McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

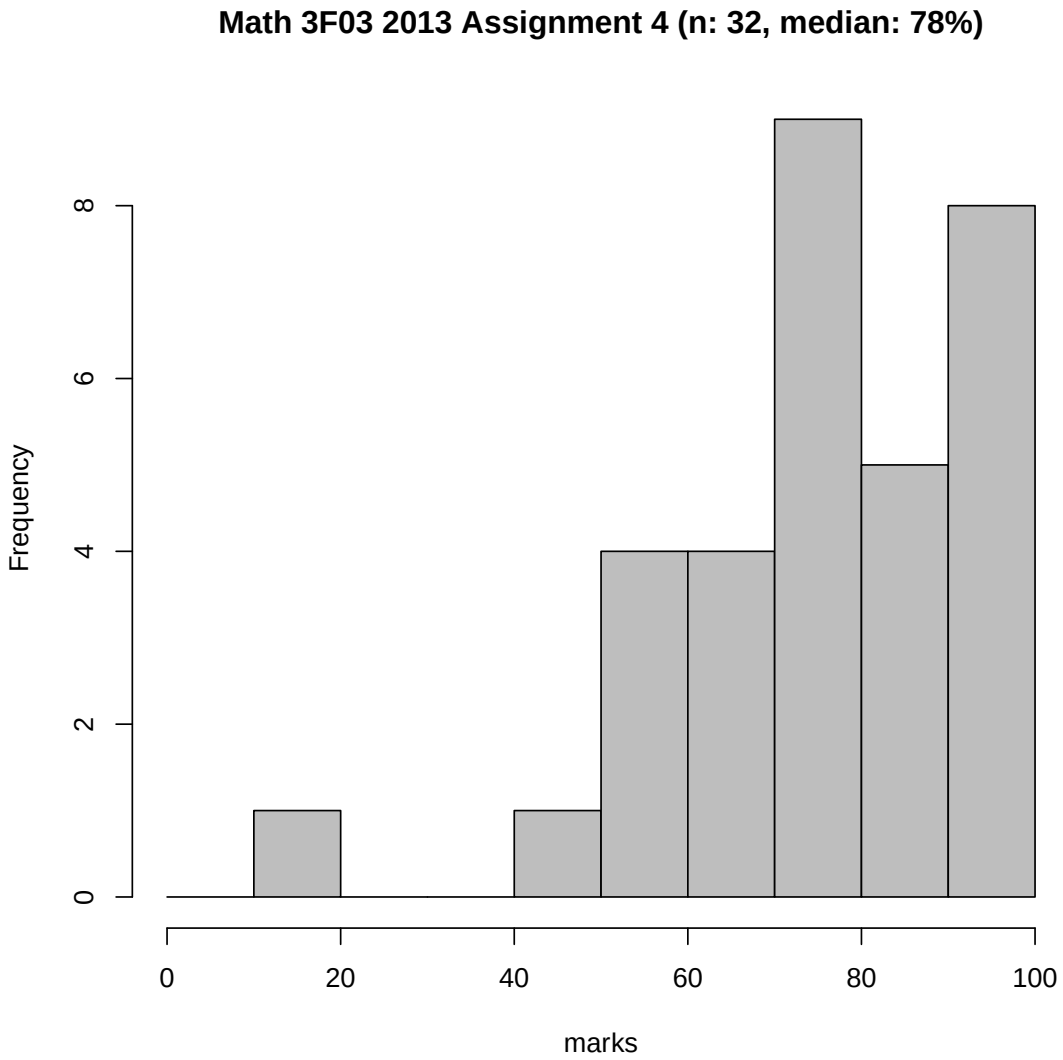
Instructor: David Earn

Lecture 35
Chaos
Wednesday 4 December 2013

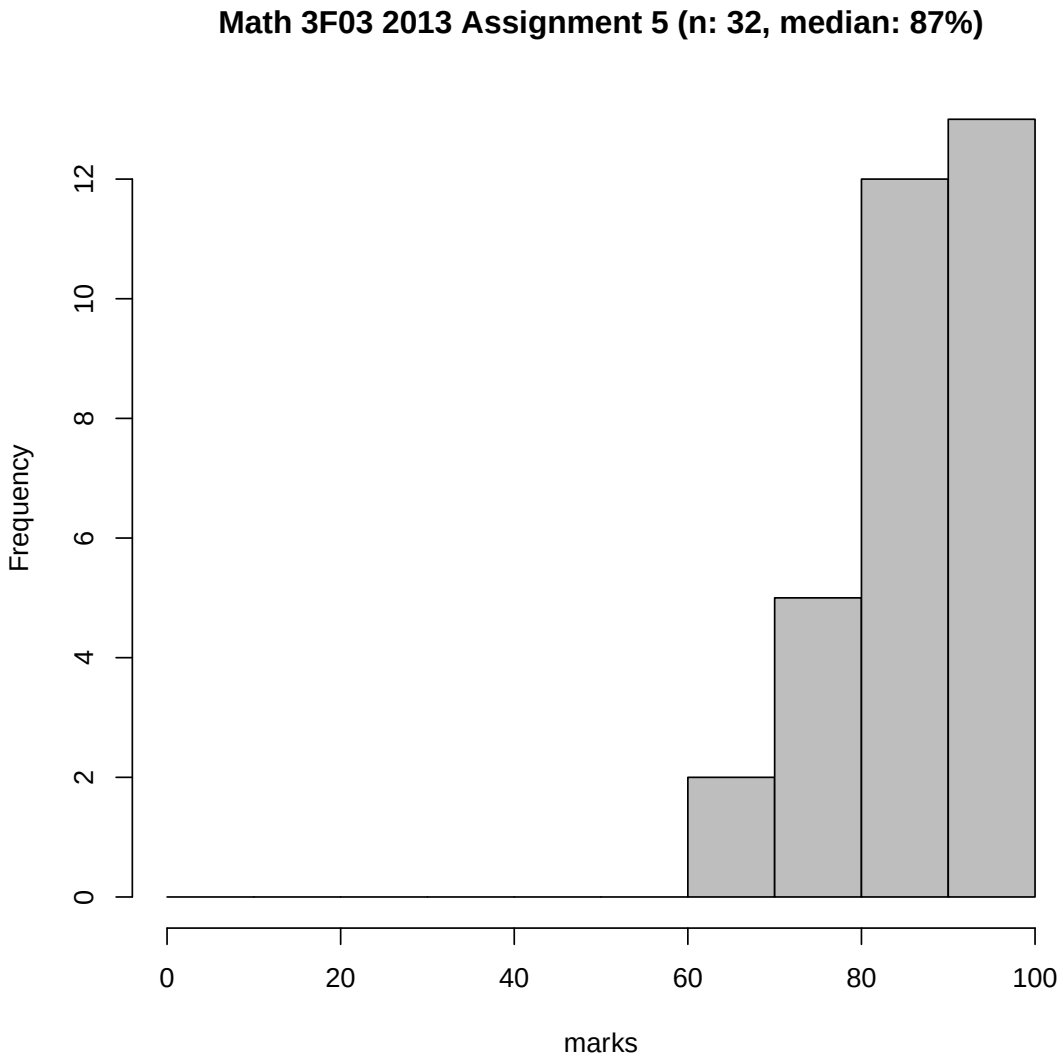
Announcements

- Dora's office hours:
  - TODAY (Wednesday 4 Dec 2013) 2:00-3:00pm
  - TOMORROW (Thursday 5 Dec 2013) 2:30-3:30pm
- Assignments 4 and 5:
  - Results excellent, but easiest problems were marked.
  - Carefully compare your solutions with instructor's for ALL problems.
- Test #2:
  - Solutions posted on course wiki: READ THEM CAREFULLY.
- Final Exam: TUESDAY 10 Dec 2013, 7:30pm to 10:30pm, Convocation Hall.
  - I will be present at the exam.
  - We will discuss the structure of the exam today.
  - Office hours: CHECK COURSE WIKI FOR ANNOUNCEMENTS.
  - **Remember that it is FUN to solve these problems. GOOD LUCK!!**

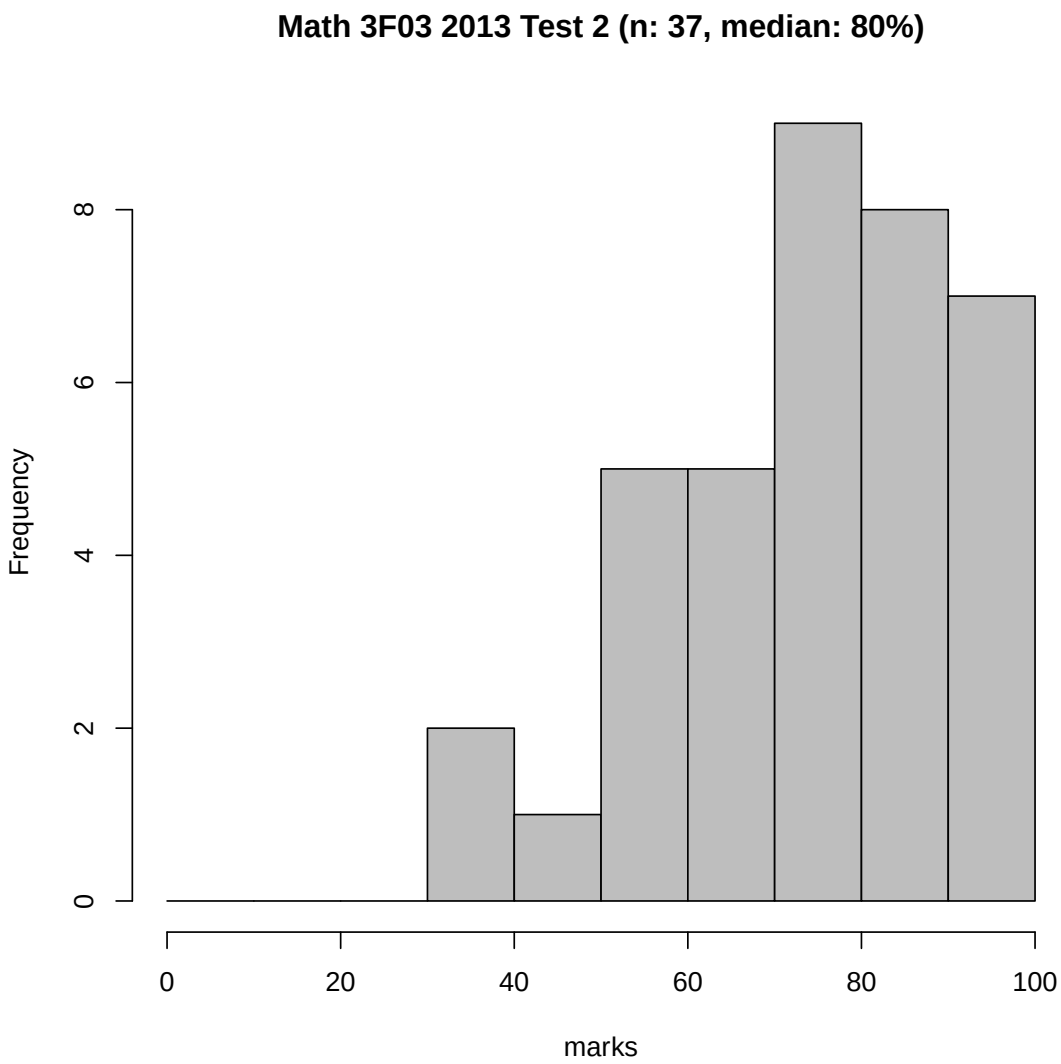
Assignment 4 Results



Assignment 5 Results



Test 2 Results



Course Evaluations close TODAY at 4:00pm!!

Online Course Evaluations

Your course evaluations are critical to future course development and instructor assessment processes.

Course Evaluations for 2013 Fall Term 1

Open: Wednesday November 20, 2013 at 10:00 a.m.*
Close: Wednesday December 4, 2013 at 4:00 p.m.

** Faculties of ENG, HUM, SOCSCI, SCI, and DSB*

- Log in with your MAC ID to evaluate your courses.
- Each evaluation will take approximately 5 to 15 minutes to complete.
- Your responses are completely anonymous.
- Evaluation results are not made available to instructors until ***after*** final marks have been submitted to the Office of the Registrar.

https://evals.mcmaster.ca

McMaster University

Chaos

Chaos: The Final Frontier

- The Lorenz System
- Chaotic Epidemics?

The Lorenz System

$$x' = \sigma(y - x)$$
$$y' = \rho x - y - xz$$
$$z' = xy - \beta z$$

- A simple ODE in $\mathbb{R}^3$
- Nonlinear, but only quadratic nonlinearities
- Equilibrium at origin

- Lorenz was studying weather
- Discovered something strange...
- Designed this simple model that produces the same strange behaviour as the full weather model
- What was so strange?...

Sensitive dependence on initial conditions

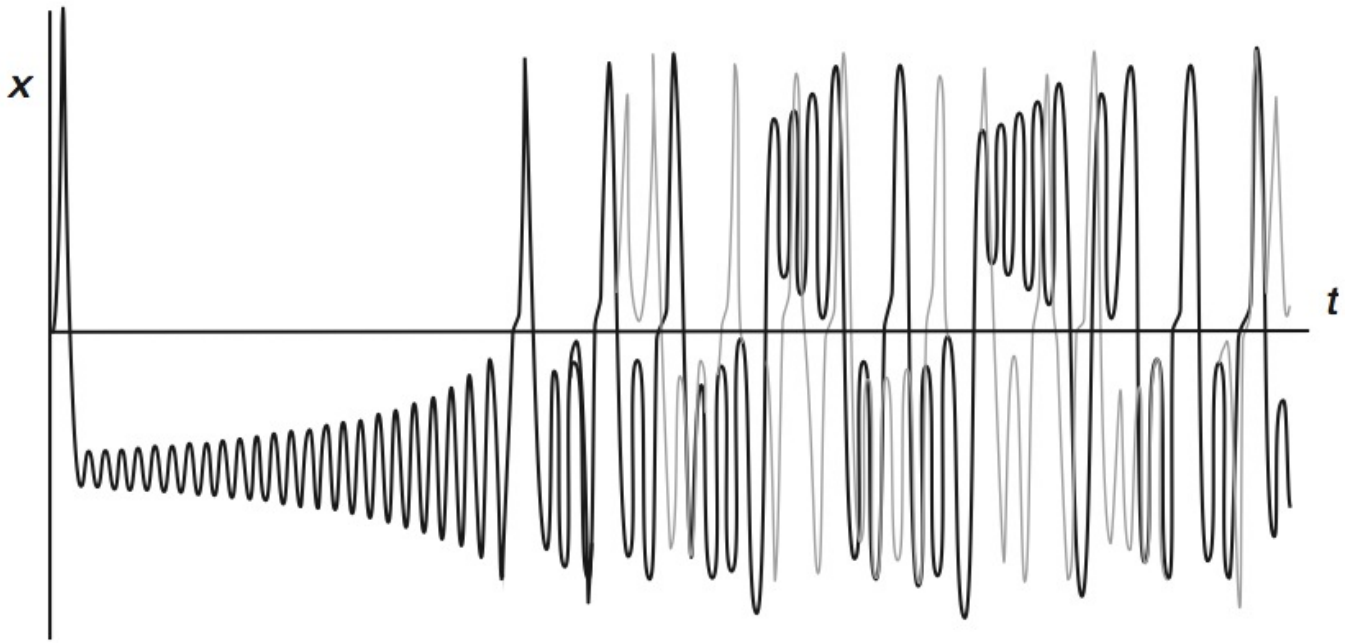


Figure 14.2 The $x(t)$ graphs for two nearby initial conditions $P_1 = (0, 2, 0)$ and $P_2 = (0, 2.01, 0)$.

- A few weeks ago, we learned that nearby trajectories of any (C^1) ODE cannot separate any faster than exponential.
- These trajectories separate exponentially fast!

Sensitive dependence on initial conditions

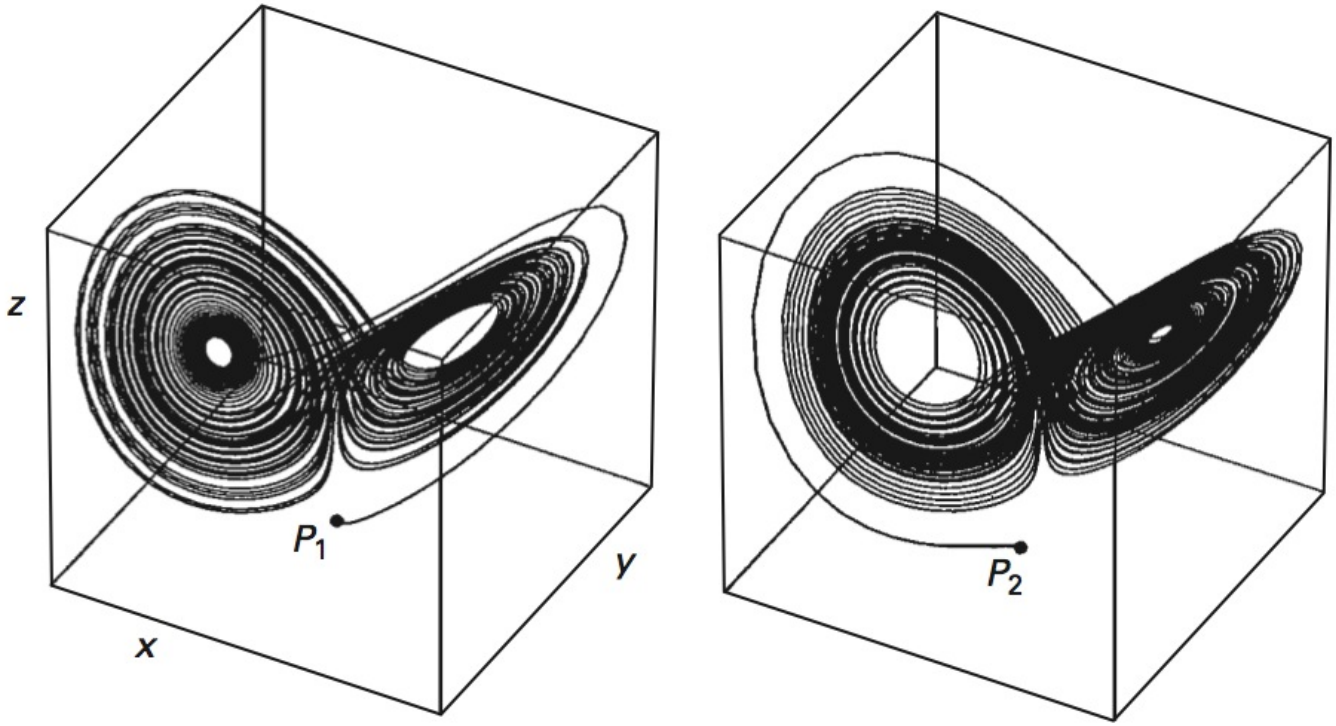
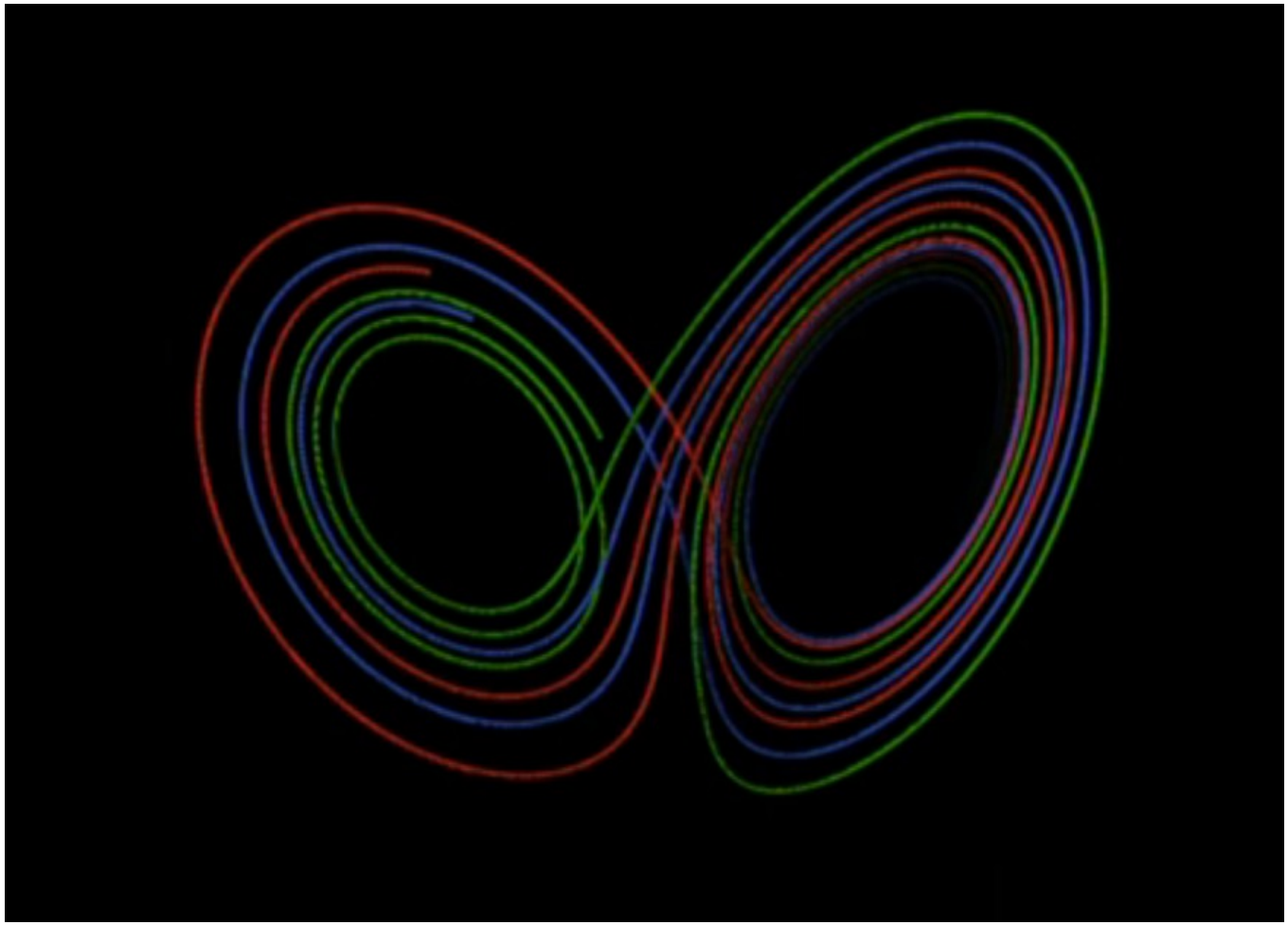


Figure 14.1 The Lorenz attractor. Two solutions with initial conditions $P_1 = (0, 2, 0)$ and $P_2 = (0, -2, 0)$.

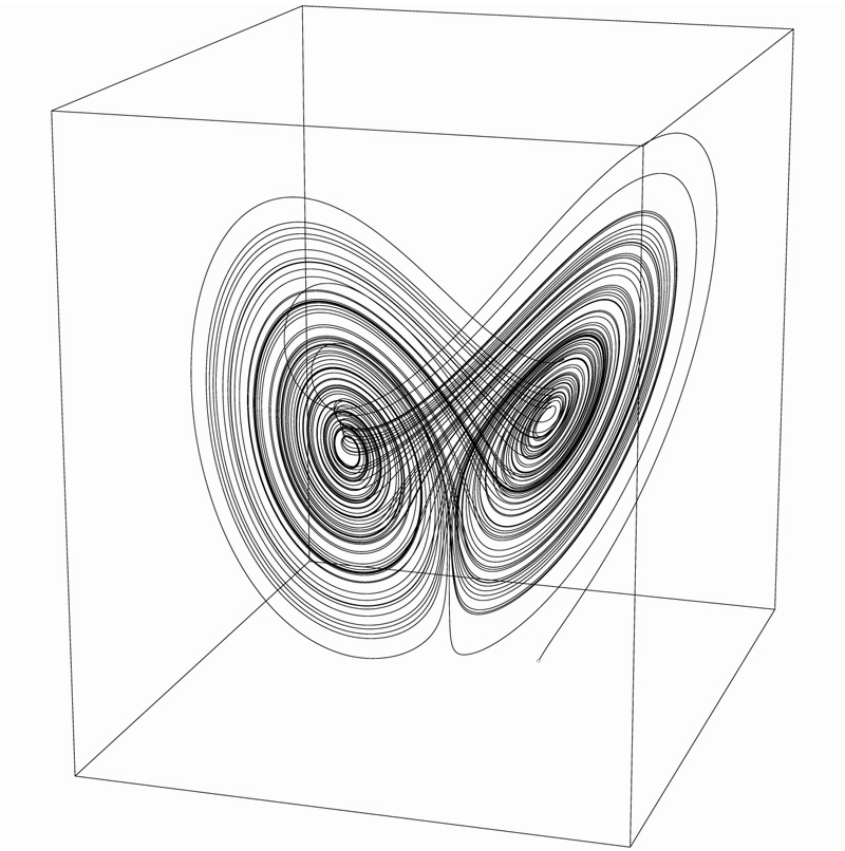
- Trajectories from P_1, P_2 go in completely different directions. Any initially close trajectories are very quickly just as far apart.
- This was shocking to Lorenz in 1963, and to the rest of the community when they clued in to his paper 10 years later.

Sensitive dependence on initial conditions

See animation at
`http://www.youtube.com/watch?v=FYE4JKAXSfY`.



ω-Limit Set is a "Strange Attractor"



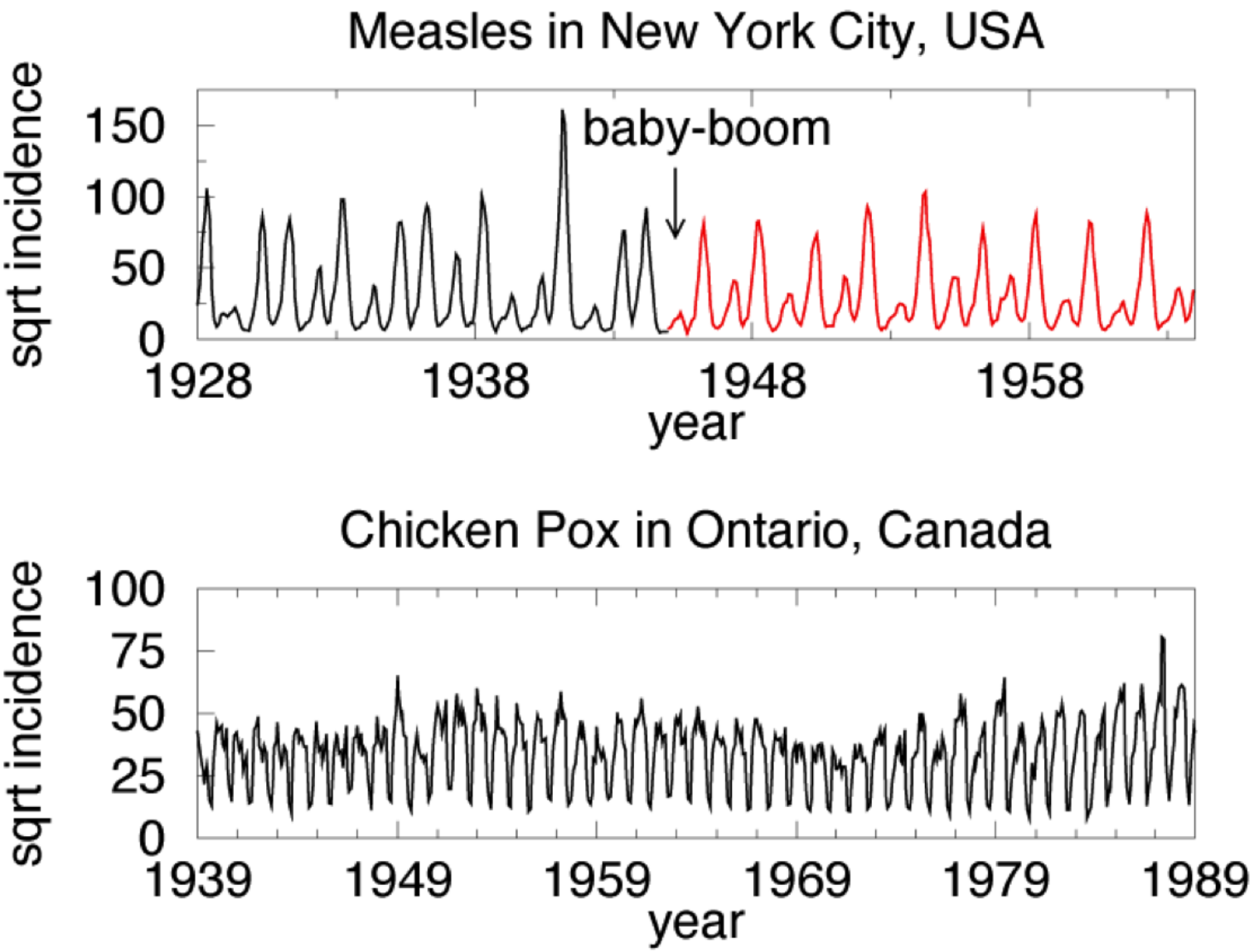
- Bounded; Contains no equilibria; Not periodic.
- Has fractional dimension (!), between 2 and 3.
- Models the "Butterfly Effect" . . . and looks like a butterfly. . .

Characteristics of Chaotic Attractors

- Sensitive dependence on initial conditions.
- Bounded.
- Aperiodic.
- Mixing.
- Typically fractal dimension.

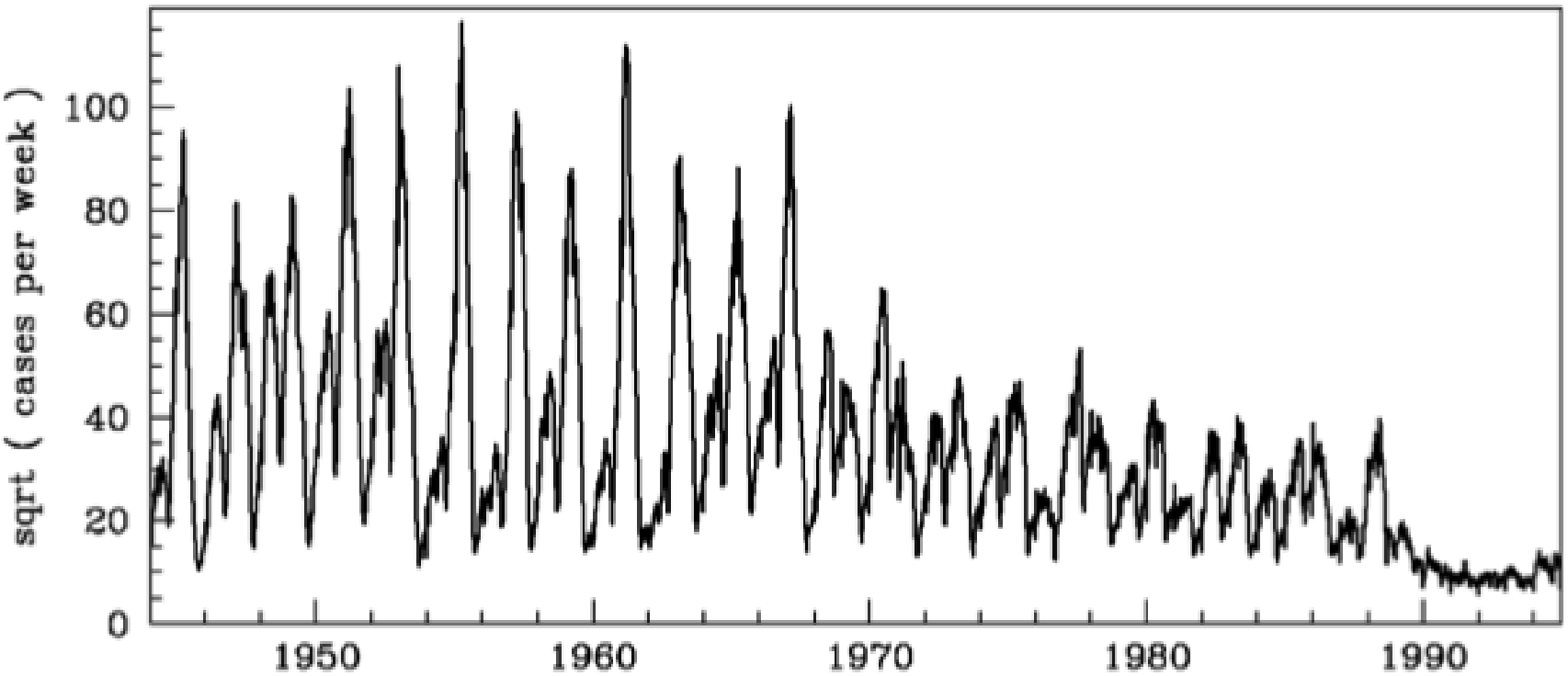
Can epidemic patterns be chaotic?

Chaotic Epidemics?



Chaotic Epidemics?

Measles incidence in England and Wales, 1945–1995

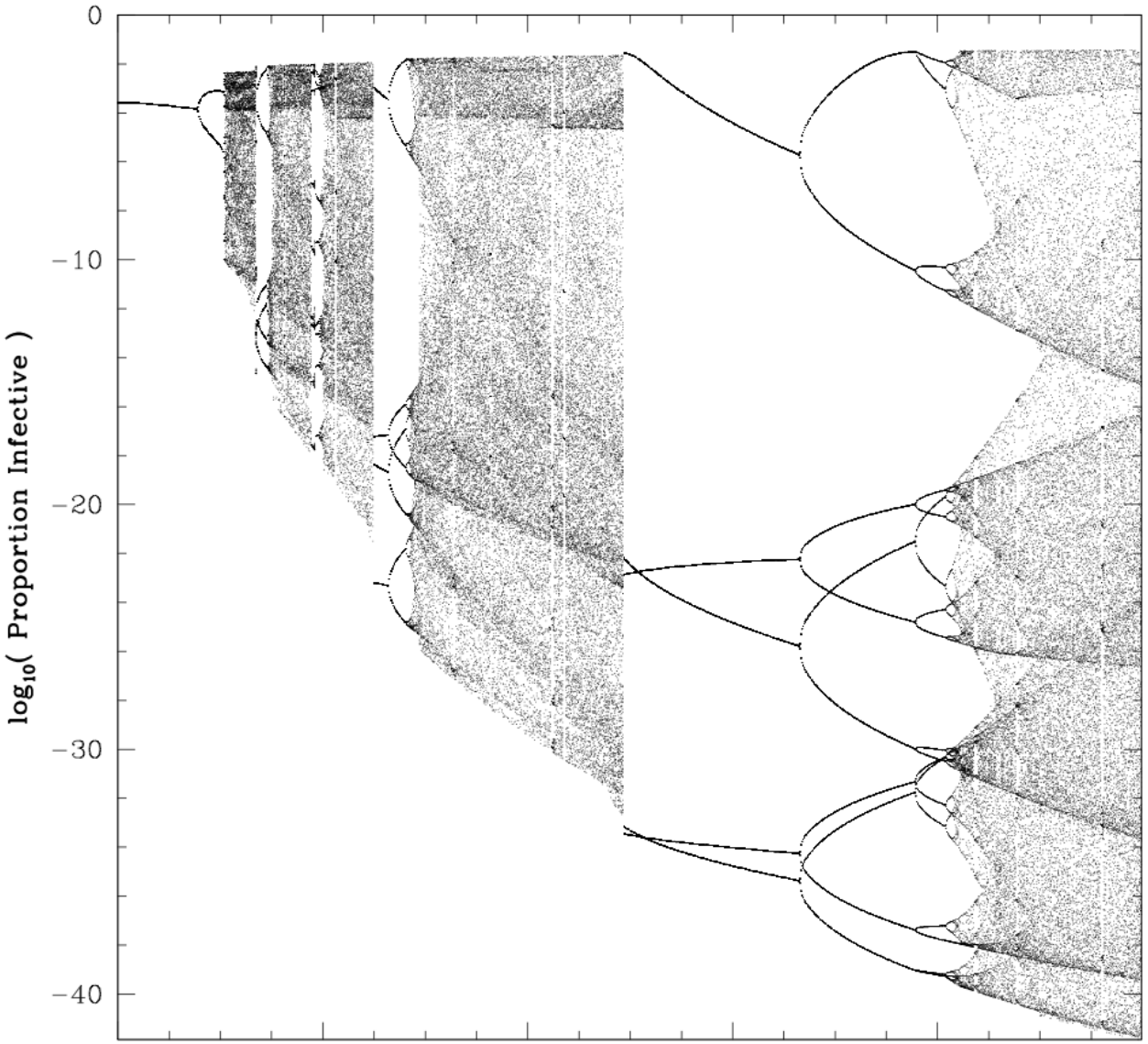


Chaotic Epidemics?

- Is measles chaotic?
- Is chicken pox chaotic?
- If so, is the SIR model the "right" model?
  - Impossible: Poincare-Bendixson prevents chaotic solutions.
- Is there an important state variable that we are missing in our model?
- Is the model wrong altogether?
- What parameter could drive the appearance of chaos?
- Are we getting ahead of ourselves? Must there be chaos?

A chaotic epidemic model

Bifurcations as a function of ????



Courses to Consider in the Future (if you liked this course)

- **Winter 2014**
  - Math 4MB3 – Earn – Mathematical Biology
    - If posted time conflicts with another course you want to take, let me know.
  - Math 4Q03 – Gully – Numerical Methods for Differential Equations (maybe PDEs)
  - Bio 3SS3 – Bolker – Population Ecology

- **MAYBE Fall 2014 or Winter 2015**
  - Math 746 – Instructor TBA – Bifurcation and Stability Theory
  - Math 747 – Instructor TBA – Topics in Mathematical Biology

The End

THANK YOU!