Game values and (sur)real numbers
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GOALS

- Describe:
  - Combinatoric games
  - Surreal numbers
  - Where the real numbers fit in
- Stay on this side of sanity

Game theory

- Classic game theory is the theory of games with *imperfect information*
- Why would that be?

Determinism

- Games with perfect information are *boring*
  - Mathematically, not practically
- Analyze the game tree; figure out who wins

Combinatorial game theory

- Except that deterministic games are not boring at all
- Conway decided to think about what it might mean to add two deterministic games together
- The result was the best thing

Resources

- * *On Numbers and Games*, Conway
- * *Surreal Numbers*, Knuth
- * *Winning Ways*, Berlekamp, Conway, Guy

Review

- We define the real numbers by:
  - Building the integers as nested sets
  - Building the rationals as equivalence classes of ordered pairs of integers
  - Building the reals as cuts of the rationals
- A lot of work, also, we're left with three definitions of the number 3 (and 2 of the number 3/2)

Axiom 1: what is a game?

- A game is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L | x^R)$
  - Options are *previously defined* games
- A game *state* is a game together with a specification of whose turn it is
- Motivation: Clearly define a wide range of deterministic games
  - in a way that's going to make it easy to add and subtract them
- Bonus: Highly inductive

Um, what?

- I have just defined a bewilderingly wonderful agglomeration of objects
  - We will need to "chop" it three times to get to the real numbers
- But is it clear that I've defined any objects at all?

What are some games?

- A set of options for the Left player, and a set of options for the Right player
- $(\emptyset|\emptyset) = (|)$
  - 0
- $(0|)$
  - 1
- $(|0)$
  - -1
- $(0|0)$
  - $*$

How to play a game?

- If it's your turn, you choose an option
- It's then the other player's turn in that game
- If you have no options than you lose

Hackenbush

- Uses a drawing with blue, red and green lines, and a "ground"
- On your turn, you remove a line
  - Lines no longer connected to ground are removed
- bLue lines can be removed by Left
- Red lines can be removed by Right
- greeN lines can be removed by aNyone

What outcomes can a game have?

- $\mathcal{O}(0) = S$ – second player wins
- $\mathcal{O}(*) = F$ – first player wins
- $\mathcal{O}(1) = L$ – Left player wins
- $\mathcal{O}(-1) = R$ – Right player wins

Axiom 2: Adding games

- To play in the game $A + B$, you move *either* in A or in B
  - $A + B = (A + B^L, A^L + B | A + B^R, A^R + B)$
- This is perfectly well defined, and beautifully inductive
  - All games are defined in terms of previously defined games
- Motivation: related to thinking about certain kinds of specific games
  - Also, turns out to be super-cool

Examples

- What happens if we add games with various outcomes?
  - $S + S = S$
  - $F + F = ?$
  - $L + L = L$
  - $L + R = ?$
  - $L + F = ?$

Some games are better

- We say $A \leq B$ if B is at least as good for the Left player as A
- Motivation:
  - * classify games by their potential additive effects
  - * put a partial ordering on the games

Definition

- The **negative** of a game reverses the roles of Left and Right
- This has a nice, recursive definition
  - $A = (A^L | A^R)$
  - $-A \equiv (-A^R | -A^L)$
- We then evaluate $A : B$ by looking at the outcome of $A - B \equiv A + (-B)$

At least as good

- A is at least as good as B (for Left) if $A - B$ has no good moves (for Right)
  - This means $\mathcal{O}(A - B) =$
    - L, or S

Mirror world

- It is sometimes useful to construct $A - B$ by imagining a mirror, and putting B on the opposite side of the mirror (Left and Right are reversed there)

Axiom 3: Partial ordering

- We say position $A - B$ is good for Left, *unless*
  - Right has a good move
- We say $A \geq B$ *unless*
  - Some $A^R \leq B$, or
  - Some $B^L \geq A$

Partial ordering

- $\mathcal{O}(A - B)$?
  - L $\implies A > B$
  - R $\implies A < B$
  - S $\implies A = B$
  - F $\implies A \sim B$

Theorem

- If $A = B$, then:
  - $\forall X, \mathcal{O}(X + A) = \mathcal{O}(X + B)$
  - $\mathcal{O}(X + A) = \mathcal{O}((X + A) + (B - A)) = \mathcal{O}((X + B) + (A - A)) = \mathcal{O}(X + B)$

Values

- We can thus define a game value as an *equivalence class* of games
  - A set of games that are linked by an equivalence relation
  - The rational numbers were defined last week in a similar way:
    - 1/2 is the equivalence class of ordered pairs (1, 2); (2, 4); ...

Numbers

- The values I've defined are a very cool group.
- But not very numerical:
  - $* + * = 0$
- Games have "numerical" value if you can count free moves, which works when moving is always bad.

Axiom 1N: what is a (surreal) number?

- Recall: a game is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L | x^R)$
  - Options are *previously defined* games
- A number is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L | x^R)$, s.t. no $x^L \geq x^R$
  - Options are *previously defined* numbers

Examples

- 1 + 1 = 2
- (0—1)
- (0—2)
- (0—3)

Simplicity theorem

- The value of $(x^L|x^R)$ is the simplest, non-prohibited value
- Prohibited: if if is larger than some x^R or less than some x^L
- Simplest: earliest created; it has no options that are not prohibited

Integers

- We create the integers as $n + 1 = (n|)$

Binary fractions

- We create the dyadic rationals as
  - $2k + 1/2^{n+1} = (k/2^n|(k + 1)/2^n)$

The limit

- What happens if we take the limit of all numbers we can make in a finite number of steps?
- We can get all the reals . . .
- plus some very weird stuff
  - $\omega = (0, 1, 2, \dots |)$

- $1/\omega = (0|1, 1/2, 1/4, \dots)$

0.999. . .

- Is 0.999. . . really equal to 1?
- Depends on your definitions
- What is 0.1111. . . (base 2) as a game?

Ordinals

- You can take as many limits as you want, and get all of the infinite ordinals, and a wide range of infinitesimals

Axiom 1R: what is a (real) number?

- Recall: a number is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L | x^R)$, s.t.:
    - no $x^L \geq x^R$
  - Options are *previously defined* numbers
- A real number is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L | x^R)$, s.t.:
    - no $x^L \geq x^R$
    - x^L has a largest element iff x^R has a smallest element
  - Options are *previously defined* real numbers

Axiom 4

- You can define multiplication
  - Motivation: $(x - x^S)(y - y^S)$ has a known sign

Theorem

- You can construct division and show that the surreal numbers are a field
  - Insane recursion that only a genius could come up with, seriously
  - Recursion simultaneously on simpler quotients, and on the quotient itself

Surreal arithmetic

- $\omega - 1$,
- $\omega/2$, $\sqrt{(}\omega)$
- Even crazier stuff: $\sqrt[3]{\omega - 1} - \pi/\omega$

Micro-infinitesimals

- If we allow values that aren't numbers, we have infinitesimals that are smaller than the smallest infinitesimal numbers

Nimbers

- We can define neutral games by identifying options for Left and Right
- This is the theory of Nim values

Hot games

- Hot games are games where there can be a positive value to moving
- Example: domineering

Conclusion

- We can define a bewildering array of games with a simple, recursive definition
- By defining addition, we can chop these into values, which form a group under sensible game addition
- By recursively requiring making a move to have a cost, we can chop these further into numbers, which contain the reals, the infinite ordinals and a consistent set of infitesimals
  - These surreal numbers form a field