McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

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Lecture 18

Matrix Exponentials

Wednesday 16 October 2013

Announcements

Test #1

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

Location: T29 / 101

- Further info may be posted on the course wiki closer to the test date.

Announcements

- **Assignment 3** due on Friday 25 Oct 2013.
  - **Check course wiki for updates!**
  - **More questions will be posted by the end of the week!**
- Tutorial this Friday, 18 Oct 2013.

Generalizing the solution of the 1D linear equation

Lemma (What you learned in Kindergarten)

If $a \in \mathbb{R}$ then the unique solution to the one-dimensional initial value problem

$$x' = ax\,, \qquad x(0) = x_0 \in \mathbb{R},$$

is $x(t) = x_0 e^{at}$.

How can we generalize this?

Generalizing the solution of the 1D linear equation

Theorem (Solution of Arbitrary Linear Systems in $\mathbb{R}^n$)

If A is an $n \times n$ real matrix then the unique solution of the IVP

$$X' = AX \,, \qquad X(0) = X_0 \in \mathbb{R}^n \,,$$

is

$$X(t) = e^{tA} X_0 \,.$$

- Wow!
- What exactly does this mean?
- How do we even **define** e^A if A is a matrix?

The Matrix Exponential

- We know meaning of matrix powers, A^k, where $k \in \mathbb{N}$.
- We know power series representation of e^x for $x \in \mathbb{R}$.
- So define the matrix exponential via the series!

Definition (Matrix Exponential e^A)

If A is an $n \times n$ matrix then

$$e^A = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} A^k \,,$$

where $A^0 \equiv I_n$, the $n \times n$ identity matrix.

- For this definition to be useful, we must prove that the matrix series *converges* for any A.
- We'll return to this later...
- Let's first look at an example.

Example: diagonal matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad A^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix}$$

$$\therefore \quad e^A = \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix}$$

Lemma (e^A for diagonal matrices)

If $A = diag(a_1, \ldots, a_n)$, with $a_i \in \mathbb{R}$ for all i, then

$$e^A = diag(e^{a_1}, \ldots, e^{a_n})\,.$$

Calculating e^A for general A

- How do we calculate e^A if A is *not* diagonal?
- Hint: $(S^{-1}BS)^k = S^{-1}B^kS$.
- $\therefore$ Change to a more convenient basis!
  Such as... a basis in which A is diagonal!
- Is that always possible?
- No.
- Best we can do in general is Jordan Canonical Form (JCF).
- After coverting to JCF, task reduces to
  - Compute e^B for each Jordan block B separately.
  - Convert back to original coordinates.

Important Properties of Matrix Exponentials

Proposition

For $n \times n$ matrices A, B, T:

(i) $e^{T^{-1}AT} = T^{-1}e^{A}T$

(ii) $AB = BA \implies e^{A+B} = e^{A}e^{B}$

(iii) e^{A} *is invertible and* $(e^{A})^{-1} = e^{-A}$

- Proof of (ii) is non-trivial.
- Is there a (non-trivial) condition on the *entries* of A and B that guarantees that $e^{A+B} = e^{A}e^{B}$?