Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

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Lecture 15

Review of Multidimensional Linear Algebra

Wednesday 9 October 2013

Announcements

- Assignment 2:
  - Solutions have been posted on the course wiki.
  - Read them carefully.
  - NOTE: there is an error in the posted solution to the spiral case in the first problem on last year's (2012) Assignment 2. A correct solution of the spiral case is given in this year's solutions!
- Assignment 3:
  - Due date delayed to Friday 25 Oct 2013.
  - Partial Assignment already posted, so you can get started.
  - Remainder of Assignment will be posted in due course.
- No class on Friday this week — Happy Thanksgiving!

Elementary Row Operations

Convert matrix A to **row reduced echelon form** to find A^{-1} or to solve the linear equations $AX = V$.
Each elementary row operation is equivalent to left multiplication of A by an **elementary matrix**.

The elementary row operations are the following (where $i \neq j$):

- Add c times row i of A to row j.
- Interchange rows i and j.
- Multiply row i by $c \neq 0$.

The associated elementary matrix is obtained applying the operation to the identity matrix.

Solutions of linear equations vs invertibility of A

Proposition

$AX = V$ has a unique solution for any $V \in \mathbb{R}^n$ iff A is invertible.

Invertibility vs linear independence of columns

Proposition

A is invertible iff the columns of A are linearly independent.

Determinant formula

Definition

The determinant of the $n \times n$ matrix $A = [a_{ij}]$ is inductively defined by "expanding along the jth row" via

$$\det A = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det A_{jk} \,,$$

where A_{jk} is the $(n-1) \times (n-1)$ submatrix obtained by deleting the jth row and kth column of A.

Determinant of triangular matrices

Proposition

The determinant of a triangular matrix is the product of its diagonal entries.

Effects of Elementary Row Operations on Determinant

- Add c times row i of A to row j $\quad (i \neq j)$
  $\implies$ $\det A$ unchanged.
- Interchange rows i and j $\quad (i \neq j)$
  $\implies$ $\det A$ changes sign.
- Multiply row i by $c \neq 0$
  $\implies$ $\det A$ multiplied by c.

Note: These facts can be extremely useful when finding eigenvalues of large matrices.

Determinant and invertibility

Proposition

A^{-1} *exists iff* $\det A \neq 0$.

Determinant of a product

Proposition

$\det(AB) = (\det A)(\det B)$.

Eigenvalues, Eigenvectors and Canonical Forms

Proposition

If A is a diagonal matrix then its eigenvalues are the diagonal elements and the standard basis is a basis of eigenvectors.

Proposition

If A is a triangular matrix then its eigenvalues are the diagonal elements.

Note: If A is triangular then there is *not necessarily* a basis of eigenvectors.
But: If A is $n \times n$ then the minimum number of eigenvectors is the number of distinct eigenvalues (so, at least 1).
i.e., the geometric multiplicity of any eigenvalue is at least 1.

Eigenvalues, Eigenvectors and Canonical Forms

Proposition

Eigenvectors of distinct eigenvalues are linearly independent.

Proposition

If A has real, distinct eigenvalues then A is similar to a diagonal matrix, i.e., $\exists T$ such that

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

This is the (Jordan) canonical form of A. The transformation matrix $T = (V_1\ V_2\ \cdots\ V_n)$, where V_i is an eigenvector associated with λ_i.

Eigenvalues, Eigenvectors and Canonical Form

Proposition

If $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigenvector V then

- $\bar{\lambda} = \alpha - i\beta$ *is an eigenvalue of A with eigenvector $\bar{V}$,*
- *The real and imaginary parts of V,*

$$W_{\mathrm{Re}} = \frac{1}{2}(V + \bar{V})\,, \tag{1}$$

$$W_{\mathrm{Im}} = -\frac{i}{2}(V - \bar{V})\,, \tag{2}$$

are linearly independent real vectors.

Note: This generalizes to any number of complex eigenvalues (the associated "W vectors" are all linearly independent).

Eigenvalues, Eigenvectors and Canonical Form

Proposition

Suppose A is a $2n \times 2n$ matrix with exactly n complex eigenvalue pairs $\lambda_1, \bar{\lambda}_1, \ldots, \lambda_n, \bar{\lambda}_n$. Define

$$T = \left(W_{1,\mathrm{Re}}\, W_{1,\mathrm{Im}}\, \cdots\, W_{n,\mathrm{Re}}\, W_{n,\mathrm{Im}} \right).$$

Then

$$T^{-1}AT = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix}$$

where $D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$ *and* $\lambda_j = \alpha_j + i\beta_j$.

Note: *This is the (real Jordan) canonical form of A.*

Eigenvalues, Eigenvectors and Canonical Form

Proposition (Canonical Form for distinct eigenvalues)

If A is an $n \times n$ matrix with distinct eigenvalues then $\exists T$ such that

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & D_1 & & \\ & & & & \ddots & \\ & & & & & D_\ell \end{pmatrix}$$

where $\lambda_j \in \mathbb{R}$ and $D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$ with $\alpha_j, \beta_j \in \mathbb{R}$, $\beta_j \neq 0$.

($\alpha_j \pm i\beta_j$ are complex eigenvalues of A.)

Other linear algebraic concepts

- Subspace: a subset that itself forms a vector space
- Span: set of all linear combinations of a given set of vectors
- Basis: a linearly independent spanning set
- Kernel: $\mathrm{Ker}\, T = \{V : TV = 0\}$
- Range: $\mathrm{Range}\, T = \{W : \exists V \text{ such that } TV = W\}$

Proposition

$\dim \mathit{Ker}\, T + \dim \mathit{Range}\, T = n.$

Proposition

$\dim \mathit{Ker}\, T = 0$ *iff* T *is invertible.*

Canonical Form for an arbitrary real matrix A

Proposition (Real Jordan Canonical Form)

If A is an $n \times n$ real matrix then there is a linear transformation T such that $T^{-1}AT = diag(B_1, \ldots, B_k)$ and each block B_j is a square real matrix of one of the following two forms:

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \qquad \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix}$$

where $C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ *with* $\beta \neq 0$ *and* $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3×3 matrix with repeated eigenvalue λ

Proposition

If A is 3×3 with repeated eigenvalue λ then there are three possible canonical forms:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Note:

- λ must be real in this situation.
- The middle case is unique up to re-ordering of the two blocks.

4×4 matrix with repeated eigenvalue λ

Note: If λ is real then the possible canonical forms are similar to the 3×3 case.

Proposition

If A is 4×4 with repeated complex eigenvalue $\lambda = \alpha + i\beta$ ($\beta \neq 0$) then there are two possible canonical forms:

$$\begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}$$

Computing Canonical Form in general

Call the original matrix A and its canonical form J.

- Find all eigenvalues and eigenvectors of A.
- Make a table listing each eigenvalue λ together with
  - its algebraic multiplicity, $\text{alg}(\lambda)$,
  - its geometric multiplicity, $\text{geom}(\lambda)$,
  - a set of linearly independent eigenvectors $\{V_{\lambda,j} : j = 1, \ldots, \text{geom}(\lambda)\}$.
- For each real λ, if $\text{geom}(\lambda) = \text{alg}(\lambda)$ then include that many copies of λ along the diagonal of J.
- For each complex conjugate pair $\lambda = \alpha + i\beta$, $\bar{\lambda} = \alpha - i\beta$, if $\text{geom}(\lambda) = \text{alg}(\lambda)$ then include $\text{geom}(\lambda)$ copies of $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ along the diagonal of J.

Computing Canonical Form in general (CONTINUED)

- For each remaining eigenvalue, we have $\text{geom}(\lambda) < \text{alg}(\lambda)$, *i.e.*, the subspace associated with λ does not have a basis of eigenvectors.

  First consider the subcase in which $\lambda \in \mathbb{R}$ and $\text{geom}(\lambda) = 1$. Then, associated with λ there is

  - a single linearly independent eigenvector (say V),
  - a single Jordan block B_λ of size $\text{alg}(\lambda) \times \text{alg}(\lambda)$, with a 1 above each λ except the first,
  - a set of linearly independent **generalized eigenvectors**,

    $$\{U_j : j = 1, \ldots, \text{alg}(\lambda) - 1\},$$

    which completes the basis of the subspace associated with this Jordan block.

Computing Canonical Form in general (CONTINUED)

Find **generalized eigenvectors** U_j (for $\lambda \in \mathbb{R}$) as follows:

- solve $(A-\lambda I)U_1 = V$ for U_1, $\implies (A-\lambda I)^2 U_1 = 0$,
- solve $(A-\lambda I)U_2 = U_1$ for U_2, $\implies (A-\lambda I)^3 U_2 = 0$,
- solve $(A-\lambda I)U_3 = U_2$ for U_3, $\implies (A-\lambda I)^4 U_3 = 0$,
- continue until process terminates, which will happen $\because$
  - The **Jordan chain** $(V, U_1, U_2, \ldots, U_{k-1})$ is a basis for the k-dimensional subspace of $\mathbb{R}^n$ associated with the block B_λ.
  - $(A-\lambda I)U_k = U_{k-1}$ has no solution U_k if $k = \text{alg}(\lambda)$.

  Proof: $\exists \{a_j\}_{j=0}^{k-1} \subset \mathbb{R}$ such that $U_k = a_0 V + \sum_{j=1}^{k-1} a_j U_j \implies$ $(A-\lambda I)U_k = \sum_{j=1}^{k-1} a_j (A-\lambda I) U_j = a_1 V + a_2 U_1 + \cdots + a_{k-1} U_{k-2}$ $\neq U_{k-1}$ because the Jordan chain is linearly independent. $\square$

- **Note:** A generalized eigenvector U satisfies $U \neq 0$ and $(A-\lambda I)^k U = 0$ for some k (eigenvector $\iff$ $k=1$).

Computing Canonical Form in general (CONTINUED)

- The construction is similar for any remaining (complex) eigenvalues for which $1 = \text{geom}(\lambda) < \text{alg}(\lambda)$, but note that:
  - $\bar{\lambda}$ will also have the same algebraic and geometric multiplicites as λ.
  - rather than multiple copies of λ (and $\bar{\lambda}$) along the diagonal of J, we require multiple copies of the 2×2 real matrix $C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and copies of $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ above all these 2×2 blocks C_2 except the first.
  - the eigenvectors and generalized eigenvectors are complex; their real and imaginary parts form a basis of the associated subspace of $\mathbb{R}^n$. (**What a pain to compute!**)

Computing Canonical Form in general (CONTINUED)

- We are now left with eigenvalues λ such that $1 < \text{geom}(\lambda) < \text{alg}(\lambda)$, which can occur only if $\text{alg}(\lambda) \geq 3$.

  We will content ourselves in this course with the possibilities described above and will not worry about how to construct a complete set of generalized eigenvectors in the most general case.

Computing Canonical Form in general (CONTINUED)

- In the case of strictly real eigenvalues, the transformation matrix T that changes from original to canonical coordinates is defined by taking its columns to be a basis of eigenvectors and generalized eigenvectors in the order computed above.
- For eigenvalues with non-zero imaginary parts, T contains columns given by the real and imaginary parts of the (complex) eigenvectors and generalized eigenvectors.

Note:

- The Jordan Canonical Form is a block diagonal matrix that is unique *up to permutation of the blocks*.
- The transformation matrix T is not unique; for example, for any $c \neq 0$ we have $(cT)^{-1}A(cT) = J$.

Practice, practice, practice, ...

Do the problems from chapter 5 in Assignment 3 from 2011 and 2012.

Study the solutions **after** *trying the problems!*