McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 20
Yet more on Matrix Exponentials
Wednesday 23 October 2013

Announcements

Test #1

Date: Wednesday 30 October 2013

Time: 11:30am to 1:20pm

Location: T29 / 101

- Further info may be posted on the course wiki closer to the test date.

Announcements

- **Assignment 3** due THIS Friday 25 Oct 2013 @ 1:30pm.
- TUTORIAL this Friday, 25 Oct 2013.
- **BASIC NOTIONS SEMINAR:**
  - Presented by: Youzhou Zhou
  - Title: **"Coin Flipping"**
  - When: Thursday, 24 October 2013, 5:30–6:30pm
  - Where: HH/312

Abstract

Coin flipping is the simplest model in probability theory. Simple as it may be, it can generate almost all the important theorems in probability theory. The model is quite straightforward: one can flip a coin independently and repeatedly. For each flipping, a head can show up with probability p; therefore, a tail shows up with probability $1 - p$. If we look at different aspects of this model, we can find different distributions. Almost all the distributions in probability theory can be obtained from this model.

In this talk, based on this model, I will talk about the law of large numbers, the Monte-Carlo method and large deviation principle. Also the Black-Scholes formula will be proved from the binomial tree model. Lastly, by considering the excursions of random walk, the Poisson-Dirichlet distribution may be mentioned. Interestingly, the number factorization can also be described by the Poisson-Dirichlet distribution, which now has found its many applications in finance, spin glass, machine learning and nonparametric Bayesian statistics.

Important Properties of Matrix Exponentials

Proposition

If λ is an eigenvalue of A with eigenvector V then e^{λ} is an eigenvalue of e^{A} with eigenvector V.

Proof.

$$AV = \lambda V \implies A^2 V = A(AV) = A(\lambda V) = \lambda AV = \lambda^2 V.$$

$$\therefore A^k V = \lambda^k V.$$

$$\therefore e^A V = \sum_{k=0}^{\infty} \frac{1}{k!} A^k V = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} V = e^{\lambda} V. \qquad \square$$

Important Properties of Matrix Exponentials

Proposition (Derivative of matrix exponential function)

$$\frac{d}{dt}\left[e^{tA}\right] = Ae^{tA} = e^{tA}A\,.$$

Important Properties of Matrix Exponentials

Proof of derivative of matrix exponential function.

$$
\begin{aligned}
\frac{d}{dt}\left[e^{tA}\right] &= \lim_{\Delta t\to 0}\frac{e^{(t+\Delta t)A}-e^{tA}}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{(t+\Delta t)A}-e^{tA}\right)\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{tA}e^{(\Delta t)A}-e^{tA}\right)\\
&= e^{tA}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{(\Delta t)A}-I\right)\\
&= e^{tA}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(I+(\Delta t)A+\frac{1}{2}(\Delta t)^2A^2+\cdots-I\right)\\
&= e^{tA}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left((\Delta t)A+\frac{1}{2}(\Delta t)^2A^2+\cdots\right)\\
&= e^{tA}\lim_{\Delta t\to 0}\left(A+\frac{1}{2}(\Delta t)A^2+\cdots\right) = e^{tA}A\,.
\end{aligned}
$$

□

Fundamental Theorem for Linear Systems

Theorem

If A is an $n \times n$ real matrix then the linear initial value problem,

$$X' = AX\,, \qquad X(0) = X_0 \in \mathbb{R}^n\,, \tag{$*$}$$

has a unique solution given by

$$X(t) = e^{tA} X_0\,.$$

Proof of Existence.

$$\frac{d}{dt}[X(t)] = \frac{d}{dt}\left[e^{tA}X_0\right] = Ae^{tA}X_0 = AX(t)\,.$$

Also, $X(0) = e^0 X_0 = X_0$. $\square$

Fundamental Theorem for Linear Systems

Proof of Uniqueness.

Suppose $Y(t)$ is also a solution of (*), *i.e.*,

$$Y' = AY\,, \qquad Y(0) = X_0\,. \tag{$**$}$$

Recall in 1D we let $z(t) = y(t)/x(t)$. How can we generalize that?

Let $Z(t) = e^{-At}Y(t)$.

$$\begin{aligned}
Z'(t) &= \left(\frac{d}{dt}\left[e^{-At}\right]\right) Y(t) + e^{-At}\left(\frac{d}{dt}\left[Y(t)\right]\right) \\
&= \big(- Ae^{-At}\big) Y(t) + e^{-At}\big(AY(t)\big) \\
&= \left[(-Ae^{-At}) + (e^{-At}A)\right] Y(t) \\
&= \left[(-A + A)e^{-At}\right] Y(t) \\
&= 0 \quad \Longrightarrow \quad Z(t) = \text{constant.}
\end{aligned}$$

Fundamental Theorem for Linear Systems

Proof of Uniqueness (CONTINUED).

So what is the constant value of $Z(t)$?

$$Z(0) = e^0 Y(0) = IX_0 = X_0 .$$

$$\therefore X_0 = Z(t) = e^{-At} Y(t) .$$

Now noting that $(e^{-At})^{-1} = e^{-(-At)} = e^{At}$, we have

$$Y(t) = e^{At} e^{-At} Y(t) = e^{At} X_0 = X(t) .$$

□