Mathematics 3F03
Advanced Differential Equations

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Lecture 2

Introduction to Existence and Uniqueness

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Last time...

- Course info on course wiki:
  `http://www.math.mcmaster.ca/earn/3F03`
- A few concepts: ODE, phase line, bifurcation, stability, ...
- Simplest interesting (linear) example: $\frac{dx}{dt} = rx$
- Quantitative vs qualitative theory

Today

1 Instructive nonlinear examples

2 Existence and uniqueness

Nonlinear example 1

$$\frac{dx}{dt} = \sqrt{x}\,, \qquad x(0) = x_0\,. \tag{1}$$

- Solve by separation of variables: $\int \frac{1}{\sqrt{x}}\frac{dx}{dt}dt = \int 1 \cdot dt$
- Integrate from time 0 to t:

$$t = \int_0^t 1 \cdot dt = \int_0^t \frac{1}{\sqrt{x}}\frac{dx}{dt}dt = \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_{x_0}^{x(t)}$$
$$= 2\Big[\sqrt{x(t)} - \sqrt{x_0}\Big]$$

- Rearrange:

$$\sqrt{x(t)} = \frac{t}{2} + \sqrt{x_0} \qquad \Longrightarrow \qquad x(t) = \Big(\frac{t}{2} + \sqrt{x_0}\Big)^2$$

Nonlinear example 1

- Equation:

$$\frac{dx}{dt} = \sqrt{x}\,, \qquad x(0) = x_0\,. \tag{1}$$

- Solution:

$$x(t) = \left(\frac{t}{2} + \sqrt{x_0}\right)^2 \tag{2}$$

- Is this correct?
- What happens if $x_0 = 0$?
  - From (1), we see that $x(t) \equiv 0$ is a solution
  - But (2) says $x(t) = t^2/4$ is also a solution! Verify by plugging into (1).
- Solution is not unique!
- Equation (1) does **not** define a *dynamical system* (a system for which the future is always well-defined).

Nonlinear example 1

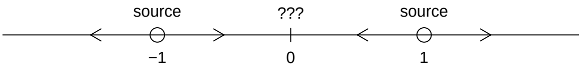
What is the "problem" with Equation (1)?

- Is our derivation of solution (2) valid if $x_0 = 0$?
  - No, but (2) really is a solution! We verified it!
- Is it that Equation (1) is not defined for $x < 0$?
  - No: same problem with $dx/dt = \sqrt{|x|}$.
  - Also same problem if we consider $x \in \mathbb{C}$.
- Problem is: RHS of Equation (1) is not differentiable...

Nonlinear example 2

$$\frac{dx}{dt} = \begin{cases} x - 1, & x \geq 0, \\ x + 1, & x < 0, \end{cases} \qquad x(0) = x_0. \tag{3}$$

- What happens at $x = 0$?
- Think *qualitatively:* direction of change.
  - If $x = 0$ then x is decreasing.
  - But if $\quad -1 < x < 0 \quad$ then x is increasing!
- There is no solution to Equation (3) that includes $x = 0$.
  - Typical if RHS is not even continuous.
- If $0 < |x_0| < 1$ then solution "stops" in finite time!
- If $|x_0| \geq 1$ then $\exists!$ solution $\forall t$.
- Phase line:



Existence and uniqueness

Roughly speaking, if $x'(t) = f(x)$ and $f(x)$ is continuously differentiable (C^1) then the initial value problem

$$x'(t) = f(x)\,, \qquad x(t_0) = x_0\,,$$

has a unique solution and therefore defines a dynamical system.

- So the Malthusian equation $x'(t) = rx$ has a unique solution for any given initial population size x_0.
- Can we prove this directly for this simple linear equation (without using the general theorem that we haven't proved)?
- Idea: if $x(t)$ and $y(t)$ are actually the same then $y(t)/x(t)$ is a constant...

Existence and uniqueness

Proposition

The one-dimensional initial value problem (IVP)

$$x'(t) = rx, \qquad x(0) = x_0,$$

has a unique solution for any $r \in \mathbb{R}$ and any $x_0 \in \mathbb{R}$.

Existence and uniqueness

Proof.

First note that $x(t) = x_0 e^{rt}$ is a solution to this IVP, hence there is **at least one** solution. Suppose $y(t)$ is another solution, *i.e.,*

$$y'(t) = ry(t), \qquad y(0) = x_0 \,.$$

Try to show that $y(t)$ must be proportional to $x(t)$, i.e., that $y(t)e^{-rt}$ is constant.

$$\frac{d}{dt}\Big(y(t)e^{-rt}\Big) = y'(t)e^{-rt} - ry(t)e^{-rt} = ry(t)e^{-rt} - ry(t)e^{-rt} = 0,$$

for all $t \in \mathbb{R}$. $\quad \therefore$ For some $c \in \mathbb{R}$, $y(t)e^{-rt} = c \;\forall t \in \mathbb{R}$.
But $y(0) = x_0 \implies c = x_0$.
$\therefore y(t) = x(t) \;\forall t \geq 0 \quad$ (and $\forall t \leq 0$),
i.e., the solution is **unique**. $\square$

Applied Question

We've proved something apparently useful about the Malthusian differential equation,

$$x' = rx\,.$$

- Is this a good population model?
  - Why or why not?