Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 6

Second Order and Planar ODEs

18 September 2013

Announcements

- Assignment 1 due THIS FRIDAY, 20 Sep 2013, before class, in the appropriately labelled locker next to HH-105.
- **Putnam Mathematical Competition:** There will be an organizational meeting on Thursday 19 Sep 2013 in HH 410.

  The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. More information can be found at `http://math.scu.edu/putnam` or on the undergraduate page of the Math department's website. This year's competition will occur on 7 Dec 2013. If you are interested in participating or learning more, drop by on Thursday 19 Sep and/or send email to Prof. Bradd Hart (`hartb@mcmaster.ca`).

Second Order ODEs

Warning: using $'$ for time derivative.

$$x'' = f(t, x, x')\,, \qquad x(0) = x_0, \quad x'(0) = v_0\,.$$

- Example: *Forced Harmonic Oscillator*

$$mx'' = -bx' - kx + f(t)$$

- Usually forcing is periodic, *e.g.*, $f(t) = \cos(2\pi t)$.

2nd order ODE equivalent to two coupled 1st order ODEs

$$x'' = f(t, x, x'), \qquad x(0) = x_0, \quad x'(0) = v_0 .$$

- Let $y = x'$, $y_0 = y(0) = x'(0)$
- $\therefore y' = x''$
- Equation becomes

$$\begin{aligned} x' &= y, & x(0) &= x_0 \\ y' &= f(t, x, y), & y(0) &= y_0 \end{aligned}$$

- We will study general ODEs in the plane:

$$\begin{aligned} x' &= f(t, x, y) & x(0) &= x_0 \\ y' &= g(t, x, y), & y(0) &= y_0 \end{aligned}$$

General Planar ODE

$$
\begin{aligned}
x' &= f(t,x,y) & \qquad x(0) &= x_0 \\
y' &= g(t,x,y)\,, & \qquad y(0) &= y_0
\end{aligned}
$$

Equivalently, in vector notation:

$$
X' = F(t,X)\,, \qquad X_0 = X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2
$$

This generalizes immediately to n dimensions. . .

General ODE in $\mathbb{R}^n$

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) & \qquad x_1(0) &= x_{1,0} \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) & \qquad x_2(0) &= x_{2,0} \\
&\vdots & &\vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n) & \qquad x_n(0) &= x_{n,0}
\end{aligned}
$$

Equivalently, in vector notation:

$$X' = F(t, X)\,, \qquad X_0 = X(0) = (x_{1,0}, x_{2,0}, \ldots, x_{n,0}) \in \mathbb{R}^n$$

Very simple planar example

$$x' = y$$
$$y' = -x$$

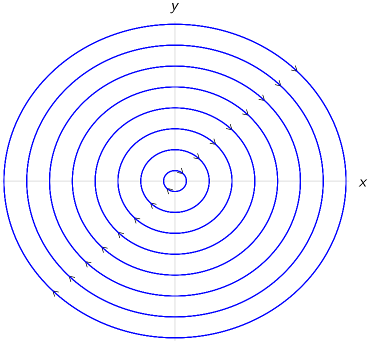
(Equivalent to $x'' = -x$)

- Solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

- Circular motion on a circle of radius a.

Solutions in the *phase plane* yield the *phase portrait*.



Interpretation as Vector Field

$$X' = F(t, X)\,, \qquad X_0 = X(0)$$

- $X =$ point in plane (or in $\mathbb{R}^n$)
- $F(t, X) =$ vector field, i.e., magnitude and direction of "flow"

- **Note:** In 1D, it is common to draw the "slope field" (a 2D drawing) since the vectors in the vector field lie on top of each other. But I find vector field a more intuitive concept. The vector field lies on the *phase line* in 1D and on the *phase plane* in 2D.

Vector Field vs Direction Field

- *Vector Field* shows both magnitude and direction of flow

- *Direction Field* shows only direction of flow
  - All vectors have the same length.
  - Usually much easier to draw and interpret.
  - Does not show speed of flow
    (though this can be represented using colour).
  - Computers are good at drawing such things...

Vector Field vs Direction Field

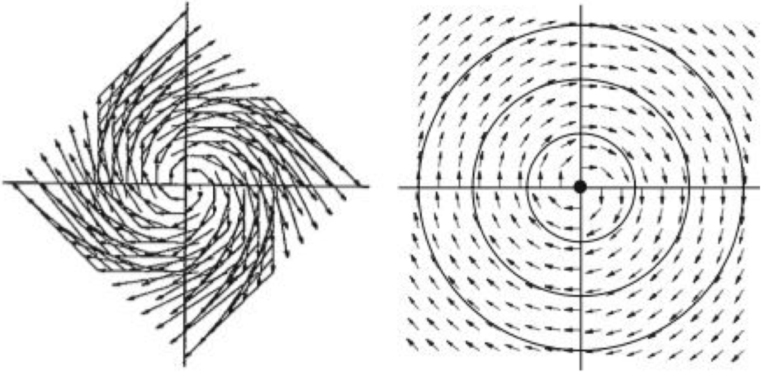


Figure 2.1 The vector field, direction field, and several solutions for the system $x' = y$, $y' = -x$.

Generalizing from the very simple planar ODE example

$$x' = y$$
$$y' = -x$$

- Equation can also be written:

$$X' = AX\,, \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- *This generalizes to any* **linear** *ODE in any number of dimensions.*
- Not surprising that linear algebra will be useful.
- Less obvious, but true: linear algebra is also critical for studying **nonlinear** ODEs.