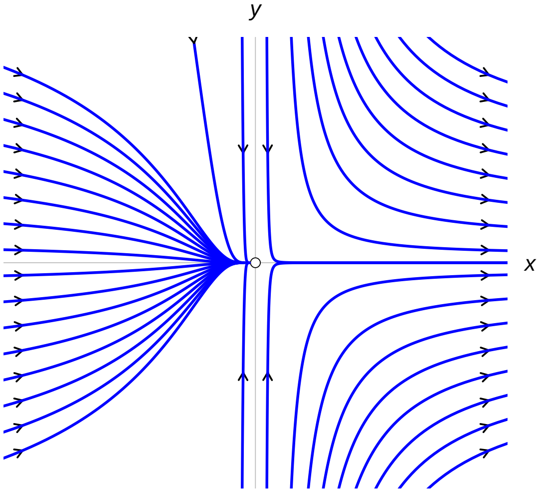
saddle-node bifurcation animation $\alpha = 0.05$



saddle-node bifurcation animation $\alpha = 0.025$



saddle-node bifurcation animation $\alpha = 0$



Announcements

- **Dora's office hours** on Wed 27 Nov 2013 are
  MOVED TO TUESDAY 26 NOV 2013, 3:30-4:30pm.

Online Course Evaluations

Your course evaluations are critical to future course development and instructor assessment processes.

Course Evaluations for 2013 Fall Term 1

Open: Wednesday November 20, 2013 at 10:00 a.m.*
Close: Wednesday December 4, 2013 at 4:00 p.m.

* *Faculties of ENG, HUM, SOCSCI, SCI, and DSB*

- Log in with your MAC ID to evaluate your courses.
- Each evaluation will take approximately 5 to 15 minutes to complete.
- Your responses are completely anonymous.
- Evaluation results are not made available to instructors until ***after*** final marks have been submitted to the Office of the Registrar.

https://evals.mcmaster.ca

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