Mathematics 3F03
Advanced Differential Equations

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Lecture 11

Systematic Qualitative Analysis
of $X' = AX$ in the Plane (continued)
30 September 2013

Announcements

- Assignment 2 due this Friday 4 October 2013.

The equation we want to understand

General linear (homogenous) planar system:

$$X' = AX\,, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}\,, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \qquad (*)$$

Various cases to consider, depending on eigenvalues of A.

Case λ_1, λ_2 complex

Proposition (Real Jordan form of $A \in \mathbb{R}^{2\times 2}$ with eigenvalues $\in \mathbb{C}$)

Suppose the 2×2 real matrix A has complex eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$. Let V be an eigenvector associated with λ, and write $V = V_1 + iV_2$, with $V_1, V_2 \in \mathbb{R}^2$. Then V_1 and V_2 are linearly independent in $\mathbb{R}^2$, and hence the real matrix $T = (V_1\ V_2)$ is invertible. Moreover,

$$J \equiv T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

The matrix J is the **real Jordan canonical form** *of A.*

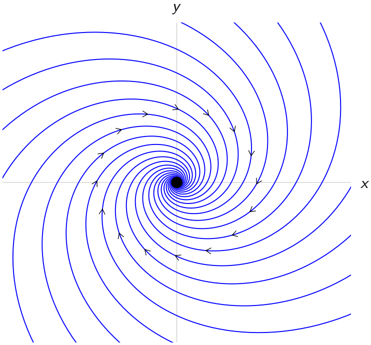
Proof.

See Example on pages 53–54 of textbook. □

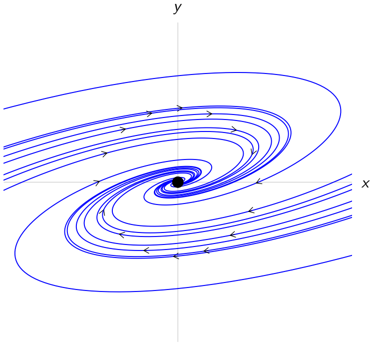
Note: The (complex) Jordan canonical form of A is $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$.

Case λ_1, λ_2 complex (Spiral Sink)

Canonical coordinates:



Original coordinates: $A_{\text{orig}} = TAT^{-1}$

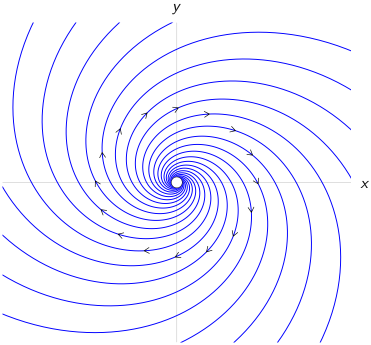


$$A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

$$T = (V_1\ V_2) = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ 1 & 0 \end{pmatrix}$$

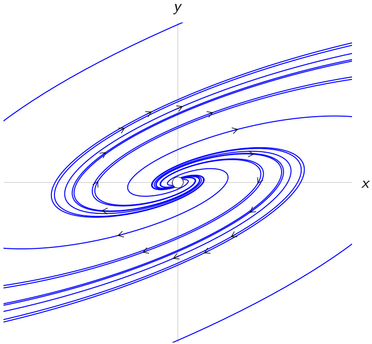
Case λ_1, λ_2 complex (Spiral Source)

Canonical coordinates:



$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

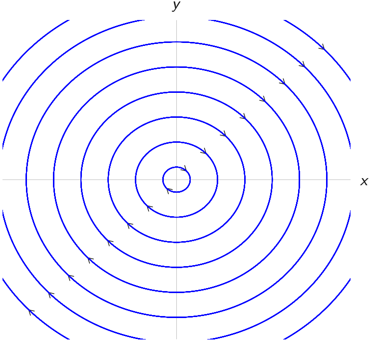
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ 1 & 0 \end{pmatrix}$$

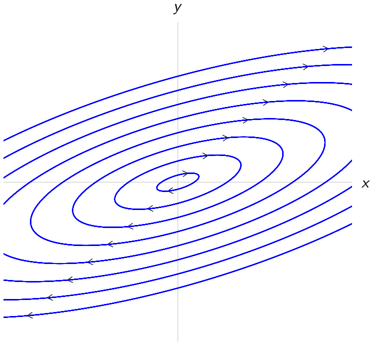
Case λ_1, λ_2 complex (Centre)

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1 \; V_2) = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ 1 & 0 \end{pmatrix}$$

Case λ_1, λ_2 complex (Exact Quantitative Solution)

Proposition (Solution of $X' = JX$ with $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$)

The general solution of

$$X' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} X \,, \qquad \alpha, \beta \in \mathbb{R}\,, \beta \neq 0,$$

is

$$X(t) = e^{\alpha t} \left(c_1 \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + c_2 \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix} \right) \,, \qquad c_1, c_2 \in \mathbb{R}.$$

Case λ_1, λ_2 complex (Exact Quantitative Solution)

Derivation, Part 1.

$\det J = \alpha^2 + \beta^2 > 0 \quad (\because \beta \neq 0) \implies$ $\exists!$ equilibrium at $(0,0)$.

Eigenvalues: $\lambda_\pm = \alpha \pm i\beta$, Eigenvectors: $V_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $V_- = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

$\therefore Z(t) = e^{\lambda_+ t} V_+$ is a (*complex*) solution (by lemma proved last week).

How do we get real solutions? Consider real and imaginary parts of $Z(t)$ separately.

$$X_{\Re}(t) = \Re(Z(t)) = \Re\left[e^{(\alpha+i\beta)t}\begin{pmatrix} 1 \\ i \end{pmatrix}\right] = e^{\alpha t}\,\Re\left[e^{i\beta t}\begin{pmatrix} 1 \\ i \end{pmatrix}\right]$$

$$= e^{\alpha t}\,\Re\left[(\cos\beta t + i\sin\beta t)\begin{pmatrix} 1 \\ i \end{pmatrix}\right]$$

$$= e^{\alpha t}\begin{pmatrix} \Re(\cos\beta t + i\sin\beta t) \\ \Re(i\cos\beta t - \sin\beta t) \end{pmatrix}$$

$$= e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}$$

Case λ_1, λ_2 complex (Exact Quantitative Solution)

Derivation, Part 2.

Similarly,

$$X_{\Im}(t) = \Im(Z(t)) = e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

So what? Why should we care about the real and imaginary parts of a complex solution?

- $X_{\Re}(t)$ and $X_{\Im}(t)$ are each (real!) solutions of $X' = JX$.
- $X_{\Re}(t)$ and $X_{\Im}(t)$ are linearly independent $\forall t \in \mathbb{R}$.

$\therefore$ General solution of $X' = JX$ is

$$X(t) = c_1 X_{\Re}(t) + c_2 X_{\Im}(t), \qquad c_1, c_2 \in \mathbb{R}.$$

□

For given initial state X_0, solve linear algebraic equations $(X(0) = X_0)$ for c_1 and c_2.

Case λ_1, λ_2 complex (Exact Quantitative Solution)

What does phase portrait look like?

Consider

$$X_{\Re}(t) = e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}$$

- $\begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}$: lies on unit circle, rotates as t increases
  - sense of rotation is determined by sign of β
  - $\beta > 0 \implies$ clockwise
  - $\beta < 0 \implies$ counter-clockwise
- $e^{\alpha t}$: magnitude of vector
  - $\alpha < 0 \implies$ spiral sink
  - $\alpha = 0 \implies$ centre
  - $\alpha > 0 \implies$ spiral source

Summary: For $\lambda = \alpha \pm i\beta$, $\beta \neq 0$, phase portrait *in canonical coordinates* is $e^{\alpha t} \times$ [circular motion].

Case λ_1, λ_2 complex (Exact Quantitative Solution)

Hang on... we considered only $X_{\Re}(t)$. What about general case?

$$X(t) = e^{\alpha t}\left(c_1\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + c_2\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}\right), \qquad c_1, c_2 \in \mathbb{R}.$$

What if $c_1 \neq 0$ and $c_2 \neq 0$?

First consider the special case $c_1 = c_2$:

$$X(t) = c_1 e^{\alpha t}\begin{pmatrix}\cos\beta t + \sin\beta t\\ \cos\beta t - \sin\beta t\end{pmatrix} = c_1 e^{\alpha t}\underbrace{\begin{pmatrix}\sqrt{2}\sin\left(\beta t + \frac{\pi}{4}\right)\\ -\sqrt{2}\cos\left(\beta t + \frac{\pi}{4}\right)\end{pmatrix}}_{\text{circular motion, radius } \sqrt{2}, \text{ period } \frac{2\pi}{\beta}}$$

For arbitrary $c_1, c_2 \in \mathbb{R}$, we have $e^{\alpha t} \times$ [sum of two vectors that each rotate at same rate β], *i.e.,* "rigid body" rotation.

$\therefore$ Always have $e^{\alpha t} \times$ [circular motion].

What happens in original coordinates?

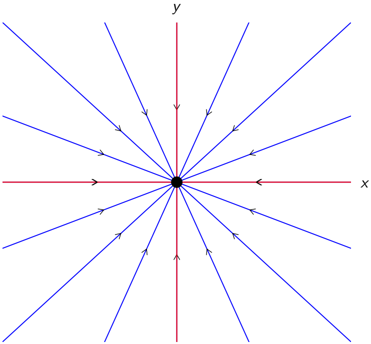
- Circles $\to$ Ellipses. $e^{\alpha t}$ has same effect.

Remaining cases

- Repeated (real) eigenvalues
  - $\lambda_1 = \lambda_2 < 0$
  - $\lambda_1 = \lambda_2 > 0$
  - $\lambda_1 = \lambda_2 = 0$

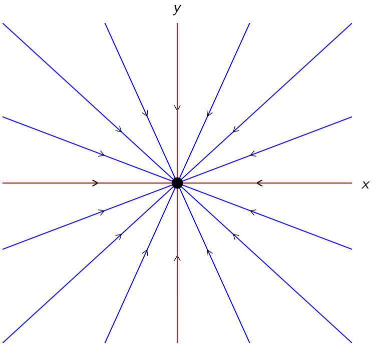
Case $\lambda_1 = \lambda_2 < 0$ ("Perfect Sink" or "Stable Star")

Canonical coordinates:



$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

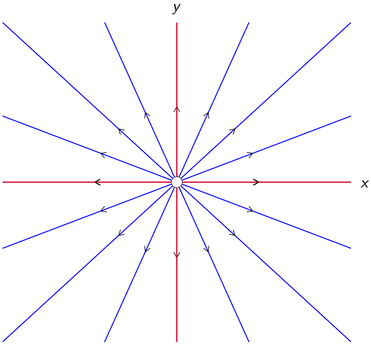
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$T =$ any invertible matrix

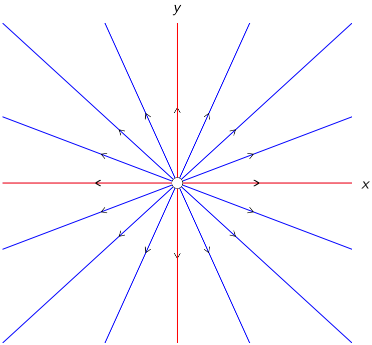
Case $\lambda_1 = \lambda_2 > 0$ ("Perfect Source" or "Unstable Star")

Canonical coordinates:



$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

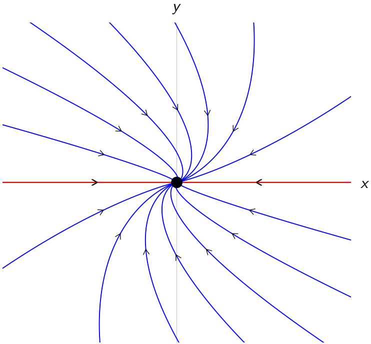
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



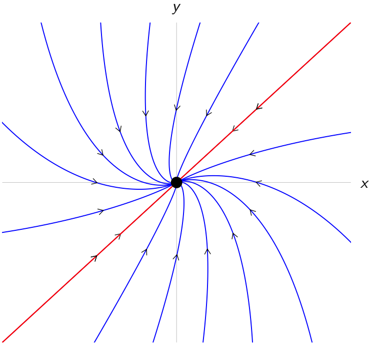
$T =$ any invertible matrix

Case $\lambda_1 = \lambda_2 < 0$, $\exists!$ eigendirection ("Improper Sink")

Canonical coordinates:



Original coordinates: $A_{\text{orig}} = TAT^{-1}$

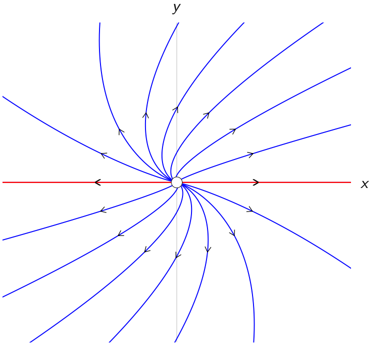


$$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$T = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

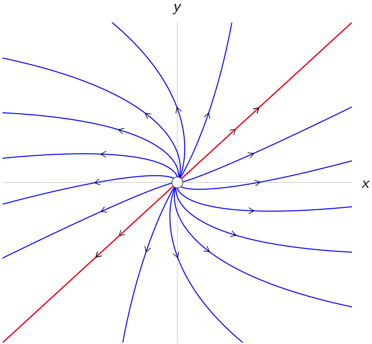
Case $\lambda_1 = \lambda_2 > 0$, $\exists!$ eigendirection ("Improper Source")

Canonical coordinates:



$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

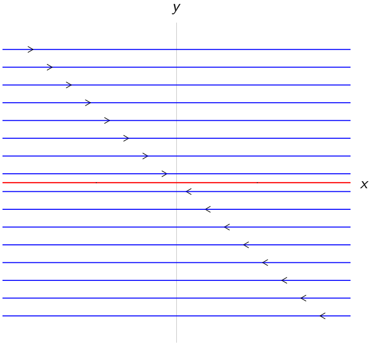
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

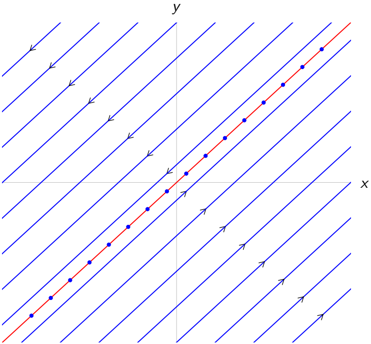
Case $\lambda_1 = \lambda_2 = 0$, $\exists!$ eigendirection ("Improper line")

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

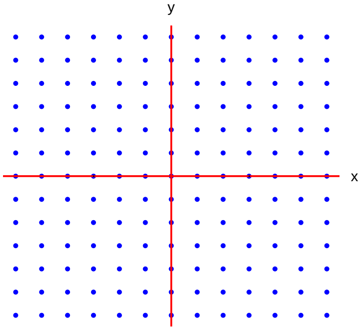
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

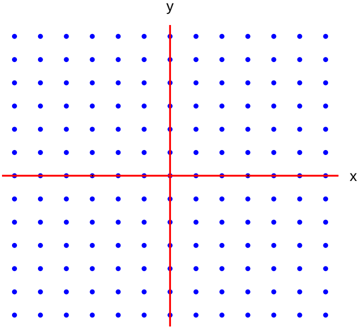
Case $\lambda_1 = \lambda_2 = 0$, ∞ eigendirections ("Dull!")

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$