Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 3

Elementary Analysis of Population Models

11 September 2013

Announcements

- NO CLASS this Friday 13 Sep 2013.
- Assignment 1 will be posted on the course wiki on Friday.
- **Putnam Mathematical Competition Organizational Meeting:** Thursday 19 Sep 2013 @ 11:30am in HH 410.

  The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. More information can be found at `http://math.scu.edu/putnam` or on the undergraduate page of the Math department's website. This year's competition will occur on 7 Dec 2013. If you are interested in participating or learning more, drop by on Thursday 19 Sep and/or send email to Prof. Bradd Hart (`hartb@mcmaster.ca`).

Mathusian model: Exponential growth

We have so far considered the Malthusian model,

$$\frac{dx}{dt} = rx\,, \qquad x(0) = x_0\,.$$

- Quantitative solution: $x(t) = x_0 e^{rt}$.
- Qualitative results:
  - There exists a unique solution for any $r, x_0 \in \mathbb{R}$.
  - There is always an equilibrium at zero population size ($x = 0$).
  - There is a bifurcation at $r = 0$:
    - $r < 0 \implies$ unique (stable) equilibrium (population crashes)
    - $r = 0 \implies$ all solutions are equilibria (neutral stability)
    - $r > 0 \implies$ unique (unstable) equilibrium (population grows exponentially forever)
- Unless the population is relatively small (x_0 small), this is usually not a good population model.
- How can we improve it?

Logistic population model: motivation

- Recall meaning of parameter r: *per capita* growth rate:

$$r = \frac{1}{x}\frac{dx}{dt}\,.$$

- Is *per capita* growth really independent of population density?
- Reproductive rate decreases if organisms are running out of space to reproduce: **density dependence**.
- Simplest form of density dependence: linear decline with density:

$$\frac{1}{x}\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)\,.$$

  Parameters:
  - r: intrinsic reproductive rate
  - K: carrying capacity

Logistic population model

- So our IVP becomes:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \qquad x(0) = x_0 . \tag{1}$$

  Parameters:
  - r: intrinsic reproductive rate
  - K: carrying capacity

- Nonlinear, but RHS is $C^1 \implies \exists!$ solution $\forall x_0$.

- *Quantitative question:* Can we find a formula for the solution?

- *Qualitative question:* Can we determine how solutions behave for different values of r, K and x_0?

Logistic population model: Quantitative solution

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \qquad x(0) = x_0.$$

- Begin by simplifying our task: change variables to eliminate new parameter K.
- Let $u = \frac{x}{K}$: density relative to carrying capacity. (In "u-space" carrying capacity is 1.)
- Logistic equation in u-space is

$$\frac{du}{dt} = ru(1-u), \qquad u(0) = u_0 \equiv \frac{x_0}{K}.$$

- How do we solve this IVP? *i.e.*, Given initial state $u(0) = u_0$, how do we find a formula that predicts the future states $u(t)$?

Logistic population model: Quantitative solution

$$\frac{du}{dt} = ru(1-u)\,, \qquad u(0) = u_0 \equiv \frac{x_0}{K}\,.$$

Let's work out the solution on the blackboard...

Logistic population model: Quantitative solution

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \qquad x(0) = x_0.$$

- Exact solution is:

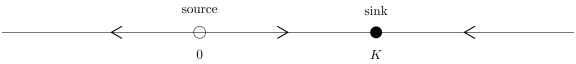
$$x(t) = \frac{K}{1 + ce^{-rt}}, \qquad c = \frac{K - x_0}{x_0}. \tag{2}$$

- What happens as $t \to \infty$?
  How does answer differ for different r, K, x_0?
- Could we have discovered the answer more easily?

Logistic population model: Qualitative solution

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \qquad x(0) = x_0.$$

- Qualitative approach:
  - Find equilibria: $\frac{dx}{dt} = 0 \iff x \in \{0, K\}$.
  - Plot phase line (assume $r, K > 0$):



- Quick and easy! More informative than formula for solution! Does not require exact solution!
- Why is "carrying capacity" a sensible name for parameter K?

Logistic population model: Qualitative solution

- We can also plot the slope field and solution curves (either quantatively or qualitatively)

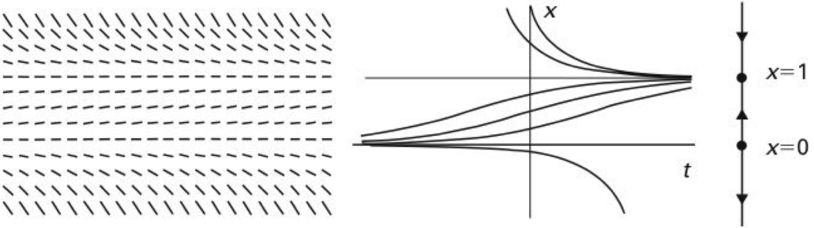


Figure 1.3 The slope field, solution graphs, and phase line for $x' = ax(1 - x)$.

- Preferable to use different symbols in phase line:

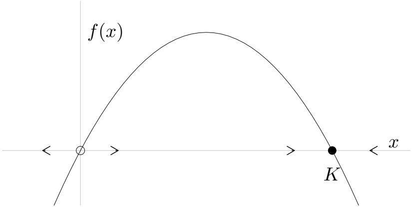
  ● stable equilibrium (sink)

  ○ unstable equilibrium (source)

Qualitative approach: helpful to plot slope function

$$\frac{dx}{dt} = f(x)\,, \qquad x(0) = x_0.$$

- $f(x)$ is the slope function.
- Plot $f(x)$ and interpret x-axis as the phase line. Sign of $f(x)$ determines direction of flow on phase line (the "phase flow").



- **Note:** This works to get qualitative behaviour *even if we cannot find the zeros of* $f(x)$ *(equilibria) analytically*.
- **Note:** Sign of $f'(x)$ (derivative of slope) at equilibrium points determines stability of equilibria.

Equilibrium stability terminology

$$\frac{dx}{dt} = f(x)\,, \qquad f(x_*) = 0\,.$$

- x_* stable (sink)
  - Sufficient condition: $f'(x_*) < 0$ $\quad$ (*not necessary!*)
- x_* unstable (source)
  - Sufficient condition: $f'(x_*) > 0$ $\quad$ (*not necessary!*)
- x_* semi-stable
  - Stable on one side of x_*, unstable on the other.
  - Derivative test *useless!*
  - $f'(x_*) = 0$ or undefined.
- **Note:** $f'(x)$ means derivative of f wrt state variable x, *NOT* derivative wrt time t (as in the differential equation).