McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 33
Poincaré-Bendixson Theorem
Wednesday 20 November 2013

Announcements

- Assignment 5 due this Friday, 22 Nov 2013, at 1:30pm.
  - Also look at Assignment 5 from 2011: includes problem on Poincaré Bendixson theorem and other relevant problems.
- Test 2 next Wednesday, 27 Nov 2013, at 11:30am.
  - Emphasis is on material covered since Test 1, including what we cover this week.
  - Location: T29 / 101
- **Dora's office hours** on Wed 27 Nov 2013 are **MOVED TO TUESDAY 26 NOV 2013, 3:30-4:30pm.**

The Poincaré-Bendixson Theorem

Before stating the theorem, recall the definitions of ω- and α-limit sets of solutions of a differential equation $X' = F(X)$.

Definition (Omega and Alpha Limit Points)

Y is an ω-limit point of the solution that passes through X if there is a sequence of times $\{t_n : n = 0, 1, 2, \ldots\}$ with $t_n \to \infty$ as $n \to \infty$ and $\phi_{t_n}(X) \to Y$ as $n \to \infty$.

(For an α-limit point, $t_n \to -\infty$.)

Definition (Omega and Alpha Limit Sets)

$\omega(X_0)$ is the set of all ω-limit points of X_0.
$\alpha(X_0)$ is the set of all α-limit points of X_0.

The Poincaré-Bendixson Theorem

Theorem

For a planar flow, if $\omega(X_0)$ is bounded and contains no equilibria then $\omega(X_0)$ is a periodic orbit. (Similarly for $\alpha(X_0)$.)

- This theorem can be used to prove that a given planar ODE has a periodic solution, by finding a set $\mathcal{S}$ that
  - is forward-invariant, *i.e.*, $X_0 \in \mathcal{S} \implies \phi_t(X_0) \in \mathcal{S},\ \forall t > 0$,
  - is bounded, *i.e.*, $\mathcal{S} \subset B$ for some ball B of finite radius,
  - contains no equilibria.
- Then $X_0 \in \mathcal{S} \implies \omega(X_0) \subset \mathcal{S}$, and hence $\omega(X_0)$ is bounded.
- The Poincaré-Bendixson Theorem then implies that $\omega(X_0)$ is a periodic orbit.

The Poincaré-Bendixson Theorem

NOTE:

- The theorem is true only in the plane.
- In $\mathbb{R}^n$ with $n \geq 3$ it is possible to have much more complicated bounded limit sets.

Example: Application of Poincaré-Bendixson Theorem

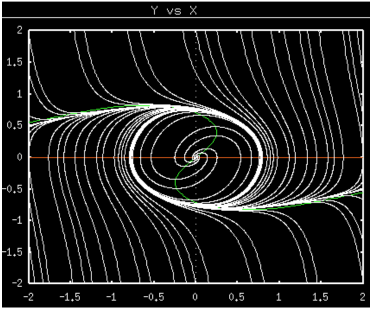
$$x' = y\,, \qquad y' = -x + y(1 - x^2 - 2y^2)$$

- Equilibria: $\exists!$ equilibrium at $(x, y) = (0, 0)$.
- If $\exists$ periodic orbit then it encloses the equilibrium (Index Theorem).
- Show $\exists$ forward-invariant annulus around the equilibrium.
- Consider $L(x, y) = \frac{1}{2}(x^2 + y^2)$; paraboloid, circular level sets.
- $\dot{L} = \nabla L \cdot (x', y') = y^2(1 - x^2 - 2y^2)$. So?. . .
- $\dot{L} \geq 0$ on circle of radius $\frac{1}{2}$ and $\dot{L} \leq 0$ on circle of radius 1.
- $\therefore \mathcal{A} = \{(x, y) : \frac{1}{4} \leq x^2 + y^2 \leq 1\}$ is forward-invariant.
- $\therefore \forall X \in \mathcal{A}$, $\omega(X) \subset \mathcal{A} \implies \omega(X)$ is bounded.
- $\therefore$ no equilibria in $\mathcal{A} \implies \omega(X)$ is a periodic orbit.

Example: Application of Poincaré-Bendixson Theorem

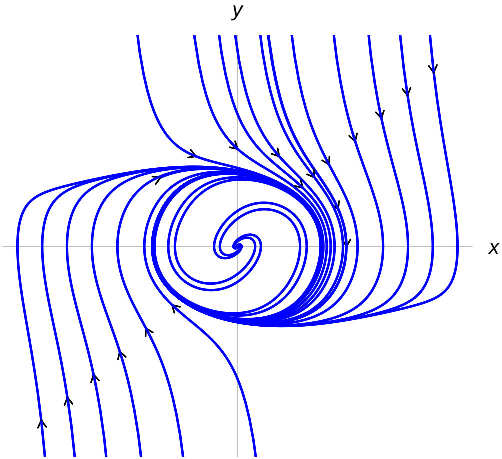
$$x' = y\,, \qquad y' = -x + y(1 - x^2 - 2y^2)$$

- Argument on previous slide shows $\exists$ periodic orbit in $\mathcal{A}$, but does NOT show $\exists!$ periodic orbit in $\mathcal{A}$ (although true for this particular system).



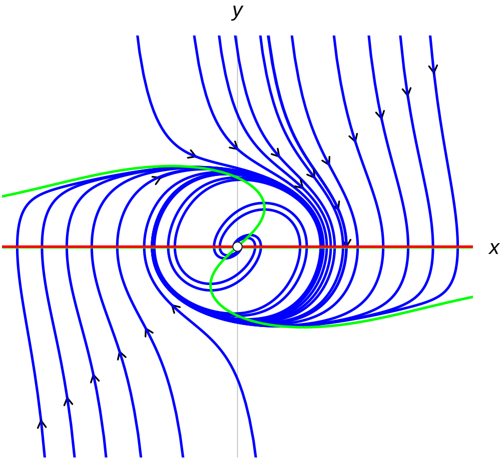
Example: Application of Poincaré-Bendixson Theorem

$$x' = y\,, \qquad y' = -x + y(1 - x^2 - 2y^2)$$



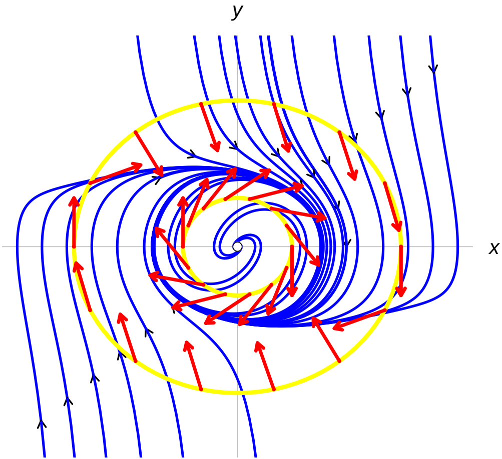
Example: $x' = y\,, \qquad y' = -x + y(1 - x^2 - 2y^2)$

with nullclines



Example: $x' = y\,, \qquad y' = -x + y(1 - x^2 - 2y^2)$

with direction field on circles for application of Poincaré-Bendixson



XPPAUT can compute phase portraits numerically

- Easy to learn to use. Free. Open source.
- Fast to produce phase portraits (also nullclines and stable and unstable manifolds).
- BUT, you need to understand the theory to know how to interpret the phase portraits (and to use the software effectively).
- XPPAUT can also calculate bifurcation diagrams numerically, which is usually the only possible way to compute bifurcation diagrams.
- Graduate course offered in Winter 2012 included learning to use this software for bifurcation analysis: Math 746 *"Bifurcation and Stability Theory"*.
  Can be taken by undergrads with permission from instructor.
  But not offerred this year.

Darn... what else would be fun next term?

- Math 4MB3 “Mathematical Biology”.
- Prerequisite is Math 3F03.
- Will involve learning to use XPPAUT
(though not to the fancy level used in Math 746).
- Will emphasize application of qualitative theory of ODEs to epidemiology (infectious disease transmission modelling).
- Fun! Fun! Fun!

Online Course Evaluations

Your course evaluations are critical to future course development and instructor assessment processes.

Course Evaluations for 2013 Fall Term 1

Open: Wednesday November 20, 2013 at 10:00 a.m.*
Close: Wednesday December 4, 2013 at 4:00 p.m.

* *Faculties of ENG, HUM, SOCSCI, SCI, and DSB*

- Log in with your MAC ID to evaluate your courses.
- Each evaluation will take approximately 5 to 15 minutes to complete.
- Your responses are completely anonymous.
- Evaluation results are not made available to instructors until ***after*** final marks have been submitted to the Office of the Registrar.

https://evals.mcmaster.ca

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