Mathematics 3F03
Advanced Differential Equations

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Lecture 10

Systematic Qualitative Analysis

of $X' = AX$ in the Plane

25 September 2013

Announcements

- Solutions to Assignment 1 are now posted on the course wiki.
- Make sure to read over the solutions carefully.
- Tutorial on Friday this week.
- Assignment 2 will be posted later this week (due Friday 4 October 2013).

The equation we want to understand

General linear (homogenous) planar system:

$$X' = AX\,, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}\,, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \qquad (*)$$

Various cases to consider, depending on eigenvalues of A.

If $\exists$ basis of eigenvectors $\{V_1, V_2\}$...

- In this basis, $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
  This is the **Jordan canonical form** of A.
- Transform to this basis via matrix $T = (V_1\ V_2)$.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = T^{-1} A_{\text{orig}}\, T = T^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} T$$

- Transform back to original coordinates via T^{-1}.
- T is invertible because V_1 and V_2 are linearly independent.
- $TE_i = V_i$, where E_i is the standard basis vector.

If $\exists$ basis of eigenvectors $\{V_1, V_2\}$...

- Simplest to analyze system in **eigencoordinates**, in which coordinate axes are along V_1 and V_2.
- Quantitative solution is very easy in eigencoordinates because system is decoupled.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \implies \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda_1 x \\ \lambda_2 y \end{pmatrix} \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{\lambda_1 t} \\ y_0 e^{\lambda_2 t} \end{pmatrix}$$

$$\textit{i.e.,} \quad X(t) = x_0 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note:

Eigendirections are along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in eigencoordinates.

Case of distinct, real, non-zero eigenvalues λ_1, λ_2

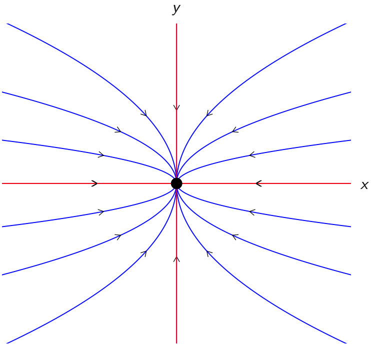
$$A_{\text{orig}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- $\exists$ a basis of eigenvectors.
- Equilibria: $\det A = \lambda_1\lambda_2 \neq 0 \implies \exists!$ equilibrium at $(0,0)$.
- Stability:
  depends on signs of λ_1, λ_2.
- Structure of direction field:
  determined by nature of equilibrium.

Case $\lambda_1 < \lambda_2 < 0$ (Sink)

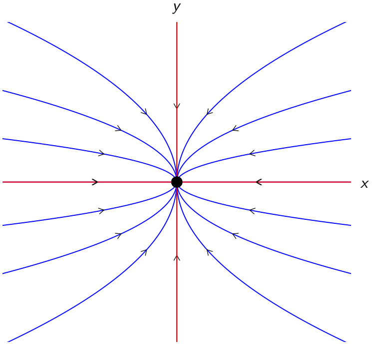
$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1 < \lambda_2 < 0$$

- All solution curves go to origin.
- But how? Direct? Spiral? Some other way?
- $\exists!$ solution for each $X_0 \implies$ no solution curves cross
- Decoupled $\implies$ solution goes monotonically to zero in each component (also clear from quantitative solution)



Case $\lambda_1 < \lambda_2 < 0$ (Sink)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1 < \lambda_2 < 0$$



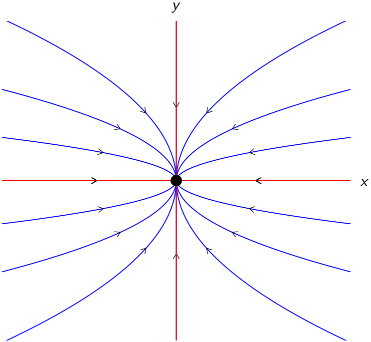
- How do we get curvature of solutions in this phase portrait?
- What is slope along unique solution through (x, y)?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\lambda_2}{\lambda_1}\frac{y}{x} = \frac{\lambda_2}{\lambda_1}\frac{y_0 e^{\lambda_2 t}}{x_0 e^{\lambda_1 t}}$$

$$= \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)}_{>0} \underbrace{\left(\frac{y_0}{x_0}\right)}_{\text{depends}} \underbrace{e^{(\lambda_2 - \lambda_1)t}}_{\to\infty \text{ as } t\to\infty}$$

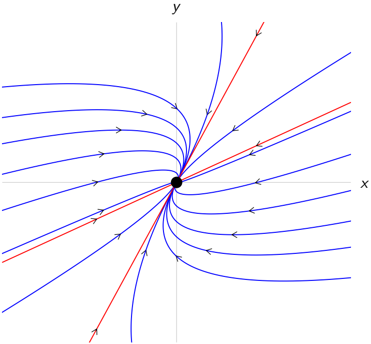
Case $\lambda_1 < \lambda_2 < 0$ (Sink)

Canonical coordinates:



$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

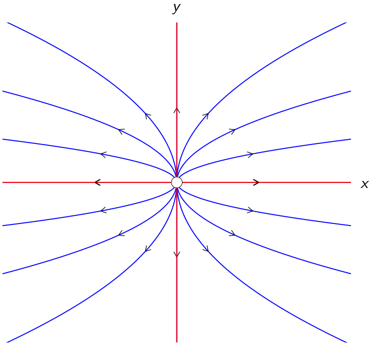
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

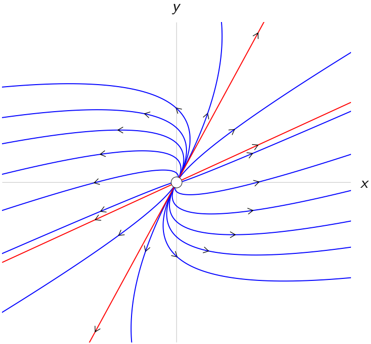
Case $0 < \lambda_2 < \lambda_1$ (Source)

Canonical coordinates:



$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

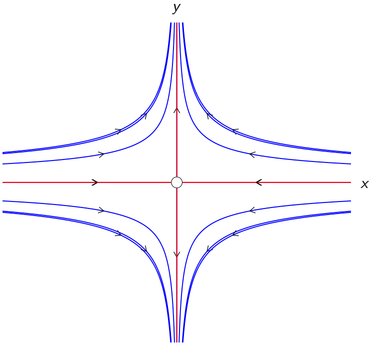
Original coordinates: $A_{\mathrm{orig}} = TAT^{-1}$



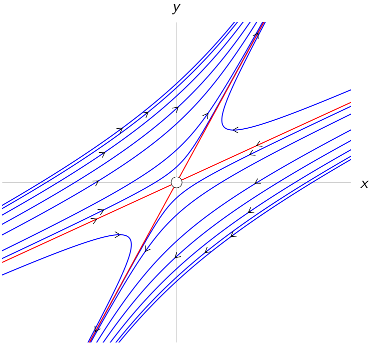
$$T = (V_1 \; V_2) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Case $\lambda_1 < 0 < \lambda_2$ (Saddle)

Canonical coordinates:



Original coordinates: $A_{\text{orig}} = TAT^{-1}$

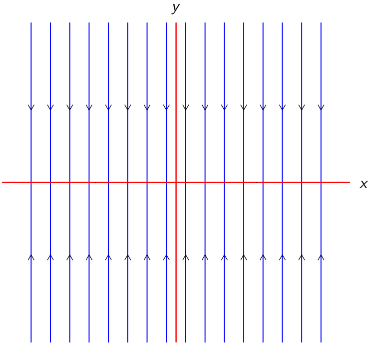


$$A = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = (V_1\ V_2) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

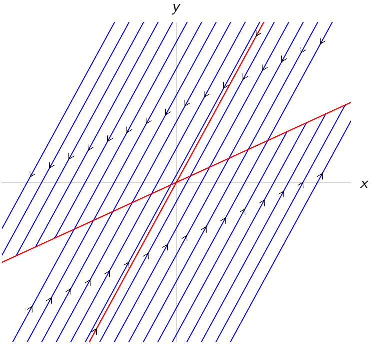
Case $\lambda_1 = 0$, $\lambda_2 < 0$ (Attracting Line)

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

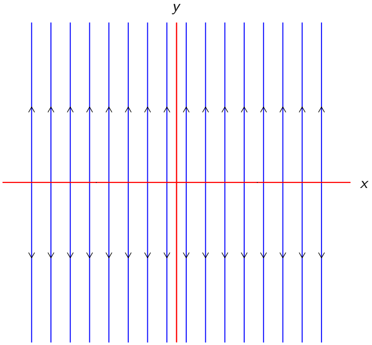
Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

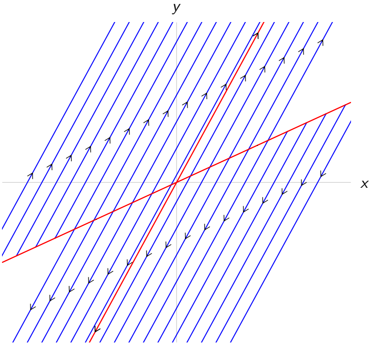
Case $\lambda_1 = 0,\ \lambda_2 > 0$ (Repelling Line)

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Case λ_1, λ_2 complex

- Can we have a mixture of real and complex eigenvalues?
  - Not in 2D: λ eigenvalue $\implies$ $\bar{\lambda}$ eigenvalue.
  - Proof 1:
    $p(\lambda) = 0 \implies \overline{p(\lambda)} = 0 \implies p(\bar{\lambda}) = 0.$
  - Proof 2:
    $AV = \lambda V \implies \overline{AV} = \overline{\lambda V} \implies \bar{A}\bar{V} = \bar{\lambda}\bar{V} \implies A\bar{V} = \bar{\lambda}\bar{V}.$
- $\therefore \Im(\lambda_1) \neq 0 \implies \lambda_2 = \bar{\lambda}_1$.
  i.e., Complex eigenvalues always come in conjugate pairs.
- **Fact:** Real and Imaginary parts of a complex eigenvector are linearly independent in $\mathbb{R}^2$. (*Good exercise.*)

Case λ_1, λ_2 complex

Proposition (Real Jordan form of $A \in \mathbb{R}^{2\times 2}$ with eigenvalues $\in \mathbb{C}$)

Suppose the 2×2 real matrix A has complex eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$. Let V be an eigenvector associated with λ, and write $V = V_1 + iV_2$, with $V_1, V_2 \in \mathbb{R}^2$. Then V_1 and V_2 are linearly independent in $\mathbb{R}^2$, and hence the real matrix $T = (V_1\ V_2)$ is invertible. Moreover,

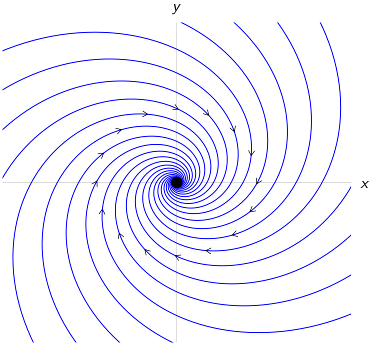
$$J \equiv T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

The matrix J is the **real Jordan canonical form** *of A.*

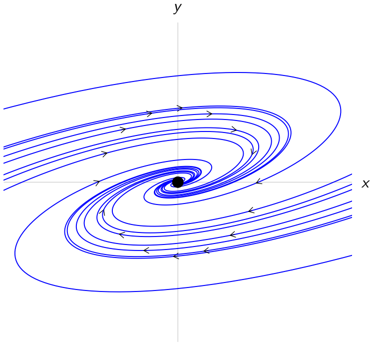
Note: The (complex) Jordan canonical form of A is $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$.

Case λ_1, λ_2 complex (Spiral Sink)

Canonical coordinates:



Original coordinates: $A_{\text{orig}} = TAT^{-1}$

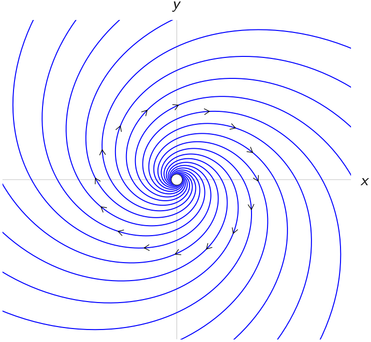


$$A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

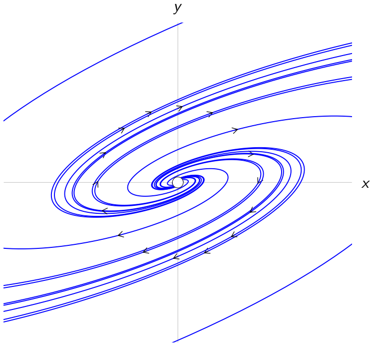
$$T = (V_1\ V_2) = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ 1 & 0 \end{pmatrix}$$

Case λ_1, λ_2 complex (Spiral Source)

Canonical coordinates:



Original coordinates: $A_{\text{orig}} = TAT^{-1}$

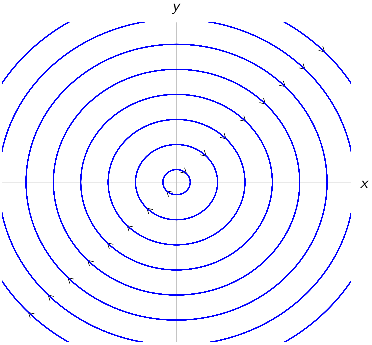


$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$T = (V_1 \; V_2) = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ 1 & 0 \end{pmatrix}$$

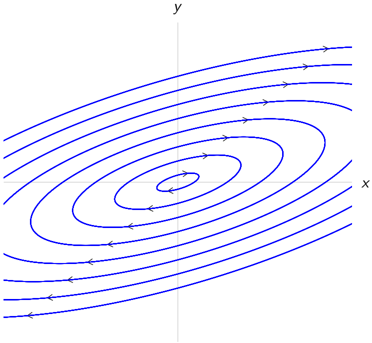
Case λ_1, λ_2 complex (Centre)

Canonical coordinates:



$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Original coordinates: $A_{\text{orig}} = TAT^{-1}$



$$T = (V_1\ V_2) = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ 1 & 0 \end{pmatrix}$$