Mathematics 3F03
Advanced Differential Equations

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Lecture 5

More on Models of Harvesting

16 September 2013

Announcements

- Assignment 1 due THIS FRIDAY, 20 Sep 2013, before class, in the appropriately labelled locker next to HH-105.
- **Putnam Mathematical Competition:** There will be an organizational meeting on Thursday 19 Sep 2013 in HH 410.

  The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. More information can be found at `http://math.scu.edu/putnam` or on the undergraduate page of the Math department's website. This year's competition will occur on 7 Dec 2013. If you are interested in participating or learning more, drop by on Thursday 19 Sep and/or send email to Prof. Bradd Hart (`hartb@mcmaster.ca`).

Constant rate harvesting

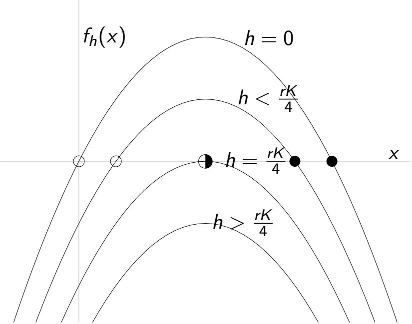
Motivation: Daily Quota, *e.g.*, maximum number of fish per day

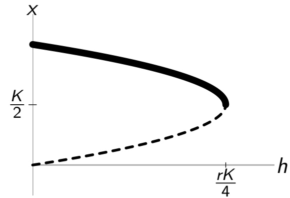
$$\frac{dx}{dt} = f_h(x) = rx\left(1 - \frac{x}{K}\right) - h\,, \qquad x(0) = x_0$$

Constant rate harvesting

Equilibria occur at $x_\pm = \frac{1}{2}\left(K \pm \sqrt{K(K-(4h/r))}\right)$.

Graphs of $f_h(x)$ $\Longrightarrow$ Bifurcation Diagram wrt h





Fold bifurcation at $h = rK/4$.

Note: Curves in bifurcation diagram are $x_-(h)$ (dashed) and $x_+(h)$ (solid). Equivalently: $h(x) = rx(1-x/K)$.

Constant Effort Harvesting

$$\frac{dx}{dt} = f_E(x) = rx\left(1 - \frac{x}{K}\right) - Ex\,, \qquad x(0) = x_0$$

- Instead of constant rate h, here we have constant *per capita* rate E.
- Mathematically sensible: $f_E(x)$ is C^1 everywhere $\implies \exists!$ solution $\forall x_0 \in \mathbb{R}$.
- Biologically well-posed: population cannot become negative $\because f_E(0) = 0$.
- What happens?

Constant Effort Harvesting

$$\frac{dx}{dt} = f_E(x) = rx\left(1 - \frac{x}{K}\right) - Ex\,, \qquad x(0) = x_0$$

How should we determine what this model predicts?

- Plot $f_E(x)$ for a range of efforts E.
- Find equilibria as functions of parameters (r, K, E).
- For each equilibrium x_*, what is sign of $f_E'(x_*)$?
(Decisive wrt stability only if $f_E'(x_*) \neq 0$.)
- Determine stability of any x_* for which derivative test is not decisive.

Do we need to start from scratch in our analysis of this model?

Constant Effort Harvesting

Actually, this is formally the same model as the original logistic model, but with different meanings to the parameters:

$$\frac{dx}{dt} = f_E(x) = (r - E)x\left(1 - \frac{r}{(r-E)K}x\right)$$

So, define

$$\tilde{r} = r - E$$
$$\tilde{K} = K\left(\frac{r-E}{r}\right)$$

Then:

$$\frac{dx}{dt} = f_E(x) = \tilde{r}x\left(1 - \frac{x}{\tilde{K}}\right).$$

So, it appears to be sufficient to study solutions of the original logistic equation!

Constant Effort Harvesting

$$\frac{dx}{dt} = f_E(x) = (r - E)x\left(1 - \frac{r}{(r-E)K}x\right)$$
$$= \tilde{r}x\left(1 - \frac{x}{\tilde{K}}\right).$$

BUT:

- $\tilde{r}, \tilde{K}$ can both be *negative* and still OK biologically. Meaning of parameters is different!
- We must be careful about the situation in which $r = E$.
  - In this case, formula for $f_E(x)$ above is WRONG.

Constant Effort Harvesting

We can still use our knowledge of solutions to the original logistic equation if we are more careful:

$$\frac{dx}{dt} = f_E(x) = \begin{cases} (r-E)x\left(1 - \frac{r}{(r-E)K}x\right) & E \neq r, \\ -\frac{r}{K}x^2 & E = r. \end{cases}$$

$$= \begin{cases} \tilde{r}x\left(1 - \frac{x}{\tilde{K}}\right) & \tilde{r} \neq 0, \\ -\frac{r}{K}x^2 & \tilde{r} = 0. \end{cases}$$

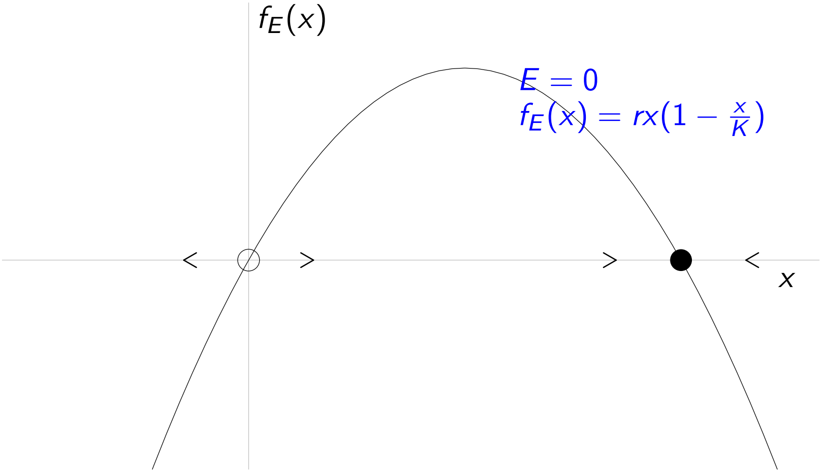
- Solutions change character as *per capita* harvesting rate (effort E) balances and then exceeds the natural growth rate (r).
- *i.e.*, $E = r$ is a bifurcation point for the model.

Constant Effort Harvesting

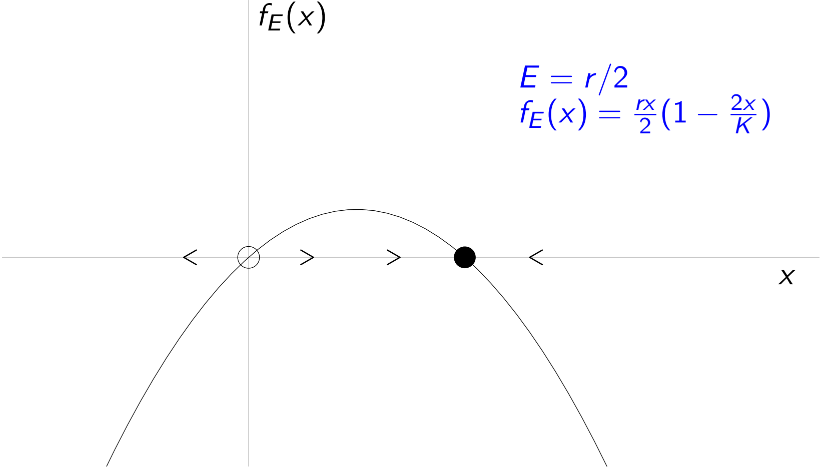
Strategy for qualitative analysis:

- Plot $f_E(x)$ and the phase line for several representative values of effort E $\quad$ (*e.g.*, $E = 0, \frac{r}{2}, r, \frac{3r}{2}$).
- Then construct bifurcation diagram:
  - Equilibria $x_*(E)$:
    - $x_* = 0$
    - $x_* = \tilde{K} = \frac{(r-E)K}{r}$
  - Use solid lines if x_* stable, dashed lines if x_* unstable.

Constant Effort Harvesting



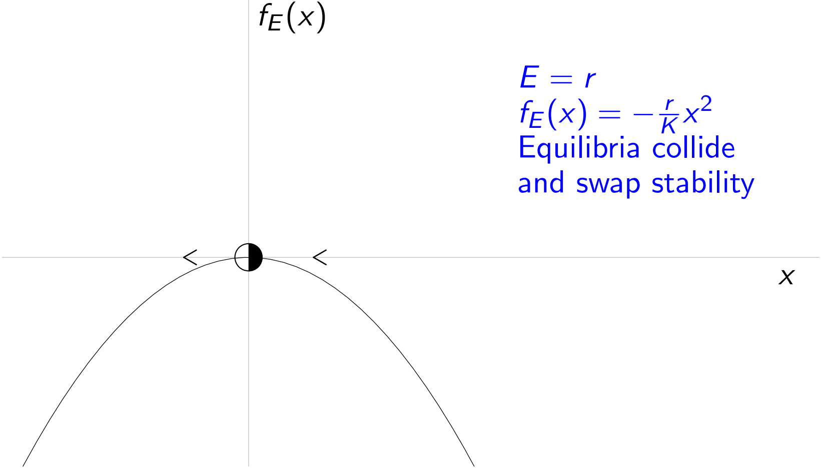
Constant Effort Harvesting



$E = r/2$

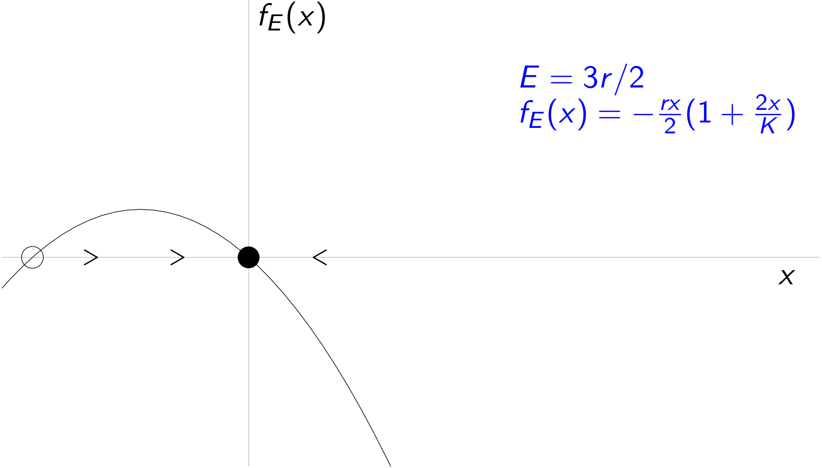
$f_E(x) = \frac{rx}{2}(1 - \frac{2x}{K})$

Constant Effort Harvesting



$E = r$
$f_E(x) = -\frac{r}{K}x^2$
Equilibria collide
and swap stability

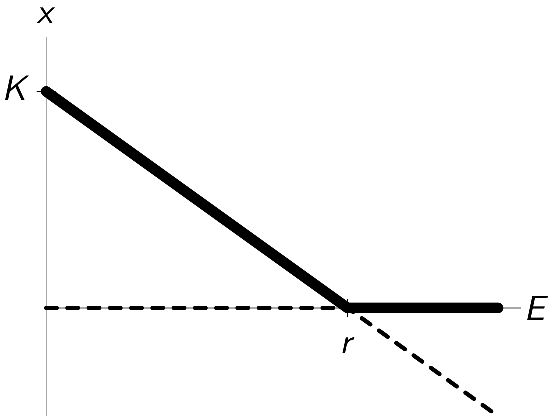
Constant Effort Harvesting



$E = 3r/2$
$f_E(x) = -\frac{rx}{2}(1 + \frac{2x}{K})$

Constant Effort Harvesting

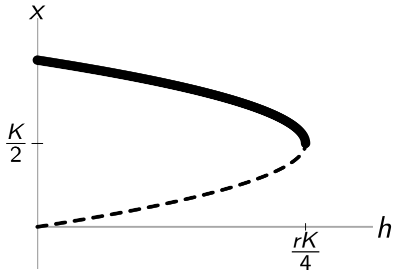
Bifurcation Diagram:



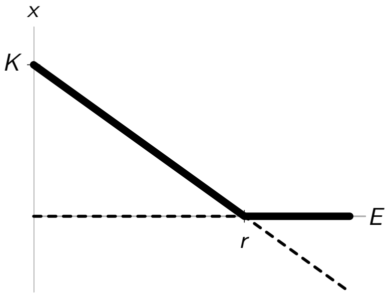
Transcritical bifurcation at $E = r$

Advice for Department of Fisheries and Oceans?

Catch quota (number of fish):



Effort quota (number of boats):



Which situation is safer for the fishery?

Reading

Read all of chapter 1.

- §1.4. Periodically forced ODE.
  - Periodic harvesting model is bad biologically (populations that start positive can become negative), but a fine example of a periodically forced dynamical system.
  - Concept of *Poincaré map* is important.
- §1.5. Computing Poincaré map.
  - Good for your soul.
  - Nice example.
  - Not critical to follow details.