McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

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Lecture 23

Yet More Foundations of Nonlinear Systems

Monday 4 November 2013

Announcements

- **Assignment 4 delayed:**
  - Will be posted at the end of this week.
  - Due Friday 15 Nov 2013.
- **Midterm Test #1:**
  - Multiple choice section marked; other sections in progress.
  - Will discuss multiple choice questions today.

Dynamical Systems

Definition

A *smooth dynamical system* on $\mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t, X) = \phi_t(X)$ satisfies

1. $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function: $\phi_0(X) = X$;
2. The composition $\phi_t \circ \phi_s = \phi_{t+s}\ \forall t, s \in \mathbb{R}$.

- Differential equations that yield unique solutions define smooth dynamical systems: we will see why today.

Existence/Uniqueness Simplified Summary

IVP: $X' = F(t,X), \quad X(t_0) = X_0, \quad (t,X) \in \mathbb{R} \times \mathbb{R}^n$

- F continuous on an open rectangle $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$
  $\implies \exists$ solution through any $(t_0, X_0) \in \mathcal{R}$
  Peano Existence Theorem

- F C^1 on $\mathcal{R} \subset \mathbb{R} \times \mathbb{R}^n$
  $\implies \exists!$ solution through any $(t_0, X_0) \in \mathcal{R}$
  Fundamental Existence and Uniqueness Theorem

- F C^1 and $\frac{\partial F}{\partial X}$ uniformly bounded on strip $\mathcal{S} = [t_1, t_2] \times \mathbb{R}^n$
  $\implies \exists!$ solution throughout the time interval in $[t_1, t_2]$
  Global Picard Theorem

Note: The summary statements above are simplified, compared to the full statements discussed in the previous lecture. Rather than "F C^1", the full statements give the weaker hypothesis that "F is C^0 in t and C^1 in X". Even this is a stronger hypothesis than necessary. We discussed **Lipschitz continuity** in the previous lecture.

Multiple Choice Question From Test

(i) For a certain autonomous differential equation, $X' = F(X)$, $X \in \mathbb{R}^n$, it is known that there is a unique solution through each point $X_0 \in \mathbb{R}^n$. Based on this fact and the Existence/Uniqueness theorems discussed in class:

(a) It follows that F is continuous.

(b) It follows that F is differentiable.

(c) It follows that F is C^1.

(d) It follows that F is C^∞.

(e) None of the above.

Continuous Dependence on Initial Conditions

Let's now restrict attention to autonomous systems, $X' = F(X)\ldots$

Theorem

Suppose F is C^1 and consider the system $X' = F(X)$. The flow of the system, $\phi(t, X)$, is a continuous function of X.

- This means that solutions starting at nearby initial conditions remain close (at least for a short time).
- Separation of solutions happens no faster than exponential (see theorem, p.147).
- Where does exponential separation come from?

Continuous Dependence on Parameters

Theorem

Let $X' = F_a(X)$ be a one-parameter family of differential equations for which F_a is C^1 as a function of X and as a function of the parameter a. Then the flow of this system is also a continuous function of a.

- Easy proof (based on a very useful trick): Expand the system by one dimension, letting the additional variable be a with associated equation $a' = 0$. Then just apply the previous theorem!

Smooth Dependence on Initial Conditions

Theorem

Suppose F is C^1 and consider the system $X' = F(X)$. The flow of the system, $\phi(t, X)$, is a C^1 function of t and X.

Proof.

See §17.6, p. 405. ☐

- This means the flow of a C^1 ODE is a smooth dynamical system.

An important type of NON-AUTONOMOUS system

Theorem

Let $A(t)$ be an $n \times n$ matrix function of t, with entries $a_{ij}(t)$ that are continuous functions for $t \in [t_1, t_2]$. Then the IVP

$$X' = A(t)X \,, \qquad X(t_0) = X_0 \,,$$

has a unique solution that is defined on the entire interval $[t_1, t_2]$.

- Note that it is enough for $A(t)$ to be continuous. It does not need to be C^1 to guarantee uniqueness.
  - This is a special case of the Global Picard Theorem, where F C^0 in t, C^1 in X and $\frac{\partial F}{\partial x_i}$ uniformly bounded in X was enough.
    Proof of uniform boundedness: The time interval $[t_1, t_2]$ is closed $\implies$ each continuous function $|a_{ij}(t)|$ has a maximum value on this interval. Let $K_{ij} = \max\{|a_{ij}(t)| : t_1 \le t \le t_2\}$ and let $K = \max_{ij} K_{ij}$. Then $\|\partial F/\partial x_i\| = \|(a_{1i}(t), \ldots, a_{ni}(t))\| = [a_{1i}^2(t) + \cdots + a_{ni}^2(t)]^{1/2} \le nK$.
- Easy to see continuity is enough in the one-dimensional case: Solve $x' = a(t)x$, $x(0) = x_0$.

Why do we care about $X' = A(t)X$?

Consider the general autonomous (**non-linear**) IVP

$$X' = F(X)\,, \qquad X(t_0) = X_0\,, \qquad X, X_0 \in \mathbb{R}^n, \qquad (\heartsuit)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Is there some $A(t)$ such that $X' = A(t)X$ can help us understand the behaviour of $(\heartsuit)$?

- Define $A(t)$ to be the Jacobian matrix of F at the point $X(t) \in \mathbb{R}^n$, i.e.,

$$A(t) = DF_{X(t)} = \left.\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}\right|_{x_1=x_1(t),\ldots,x_n=x_n(t)}$$

$$= \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\right|_{x_1=x_1(t),\ldots,x_n=x_n(t)}$$

Why do we care about $X' = A(t)X$?

If F is C^1 then $A(t) = DF_{X(t)}$ is a continuous matrix function of time.

$\therefore$ The **variational equation**

$$U' = A(t)U\,, \qquad U \in \mathbb{R}^n,$$

has a unique solution for any initial condition $U(t_0) = U_0$.

Why should we care?

ANSWER: If $X(t)$ is a solution of $X' = F(X)$ and U_0 is small ($\|U_0\| \ll 1$) then $X(t) + U(t)$ is a good approximation to the solution that starts at $X_0 + U_0$ (at least for a short time).

Why do we care about $X' = A(t)X$?

Where does exponential divergence come from?

Consider $|t - t_0| \ll 1$ and $\|X_0 - Y_0\| \ll 1$. Then:

- $DF_{X(t)} \approx$ constant matrix $\equiv A$
  $\implies X' \approx X_0 + AX$ and $Y' \approx Y_0 + AY$
- Let $U = X - Y$. Then

$$U' = X' - Y' \approx (X_0 - Y_0) + A(X - Y) \approx A(X - Y) = AU$$

$U' = AU$ **is the equation we've been studying for weeks!**

- Solution of $U' \approx AU$ is $U(t) \approx e^{tA}U_0$

i.e., $X(t) - Y(t) \approx e^{tA}(X_0 - Y_0)$

$\implies$ exponential divergence "at worst"

Why do we care about $X' = A(t)X$?

- VERY IMPORTANT POINT: If $X(t)$ is an equilibrium solution, then $DF_{X(t)} \equiv A$, a constant matrix, so **the variational equation is EXACTLY the linear equation we have been studying for the last few weeks.**

- CONSEQUENTLY: near (almost all) equilibria, the "linearized" equation behaves like the full non-linear equation. (We'll see later exactly when this is true.)