McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 30
Limit Sets
Friday 15 November 2013

Announcements

- **Assignment 4:**
  - Was due today, 15 Nov 2013, 1:30pm.
- **Test 1:**
  - Solutions were posted on Wednesday.
  - Compare with your solutions and make sure you understand the differences.
- **Assignment 5:**
  - Due NEXT Friday 22 Nov 2013, 1:30pm.
  - To be posted by the end of today.
- **Test 2:**
  - Wednesday 27 November 2013, 11:30–1:20.
- **Basic Notions Seminar:** "A gentle intro to graph theory"
  - Thurs 28 Nov 2013 @ 5:30pm in HH-312
  - Lauren DeDieu will talk about its history a little bit, discuss the 4-colour problem, the 3-colour problem, and then talk a little bit about list colouring, and harmonious colouring.

Equilibria

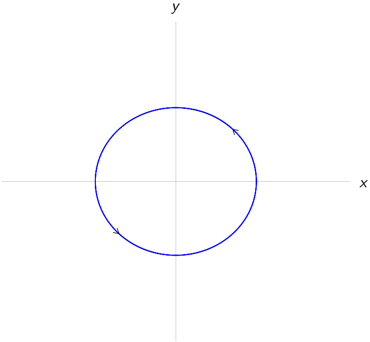
- Single points that are fixed for all time.

Periodic Orbits

- A set $\mathcal{P} \subset \mathbb{R}^n$ is a *periodic orbit* if $\exists \tau > 0$ such that $\forall X \in \mathcal{P}$

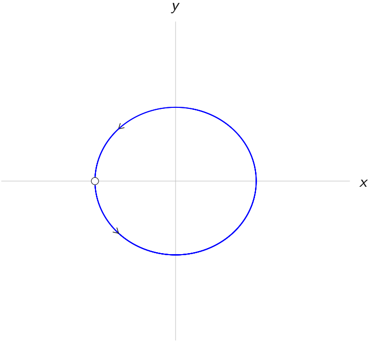
$$\phi_{t+\tau}(X) = \phi_t(X), \quad \forall t \in \mathbb{R}.$$

- Period = mininum τ for which this is true.
- ($\tau > 0$, else it is an equilibrium)



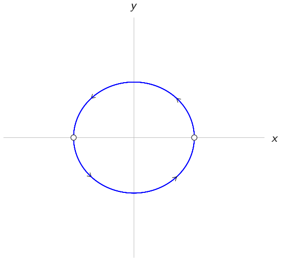
Homoclinic Orbits

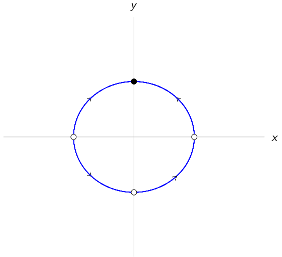
- Begin and end at an equilibrium X_*.
- A solution that tends to a given equilibrium X_* in both forward and backward time.
- $\lim_{t\to\infty} X(t) = \lim_{t\to-\infty} X(t) = X_*$.



Heteroclinic Orbits

- A solution that tends to one equilibrium X_*^1 in backward time and a different equilibrium X_*^2 in forward time.
- $\lim_{t\to-\infty} X(t) = X_*^1$.
- $\lim_{t\to\infty} X(t) = X_*^2$.
- Several heteroclinic orbits can link together to form a chain, which may or may not be closed.





Other complex invariant sets

- Arbitrarily complicated invariant sets made up of homoclinic and heteroclinic orbits can occur.
- Other "strange" sets that we'll discuss later.

- All such invariant sets can occur as *limit sets* of a nonlinear system.

ω-limit point

Definition (ω-limit point)

Suppose $\phi_t(X)$ is a the time-t map associated with the flow of a differential equation $X' = F(X)$. If $\exists$ a sequence of times $\{t_0, t_1, t_2, \ldots\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \phi_{t_n}(X) = Y$$

then Y is said to be an **ω-limit point** of the solution through X.

Note: We need a discrete set of times $\{t_n\}$ since $\lim_{t\to\infty} \phi_t(X)$ may not exist.

ω-limit set

Definition (ω-limit set)

Given a differential equation $X' = F(X)$, the **ω-limit set** of a point $X \in \mathbb{R}^n$ is

$$\omega(X) = \{Y \in \mathbb{R}^n : Y \text{ is an } \omega\text{-limit point of the solution through } X\}.$$

Note: We similarly define **α-limit points** for backwards time limits ($t_n \to -\infty$). The **α-limit set** of X, $\alpha(X)$, is the set of all α-limit points of X.

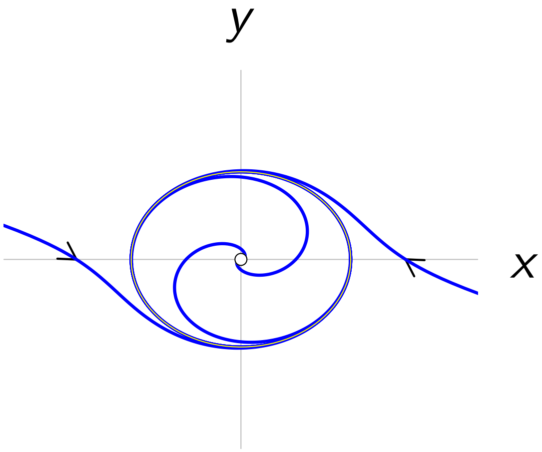
Limit set example

Example

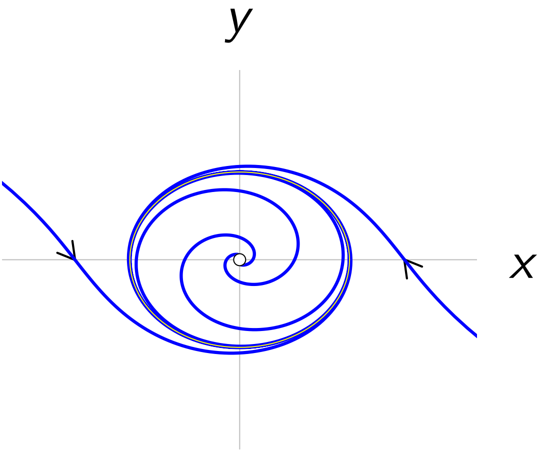
$$
\begin{aligned}
r' &= r(1-r^2) \\
\theta' &= 1
\end{aligned}
$$

- Unit circle $\mathcal{C} = \{(r,\theta) : r = 1\}$ is a periodic orbit.
- Origin $E = (0,0)$ is an equilibrium.
- $\forall X \neq E$, $\omega(X) = \mathcal{C}$.
- $\forall X \in \{(r,\theta) : 0 < r < 1\}$, $\alpha(X) = E$.
- $\forall X \in \{(r,\theta) : r > 1\}$, $\alpha(X) = \emptyset$.

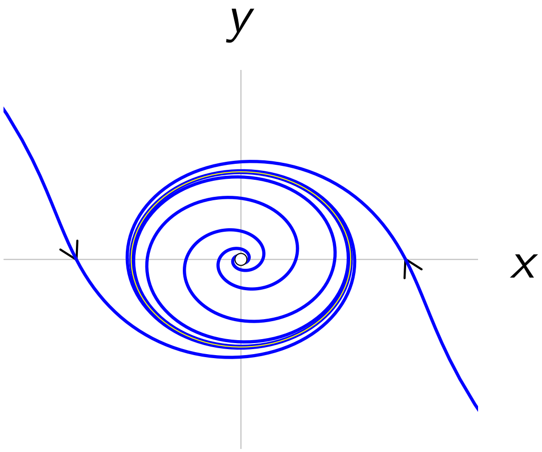
Limit set example: $r' = r(1 - r^2), \quad \theta' = 1$



Limit set example: $r' = r(1 - r^2), \quad \theta' = 2$



Limit set example: $r' = r(1 - r^2), \quad \theta' = 3$



Stable and Unstable Manifolds

Next few slides would have been more natural in Lecture 26

The Stable Curve Theorem

Is there an analogue of "eigendirections" in the vicinity of a nonlinear saddle?

- In general, for a nonlinear saddle:
  - $\nexists$ stable invariant line
  - $\nexists$ unstable invariant line
- However:
  - $\exists$ stable invariant *curve*
  - $\exists$ unstable invariant *curve*
  - These invariant curves meet at the equilibrium point.

Theorem (Stable Curve Theorem)

Consider a smooth planar system $X' = F(X)$, with a hyperbolic saddle equilibrium X_*. There is an open ball B containing X_* and a smooth curve $\gamma_s \subset B$ such that

(i) $X_* \in \gamma_s$;

(ii) *All solutions with initial conditions that lie on γ_s remain on γ_s for all $t \geq 0$ and tend to X_* as $t \to \infty$;*

(iii) *The curve γ_s passes through X_* tangent to the stable eigendirection of the linearized system $(X - X_*)' = DF_{X_*}(X - X_*)$;*

(iv) *All other solutions with initial conditions that lie in B leave B in finite time.*

Proof.

See HSD §8.3. □

Stable and Unstable Manifolds

- There is similarly an *Unstable Curve Theorem*.
- The theorems refer to *local* stable and unstable curves, but they connect to *complete* stable and unstable curves.
- The complete stable curve $W_{\mathrm{s}}(X_*)$ is found by following any point on γ_{s} backward in time ($t \to -\infty$).
- The complete unstable curve $W_{\mathrm{u}}(X_*)$ is found by following any point on γ_{u} forward in time ($t \to \infty$).
- These notions (and theorems) generalize to nonlinear saddles in $\mathbb{R}^n$. $W_{\mathrm{s}}(X_*)$ and $W_{\mathrm{u}}(X_*)$ becomes surfaces or hypersurfaces, known as the stable and unstable *manifolds*.