Mathematics 3F03
Advanced Differential Equations

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Lecture 9

Solution of $X' = AX$ with Distinct Eigenvalues

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Announcements

- Solutions to Assignment 1 are now posted on the course wiki.
- Make sure to read over the solutions carefully.

General solutions of linear systems of ODEs

Theorem

Suppose a 2×2 real matrix A has distinct (real) eigenvalues, λ_1 and λ_2, and associated eigenvectors, V_1 and V_2. Consider the initial value problem

$$X' = AX, \qquad X(0) = X_0 \,. \tag{$**$}$$

Then there exist $\alpha_1^0, \alpha_2^0 \in \mathbb{R}$ such that $X_0 = \alpha_1^0 V_1 + \alpha_2^0 V_2$ and

$$X(t) = \alpha_1^0 e^{\lambda_1 t} V_1 + \alpha_2^0 e^{\lambda_2 t} V_2$$

is the unique solution to $(**)$.

Proof.

On the blackboard. . . $\square$

General solutions of linear systems of ODEs

Proof Step 1: Eigenvectors V_1 and V_2 are linearly independent.

Suppose

$$\beta_1 V_1 + \beta_2 V_2 = 0\,. \qquad (\heartsuit)$$

Apply A to both sides of $(\heartsuit)$ to obtain:

$$0 = A0 = \beta_1 AV_1 + \beta_2 AV_2 = \beta_1 \lambda_1 V_1 + \beta_2 \lambda_2 V_2\,. \qquad (\spadesuit)$$

Multiply $(\heartsuit)$ by the scalar λ_1 to find:

$$0 = \lambda_1 0 = \beta_1 \lambda_1 V_1 + \beta_2 \lambda_1 V_2\,. \qquad (\clubsuit)$$

Subtract $(\clubsuit)$ from $(\spadesuit)$ to get $\beta_2(\lambda_2 - \lambda_1)V_2 = 0$. But V_2 is an eigenvector, so $V_2 \neq (0,0)$, and $\lambda_1 \neq \lambda_2 \implies (\lambda_2 - \lambda_1) \neq 0$. Therefore $\beta_2 = 0$. Similarly, multiply $(\heartsuit)$ by λ_2 to yield $\beta_1 = 0$. Therefore, V_1 and V_2 are linearly independent. $\square$

General solutions of linear systems of ODEs

Proof Step 2: $\exists \alpha_1^0, \alpha_2^0 \in \mathbb{R}$ such that $X_0 = \alpha_1^0 V_1 + \alpha_2^0 V_2$.

V_1 and V_2 linearly independent

$\implies \{V_1, V_2\}$ is a basis of $\mathbb{R}^2$

$\implies \exists \alpha_1^0, \alpha_2^0 \in \mathbb{R}$ such that $X_0 = \alpha_1^0 V_1 + \alpha_2^0 V_2$. $\square$

General solutions of linear systems of ODEs

Proof Step 3: Existence of solution.

Simply verify that

$$X(t) = \alpha_1^0 e^{\lambda_1 t} V_1 + \alpha_2^0 e^{\lambda_2 t} V_2$$

is a solution of the initial value problem (**).

- From lemma, we know that $e^{\lambda_1 t} V_1$ and $e^{\lambda_2 t} V_2$ are solutions of $X' = AX$.
- $X' = AX$ is a linear differential equation, so linear combinations of solutions are also solutions.
  $\therefore X(t) = \alpha_1^0 e^{\lambda_1 t} V_1 + \alpha_2^0 e^{\lambda_2 t} V_2$ is a solution of $X' = AX$.
- Moreover, $X(0) = X_0 \implies X(t)$ solves the IVP (**).

$\square$

General solutions of linear systems of ODEs

Proof Step 4: Uniqueness of solution.

Suppose $Y(t)$ is another solution of the IVP (**). Since $\{V_1, V_2\}$ is a basis of $\mathbb{R}^2$, for each time $t \in \mathbb{R}$, $\exists \alpha_1(t), \alpha_2(t) \in \mathbb{R}$ such that

$$Y(t) = \alpha_1(t) V_1 + \alpha_2(t) V_2 \, .$$

N.B. The functions $\alpha_i(t)$ are differentiable because they are components (wrt the basis $\{V_1, V_2\}$) of the differentiable vector function $Y(t)$. (*A linear change of coordinates never changes the smoothness of a function.*)

$$\therefore \; Y'(t) = \alpha_1'(t) V_1 + \alpha_2'(t) V_2 \, .$$

But $Y(t)$ is a solution of (**), *i.e.*, $Y'(t) = AY(t)$, hence. . . $\square$

General solutions of linear systems of ODEs

Proof Step 4 (CONTINUED): Uniqueness of solution.

$$\alpha_1'(t)V_1+\alpha_2'(t)V_2 = Y'(t) = AY(t) = A\big(\alpha_1(t)V_1 + \alpha_2(t)V_2\big)$$
$$= \alpha_1(t)AV_1 + \alpha_2(t)AV_2 = \alpha_1(t)\lambda_1 V_1 + \alpha_2(t)\lambda_2 V_2$$

But $\{V_1, V_2\}$ linearly independent, hence $\alpha_1'(t) = \lambda_1\alpha_1(t)\ \forall t$ and $\alpha_2'(t) = \lambda_2\alpha_2(t)\ \forall t$. Also,

$$Y(0) = \alpha_1(0)V_1 + \alpha_2(0)V_2 = \alpha_1(0)^0 V_1 + \alpha_2^0 V_2 = X_0\,,$$

so we have $\quad \alpha_i'(t) = \lambda_i\alpha_i(t)\,, \qquad \alpha_i(0) = \alpha_i^0\,, \qquad i = 1, 2.$

$$\therefore \alpha_i(t) = \alpha_i^0 e^{\lambda_i t},\ i = 1, 2, \qquad \textit{i.e.,}\ Y(t) = X(t)\,\forall t.$$

$\square$

General solutions of linear systems of ODEs

- Theorem generalizes immediately to $\mathbb{R}^n$.
  (Proof is identical.)

- We assumed that A has n distinct real eigenvalues.
  Is this necessary?
  - No. The proof depends on existence of a basis of eigenvectors. If some of the eigenvectors are associated with the same eigenvalue, it doesn't matter.

Example

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Solution:

- Characteristic equation: $p(\lambda) = (\lambda - 3)(\lambda + 1) = 0$.
- Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = -1$.
- Eigenvectors: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
- Express initial condition in terms of eigenvectors:

$$X_0 = \frac{1}{4} V_1 - \frac{1}{4} V_2 .$$

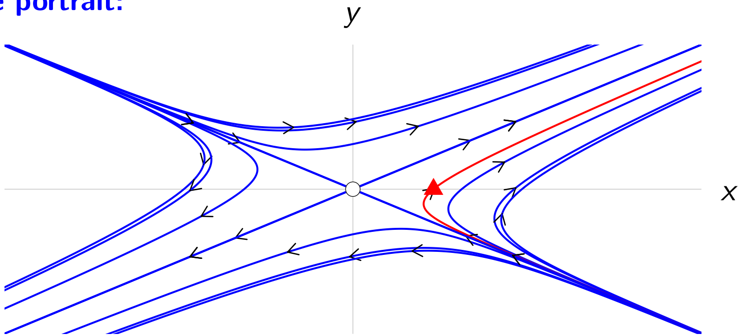
- Infer solution to IVP:

$$X(t) = \frac{1}{4} e^{3t} V_1 - \frac{1}{4} e^{-t} V_2 , \qquad \textit{i.e.,} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} e^{3t} + e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix} .$$

Example

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Phase portrait:



$$\blacktriangle = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Drawing Phase Portraits for linear systems, $X' = AX$

Rather than drawing many exact solutions based on quantitative solution formula:

- Draw all equilibria, indicating their stability.
- Draw eigendirections, indicating direction of motion.
- Draw direction field.
- Fill in phase portrait based on solutions always being parallel to direction field (and never crossing).
- Approach also works for nonlinear systems, *except* eigendirections do not necessarily correspond to solutions.