McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

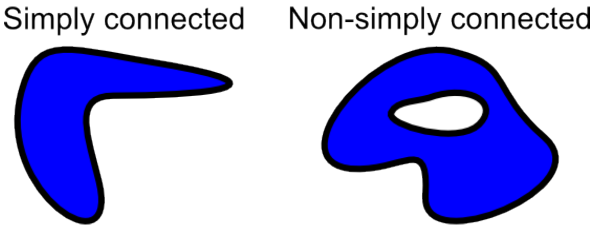
Instructor: David Earn

Lecture 31
Non-Existence of Periodic Orbits
Monday 18 November 2013

Announcements

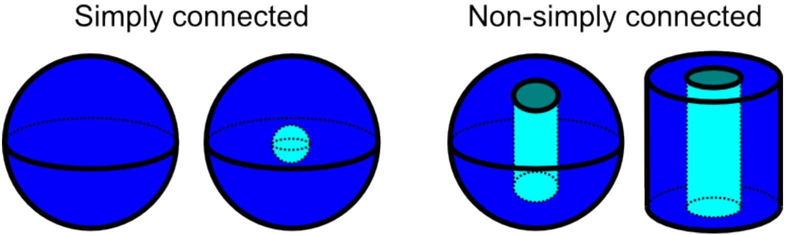
- Assignment 4 solutions were posted on Saturday.
- Assignment 5 was posted on Saturday
(due this Friday, 22 Nov 2013, at 1:30pm).
    - Also look at Assignment 5 from 2011: includes problem on Poincaré Bendixson theorem and other relevant problems.
- Test 2 next Wednesday, 27 Nov 2013, at 11:30am.
    - Emphasis is on material covered since Test 1, including what we cover this week.
    - Location: T29 / 101

Simply Connected Domains in the plane



- Any two points are connected by a path.
- Every closed curve can be continuously shrunk to a point.

Simply Connected Domains in $\mathbb{R}^3$



- More subtle than in 2D.
- We will be using this concept only in 2D.

Bendixson Negative Criterion

Theorem (Bendixson Negative Criterion)

Consider the planar differential equation $X' = F(X)$*, i.e.,*

$$\begin{aligned} x' &= f(x,y), \\ y' &= g(x,y), \end{aligned}$$

where $f, g : D \to \mathbb{R}$ *are* C^1 *functions on a simply connected domain* $D \subset \mathbb{R}^2$*. If* $\operatorname{div} F \equiv \nabla \cdot F > 0$ *everywhere in* D *(or* < 0 *everywhere in* D*) then* $\nexists$ *periodic orbits lying entirely in* D*.*

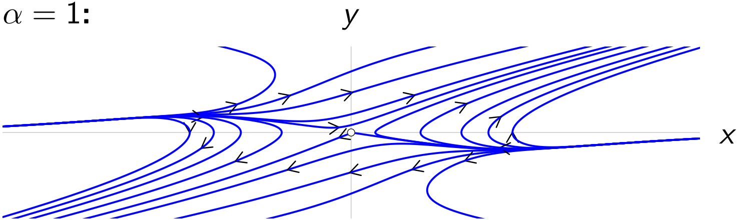
Example: non-existence of periodic orbits

$$x' = xy^2 + \sin y$$
$$y' = x^2 y + \sin x + \alpha y$$

Does this system have periodic solutions?

- $\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (x', y') = y^2 + x^2 + \alpha$
- $\alpha > 0 \implies \nabla \cdot F > 0 \quad \forall X \in \mathbb{R}^2 \implies \nexists$ periodic orbits.

$\alpha = 1$:



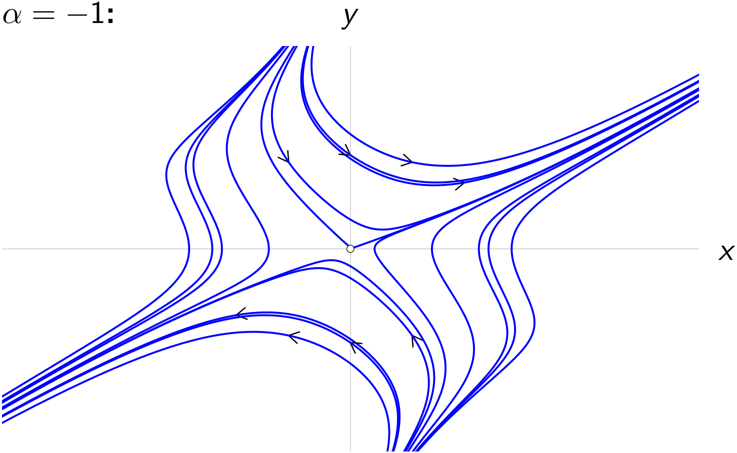
Example: non-existence of periodic orbits

$$x' = xy^2 + \sin y$$
$$y' = x^2 y + \sin x + \alpha y$$

- $\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (x', y') = y^2 + x^2 + \alpha$
- $\alpha < 0 \implies \nabla \cdot F < 0$ iff $\|X\| < \sqrt{|\alpha|}$
  $\implies \nexists$ periodic orbits inside the disk of radius $\sqrt{|\alpha|}$.
- Also $\nexists$ periodic orbits lying entirely in any simply connected domain outside the disk D.
- In principle, there might be periodic orbits on the boundary or encircling the disk D.
- Computer-generated phase portrait (next slide) suggests this is implausible, *but that is not a proof*.

Example: non-existence of periodic orbits

$\alpha = -1$:



Example: non-existence of periodic orbits

$$
\begin{aligned}
x' &= xy^2 + \sin y \\
y' &= x^2 y + \sin x + \alpha y
\end{aligned}
$$

- $\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (x', y') = y^2 + x^2 + \alpha$
- $\alpha = 0 \implies \nabla \cdot F > 0 \quad \forall X \neq (0,0) \implies \nexists$ periodic orbits lying entirely in any simply connected domain that does not include the origin.
- In principle, there might be periodic orbits that enclose the origin (or that pass through it).

Bendixson-Dulac Negative Criterion

Theorem (Bendixson-Dulac Negative Criterion)

Consider the planar differential equation $X' = F(X)$, i.e.,

$$x' = f(x, y), \qquad y' = g(x, y),$$

where $f, g : D \to \mathbb{R}$ are C^1 functions on a simply connected domain $D \subset \mathbb{R}^2$. In addition, consider a C^1 scalar function $h : D \to \mathbb{R}$. If $\mathrm{div}(hF) \equiv \nabla \cdot (hF) > 0$ everywhere in D (or < 0 everywhere in D) then $\nexists$ periodic orbits lying entirely in D and h is said to be a "Dulac function".

- $h \equiv 1$ yields the original Bendixson negative criterion.
- As with Lyapunov functions, there is no algorithm for discovering a Dulac function.

Example: Does this system have periodic orbits?

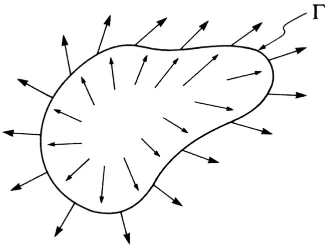
$$x' = \frac{1}{x^2+y^2+1}, \qquad y' = \frac{y}{x^2+y^2+1}$$

- $\nabla \cdot F = \frac{x^2-y^2-2x+1}{(x^2+y^2+1)^2}$
  - At $(x,y) = (0,0)$, $\nabla \cdot F = 1 > 0$.
  - At $(x,y) = (0,2)$, $\nabla \cdot F = -3/25 < 0$.
  - $\therefore$ Bendixson's criterion rules out periodic orbits in some simply connected regions of the plane, *but not in the whole plane*.
- Dulac function?
  - Try $h(x,y) = x^2 + y^2 + 1$.
  - $\nabla \cdot (hF) = 1 > 0 \;\; \forall (x,y) \in \mathbb{R}^2$.
  - $\therefore \;\; \nexists$ periodic orbits anywhere in $\mathbb{R}^2$.

Index Theory

Index Theory: Concept

Consider a simply connected domain $D \subset \mathbb{R}^2$ *and a closed loop* $\Gamma \subset D$ *that contains no fixed points (equilibria) of* $X' = F(X)$*.*



- Arrows represent values of the vector field F.

- As you slide all the way along Γ, the angle ϕ of the vector field changes by an integer number of full rotations: $\Delta\phi = 2k\pi$ $\exists k \in \mathbb{Z}$.
- k is the **index** of Γ.

Index Theory: Formal Definition

Definition (Index of a closed curve Γ)

Consider a smooth, planar vector field, $(x', y') = (f(x, y), g(x, y))$, defined in a simply connected domain $D \subset \mathbb{R}^2$. Suppose Γ is a closed loop in D ($\Gamma \subset D$) and that Γ contains no fixed points of the vector field (*i.e.*, there is no point $(x_*, y_*) \in \Gamma$ such that $f(x_*, y_*) = g(x_*, y_*) = 0$). Then the index of Γ is

$$k = \frac{1}{2\pi} \oint_\Gamma d\phi \,,$$

which can be calculated in Cartesian coordinates via

$$k = \frac{1}{2\pi} \oint_\Gamma d\left(\arctan \frac{g(x, y)}{f(x, y)}\right) = \frac{1}{2\pi} \oint_\Gamma \frac{f\, dg - g\, df}{f^2 + g^2} \,.$$

Index Theory: Properties

- The index is the same if Γ is smoothly deformed, as long as it is not deformed through some fixed point of the vector field.

- The index of a fixed point is defined to be the index of a closed curve that contains only this one fixed point, and where no fixed points are on the closed curve.