Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 7

Review of Planar Linear Algebra

18 September 2013

Announcements

- Assignment 1 due THIS FRIDAY, 20 Sep 2013, before class, in the appropriately labelled locker next to HH-105.
- **Putnam Mathematical Competition:** There will be an organizational meeting on Thursday 19 Sep 2013 in HH 410.

  The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. More information can be found at `http://math.scu.edu/putnam` or on the undergraduate page of the Math department's website. This year's competition will occur on 7 Dec 2013. If you are interested in participating or learning more, drop by on Thursday 19 Sep and/or send email to Prof. Bradd Hart (`hartb@mcmaster.ca`).

Solving Linear Algebraic Equations

When can we solve

$$AX = B$$

for $X \in \mathbb{R}^n$?

In the plane, when can we solve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$?.

Solving Linear Algebraic Equations

$AX = B$
When is there a unique solution?

$$
\begin{aligned}
\exists! \text{ solution} \iff & \det A \neq 0 \\
\iff & A^{-1} \text{ exists} \\
\iff & X = A^{-1}B
\end{aligned}
$$

Solving Linear Algebraic Equations

$AX = B$. Is there necessarily a solution?

Examples:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- no solution

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- no solution

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

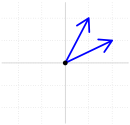
- $\forall y \in \mathbb{R}$, $X = \begin{pmatrix} 1 \\ y \end{pmatrix}$ is a solution

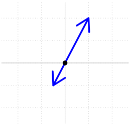
Solving Linear Algebraic Equations

$AX = B$
What are the possibilities?

- 0, 1 or ∞ solutions
- If $\det A \neq 0$ then $\exists!$ solution
- If $\det A = 0$ then either
  - no solutions
  - ∞ solutions

Linear Independence





$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \; W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$V, W \in \mathbb{R}^2$ are **linearly independent** if and only if any of the following is true:

- V and W do not lie along the same line through the origin
- V and W are not proportional
- $\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0$

Basis

A **basis** of $\mathbb{R}^n$ is a set of n linearly independent vectors.

The **standard basis** is $\{E_i : i = 1, \ldots, n\}$, where

$$E_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad E_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

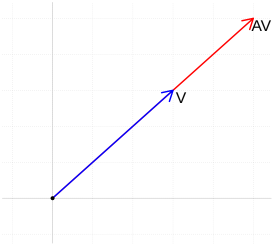
Eigenvectors and Eigenvalues

Eigenvectors of a matrix A are vectors in special directions in which A acts like scalar multiplication.

The following are equivalent (assuming $V \neq 0$):

- V is an eigenvector of A
- $\exists\lambda$ such that $AV = \lambda V$
- $\exists\lambda$ such that $(A - \lambda I)V = 0$

Such a λ is called an **eigenvalue** of A.



Eigenvectors and Eigenvalues

To find eigenvalues and eigenvectors, note that

$$V \neq 0 \;\text{ and }\; (A - \lambda I)V = 0 \quad \Longrightarrow \quad \det(A - \lambda I) = 0$$

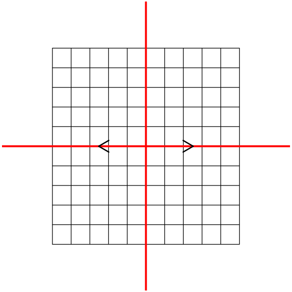
$\therefore$ Solutions λ of the **characteristic equation** $\det(A - \lambda I) = 0$ are eigenvalues.

For each eigenvalue, find eigenvectors by solving $AV = \lambda V$ for V.

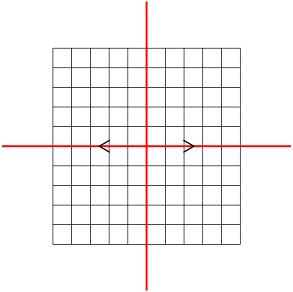
Action of A is easiest to understand if $\exists$ basis of eigenvectors.

Action of a matrix A

Original Grid



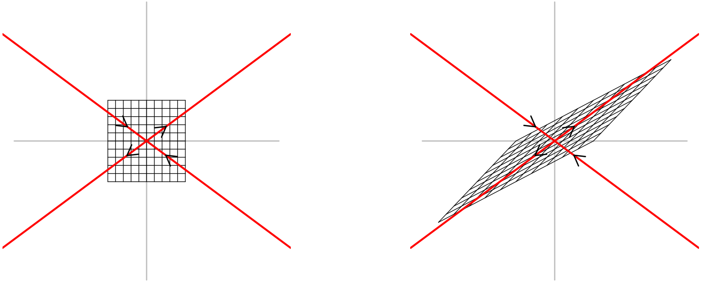
After applying $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$$\lambda_1 = 1, \quad \lambda_2 = 1$$

Action of a matrix A

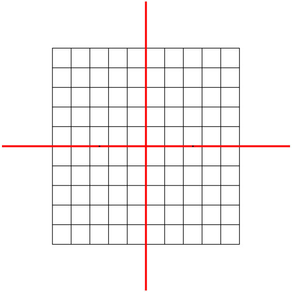
Original Grid **After applying** $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$



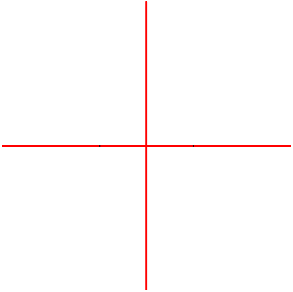
$\lambda_1 = 2.41421, \quad \lambda_2 = -0.414214$

Action of a matrix A

Original Grid



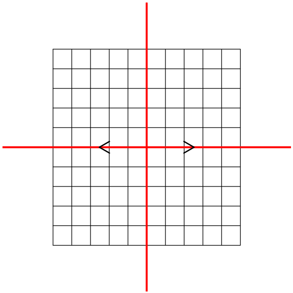
After applying $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$



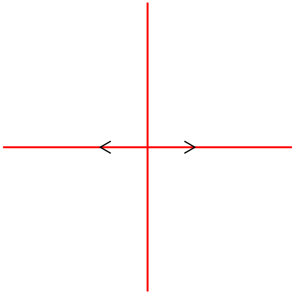
$\lambda_1 = 0, \quad \lambda_2 = -0$

Action of a matrix A

Original Grid



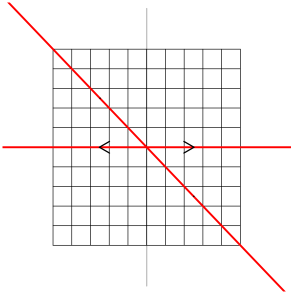
After applying $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



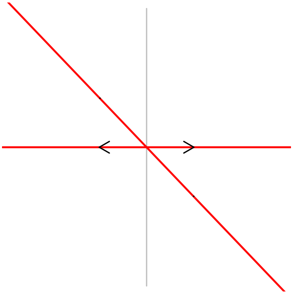
$$\lambda_1 = 1, \quad \lambda_2 = 0$$

Action of a matrix A

Original Grid



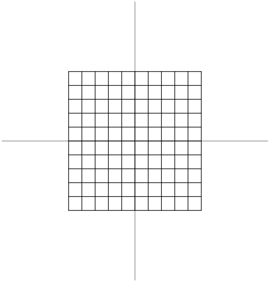
After applying $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$



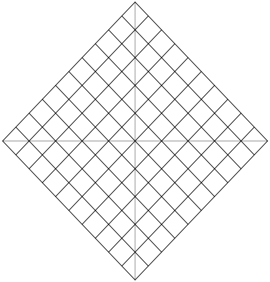
$$\lambda_1 = 1, \quad \lambda_2 = 0$$

Action of a matrix A

Original Grid



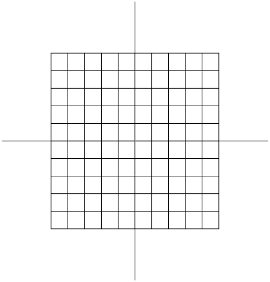
After applying $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$



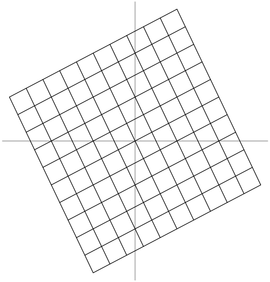
$\lambda_{\pm} = 1 \pm i, \quad |\lambda| = 1.414, \quad \theta = 0.785$

Action of a matrix A

Original Grid



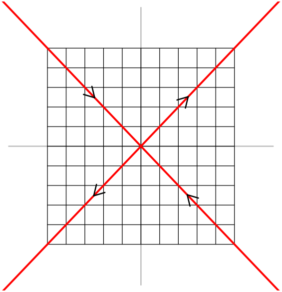
After applying $A = \begin{pmatrix} 1 & -0.5 \\ 0.5 & 1 \end{pmatrix}$



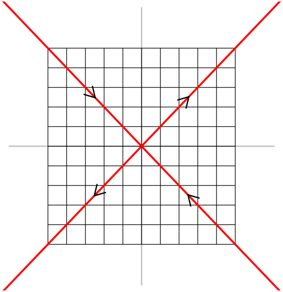
$$\lambda_\pm = 1 \pm 0.5i, \quad |\lambda| = 1.118, \quad \theta = 0.464$$

Action of a matrix A

Original Grid



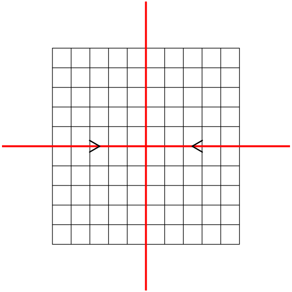
After applying $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



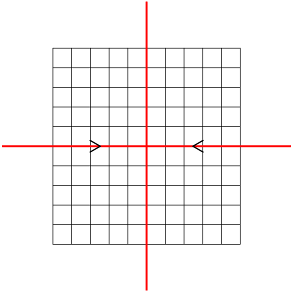
$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Action of a matrix A

Original Grid



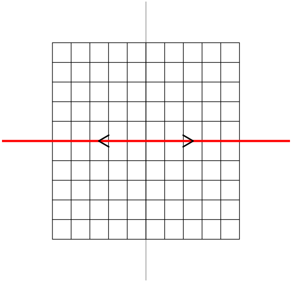
After applying $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



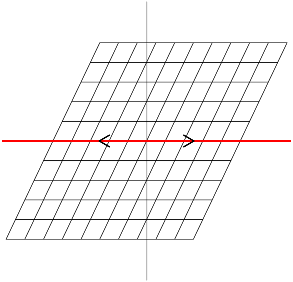
$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Action of a matrix A

Original Grid



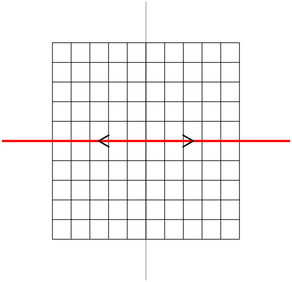
After applying $A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$



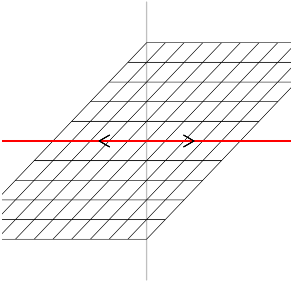
$$\lambda_1 = 1, \quad \lambda_2 = 1$$

Action of a matrix A

Original Grid



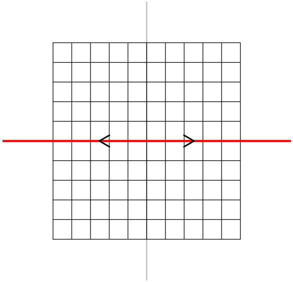
After applying $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



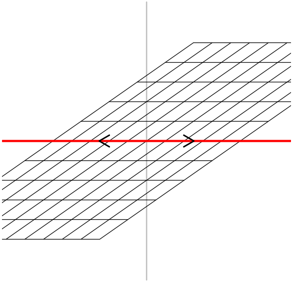
$$\lambda_1 = 1, \quad \lambda_2 = 1$$

Action of a matrix A

Original Grid



After applying $A^3 = \begin{pmatrix} 1 & 1.5 \\ 0 & 1 \end{pmatrix}$



$\lambda_1 = 1, \quad \lambda_2 = 1$