# Game values and (sur)real numbers

Jonathan Dushoff, McMaster University
http://lalashan.mcmaster.ca/DushoffLab

McMaster Math 3A

December 2017

# GOALS

- Describe:

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  - Combinatoric games

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  - Where the real numbers fit in

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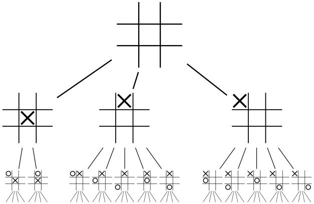

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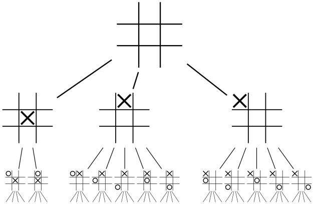

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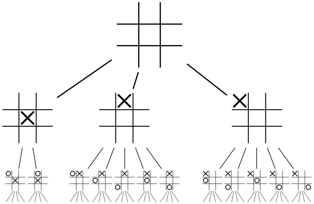

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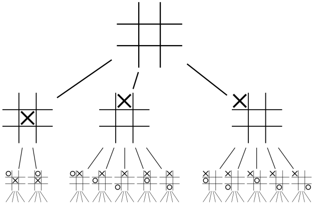

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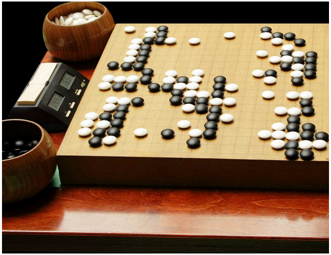

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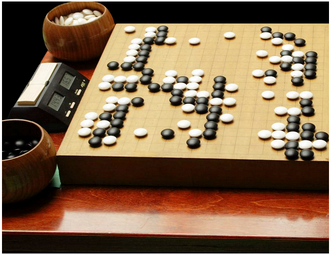

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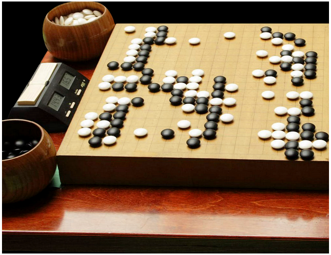

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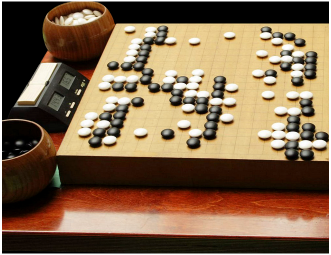

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